Commentary on “Facility Location in the Presence of Congested Regions with the Rectilinear Distance Metric”

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Abstract
This paper is a commentary on the work of Butt and Cavalier [2], a paper that was published in an earlier issue of this journal. With the aid of an example problem, we demonstrate that the set of gridlines proposed by them to find the rectilinear least cost path between two points in the presence of convex polygon congested regions is inadequate. We proceed to prove its adequacy for the case of rectangular congested regions in which the edges of the rectangles are parallel to the travel directions. In light of the difficulties of the general problem, we consider a specific example of a convex quadrilateral congested region and a pair of external origin and destination points. Finally, we revisit the example shown in Butt and Cavalier’s paper and present a Mixed Integer Linear Programming formulation that determines the optimal locations of the entry and exit points for this example.

1 Introduction
Location problems which impose restrictions on locating new facilities and/or travel through are typically referred to as constrained or restricted location problems. Such problems have the following two topographical properties. (1) The new facilities cannot be located within certain predescribed restricted areas in the plane. (2) It is not always necessary that any two points in the plane would be “simply communicating”, i.e., the minimum travel distance between any two points in the plane may be made longer by the presence of the restricted regions.

Restricted location problems have been studied by Larson and Sadiq [4] and Batta, Ghose and Palekar [1]. Larson and Sadiq examine the rectilinear $p$-median problem
with arbitrarily shaped barriers (bounded areas in $\mathbb{R}^2$ which neither allow location nor travel through). Batta et al. examine the $p$-median problem in the presence of arbitrarily shaped barriers and convex forbidden regions (bounded areas in $\mathbb{R}^2$ which do not allow location but allow travel through at no extra cost) under the rectilinear distance metric. Butt and Cavalier [2] consider such a restricted location problem in which the restriction comes in the form of a congested region. Congested regions are defined by them as closed and bounded areas in $\mathbb{R}^2$ in which facility location is prohibited but traveling through is allowed at a possible additional cost. The authors introduce the concept of least cost paths and conclude that a rectilinear least cost path between two points in the presence of congested regions may not necessarily be the path of shortest length. They formulate the problem of calculating the cost of a least cost path as a linear program. Based on the results obtained, the authors propose an extension of the grid construction procedure for the corresponding barrier problem considered by Larson and Sadiq. They claim that at least one least cost path will always coincide with segments of the grid obtained by drawing horizontal and vertical lines through the existing facilities and the vertices of the congested region. Based on such a grid construction procedure, the authors transform the constrained form of the planar $p$-median problem to an unconstrained $p$-median problem on a network where an optimal set of new facility locations is chosen from a finite set of candidate points.

The remainder of this paper is organized as follows. In Section 2, we critique the work of Butt and Cavalier and demonstrate that their proposed grid is incorrect. In Section 3 we consider rectangular congested regions whose edges are parallel to the travel axes and prove the optimality of the Butt and Cavalier grid structure for this special case. In Section 4 we revisit the example presented in the Butt and Cavalier paper and present a Mixed Integer Linear Programming (MILP) formulation that determines the least cost path for this example. Finally in Section 5, we present our conclusions and directions for further research.

2 Critique of Butt and Cavalier’s Paper

2.1 Some Definitions and Assumptions

Butt and Cavalier define a congested region as a closed and bounded area in $\mathbb{R}^2$ in which a new facility cannot be located but traveling through is allowed at an additional cost per unit distance. This additional cost per unit distance is called
the congestion factor of the congested region and is denoted by $\alpha$, $0 \leq \alpha < \infty$. Thus if $w$ is the cost of travel per unit distance between two points lying outside a congested region, then the cost of travel between the same points when lying inside the congested region would be $(1 + \alpha)w$.

The authors assume the following in their work:

- A congested region is the interior of a convex polygon that is defined by a finite number of vertices. This implies that there is no congestion along the boundary of the congested region. Thus traveling along the boundary of a congested region would not result in an increase in the cost per unit distance.

- The congested regions are non-intersecting and share no common boundaries.

- No existing facility is located inside a congested region.

Note that barriers and forbidden regions can be considered to be special cases of congested regions. Barriers can be considered to be congested regions with $\alpha = \infty$ (because they do not allow through travel) whereas forbidden regions can be considered as congested regions with $\alpha = 0$ (because they allow through travel at no additional cost).

2.2 Least Cost Paths in the Presence of Congested Regions

Larson and Sadiq have proposed a grid structure to solve the rectilinear $p$-median problem in the presence of arbitrarily shaped barriers to travel. The grid consists of tangential $X$ and $Y$ lines drawn through the existing facilities and the vertices of the barriers (barrier vertices are points of tangency lying on the barrier boundary thorough which horizontal or vertical line segments can be passed). The resulting set of lines, called node traversal lines are terminated when they intersect barriers. With congested regions, one may want to travel through (or slip out as in the case of barriers) depending on the location of the origin and destination points with respect to the congested region and also the congestion factor. Hence Butt and Cavalier extend the node traversal lines of Larson and Sadiq to pass through congested regions. However it is not necessary that the rectilinear least cost path between two points in the presence of a congested region is the shortest rectilinear path between the two points. This is evident from the four scenarios depicted in Fig.(1).

Fig.(1) considers a congested region $ABCD$ with congestion factor $\alpha$, an origin $X$ and a destination $P$. We assume without loss of generality that $w = 1$. In Fig.(1)a, $X$
and $P$ are “simply communicating”. Hence the least cost path between $X$ and $P$ will never enter $ABCD$. Infinitely many such paths can be conceived. In Fig.(1)b, $X$ and $P$ are not visible in the rectilinear sense. A possible path between them (as shown by the continuous line) has a cost $d(X, K) + (1 + \alpha)d(K, M) + d(M, P)$. However the rectilinear least cost path between $X$ and $P$ could be a path $XLMP$ (shown by dotted line) with a cost $d(X, L) + (1 + \alpha)d(L, M) + d(M, P)$ if:

$$d(X, L) + (1 + \alpha)d(L, M) + d(M, P) < d(X, K) + (1 + \alpha)d(K, M) + d(M, P)$$

where $d(A, B)$ denotes the length of the shortest rectilinear path between points $A$ and $B$. Fig.(1)c shows two possible paths between $X$ and $P$. However the costs of the two paths are not necessarily equal even though both are shortest rectilinear distance paths between $X$ and $P$. The cost of the two paths would depend on the distance traveled outside and inside the congested region and $\alpha$. Finally Fig.(1)d emphasizes that a least cost path between $X$ and $P$ can enter and exit a congested region more than once, thereby incurring savings in cost. However as $\alpha \to \infty$, the least cost path will be gradually forced out of the congested region. We call the threshold value of $\alpha$ for which a least cost path bypasses a congested region the “break-point” of $\alpha$. 

Figure 1: Least cost paths in the presence of a congested regions: Different scenarios, adapted from Butt and Cavalier
Butt and Cavalier define any point where a path enters and leaves a congested region as an entry point and an exit point respectively. They proceed to formulate the problem of calculating the cost of a least cost path in the presence of congested regions as a linear programming problem. The linear program determines the optimal location of a single entry and a single exit point of a least cost path. Based on the solution of the linear program, Butt and Cavalier conclude that at least one optimal least cost path between two points will coincide with segments of the horizontal and vertical lines drawn through the two points and the vertices of the congested region.

To solve the $p$-median problem in the presence of congested regions, the authors devise a grid construction procedure in which they pass horizontal and vertical lines through each congested region vertex and each existing facility location. The resulting grid divides the feasible region into cells. The main results claimed by Butt and Cavalier are:

1. The optimal 1-median in a given cell must coincide with a cell corner.

2. Based on the grid construction procedure developed, there is at least one optimal solution to the rectilinear $p$-median problem in the presence of congested regions where each new facility location coincides with a cell corner of the grid.

2.3 A Contradictory Example

Fig.(2)a shows a four-sided congested region $ABCD$ with vertices $A(1,11)$, $B(13,8)$, $C(11,2)$ and $D(2,5)$ and two existing facility locations $X(4,3)$ and $P(9,10)$. According to the formulation presented in Butt and Cavalier, the rectilinear least cost path from $X$ to $P$ should enter $ABCD$ at $E_1(4,4.33)$ and exit $ABCD$ at $E_2(5,10)$. The least cost path, shown by a bold line coincides with the grid obtained by passing horizontal and vertical lines through $X$, $P$, $A$, $B$, $C$ and $D$. The cost incurred by traveling on this path for a congestion factor $\alpha = 0.3$ is $14.0$ units.

However in Fig.(2)b, using the entry point $E'_1(5,4)$ and the exit point $E_2(5,10)$, the cost is $13.8$ units for the same congestion factor. We conclude that the grid constructed by Butt and Cavalier to solve the infinitesimal facility location problem in the presence of congested regions is inadequate and their contention regarding construction of a grid is incorrect. It does not suffice to draw horizontal and vertical lines thorough the existing facilities and the vertices of a congested region and extending them to pass through the congested region. In fact some other gridlines, as shown in Fig.(2)b are necessary for completion of the grid. The precise set of gridlines that
need to be drawn is not immediately obvious and is suggested as a future research direction.

3 Analysis for Rectangular Congested Regions

When a congested region is a convex polygon, the locations of the entry and exit points determine the distance traversed inside (and also outside) the region. Entering the congested region at some point (viz. point $E_1'$ in Fig. (2)b) rather than another (viz. point $E_1$ in Fig. (2)b) may result in a reduction of the total cost. However this issue will not arise at all if the distance traversed inside a congested region is unaffected by the location of the entry and exit points of a rectilinear least cost path. This is possible if the congested regions are squares or rectangles with their edges being parallel to the travel axes. This observation serves as our motivation to study the problem of developing a precise grid construction procedure for rectangular congested regions.

We assume that a congested region is a closed region in $\mathbb{R}^2$ with a finite area and a continuous closed boundary. Let $C$ (an open set) denote the set of points $(x, y) \in \mathbb{R}^2$ contained strictly within the congested region. We also define $\bar{C} = C \cup \{\text{boundary of congested region}\}$, a closed set. Thus $\bar{C}$ is a congested region (viz.
region $ABCD$ in Fig.(3)). Furthermore, we assume here that all congested regions are disjoint and have rectangular shapes, with their sides being parallel to the travel axes.

We first prove (Lemma 3.1) that the Butt and Cavalier grid structure works for the case of a single rectangular congested region. We then demonstrate (Theorem 3.1) that this result also holds for multiple rectangular congested regions.

To facilitate our analysis, we define two points to be *simply communicating* if the presence of the congested regions causes no net increase in the minimum travel distance between two points. If it does cause an increase, the points are not simply communicating. We also assume, without loss of generality, that $w = 1$.

![Figure 3: A Rectangular Congested Region $\bar{C}$, used for Proof of Lemma 3.1](image)

**Lemma 3.1** The grid structure proposed in the Butt and Cavalier paper works for the case of a single rectangular congested region, when the edges of the rectangle are parallel to the travel axes.

**Proof:** We consider the following two cases with points $X$ and $P$ in the presence of a rectangular congested region $ABCD$ as shown in Fig.(3).

**Case 1:** $X$ and $P$ are simply communicating

In this case a least cost path cannot penetrate a congested region (otherwise its cost would increase). Thus the congested region can be thought of as a barrier to travel and the grid structure of Larson and Sadiq would apply to this problem. However, the grid structure of Larson and Sadiq is a subset of the grid structure of Butt
and Cavalier (because their grid lines terminate when they intersect a barrier). Thus the Butt and Cavalier grid structure would suffice as well.

**Case 2 : X and P are not simply communicating**

Consider the congested region $\tilde{C}$ shown in Fig.(3). We divide the region $\mathbb{R}^2 - \tilde{C}$ into regions $E, W, N, S \in \mathbb{R}^2$ as shown in Fig.(3) and note the following for a point $(x, y)$:

- $(x, y) \in E$ if $x > x_c$ and $y_c < y < y_b$
- $(x, y) \in W$ if $x < x_d$ and $y_d < y < y_a$
- $(x, y) \in N$ if $x_a < x < x_b$ and $y > y_a$
- $(x, y) \in S$ if $x_d < x < x_c$ and $y < y_d$

We consider the sample case where $P \in E$ and $X \in W$. Other situations for $X$ and $P$ can be analyzed in a similar manner.

For traveling from $X$ to $P$ through $ABCD$, the total distance traveled along the path $XE_1ZE_2P$ is $(p + d(1 + \alpha) + q + a)$, as is evident from Fig.(3). For traveling from $X$ to $P$ bypassing $ABCD$ but along the edge $DC$, the total distance traveled is $(a + a_2 + p + d + q + a_2)$. Thus we would travel through $ABCD$ as long as $\alpha < \frac{2a_2}{d}$. The congested region $ABCD$ could also be bypassed by traveling along the edge $AB$. In that case, it can be similarly shown that $\alpha < \frac{2a_2}{d}$. We conclude that $\alpha < \frac{2\min(a_1, a_2)}{d}$ implies that we travel through $ABCD$, and that $\alpha \geq \frac{2\min(a_1, a_2)}{d}$ implies that we bypass $ABCD$. We now consider the following two subcases.

**Subcase 2a :** $\alpha \geq \frac{2\min(a_1, a_2)}{d}$

In this situation, we can treat the congested region as a barrier and following the reasoning in Case 1, we can conclude that the Butt and Cavalier grid structure suffices.

**Subcase 2b :** $\alpha < \frac{2\min(a_1, a_2)}{d}$

Since $\alpha > 0$, we would need to minimize the length of the path that passes through the congested region. This is achieved by traveling along the path $XE_1ZE_2P$. The Butt and Cavalier grid structure would work as it contains this path.

The lemma follows. $\square$

**Theorem 3.1** The grid structure proposed in the Butt and Cavalier paper works for the case of multiple rectangular congested regions, when the edges of the rectangles are parallel to the travel axes.
**Proof**: As in the proof of Lemma 3.1, consider points $X$ and $P$ as shown in Fig. (3), along with one congested region $CR_1$. Applying Lemma 3.1 to this case would result in lines 1 through 8 as shown in Fig. (3).

Now consider adding a second congested region $CR_2$. If the least cost path from $X$ to $P$ enters $CR_1$, then without loss of generality, we can assume that its entry point is $d$ and its exit point is $f$. If the least cost path from $f$ to $P$ enters $CR_2$, without loss of generality its entry point will be $b$. But this point has already been defined due to the earlier application of Lemma 3.1.

On the other hand, if the least cost path from $X$ to $P$ bypasses $CR_1$, then its exit point without loss of generality is either $e$ or $g$. Again if the least cost path from $e$ (or $g$) to $P$ enters $CR_2$, its entry point without loss of generality will be $a$ (or $c$). But these points have also been defined earlier.

We conclude that the only additional lines are those necessary to bypass $CR_2$. These are the lines 9, 10, 11 and 12.

By similar reasoning, for each additional congested region that is present, the only new lines that need to be introduced are those created by its edges.

The theorem follows. □
4 The Butt and Cavalier Example: Revisited

Consider the example in the Butt and Cavalier paper, as shown in Fig.4. For this example we can easily verify the following facts, some of which are presented as lemmas. Note that these lemmas may not hold for all possible shapes of the congested region, including other quadrilaterals.

1. The least cost path from \( X \) (origin) to \( P \) (destination) either bypasses the congested region \( ABCD \) or enters it.

2. **Lemma 4.1** If the least cost path bypasses \( ABCD \), the path is either \( XDAP \) or \( XCBP \).

   **Proof:** If the least cost path bypasses \( ABCD \), the congested region can be thought of as a barrier to travel. In that case, the least cost path between \( X \) and \( P \) would be the shortest path between \( X \) and \( P \) that bypasses \( ABCD \). The lemma follows from Theorem 2 of Larson and Li. [3] □

3. **Lemma 4.2** If the least cost path enters \( ABCD \), the first entry point is either on edge \( AD \), \( DC \) or \( BC \).

   **Proof:** Assume the contrary, i.e. the entry point to \( ABCD \) lies on edge \( AB \). For the first entry point to be on \( AB \), we must already have passed through either \( A \) or \( B \) (because if we came from within the congested region it would not be the first entry point). The lemma follows from the fact that \( A \) and \( B \) simply communicate with \( P \). □

4. **Lemma 4.3** Given exactly one entry point of the least cost path, the exit point must either be any point on edge \( AB \) excluding \( A \) and \( B \), or the path from this exit point to \( P \) must pass through either \( A \) or \( B \).

   **Proof:** If the path enters \( ABCD \) exactly once, it must exit it exactly once. If this exit point lies either on edges \( AD \), \( DC \) or \( CB \), then the least cost path from it to \( P \) must go through either \( A \) or \( B \). The other case is the situation where this exit point is any point on \( AB \) excluding \( A \) and \( B \). The lemma follows. □

5. **Lemma 4.4** Given exactly two entry points of the least cost path: (i) the second exit point must be on \( AB \) (excluding \( A \), \( B \)), or the path from this exit point to \( P \) must pass through \( A \) or \( B \); (ii) the first exit point must lie on \( AD \) (excluding \( A \), \( D \)) or on \( BC \) (excluding \( B \), \( C \)).
Proof: Assertion (i) follows from the arguments in Lemma 4.3. Assertion (ii) follows from the observation that if the first exit point is on $AB$ (excluding $A, B$), then this point simply communicates with $P$. Consequently there would not be a second entry point. Also, the first exit point cannot be on edge $DC$, since we could directly go on a simply communicating path from $X$ to this point and reduce the cost. The lemma follows. □

6. Lemma 4.5 The least cost path from $X$ to $P$ will not enter the congested region $ABCD$ more than two times.

Proof: From Lemma 4.2, we know that the first entry point is either on edge $AD, DC$ or $BC$. If we enter on $DC$, then the first exit is either on edge $AD$, $BC$ or $AB$. If we exit on $AB$, there are no more entries into $ABCD$, from arguments in Lemmas 4.3 and 4.4. If we exit on $AD$, then we may reenter on $AD$ at a point with a higher $y$-coordinate. In this case, our second exit must be on $AB$. This follows from the fact that edge $AD$ is a straight line. A similar reasoning applies for edge $BC$. The lemma follows. □

For a single entry/single exit case, $E_2$ and $E_3$ will be coincident either $E_1$ or $E_4$. Hence we allow $E_2$ and $E_3$ to lie on $CD$. Based on these lemmas, we present a Mixed Integer Linear Programming (MILP) formulation. Here $E_1$ represents the first entry
point, $E_2$ the first exit point, $E_3$ the second entry point and $E_4$ the second exit point. If the least cost path bypasses $ABCD$, then $E_1$ is coincident with $C$ or $D$, $E_4$ is coincident with $A$ or $B$ and $E_2$ and $E_3$ are both coincident with $E_1$ or $E_4$. We note here that if either $E_1$ or $E_2$ or $E_3$ or $E_4$ are coincident with any vertex of $ABCD$, we no longer consider them as entry/exit points. The formulation outputs the optimal locations of $E_1$, $E_2$, $E_3$ and $E_4$. Binary variables are needed to capture the edges on which $E_1$, $E_2$ and $E_3$ could lie. For simplicity in presentation, we label the edges $CD$, $AD$, $AB$ and $BC$ as 1, 2, 3 and 4 respectively. The formulation is as follows. We note that the values of $a_i$, $b_i$, $c_i$, $x_i^l$, $x_i^r$, $x_n$, $y_n$, $x_p$ and $y_p$ are obtained from Fig.(4).

\[
\begin{align*}
\text{minimize} & \quad |x_n - x_1| + |y_n - y_1| + (1 + \alpha)(|x_1 - x_2| + |y_1 - y_2|) + |x_2 - x_3| + |y_2 - y_3| + (1 + \alpha)(|x_3 - x_4| + |y_3 - y_4|) + |x_4 - x_p| + |y_4 - y_p| + \theta \\
\text{subject to} & \\
\begin{align*}
& a_i x_1 + b_i y_1 + c_i + (1 - z_i)M \geq 0 \quad i = 1, 2, 4 \quad (1) \\
& a_i x_1 + b_i y_1 + c_i + (z_i - 1)M \leq 0 \quad i = 1, 2, 4 \quad (2) \\
& z_1 + z_2 + z_4 = 1 \quad (3) \\
& a_i x_2 + b_i y_2 + c_i + (1 - u_i)M \geq 0 \quad i = 1, 2, 4 \quad (4) \\
& a_i x_2 + b_i y_2 + c_i + (u_i - 1)M \leq 0 \quad i = 1, 2, 4 \quad (5) \\
& u_1 + u_2 + u_4 = 1 \quad (6) \\
& a_i x_3 + b_i y_3 + c_i + (1 - w_i)M \geq 0 \quad i = 1, 2, 4 \quad (7) \\
& a_i x_3 + b_i y_3 + c_i + (w_i - 1)M \leq 0 \quad i = 1, 2, 4 \quad (8) \\
& w_1 + w_2 + w_4 = 1 \quad (9) \\
& x_i^l \leq x_1 + (1 - z_i)M \quad i = 1, 2, 4 \quad (10) \\
& x_1 + (z_i - 1)M \leq x_i^r \quad i = 1, 2, 4 \quad (11) \\
& x_i^l \leq x_2 + (1 - u_i)M \quad i = 1, 2, 4 \quad (12) \\
& x_2 + (u_i - 1)M \leq x_i^r \quad i = 1, 2, 4 \quad (13) \\
& x_i^l \leq x_3 + (1 - w_i)M \quad i = 1, 2, 4 \quad (14) \\
& x_3 + (w_i - 1)M \leq x_i^r \quad i = 1, 2, 4 \quad (15) \\
& w_i \geq z_i + u_i - 1 \quad i = 1, 2, 4 \quad (16) \\
& a_2 x_4 + b_2 y_4 + c_2 = 0 \quad (17) \\
& x_a \leq x_4 \leq x_b \quad (18) \\
& \theta \geq |x_n - x_c| + |y_n - y_c| + |x_c - x_2| + |y_c - y_2| - (1 - z_4)M \quad (19)
\end{align*}
\end{align*}
\]
\[ z_i, u_i, w_i \in \{0, 1\} \tag{20} \]

\( z_i, u_i \) and \( w_i \) are the binary variables associated with the entry/exit points \((x_1, y_1), (x_3, y_3)\) and \((x_4, y_4)\) respectively for sides 1, 2 and 4.

\[
z_i = \begin{cases} 
1, & \text{if } E_1 \text{ lies on edge } i \text{ of } ABCD \quad i = 1, 2, 4 \\
0, & \text{otherwise}
\end{cases}
\]

\[
u_i = \begin{cases} 
1, & \text{if } E_2 \text{ lies on edge } i \text{ of } ABCD \quad i = 2, 4 \\
0, & \text{otherwise}
\end{cases}
\]

\[
w_i = \begin{cases} 
1, & \text{if } E_3 \text{ lies on edge } i \text{ of } ABCD \quad i = 2, 4 \\
0, & \text{otherwise}
\end{cases}
\]

\( x_i^j \leq x_r^i \quad \forall i = 1, 2, 4 \) denote the \( x \)-coordinate of the left and right vertices of any side \( i \) of a congested region.

For a single entry/single exit case, \( E_2 \) and \( E_3 \) will be coincident with either \( E_1 \) or \( E_4 \). Hence we allow \( E_2 \) and \( E_3 \) to lie on \( CD \). Constraints 1, 2 and 3 ensure that \( E_1 \) lies on exactly one of the sides 1, 2 or 4. Here \( M \) is a large scalar. Constraints 4, 5 and 6 ensure that \( E_2 \) lies on exactly one of the sides 1, 2 or 4. Similarly constraints 7, 8 and 9 ensure that \( E_3 \) lies on exactly one of the sides 1, 2 or 4. Constraints 10 and 11 provide the bounds on the \( x \)-coordinates of \( E_1 \). Similarly constraints 12, 13 and 14, 15 provide the bounds on the \( x \)-coordinates of \( E_2 \) and \( E_3 \) respectively. Constraint 16 ensures that if the points \( E_1 \) and \( E_2 \) lie on the same edge of \( ABCD \), then \( E_3 \) must also lie on that same edge. Constraints 17 and 18 ensure that the final exit point \( E_4 \) lies on edge \( AB \) of \( ABCD \). Constraint 19 takes care of the extra distance that is traversed if the rectilinear least cost path goes through the vertex \( C \) of \( ABCD \). Constraint 20 represents the binary variables.

We used the solver LINDO 6.1 to obtain solutions for different values of \( \alpha \) (the solution times in all cases were less than a second). Our results show two possible paths as illustrated in Fig.(5)a by bold lines. For \( 0 < \alpha \leq 1.33 \), the path is \( XE_1E_4P \) \([E_1 = (5, 4), E_2 = (5, 4), E_3 = (5, 4), E_4 = (5, 10)]\). For \( \alpha > 1.33 \), path is \( XDAP \) \([E_1 = (2, 5), E_2 = (2, 5), E_3 = (1, 11) \) and \( E_4 = (1, 11)\] with an objective function value of 20. The case for two entry and exit points is not demonstrated in this case.

To demonstrate this case, we consider a new example in which everything is unchanged except that \( P = (12.5, 10) \) as illustrated in Fig.(5)b. We find that for \( 0 < \alpha \leq 0.59 \), the least cost path between \( X \) and \( P \) is \( XE_1E_2E_3E_4P \) \([E_1 = (8, 3), E_2 = (11.33, 3), E_3 = (12.5, 6.5), E_4 = (12.5, 8.125)]\). For \( \alpha = 0.60 \) and \( \alpha = 0.61 \), the path is \( XCE_3E_4P \) \([E_1 = (11, 2), E_2 = (11, 2), E_3 = (12.5, 6.5), E_4 = (12.5, 8.125)]\). For
\( \alpha > 0.61 \), the rectilinear least cost path is \( XCBP \) \([E_1 = (11, 2), E_2 = (11, 2), E_3 = (13, 8), E_4 = (13, 8)]\) with an objective function value of 18.5.

It is pertinent to mention here that the MILP formulation can also act as an useful tool to determine the break-point of \( \alpha \) for any congested region.

## 5 Conclusion and Future Work

We conclude that Butt and Cavalier’s contention that at least one rectilinear least cost path will always coincide with segments of the grid formed by drawing horizontal and vertical lines through each existing facility and the vertices of a convex polygonal congested region is incorrect. A straightforward “barrier” extension of the grid structure proposed by Larson and Sadiq is inadequate. Nevertheless the grid suffices for rectangular congested regions because through the point of intersection of a node traversal line with the edge of a rectangle, no additional \( X \) or \( Y \) node traversal line can be drawn, as the edge of the rectangle is already perpendicular at that point. When the congested regions are convex polyhedra, we conjecture that such a grid would have an extended set of node traversal lines that are perpendicular to a traditional node traversal line at its point of incidence to a congested region with \( \alpha < \infty \). The completeness and optimality of such a grid structure needs to be proven.
The composition of such a grid structure is also not immediately apparent.

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References


