Online Appendix to
"Search Equilibrium with Unobservable Investment"

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Abstract

The online appendix to the paper is organized as follows. Section A formally identifies the search behavior that is socially optimal. Section B discusses the properties of the equilibrium wage and skill distributions when investment is unobservable. Section C illustrates the equilibrium with a numerical example. Section D discusses the decision to participate in the labor force when there is a direct cost of search.

A Efficient Search Policy

This section demonstrates that the present value of payoff flows generated by each worker in the labor market is uniquely maximized by an action path in which each worker accepts every job offer. Let $\{l_k\}_{k=1}^{\infty}$ be an increasing sequence of nonnegative jump times of the Poisson process representing the arrival of jobs. Let $\{d_k\}_{k=1}^{\infty}$ be an increasing sequence of nonnegative jump times of the Poisson process representing the destruction of jobs. Note that a Poisson process will almost surely have infinitely many jump times in the interval $[0, \infty)$. Let S denote the set of all infinite increasing sequences of nonnegative real numbers.

The action space in the model is $\{a_Y, a_N\}$, where a_Y and a_N respectively stand for accepting and rejecting a job. At each time t for which there exists a positive integer k such that $t = l_k$, an action denoted by e_k is selected.

The employment status of a worker at each time is defined as follows. For any $t \in \mathbb{R}_+$, let $\tilde{q}(\{d_k\}_{k=1}^{\infty}, t)$ denote the supremum of the set consisting of every τ for which there exists a positive integer k such that $\tau = d_k < t$, where the supremum is $-\infty$ if the set is empty. For any $t \in \mathbb{R}_+$, let $\hat{q}(\{l_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t)$ denote the supremum of the set consisting of every τ for which there exists a positive integer k such that $\tau = l_k < t$ and $e_k = a_Y$, where the supremum is $-\infty$ if the set is empty. For any $t \in \mathbb{R}_+$, define the function $r(\{l_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t)$ as follows:

$$r(\{l_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t) = \begin{cases} 1, & \text{if } \tilde{q}(\{d_k\}_{k=1}^{\infty}, t) < \hat{q}(\{l_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t) \\ 0, & \text{if } \tilde{q}(\{d_k\}_{k=1}^{\infty}, t) \geq \hat{q}(\{l_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t) \end{cases}. \tag{A.1}$$

The function $r(\{l_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, t)$ is an indicator for being employed at time t.

An action path $\{e_k\}_{k=1}^{\infty}$ is feasible if it satisfies the following two conditions. First, $e_j = a_Y$ for every positive integer j such that $r(\{l_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, l_j) = 1$. Second, $e_j \in \{a_Y, a_N\}$ for every positive integer j such that $r(\{l_k\}_{k=1}^{\infty}, \{d_k\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty}, l_j) = 0$. That is, a worker who is already employed at the time of a job offer remains employed, whereas a worker who is unemployed at the time of a job offer faces a choice between employment and unemployment. Let F denote the set of all feasible action paths.

The payoff function can now be specified. For any $(l, d) \in S^2$, $e \in F$ and $(h, t) \in \mathbb{R}_+^2$, define the function $v(l, d, e, h, t)$ as follows:

$$v(l, d, e, h, t) = \begin{cases} b \exp(-\rho t), & \text{if } r(l, d, e, t) = 0 \\ [p - c(h)] \exp(-\rho t), & \text{if } r(l, d, e, t) = 1 \end{cases}. \tag{A.2}$$

Given any $(l, d) \in S^2$ and $e \in F$, the present value of the payoff flows generated by a worker with skill level $h \geq 0$ in the labor market is equal to:

$$\psi(l, d, e, h) = \int_0^\infty v(l, d, e, h, t)dt. \tag{A.3}$$

The result below formalizes the claim at the beginning of this section. It shows that a feasible action path $\{e_k\}_{k=1}^\infty$ that maximizes $\psi(\{l_k\}_{k=1}^\infty, \{d_k\}_{k=1}^\infty, \{e_k\}_{k=1}^\infty, h)$ exists, is unique, and satisfies $e_k = a_Y$ for every positive integer k.

Lemma A.1 *For any $(l, d) \in S^2$ and $h \geq 0$,* $\operatorname{argmax}_{\{e_k\}_{k=1}^\infty \in F} \psi(l, d, \{e_k\}_{k=1}^\infty, h) = \{a_Y\}_{k=1}^\infty$.

Proof Let $\{\tilde{e}_k\}_{k=1}^\infty$ be any feasible action path such that there exists a positive integer j for which $\tilde{e}_j = a_N$. Let J denote the number of distinct values of j such that $\tilde{e}_j = a_N$. For each positive integer $k \leq J$, let $m(k)$ be the k^{th} lowest value of j such that $\tilde{e}_j = a_N$, and let τ_k be the least time $t > l_{m(k)}$ such that there exists a positive integer j for which $d_j = t$ or $l_j = t$. Then the present value of payoff flows generated by a worker with skill level $h \geq 0$ satisfies:

$$\begin{aligned} \psi(l, d, \{a_Y\}_{k=1}^\infty, h) - \psi(l, d, \{\tilde{e}_k\}_{k=1}^\infty, h) &= \sum_{k=1}^{J} \int_{l_{m(k)}}^{\tau_k} [p - c(h) - b] \exp(-\rho t) dt \\ &= \sum_{k=1}^{J} \rho^{-1} [p - c(h) - b][\exp(-\rho l_{m(k)}) - \exp(-\rho \tau_k)] > 0 \end{aligned} . \tag{A.4}$$

Hence, $\psi(l, d, \{a_Y\}_{k=1}^\infty, h) > \psi(l, d, \{\tilde{e}_k\}_{k=1}^\infty, h)$ for any feasible action path $\{\tilde{e}_k\}_{k=1}^\infty \neq \{a_Y\}_{k=1}^\infty$. ∎

Note that $\hat{U}(h) = \mathbb{E}[\psi(l, d, \{a_Y\}_{k=1}^\infty, h)]$ for any $h \geq 0$, where $\hat{U}(h)$ represents the expected present value of payoff flows generated by an unemployed worker with skill level h, and the expectation is taken over the Poisson jump times l and d.

B Properties of Equilibrium

This section examines some technical properties of the equilibrium under unobservable investment, which is presented in theorem 1. We start with the human capital distribution. The distribution function K is continuous except at the supremum of its support. Given that K has an atom at h^u, even a firm offering what is essentially the lowest wage has positive employment since $w^l = R(h^u)$, which results in positive profits as $w^l < w^u < p$. Because of the discontinuity in K at h^u, the equilibrium skill distribution does not admit a density. Nonetheless, given the properties of c in section 2.1, a partial probability density function k satisfying $K(h) = \int_0^h k(z)dz$ for all $h \in [0, h^u)$ can be defined by letting $k(z) = K^{'}(z)$ for all $z \in (0, h^u)$. The following corollary of theorem 1 basically states that the density is decreasing on the part of the support of K where the distribution is continuous.

Corollary B.1 *In equilibrium, $dK(h')/dh > dK(h'')/dh$ for all $h', h'' \in [0, h^u)$ with $h' < h''$.*

Proof From the first fundamental theorem of calculus, the derivative of K for $h \in [0, h^u)$ is given by:

$$K^{'}(h) = \frac{[p - R(h^u)]S^{'}(h)}{[p - R(h^u)]S(h^u) + [p - R(h^l)](\delta + \lambda)}, \tag{B.1}$$

where the derivative of S is equal to $S^{'}(h) = [p - R(h^l)][\rho^2 - \delta c^{'}(h)]/[p - R(h)]^2$. Because $R(h)$ with $R(0) < p$ is decreasing in h and $c^{'}(h)$ with $\lim_{h\uparrow\infty} c^{'}(h) = 0$ is increasing in h, $S^{'}(h)$ and hence $K^{'}(h)$ are decreasing for all $h \in [0, h^u)$. ∎

We now turn to the equilibrium wage offer distribution, which has a connected support like the equilibrium skill distribution. The distribution function F is continuous except possibly at the supremum of its support. If $c^{'+}(0) = -\infty$, then F is continuous everywhere. However, F has a discontinuity at w^u when $c^{'+}(0) > -\infty$. Intuitively, if $c^{'+}(0) = -\infty$, then an infinitesimal investment by a worker substantially reduces the disutility of labor, causing a significant increase in the number of firms whose wage offer a worker is willing to accept. When $c^{'+}(0) > -\infty$, an atom at w^u is needed to produce the same effect.

Given only the properties of c in section 2.1, the equilibrium wage distribution may not admit a density function, regardless of whether F has an atom at the supremum of its support.[1] In order to ensure the existence of a density, a further restriction on c is needed, so that F is absolutely continuous. The wage offer density, if it exists, may not be monotonic.[2]

[1]This situation arises, for instance, if c is given by:

$$c(h) = \begin{cases} 1 - \sqrt{h} - h + \int_0^h \theta(z)\,dz, & \text{for } h \in [0,1] \\ 1/2 - \sqrt{h}, & \text{for } h > 1 \end{cases}, \tag{B.2}$$

where θ is the Cantor function, which is everywhere continuous and almost everywhere differentiable, but not absolutely continuous.

[2]For example, suppose that c is given by:

$$c(h) = \begin{cases} (41621/41472 - \sqrt{2}/8) - (1 + 43\sqrt{2}/3456)\sqrt{h}, & \text{for } h \in [0, 1/72) \\ (731/729 - \sqrt{2}/6) - (11/108 + 3\sqrt{2})h + h^2 - h^3, & \text{for } h \in [1/72, 1/18) \\ 1 - 2\sqrt{h}, & \text{for } h \in [1/18, \infty) \end{cases}. \tag{B.3}$$

The corollary below provides a condition that helps to determine the convexity or concavity of F in the case where the third derivative of c exists. It relates the sign of the second derivative of F to c and its first three derivatives.

Corollary B.2 *The equilibrium wage offer distribution satisfies the following for any $w \in (w^l, w^u)$ such that $c^{'''}[R^{-1}(w)]$ exists:*

$$\operatorname{sgn}[F^{''}(w)] = \operatorname{sgn}\big(\{\rho + c^{'}[R^{-1}(w)]\}c^{'''}[R^{-1}(w)] - 3\{c^{''}[R^{-1}(w)]\}^2\big), \tag{B.4}$$

where R is as specified in the statement of theorem 1.

Proof Twice differentiating the expression for F in the statement of theorem 1 yields the following for $w \in (w^l, w^u)$:

$$F^{''}(w) = \frac{\rho(\delta+\rho)\big(3\{c^{''}[R^{-1}(w)]\}^2 - \{\rho + c^{'}[R^{-1}(w)]\}c^{'''}[R^{-1}(w)]\big)}{\lambda\{\rho + c^{'}[R^{-1}(w)]\}^5}, \tag{B.5}$$

where R is as specified in the statement of theorem 1. The sign of $F^{''}(w)$ is the opposite of the sign of the expression in large parentheses in the numerator of the right-hand side of the preceding equation. ∎

C Numerical Example

This section presents a numerical example in which an explicit solution to the model can be obtained when investment is unobservable. Assume that the labor market parameters take on the values $p = 2$, $b = 0$, and $\delta = \lambda = \rho = 1$. Let the disutility of labor be $c(h) = 1 - 2\sqrt{h}$ for all $h \in \mathbb{R}_+$. Note that this functional form satisfies the assumptions on c in section 2.1. We use theorem 1 to find R, K, and F in equilibrium.

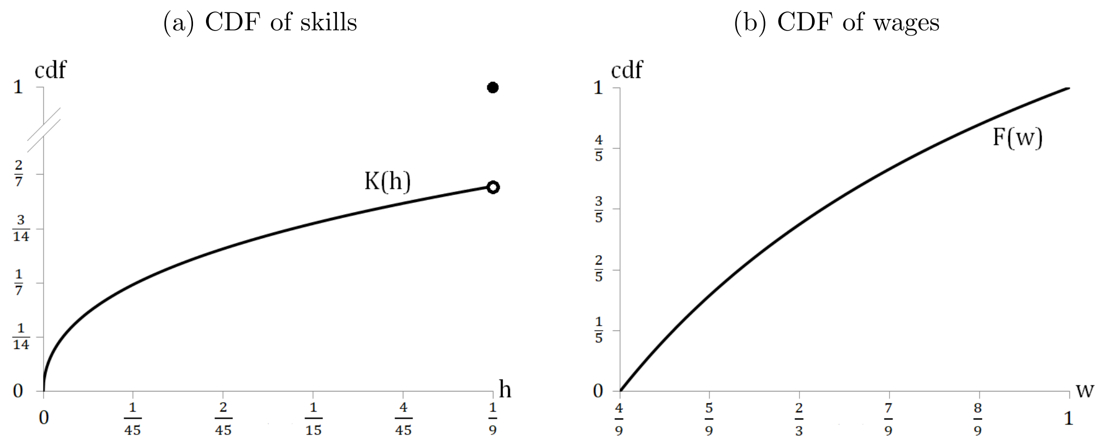
The reservation wage is given by:

$$R(h) = \begin{cases} 1 - 2\sqrt{h} + h, & \text{for } h \in [0, \frac{1}{9}] \\ \frac{8}{9} - \frac{4}{3}\sqrt{h}, & \text{for } h > \frac{1}{9} \end{cases}. \tag{C.1}$$

Note that $R(h)$ is decreasing in the skill level h.

Letting $\delta = \lambda = \rho = 1$ and $b = 0$, the model has a nondegenerate equilibrium in which $F^{'}$ is decreasing in a neighborhood of $1/144 + c(1/144)$ and increasing in a neighborhood of $1/36 + c(1/36)$.

Figure C.1: Equilibrium Wage Offer and Human Capital Distributions with Unobservable Investment



Disutility Function: $c(h) = 1 - 2\sqrt{h}$. Market Parameters: $p = 2$, $b = 0$, $\delta = 1$, $\lambda = 1$, $\rho = 1$.

The human capital distribution is supported on the interval $[0, \frac{1}{9}]$ and is as follows:

$$K(h) = \begin{cases} \dfrac{7\left\{\frac{2\sqrt{h}}{1+2\sqrt{h}-h} + \sqrt{2}\left[\operatorname{arcoth}(\sqrt{2}) + \operatorname{artanh}\left(\frac{\sqrt{h}-1}{\sqrt{2}}\right)\right]\right\}}{21 + 7\operatorname{artanh}\left(\frac{4\sqrt{2}}{9}\right)/\sqrt{2}}, & \text{for } h \in [0, \frac{1}{9}] \\ 1, & \text{for } h \geq \frac{1}{9} \end{cases}, \tag{C.2}$$

where $\operatorname{artanh}(z)$, which is equal to $\frac{1}{2}\ln[(1+z)/(1-z)]$ for $z \in (-1, 1)$, and $\operatorname{arcoth}(z)$, which is equal to $\frac{1}{2}\ln[(z+1)/(z-1)]$ for $z < -1$ and $z > 1$, are the inverse hyperbolic tangent and cotangent, respectively. The distribution function K is depicted in the left panel of figure C.1. As is true in general, K has a connected support and is continuous except at the supremum of its support, where it has an atom. In accordance with corollary B.1, the slope of K is decreasing on the interior of its support.

The wage offer distribution, which has the support $[\frac{4}{9}, 1]$, is given by:

$$F(w) = \begin{cases} 0, & \text{for } w \leq \frac{4}{9} \\ 3 - 2/\sqrt{w}, & \text{for } w \in (\frac{4}{9}, 1) \\ 1, & \text{for } w \geq 1 \end{cases}. \tag{C.3}$$

The right panel of figure C.1 plots the distribution function F, which has a connected support, as is generally the case. Noting that $c'^{+}(0) = -\infty$ given the specified functional form, F is continuous everywhere, including the supremum of its support. In addition, F is concave on the interior of its support, as indicated by the condition in corollary B.2.

D Participation Decision

This section considers the decisions of individuals about whether to participate in the labor market. An individual essentially chooses between incurring some expense to look for a job and withdrawing from the labor force to avoid this search cost. In order to sustain a nondegenerate equilibrium, the gains from search must at least equal the cost of search, which depends on the selected level of human capital investment, as can the payoff from labor force withdrawal. We show that the analysis of the basic model with unobservable investment can be extended to accommodate this situation under certain regularity conditions.

The setup is the same as in section 2, except that an individual faces an explicit search cost while unemployed and actively decides whether or not to enter the labor market. Let $s(h)$ with $s(0) = 0$ denote the out-of-pocket search cost of a worker with skill level h. The flow utility while unemployed becomes $b - s(h)$. Define a transformed skill level $g(h) = h + s(h)/\rho$. Assume that the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and surjective. Denote the difference between the disutility of labor and the cost of search by $\tilde{c}(x) = c[g^{-1}(x)] - s[g^{-1}(x)]$. Assume that the function $\tilde{c}$ has the same properties as the function c in section 2.1.[3]

Let $\tilde{F}$ denote the distribution of wage offers across firms. Let $\tilde{U}(x)$ represent the expected present value of lifetime utility flows for an unemployed worker whose transformed skill level is x. Let $\tilde{V}(w, x)$ be the expected present value of lifetime utility flows for a worker with transformed skill level x who is employed at wage w. The Bellman equation for an employed worker with transformed skills x receiving wage w is:

$$\rho\tilde{V}(w, x) = \{w - c[g^{-1}(x)]\} + \delta[\tilde{U}(x) - \tilde{V}(w, x)], \tag{D.1}$$

and the Bellman equation for an unemployed worker with transformed skills x is:

$$\rho\tilde{U}(x) = \{b - s[g^{-1}(x)]\} + \lambda \int_{-\infty}^{\infty} \max[\tilde{V}(w, x) - \tilde{U}(x), 0]\, d\tilde{F}(w). \tag{D.2}$$

As in the original model, the optimal strategy of a job seeker can be expressed in terms of

[3]For example, if $s(h) = \sqrt{h}$ for all $h \in \mathbb{R}_+$, then g is increasing and surjective, and if $c(h) = 1 - \sqrt{h}$ as well, then $\tilde{c}$ satisfies the restrictions on c in section 2.1.

a reservation wage, which is denoted by $\tilde{R}(x)$ in this case and is such that $\tilde{V}(w,x) < \tilde{U}(x)$ for $w < \tilde{R}(x)$, $\tilde{V}(w,x) > \tilde{U}(x)$ for $w > \tilde{R}(x)$, and $\tilde{V}[\tilde{R}(x),x] = \tilde{U}(x)$. Equations (D.1) and (D.2) yield the following analogue of equation (6):

$$\tilde{R}(x) - \tilde{c}(x) = b + \frac{\lambda}{\delta+\rho}\int_{\tilde{R}(x)}^{\infty} w - \tilde{R}(x)\, d\tilde{F}(w). \tag{D.3}$$

The expected payoff of a worker making a transformed investment x in human capital is given by:

$$\begin{aligned} \tilde{Y}(x) &= \tilde{U}(x) - g^{-1}(x) = \tilde{R}(x)/\rho - c[g^{-1}(x)]/\rho - g^{-1}(x) \\ &= \tilde{R}(x)/\rho - \tilde{c}(x)/\rho - s[g^{-1}(x)]/\rho - g^{-1}(x) = \tilde{R}(x)/\rho - \tilde{c}(x)/\rho - x \end{aligned}, \tag{D.4}$$

where the second equality follows from applying the Bellman equation for an employed worker in the case where $w = \tilde{R}(x)$, and the third and fourth equalities follow respectively from the definitions of $\tilde{c}$ and g.

Let $\tilde{Q}(x)$ denote the expected present value of lifetime utility flows for an individual with transformed skills x who remains out of the labor force. Assume that this function satisfies $\tilde{Q}(x) \leq b/\rho + g^{-1}(x)$ for all $x \in \mathbb{R}_+$, with equality if $x = 0$. The first term on the right-hand side of the preceding inequality represents a payoff derived from leisure or government assistance. It is equal to the present value of benefits received by a worker in the labor market that remains unemployed in perpetuity. The second term on the right-hand side allows for skill acquisition to improve productivity or enjoyment when out of the labor force, as human capital may enhance the value of leisure or the output from home production. The expected payoff to an individual that invests the transformed amount x in skills but stays out of the labor force is given by $\tilde{Z}(x) = \tilde{Q}(x) - g^{-1}(x)$.

Denote by $\tilde{e} \in [0,n]$ the measure of individuals that participate in the labor market. Let $\tilde{K}_I$ and $\tilde{K}_O$ be the respective distributions of the transformed level of human capital among workers in and out of the labor force. The measure of individuals who are unemployed and have transformed skills no greater than $x \geq 0$ can be written as:

$$\tilde{m}(x) = \tilde{e}\left(\int_0^x \tilde{u}(z)\, d\tilde{K}_I(z) + \tilde{u}(0)\tilde{K}_I(0)\right), \tag{D.5}$$

where the unemployment rate $\tilde{u}(x)$ in steady state among workers with transformed skill level x is given by:

$$\tilde{u}(x) = \delta/\big(\delta + \lambda\{1 - \tilde{F}[\tilde{R}(x)^{-}]\}\big). \tag{D.6}$$

The total measure of unemployed workers is represented by $\tilde{m}(\infty) = \lim_{x\uparrow\infty} \tilde{m}(x)$. Defining $\tilde{J}(x) = \tilde{m}(x)/\tilde{m}(\infty)$, a firm offering wage w in steady state has the employment:

$$\tilde{\ell}(w) = \begin{cases} \lambda\tilde{m}(\infty)\{1 - \tilde{J}[\tilde{R}^{-1}(w)^{-}]\}/\delta, & \text{for } \ w < \tilde{R}(0) \\ \lambda\tilde{m}(\infty)/\delta, & \text{for } \ w \geq \tilde{R}(0) \end{cases}, \tag{D.7}$$

and the profit flow:

$$\tilde{\pi}(w) = (p - w)\tilde{\ell}(w), \tag{D.8}$$

which are counterparts of equations (10) and (11).

The definition of equilibrium is formally extended below to incorporate a decision about labor force participation. First, the reservation wage satisfies equation (D.3), meaning that unemployed workers search optimally. Second, workers decide between entering the labor market to search for a job and remaining out of the labor force by comparing the expected payoffs from these two options. Third, workers invest in human capital so as to maximize their expected payoff given their participation decision. Fourth, firms offer wages that maximize their profits.

Definition D.1 *A wage posting and skill investment equilibrium with participation costs is a quintuple* $(\tilde{R}, \tilde{e}, \tilde{K}_O, \tilde{K}_I, \tilde{F})$ *such that the following hold:*

a) $\tilde{R}(x)$ *satisfies equation (D.3) for all* $x \in \mathbb{R}_+$.

b) $\tilde{e} = n$ *if* $\max_{x\in\mathbb{R}_+} \tilde{Y}(x) > \max_{x\in\mathbb{R}_+} \tilde{Z}(x)$, $\tilde{e} = 0$ *if* $\max_{x\in\mathbb{R}_+} \tilde{Y}(x) < \max_{x\in\mathbb{R}_+} \tilde{Z}(x)$, *and* $\tilde{e} \in [0, n]$ *if* $\max_{x\in\mathbb{R}_+} \tilde{Y}(x) = \max_{x\in\mathbb{R}_+} \tilde{Z}(x)$.

c) $x' \in \operatorname{argmax}_{x\in\mathbb{R}_+} \tilde{Y}(x)$ *for all* x' *on the support of* $\tilde{K}_I$, *and* $x' \in \operatorname{argmax}_{x\in\mathbb{R}_+} \tilde{Z}(x)$ *for all* x' *on the support of* $\tilde{K}_O$.

d) $w' \in \operatorname{argmax}_{w\in\mathbb{R}} \tilde{\pi}(w)$ *for all* w' *on the support of* $\tilde{F}$.

The result below demonstrates the existence of an equilibrium when participation decisions are taken into account. It relates the solution to the model in this section to the unique equilibrium of the basic model that is analyzed in section 3.3.

Theorem D.1 *For any* $\tilde{e} \in [0, n]$, *there exists* $\tilde{R}$, $\tilde{K}_I$, $\tilde{K}_O$, *and* $\tilde{F}$ *such that the quintuple* $(\tilde{R}, \tilde{e}, \tilde{K}_O, \tilde{K}_I, \tilde{F})$ *is a wage posting and skill investment equilibrium with participation costs. Moreover, in any equilibrium with* $\tilde{e} \in (0, n]$, $\tilde{R}$, $\tilde{K}_I$, *and* $\tilde{F}$ *are respectively the same as* R, K, *and* F *specified in theorem 1 except that* c *is replaced by* $\tilde{c}$.

Proof We first demonstrate the existence of an equilibrium. Let $\tilde{R}$, $\tilde{K}_I$, and $\tilde{F}$ respectively be the same as R, K, and F given in theorem 1 but with c replaced by $\tilde{c}$.

The reservation wage equation (D.3) is the same as the corresponding equation (6) in the original model except that R, F, and c are replaced by $\tilde{R}$, $\tilde{F}$, and $\tilde{c}$, respectively. Since R in theorem 1 solves equation (6), $\tilde{R}$ satisfies equation (D.3).

The expected payoff function $\tilde{Y}$ is the same as the corresponding function Y in the original model except that R and c are replaced by $\tilde{R}$ and $\tilde{c}$, respectively. Since K in theorem 1 is such that $h' \in \operatorname{argmax}_{h\in\mathbb{R}_+} Y(h)$ for all h' on the support of K, we have $x' \in \operatorname{argmax}_{x\in\mathbb{R}_+} \tilde{Y}(x)$ for all x' on the support of $\tilde{K}_I$.

If $\tilde{e} = 0$, then $\tilde{\pi}(w) = 0$ for all $w \in \mathbb{R}$, and so $w' \in \operatorname{argmax}_{w\in\mathbb{R}} \tilde{\pi}(w)$ for all w' on the support of $\tilde{F}$. If $\tilde{e} \in (0, n]$, then the profit function $\tilde{\pi}$ is the same as the corresponding function π in the original model except that R, K, F, and n are replaced by $\tilde{R}$, $\tilde{K}_I$, $\tilde{F}$, and $\tilde{e}$, respectively. Since $w' \in \operatorname{argmax}_{w\in\mathbb{R}} \pi(w)$ for all w' on the support of F, we have $w' \in \operatorname{argmax}_{w\in\mathbb{R}} \tilde{\pi}(w)$ for all w' on the support of $\tilde{F}$, noting that R, K, and F in theorem 1 do not depend on n.

Let $\tilde{K}_O$ be the distribution that puts mass one at 0. Because $\tilde{Q}(x) \leq b/\rho + g^{-1}(x)$ for all $x \in \mathbb{R}_+$, we have $\tilde{Z}(x) \leq b/\rho = \tilde{Z}(0)$ for all $x \in \mathbb{R}_+$, so that $x' \in \operatorname{argmax}_{x\in\mathbb{R}_+} \tilde{Z}(x)$ for all x' on the support of $\tilde{K}_O$.

Since both $\max_{x\in\mathbb{R}_+} \tilde{Y}(x) = \tilde{Y}(0) = b/\rho$ and $\max_{x\in\mathbb{R}_+} \tilde{Z}(x) = \tilde{Z}(0) = b/\rho$, we have $\max_{x\in\mathbb{R}_+} Y(x) = \max_{x\in\mathbb{R}_+} Z(x)$.

Thus, for any $\tilde{e} \in [0, n]$, the quintuple $(\tilde{R}, \tilde{e}, \tilde{K}_O, \tilde{K}_I, \tilde{F})$ meets all the requirements for a wage posting and skill investment equilibrium with participation costs.

We next consider the uniqueness of the equilibrium. As noted above, the reservation wage equation, expected payoff function of workers, and profit function for $\tilde{e} \in (0, n]$ are the same as in the original model except that R, K, F, and n are replaced by $\tilde{R}$, $\tilde{K}_I$, $\tilde{F}$, and $\tilde{e}$, respectively. From theorem 1, there is a unique triple (R, K, F) meeting the three conditions for a wage posting and skill investment equilibrium in definition 1. Hence, for $\tilde{e} \in (0, n]$, there is a unique triple $(\tilde{R}, \tilde{K}_I, \tilde{F})$ satisfying the corresponding requirements of definition D.1 that $\tilde{R}$ satisfy equation (D.3), $x' \in \operatorname{argmax}_{x\in\mathbb{R}_+} \tilde{Y}(x)$ for all x' on the support of $\tilde{K}_I$, and $w' \in \operatorname{argmax}_{w\in\mathbb{R}} \tilde{\pi}(w)$ for all w' on the support of $\tilde{F}$. Thus, $\tilde{R}$, $\tilde{K}_I$, and $\tilde{F}$ are as specified above in any wage posting and skill investment equilibrium with participation costs. ∎

Once the model with participation decisions by workers has been appropriately transformed, the reservation wage equation, expected payoff function of workers, and profit function are essentially identical to those in the basic model. Thus, the original solution can be applied to the current setting, so as to show the existence of an equilibrium and, when labor force participation is positive, the uniqueness of the behavior of firms and workers.

Regarding $\tilde{K}_O$, there is always an equilibrium in which it puts mass one on zero. If and only if the inequality $\tilde{Q}(x) \leq b/\rho + g^{-1}(x)$ holds strictly for all $x > 0$, this is the unique equilibrium skill distribution among nonworkers. If $\tilde{e}$ is zero, then the profit function is simply zero. In this case, there are multiple values of $\tilde{R}$, $\tilde{K}_I$, and $\tilde{F}$ that can be supported in equilibrium. One possibility is that $\tilde{R}$, $\tilde{K}_I$, and $\tilde{F}$ are respectively given by the equilibrium values of R, K, and F in theorem 1 except that $\tilde{c}$ is substituted for c.[4]

The solution to the untransformed model can easily be derived from the preceding results. The reservation wage $R(h)$ of a worker with skill level $h \in \mathbb{R}_+$ is given by $\tilde{R}[g(h)]$. Letting K_I and K_O denote the respective distributions of human capital among labor force participants and nonparticipants, we have $K_I(h) = \tilde{K}_I[g(h)]$ and $K_O(h) = \tilde{K}_O[g(h)]$ for all $h \in \mathbb{R}_+$.

The analysis in this section demonstrates that solution to the model is robust to the inclusion of out-of-pocket search costs. Every worker may incur a direct cost of search that is positive, except those who invest nothing in skills. However, such workers have measure zero because the human capital distribution does not have an atom at the infimum of its support. Thus, even if all workers effectively face positive search costs, a nondegenerate wage distribution may still be supported in equilibrium. Nonetheless, given the properties of s and g, it must be that for any $\epsilon > 0$, a positive measure of workers have a direct cost of search less than ϵ. This requirement follows intuitively from the basic model in which the gains from search tend to zero as investment becomes arbitrarily small.

[4]Another possibility is that $\tilde{R}$ is given by:

$$\tilde{R}(x) = \begin{cases} b + \tilde{c}(x), & \text{if } x \leq x^e \\ b + [\lambda\tilde{c}(x^e) + (\delta+\rho)\tilde{c}(x)]/(\delta+\lambda+\rho), & \text{if } x > x^e \end{cases}, \tag{D.9}$$

and $\tilde{K}_I$ and $\tilde{F}$ respectively consist of a single atom at 0 and $b + \tilde{c}(x^e)$, where $x^e = \tilde{c}'^{-1}[-\rho(\delta+\lambda+\rho)/\lambda]$.