Assignment and Matching

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Introduction

Assignment Problems: Deals with Optimal Pairing or Matching of objects in two distinct sets.

Bipartite Matching Problems: Two versions are as follows -

• Cardinality Problem - Find a matching containing the max. no. of arcs

•Weighted Problem - Find a matching with the largest overall weight

Bipartite Cardinality Matching

- Identify a matching of Maximum Cardinality in a bipartite undirected network.
- Transform into Maximum Flow Problem (have an equivalence relationship)
- Worst case bound for solving is O($\sqrt{n}m$)

Bipartite Weighted Matching

- Also known as the Assignment Problem
- Find a perfect matching of min. weight; adapt algorithms for minimum cost flow
- Successive Shortest Path Algorithm
  - Hungarian Algorithm (primal-dual)
  - Relaxation Algorithm
  - Cost Scaling Algorithm

Stable Marriage Problem

- Consider n men and n women.
- Consists of two n X n matrices; one gives each man's ranking of women; second gives each woman's ranking of men.
- Man-Woman pair is unstable if
  - Not married to each other but prefer each other to their current spouses

Stable Marriage Problem(Contd.)

- Each person have ranks from 1 to n; form a priority list.
- Total time is O(n^2) - Bucket sort algorithm
- LIST: Set of unassigned men
- *Current-woman:* index of each man

Stable Marriage Problem(Contd.)

Algorithm:

- Initialize-
  LIST = N_1 , set of all men
  current-woman = first woman
- Select any man (Bill) from LIST, propose to his *current-woman* (Helen); engaged or go to next *current-woman* ;
- Repeat till LIST = ϕ

Non-Bipartite Matching Problem

Few Definitions

Matched Arcs and Nodes: A matching M of a graph G is a subset of arcs with the property that no two arcs are incident to the same node. Such arcs are called Matched Arcs and such nodes are called Matched Nodes

Alternating Paths: A path P = $i_1 - i_2 - ..... - i_k$, is an alternating path wrt a matching M if every consecutive pair of arcs contains one matched and one unmatched arc.

Two types: Odd and Even

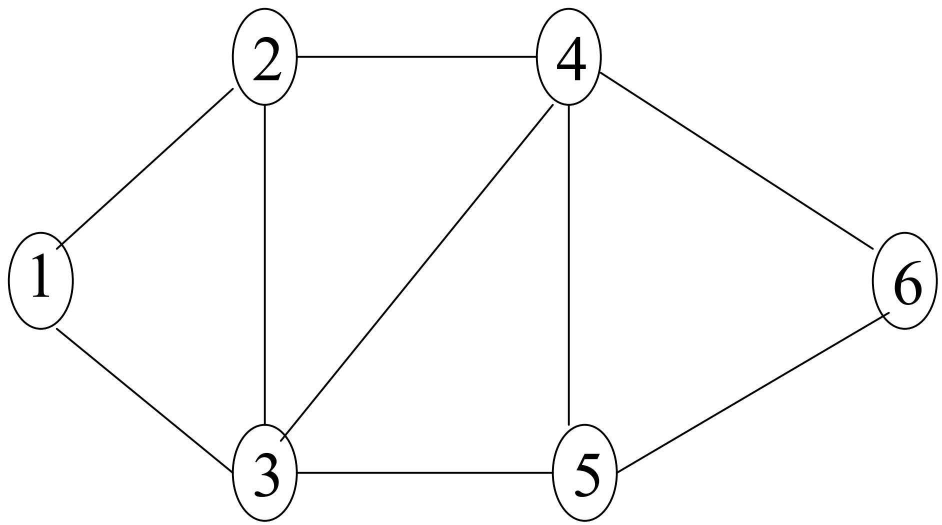
Non-Bipartite Matching Problem

Augmenting Paths: Odd alternating path P w.r.t. a matching M if the first and the last nodes are unmatched.

Symmetric Difference: Let S_1 and S_2 be two sets; the symmetric difference of these sets, denoted by

$$S_1 \oplus S_2 = (S_1 \bigcup S_2) - (S_1 \bigcap S_2)$$

Non-Bipartite Matching Problem



An Example

Bipartite Matching Algorithm

Augmented Path Theorem: If a node p is unmatched in a matching M, and this matching contains no augmenting path that starts at node p, then node p is unmatched in some maximum matching.

Bipartite Matching Algorithm

Start with a feasible matching M(which might be null); identify an augmenting path starting at any unmatched node p. If such path exists, replace M with $M \oplus P$; else delete node p and all the arcs incident to it. Repeat for every such unmatched node p.

Flower and Blossoms

A *flower*, defined wrt a matching M and a root node p, is a subgraph with two components:

• *Stem*: Even Alternating Path with starting node as p and terminating at some w. If p = w; stem is empty.

•*Blossom*: Odd Alternating cycle with starting and terminating at node w, which is the terminal of a stem.

Non-Bipartite Algorithm

- Similar to Bipartite Matching Algorithm; different in its search procedure
- Search procedure assigns an even or odd label to the nodes
- Never relabels the nodes; if yes, then suspend the procedure.
- Runs for O(n^3) time

Nonbipartite Matching Algorithm

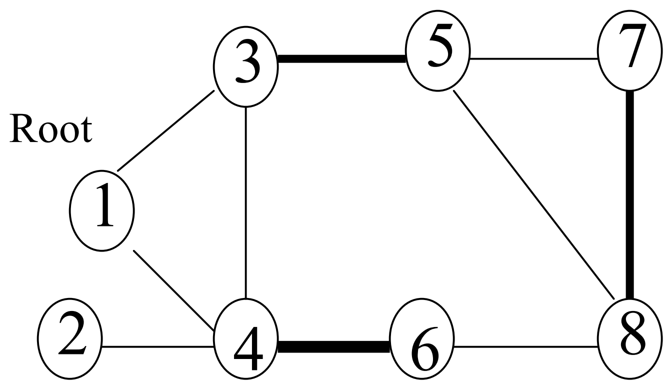


Fig. A

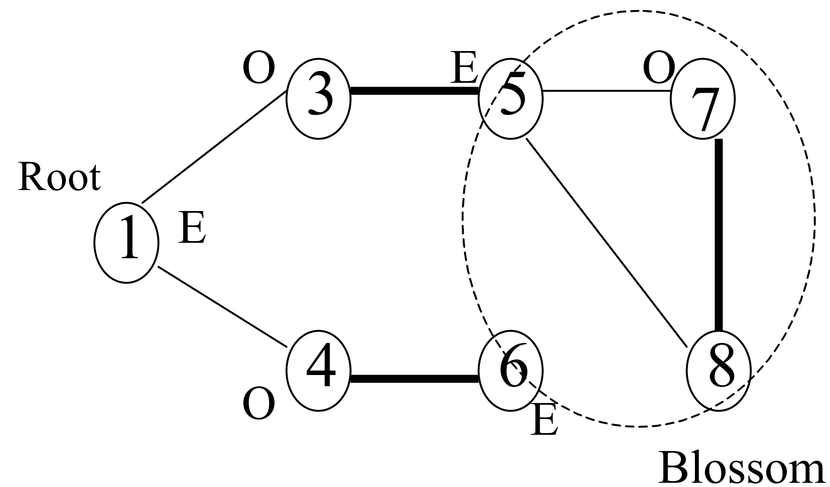


Fig. B

An Example

Nonbipartite Matching Algorithm

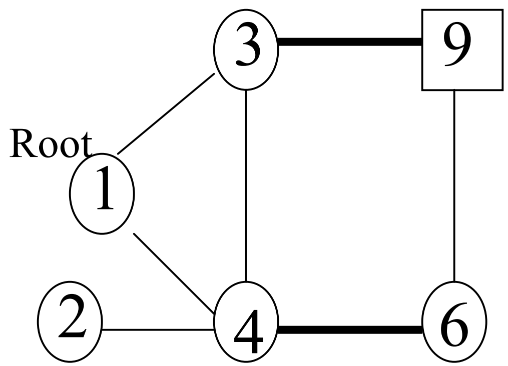


Fig. C

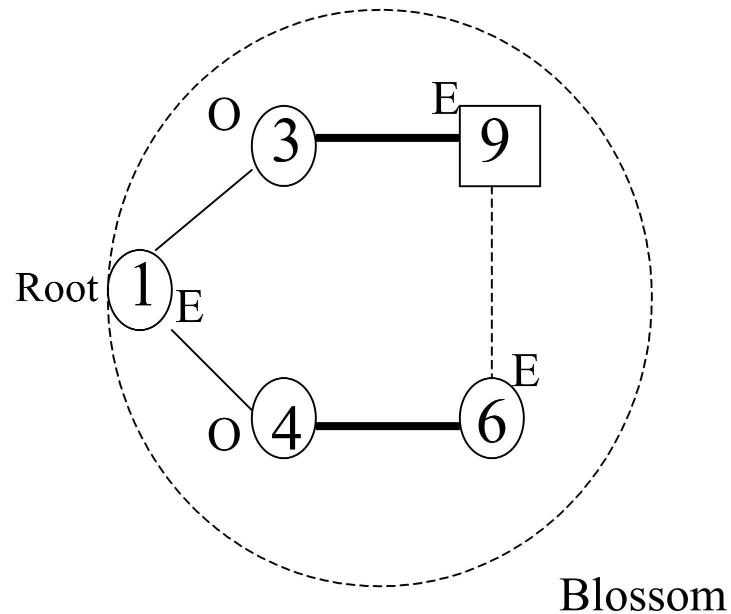


Fig. D

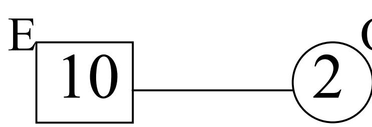


Fig. E

Matchings and Paths

- Shortest Paths in Directed Networks
  - Transform the problem into an assignment problem
- Shortest Paths in Undirected Networks
  - Solving a non-bipartite weighted matching problem

Summary

- Matching are important class of Optimization models
- Lie between Network Flows and Combinatorial Optimization
- Matching Problems related to Shortest Path problems