Unimodularity and Total Unimodularity

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Definitions

Definitions:

- A square, integer matrix B is *unimodular* (UM) if its determinant is 1 or –1.
- An integer matrix A is called *totally unimodular* (TUM) if every square, nonsingular submatrix of A is UM.
- Every TUM is a UM but the converse is not necessarily true.

Total Unimodularity

Theorem1: If A is TUM, then all the vertices of the convex polytope defined by the constraints $Ax = b, x \geq 0,$ are integral for any integer vector b.

Proof : Let B be a basis matrix of the feasible set $Ax = b$. Then,
$Bx_B + Nx_N = b$, where (x_B, x_N) is an extreme point of the convex polytope.
Clearly, $Bx_B = b \Rightarrow x_B = B^{-1}b$. By Cramer's rule, det $B = \pm 1$ implies that B^{-1} is integral. Since b is an integer matrix, it follows that x_B is integral. Hence all the vertices of the convex polytope are integral.

Corollary1: If A is TUM, then all the vertices of the convex polytope defined by the inequality constraints $Ax \leq b, x \geq 0,$ are integral for any integer vector b.

Necessary Condition

<u>Eulerian Matrix</u>: A matrix is said to be Eulerian if the sum of the elements in each row and each column is even.

<u>Theorem1</u>: (Camion) A (0,+1,-1) matrix is totally unimodular if and only if the sum of elements in each Eulerian square submatrix is a multiple of four.

Example:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ is not TUM, since } \det A = -2. \text{ This can also be verified by}$$

the above theorem. A is an Eulerian matrix since the sum of elements in each row and each column is even. Now, the sum of the elements of A is 2 which is not a multiple of 4. Hence A is not TUM.

Sufficient Conditions

Theorem2: A (0,+1,-1) matrix A is totally unimodular if both of the following conditions are satisfied:

- Each column contains at most two nonzero elements
- The rows of A can be partitioned into two sets A_1 and A_2 such that two nonzero entries in a column are in the same set of rows if they have different signs and in different sets of rows if they have the same sign.

Corollary2: A (0,+1,-1) matrix A is totally unimodular if it contains no more than one +1 and no more than one –1 in each column.

Sufficient Conditions(contd)

Proof of Theorem2 : A submatrix of a $(0, +1, -1)$ matrix satisfying the conditions of the theorem must also satisfy the same conditions. Hence it is sufficient to prove that det $A = 0, \pm 1$, for all square matrices satisfying the conditions. For any 1×1 matrix A, clearly det $A = 0, \pm 1$. Now suppose, by inductive hypothesis, that det $A = 0, \pm 1$ for all $(n-1) \times (n-1)$ matrices A. Let A be $n \times n$. If A contains a zero column, det $A = 0$. If some column of A contains exactly one nonzero entry, then det $A = \pm$ det $A' = 0, \pm 1$, where A' is the cofactor of that entry. If every column of A contains exactly two nonzero entries, then

$$\sum_{i \in A_1} a_{ij} = \sum_{i \in A_2} a_{ij}, for j = 1,2,\ldots,n.$$

This implies that det $A = 0$ and the proof is complete.

<u>Theorem3</u>: A matrix A is totally unimodular if any one of the matrices A^T, -A, (A,A), (A,I) is totally unimodular.

Examples

Example1: Any LP in standard or canonical form whose constraint matrix A is either

1. The node-arc incidence matrix of a directed graph, or
2. The node-edge incidence matrix of an undirected bipartite graph

has only integer optimal vertices. This includes the LP formulations of shortest path, max-flow and weighted bipartite matching.

Example2:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

A is a totally unimodular matrix from Theorem2.

NOTE: The sufficient conditions stated above can only prove whether a given matrix is TUM but cannot prove that a given matrix is not TUM.

The Shortest Path Problem

Lp formulation of the SPP :

$$Min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

Subject to :

$$\sum_{\{j:(i,j)\in A\}} x_{ij} - \sum_{\{j:(j,i)\in A\}} x_{ij} = \begin{cases} n-1 & \text{for } i = s \\ -1 & \text{for all } i \in N-\{s\} \end{cases}$$

$x_{ij} \geq 0$ for all $(i,j) \in A$

The constraint is of the form $Ax = b$ where b is a matrix of integers. The matrix A is composed of $+1$,0 and -1. Each column of A corresponds to an arc in the graph G. And note that each column has entries of exactly one $+1$ and one -1 and the rest zeros. It follows by Corollary 2 that the matrix A is totally unimodular. Hence the LP problem above has integral solutions.