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Benders' Decomposition (Benders, 1962)

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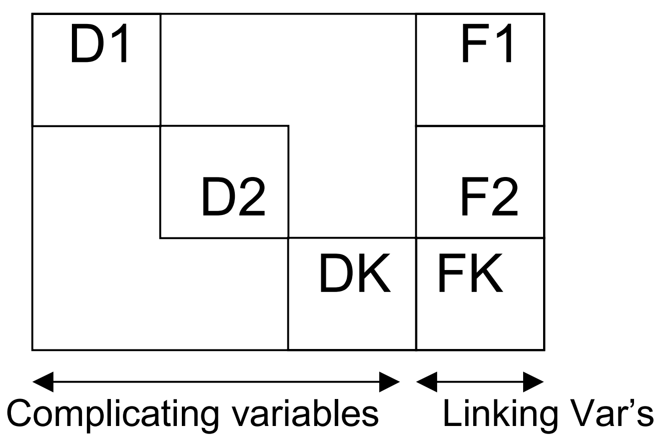
Introduction to Benders Decomposition

- Solves large scale linear programs.
- Problems have a special structure called Block Diagonal structure.
- Constraint set in Dantzig Wolfe is sub-divided into complicating and special constraints, similarly variable set in Benders is subdivided into "Linking" and "Complicating" variables.



Block diagonal structure with linking variables

The Decomposition Algorithm

Let us consider the following problem (P0):

$$\text{Min } c^1y + c^2z$$

$$\text{s.t. } A^1y + A^2z \geq b$$

$$y \in Y,\ z \geq 0$$

y_j represent the complicating variables.

For fixed y, the above problem is a linear program over z.

The Primal Partitioning Algorithm

For some fixed $y \in Y$, the previous problem (P0) reduces to the following LP (say P1):

$$\min\ c^2 z + c^1 y$$
$$\text{s.t. } A^2 z \geq b - A^1 y$$
$$z \geq 0$$

Each LP 'P1' has a dual (PD) as follows :

$$\max\ u(b - A^1 y) + c^1 y$$
$$\text{s.t. } u A^2 \leq c^2 \qquad (1)$$
$$u \geq 0 \qquad (2)$$

Benders' Masters Problem

We now have Benders' Masters Problem as follows :

$$\min \beta$$

$$\text{s.t. } \beta \geq c^1 y + u\,(b - A^1 y) \quad \forall u \in U$$

$$0 \geq v\,(b - A^1 y) \qquad \forall v \in V$$

$$y \in Y,$$

where:

U = extreme points of the set of u defined by (1) and (2)

and

V = extreme directions of the set of u defined by (1) and (2)

Extreme Point & Extreme Direction

- Extreme Point – p is an extreme point of a convex set C, if

  $p = \lambda x^1 + (1-\lambda) x^2$, informally extreme points of convex sets are those that do not lie within the line segment between any two other points of the set.
- Extreme Direction – is a direction which cannot be expressed as a positive linear combination of two distinct directions. Thus d is an extreme of a set S if :

  $d = \lambda_1 d^1 + \lambda_2 d^2$; $\lambda_1, \lambda_2 > 0$, d^1, d^2 are directions of S.

The Decomposition Algorithm

Step 0: Initialization: set $\beta^1 \leftarrow -\infty$, the iteration counter $k \leftarrow 1$, an extreme point subset $U^0 \leftarrow \emptyset$ and an extreme direction subset $V^0 \leftarrow \emptyset$. Also choose any $y^1 \in Y$.

Step 1: Subproblem Apply some form of the simplex method tol PD(k). If PD(k) is unbounded, let v^k be the extreme direction obtained from the simplex algorithm and go to step 2. If PD(k) is bounded, let u^k be an optimal extreme point solution and go to step 3.

Step 2 : Extreme direction processing Add v^k to the extreme direction subset by $V^k \leftarrow V^{k-1} \cup \{v^k\}$, set $U^k \leftarrow U^{k-1}$ and go to step 4.

Step 3: Extreme point processing

if $\beta^k \geq c^1 y^k + u^k (b - A^1 y^k)$, solve $P1(y^k)$ for z^k and stop;

(y^k, z^k) is optimal in P0; otherwise, add u^k to the extreme point subset by $U^k \leftarrow U^{k-1} \cup \{u^k\}$, set $V^k \leftarrow V^{k-1}$

and go to step 4.

The Decomposition Algorithm- contd.

Step 5: Master Problem(M) Attempt to solve

$$\min \beta$$

$$\text{s.t. } \beta \geq c^1 y + u(b - A^1 y) \quad \forall u \in U^k$$

$$0 \geq v(b - A^1 y) \quad \forall v \in V^k$$

$$y \in Y$$

If M is infeasible, stop; P0 is infeasible.

If M is bounded, let (β^{k+1}, y^{k+1}) be any optimal solution.

If M is unbounded, choose $\beta^{k+1} = \beta^k$ and y^{k+1} = any y satisfying

$0 \geq v(b - A^1 y) \ \forall v \in V^k$.

Then advance $k \leftarrow k+1$ and return to step 1.

Theorem on the convergence of the Benders algorithm

Theorem: Suppose P0 is the partitionable problem along with feasible, bounded-value LP relaxation P0. Then after finitely many of its steps, the previous algorithm finds an optimal solution to P0 or proves that none exists.

An Example

Consider the simple mixed 0-1 program

$$\min\ 42y_1 + 18y_2 + 33y_3 - 8z_1 - 6z_2 + 2z_3$$

$$\text{s.t.}\quad 10y_1 + 8y_2 \qquad -2z_1 - z_2 + z_3 \geq 4$$

$$5y_1 \qquad + 8y_3 \ -z_1 - z_2 - z_3 \geq 3$$

$$y_1\ ,\ y_2\ ,\ y_3 \in \{0,1\}, \qquad z_1,\ z_2\ , z_3 \geq 0$$

Dual subproblems of the Benders algorithm have the form:

$$\text{Max } (4 - 10y_1 - 8y_2)\ u_1 + (3 - 5y_1 - 8y_3)\ u_2 + (42y_1 + 18y_2 + 33y_3\ )$$

$$\text{s.t.} -2u_1 - u_2 \leq -8$$

$$-u_1 - u_2 \leq -6$$

$$u_1 - u_2 \leq 2$$

$$u_1\ ,\ u_2 \geq 0$$

Feasible space of duals as shown in attached figure.

Algorithm Execution -- Summary

K	y^k	U^k	V^k	β^k	$c^1y^k + u\,(b - A^1y^k)$
0	--	--	$\varnothing$	$\varnothing$	--
1	(1,1,1)	(2,4)	$\varnothing$	$-\infty$	25
2	(0,0,0)	(2,4)	(0,1)	20	20**
3	(0,0,1)	(4,2)	(0,1)	21	39
4	(1,0,0)	(0,8)	(0,1)	22	26
5	(1,0,0)	(0,8)	(0,1)	26	26

For k=5, $\beta^k \geq c^1y^k + u^k\,(b - A^1y^k)$ is satisfied.

We conclude $y^* = y^5 = (1,0,0)$, $\beta^5 = 26$

Corresponding $z^* = (2,0,0)$ found by solving $P1(y^5)$.

References

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