Branch and Cut algorithms for Combinatorial Optimization Problems

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Introduction to Branch & Cut

- Branch & Cut algorithms modify the basic Branch & Bound strategy by attempting to strengthen the linear programming relaxation(LPR) of an IP with new inequalities before branching a partial solution.
- Basically, Branch & Cut = Branch & Bound + Cutting Planes
- Pure Branch & Bound can be considerably sped up by employing cutting planes either at the top of a Branch & Bound tree or at every node of the tree, because cutting planes considerably reduce the size of the tree.
- Branch & Cut can be used in conjunction with heuristics to obtain a lower bound on the optimal value, using the Branch & Bound algorithm.

A Simple Example

Consider the following IP :

P0: $\min z = -6x_1 - 5x_2$

subject to $3x_1 + x_2 \leq 11$

$-x_1 + 2x_2 \leq 5$

$x_1, x_2 \geq 0$, integer (refer to fig 1 here)

- Ignoring integrality constraints, we solve the LPR.
- Optimal solution is $(2\frac{3}{7}, 3\frac{5}{7})$, $z_{opt} = -33\frac{1}{7}$

Sub-problems generated by branching on x_1

P1: min $z = -6x_1 - 5x_2$

subject to
$3x_1 + x_2 \le 11$
$-x_1 + 2x_2 \le 5$
$\boldsymbol{x_1 \ge 3}$
$x_1, x_2 \ge 0$, integer

<u>Optimal Solution of LPR_{P1}</u>

(3,2), $z_{opt} = -28$

(refer to fig 2 here)

P2: min $z = -6x_1 - 5x_2$

subject to
$3x_1 + x_2 \le 11$
$-x_1 + 2x_2 \le 5$
$\boldsymbol{x_1 \le 2}$
$x_1, x_2 \ge 0$, integer

<u>Optimal Solution of LPR_{P2}</u>

(2,3.5), $z_{opt} = -29.5$

(refer to fig 3 here)

Add a cut to P2

P3: $\min z = -6x_1 - 5x_2$

subject to

$3x_1 + x_2 \leq 11$

$-x_1 + 2x_2 \leq 5$

$x_1 \leq 2$

$2x_1 + x_2 \leq 7$ (added cut)

$x_1, x_2 \geq 0$, integer

Optimal Solution of LPR_{P3}

$(1.8, 3.4)$, $z_{opt} = -27.8$

(refer to fig 4 here)

Questions that arise :

- Is the added inequality a valid inequality?
- How to generate a valid inequality?
- Whether to branch or to cut?

Progress of Branch & Cut on a 2D IP

PROBLEM P0

SOLN TO RELAXATION:$(2\frac{3}{7}, 3\frac{5}{7})$, $z_{opt} = -33$

Branch on x_1

$x_1 \geq 3$

$x_1 \leq 2$

Problem P1

SOLN TO RELAXATION

$(3,2)$, $z_{opt} = -28$

Problem P2

SOLN TO RELAXATION

$(2,3.5)$, $z_{opt} = -29.5$

Add cut $\boldsymbol{2x_1 + x_2 \leq 7}$

Problem P3

SOLN TO RELAXATION

$(1.8,3.4)$, $z=-27.8$

Branch & Cut algorithm

1. *Initialization:* Denote the initial IP problem by ILP^0 & set the active nodes to be $L=\{ILP^0\}$. Set the upper bound to be $\overline{z}=+\infty$. Select one problem $l \in L$ and set its lower-bound on the optimal value, $\underline{z}_l = -\infty$.
2. *Termination:* If $L=\emptyset$, then the solution x^* which yielded the incumbent objective value z is optimal. If no such x^* exists, i.e. $z=+\infty$ the ILP is infeasible.
3. *Problem selection* : Select and delete a problem ILP^l from L.
4. *Relaxation* : Solve the LPR of ILP^l . If the relaxation is infeasible, set $\underline{z}_l = +\infty$ and go to step 6. Let z_l denote the optimal objective value of the relaxation, if it is finite and let x^{lR} be an optimal solution; otherwise, set $\underline{z}_l = -\infty$.
5. *Add cutting planes* : If desired, search for cutting planes, if any are found, add them to the relaxation and return to step 4.
6. *Fathoming & Pruning* :

   (a) If $\underline{z}_l \geq \overline{z}$ go to step 2

   (b) If $\underline{z}_l < \overline{z}$ and x^{lR} is integral feasible, update $\overline{z}=z_l$, delete from L all problems with $\underline{z}_l \geq \overline{z}$, and go to step 2.
7. *Partitioning* **:** Let $\left\{S^{lj}\right\}_{j=1}^{j=k}$ be a partition of the constraint set S^l of problem ILP^l. Add problems $\left\{ILP^{lj}\right\}_{j=1}^{j=k}$ to L, where ILP^{lj} is ILP^l with feasible region restricted to S^{lj} and $\underline{z}_{lj}$ for $j=1,\ldots k$ is set to the value of $\underline{z}_l$ for the parent problem l. Go to step 2.

Generating Cutting Planes

- Take a weighted combination of the inequalities from the current LPR
- Exploit the fact that variables must be integral, process known as integer rounding
- Cutting planes generated in this way are called Chvatal-Gomory cutting planes.
- Example:

$$\tfrac{1}{6}(3x_1 + x_2 \leq 11) + \tfrac{5}{12}\ (-x_1 + 2x_2 \leq 5)$$

gives $$\tfrac{1}{12}\, x_1 + x_2 \leq 3\,\tfrac{11}{12}$$

LHS of the inequality is rounded down, which gives

$$x_2 \leq 3\,\tfrac{11}{12}$$

In any feasible solution to an IP, the LHS must take an integer value, so the RHS is rounded down.

Finally we have the valid inequality : $x_2 \leq 3$

When to Generate Cuts?

- If there is a situation, in which the cutting plane loop in steps 4 & 5 tails off, i.e. the solution to a current LPR is not much better than the solutions to recent previous LPRs, then a node can be fathomed.

A Generalized Assignment Problem with SOS2

- Problem uses Special Ordered Sets of Type 2.
- Each set has 20 variables.
- 120 – 220 constraints, excluding non-negativity.
- Problem solved using:
  - (A) IBM RS6000/590
  - (B) MINTO 3.0 as branch-and-bound algorithm
  - (C) CPLEX 6.0 as LP solver
- Branching tree limited to 50,000 nodes

Comparison of CPU time(secs), using B&B and B&C(Defarias et al.)

			OPTIMALITY		FIRST SOLUTION	
SETS	CONS	VARS	TIME(NO CUTS)	TIME(W/CUTS)	TIME(NO CUTS)	TIME(W/CUTS)
100	120	2000	2833	90	2590	56
110	130	2200	3312	2290	2999	1063
120	140	2400	2397	83	2241	73
130	150	2600	2311	104	2204	65
140	160	2800	2314	227	2278	206
150	170	3000	956	73	457	50
160	180	3200	2998	731	2913	554
170	190	3400	1967	158	1561	108
200	220	4000	1310	43	982	37

Comparison of nodes evaluated with & without cuts

	OPTIMALITY			FIRST SOLUTION		
SETS	NODES(NO CUTS)	NODES(W/CUTS)	CUTS	NODES(NO CUTS)	NODE(W/CUTS)	CUTS
100	38967	881	138	32351	456	135
110	33933	15429	641	31251	7151	510
120	30523	1067	130	26662	898	101
130	27679	1147	151	25094	654	121
140	24191	1735	143	23674	1499	96
150	18180	792	117	8243	421	110
160	29708	4645	201	28020	3239	118
170	21707	1462	138	20532	900	95
200	14260	344	107	10437	276	103

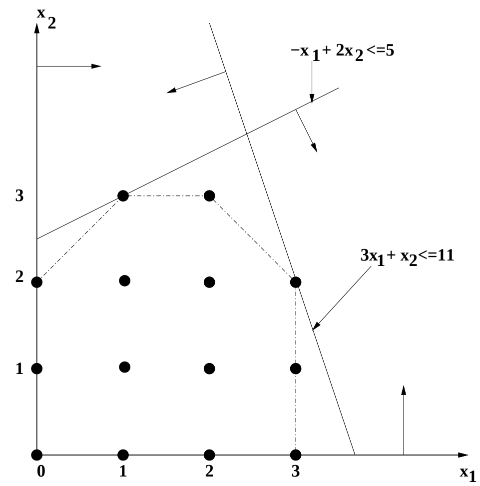

Figure 1:

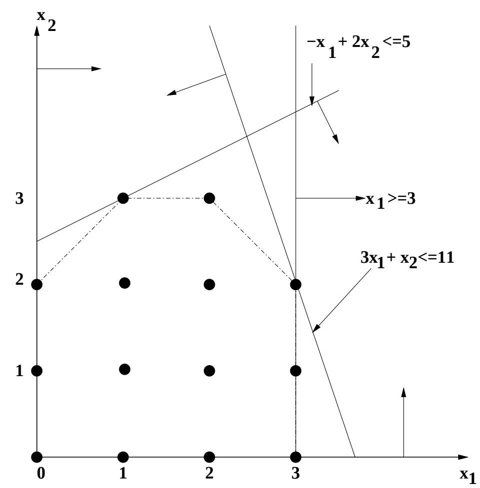

Figure 2:

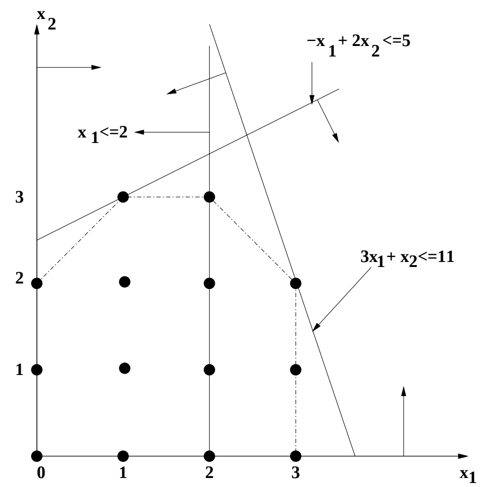

Figure 3:

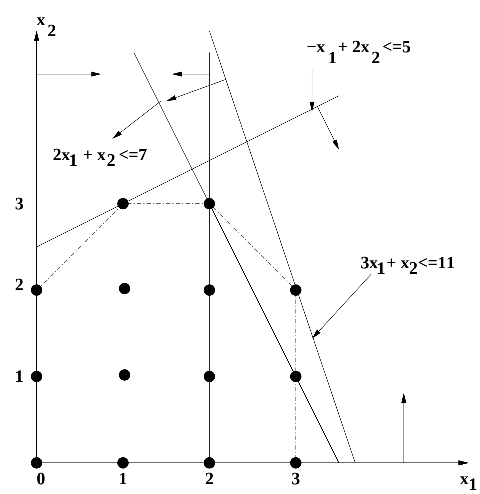

Figure 4: