Notations

f_{ij}: the flow from department i to j;

d_{ik}: the shortest distances from department i to connection k;

α: the net present value of per unit distance per unit flow evaluated over the system's lifetime;

F_k: the installation cost related to location;

x_{ijk}: the fraction of flow i-j by way of connection $k \in K$;

$y_k = 1$ if a connection is located at candidate site k, and 0 otherwise;

c_k: the capacity of connection k;

c_{kl}: the l-th capacity option of a connection at candidate site k if a connection is located there;

F_{kl}: the fixed cost of locating a connection at candidate site k with capacity option l;

$y_{kl} = 1$ if a connection is located at candidate site k with capacity option l, and 0 otherwise.

The continuous capacity case

$$(P_1) \qquad \min_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}} Z^{P_1}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}) = \sum_{(i,j)\in A}\sum_{k\in K} \alpha f_{ij}(d_{ik}+d_{kj})x_{ijk} + \sum_{k\in K} y_k(F_k + vc_k) \qquad (1)$$

$$\text{subject to} \qquad \sum_{k\in K} x_{ijk} = 1, \quad \forall\,(i,j)\in A, \qquad (2)$$

$$\sum_{(i,j)\in A} f_{ij}x_{ijk} \le c_k y_k, \quad \forall\, k\in K, \qquad (3)$$

$$x_{ijk} \ge 0, \quad y_k \in \{0,1\}, \quad c_k \ge 0, \quad \forall\,(i,j)\in A, k\in K. \qquad (4)$$

The continuous capacity case

Property 1. If $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c})$ is an optimal solution for the problem (P_1), then $\forall k \in K$, $\Sigma_{(i,j)\in A} f_{ij} x_{ijk} = c_k y_k$.

We can infer from this property that the optimal total capacity should be equal to the total flow, that is, $\Sigma_{k\in K} \Sigma_{(i,j)\in A} f_{ij} x_{ijk} = \Sigma_{(i,j)\in A} f_{ij} = \Sigma_{k\in K} c_k y_k$. Thus the term $v \Sigma_{k\in K} c_k y_k$ in the objective function is a constant if the solution is optimal.

Property 2. There exists an optimal solution for the problem (P_1) in which $x_{ijk} = 0$ or 1, $\forall (i,j) \in A, k \in K$.

According to property 2, any flow will be fully assigned to the chosen open connection. Thus, the assignment variables, x_{ijk}, will naturally assume integer values.

The continuous capacity case

Two step procedures:

Step 1. Solve the following uncapacitated fixed charge connection location:

$$(P_1^{'}) \qquad \min_{\boldsymbol{x},\boldsymbol{y}} Z^{P_1^{'}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{(i,j)\in A}\sum_{k\in K} \alpha f_{ij}(d_{ik}+d_{kj})x_{ijk} + \sum_{k\in K} F_k y_k,$$

$$\text{subject to} \qquad \sum_k x_{ijk} = 1, \qquad \forall\,(i,j)\in A,$$

$$x_{ijk} \le y_k, \qquad \forall\, k \in K,$$

$$x_{ijk} \ge 0, \qquad y_k \in \{0,1\} \qquad \forall\,(i,j)\in A, k\in K.$$

This can also be viewed as an uncapacitated fixed charge facility location problem if we treat each flow as a demand node and each connection as a facility.

Step 2. Set $c_k = \Sigma_{(i,j)\in A}\, f_{ij}x_{ijk}$, $\forall k$.

Theorem 1. The solution of the two-step algorithm is optimal for the problem (P_1).

The discrete capacity case

$$(P_2) \qquad \min_{\boldsymbol{x},\boldsymbol{y}} Z^{P_2}(\boldsymbol{x},\boldsymbol{y}) = \sum_{(i,j)\in A}\sum_{k\in K} \alpha f_{ij}(d_{ik}+d_{kj})x_{ijk} + \sum_{k\in K}\sum_{l\in L} y_{kl}F_{kl} \qquad (5)$$

$$\text{subject to} \qquad \sum_{k\in K} x_{ijk} = 1, \qquad \forall\,(i,j)\in A, \qquad (6)$$

$$\sum_{(i,j)\in A} f_{ij}x_{ijk} \le \sum_{l\in L} c_{kl}y_{kl}, \qquad \forall\,k\in K, \qquad (7)$$

$$\sum_{l\in L} y_{kl} \le 1, \qquad \forall k\in K, \qquad (8)$$

$$x_{ijk}\ge 0, \qquad y_{kl}\in\{0,1\}, \qquad \forall\,(i,j)\in A, k\in K, l\in L. \qquad (9)$$

We note the following: if we are given y_{kl} and $\Sigma_{k\in K}\Sigma_{l\in L}\, y_{kl}c_{kl} \ge \Sigma_{(i,j)\in A}\, f_{ij}$, then the optimal assignment of flows to connections can be found by solving the following transportation problem:

$$\min_{\boldsymbol{x}} \sum_{(i,j)\in A}\sum_{k\in K} \alpha f_{ij}(d_{ik}+d_{kj})x_{ijk}$$

$$\text{subject to:} \quad \sum_{(i,j)\in A} x_{ijk} = 1 \qquad \forall\,(i,j)\in A,$$

$$\sum_{(i,j)\in A} f_{ij}x_{ijk} \le c_{kl} \qquad \forall k,l \text{ such that } y_{kl}=1,$$

$$x_{ijk} \ge 0.$$

Lagrangian Relaxation Algorithm – Basic Idea

Consider problem (P):

$$(P) \qquad \min_{\boldsymbol{x}} c\boldsymbol{x}$$

$$\text{subject to: } A\boldsymbol{x} \ \geq \ b,$$

$$B\boldsymbol{x} \ \geq \ d,$$

$$\boldsymbol{x} \in \{0, 1\}.$$

By introducing a Lagrange multiplier vector $\lambda \geq \mathbf{0}$,

$$(P^{'}) \qquad \min_{\boldsymbol{x}} c\boldsymbol{x} + \boldsymbol{\lambda}(b - A\boldsymbol{x})$$

$$\text{subject to: } B\boldsymbol{x} \ \geq \ d,$$

$$\boldsymbol{x} \in \{0, 1\}.$$

Fisher, M. L. (1981) The Lagrangian Relaxation Method for Solving Integer Programming Problem. *Management Science*, **27**, 1-18.

The discrete capacity case

We choose to relax the capacity constraints (7) to obtain the following Lagrangian relaxation problem:

$$(P_2^{'}) \qquad \min_{\boldsymbol{x},\boldsymbol{y}} Z^{P_2^{'}}(\boldsymbol{\mu}) = \sum_{(i,j)\in A} \sum_{k\in K} f_{ij}(\alpha(d_{ik}+d_{kj})+\mu_k)x_{ijk} + \sum_{k\in K} \sum_{l in L} y_{kl}(F_{kl}-\mu_k c_{kl}) \quad (10)$$

subject to (6), (8), and (9)

The ideal choice of multipliers is such that they solve the Lagrangian dual problem:

$$(DP) \qquad \max_{\boldsymbol{\mu}} Z^{DP}(\boldsymbol{\mu}) \qquad (11)$$

The optimal value of the above problem (DP) provides the "best" lower bound (using the Lagrangian method).

For fixed values of the Lagrange multipliers μ_k, $(P_2^{'})$ can be solved by independently solving the following two sets of problems: a set of $|A|$ problems (SP_1) and a set of n_K problems (SP_2), where:

$$(SP_1) \qquad \min_{\boldsymbol{x}} Z^{SP_1}(\boldsymbol{\mu}, i, j) = \sum_{k \in K} f_{ij}\alpha(d_{ik} + d_{kj}) + \mu_k)x_{ijk},$$

$$\text{subject to} \quad \begin{aligned} \sum_{k \in K} x_{ijk} &= 1, \qquad \forall (i,j) \in A, \\ x_{ijk} &\geq 0, \qquad \forall (i,j) \in A, k \in K, \end{aligned}$$

(which is decomposed on the indicies i and j), and

$$(SP_2) \qquad \min_{\boldsymbol{y}} Z^{SP_2}(\boldsymbol{\mu}, k) = \sum_{l \in L} y_{kl}(F_{kl} - \mu_k c_{kl})$$

$$\text{subject to} \quad \begin{aligned} \sum_{l \in L} y_{kl} &\leq 1, \qquad \forall k \in K, \\ y_{kl} &\in \{0,1\}, \qquad \forall k \in K, l \in L. \end{aligned}$$

(which is decomposed on the index k).

(SP_1) can be solved easily by setting $x_{ijs} = 1$ for the s value that minimizes $f_{ij}\alpha(d_{ik} + d_{kj}) + \mu_k)$ over all k. The optimal value for (SP_1) is

$$Z^{SP_1}(\boldsymbol{\mu}, i, j) = \min_{k \in K}\{f_{ij}\alpha(d_{ik} + d_{kj}) + \mu_k\}. \tag{12}$$

(SP_2) is a 0-1 knapsack problem. It can be solved by setting $y_{kl} = 1$ if $F_{kl} - \mu_k c_{kl} \leq 0$ and is chosen such that $F_{kl} - \mu_k c_{kl}$ is minimized. The optimal value for (SP_2) is

$$Z^{SP_2}(\boldsymbol{\mu}, k) = \min\{\min_{l \in L}\{F_{kl} - \mu_k c_{kl}\}, 0\}. \tag{13}$$

Therefore, the optimal value for problem (P_2') is

$$Z^{SP_2'}(\boldsymbol{\mu}) = \sum_{(i,j) \in A} Z^{SP_1}(\boldsymbol{\mu}, i, j) + \sum_{k \in K} Z^{SP_2}(\boldsymbol{\mu}, k). \tag{14}$$

The dual problem can be solved by the subgradient approach (Fisher, 1981). Given an initial value $\boldsymbol{\mu}^0$, a sequence of values $\boldsymbol{\mu}^n$ is generated by the rule:

$$\mu_k^{n+1} = \max\{0, \mu_k^n + t^n(\sum_{(i,j)\in A} f_{ij}x_{ijk}^n - \sum_{l\in L} c_{kl}y_{kl}^n)\}, \qquad (15)$$

where the values of $\boldsymbol{x}^n$ and $\boldsymbol{y}^n$ are the optimal solution to $(P_2^{'})$ for fixed $\boldsymbol{\mu}^n$. Here t^n is a positive scalar step size. It can be computed as follows:

$$t^n = \frac{\beta^n(UB - Z^{P_2^{'}}(\boldsymbol{\mu}^n))}{\Sigma_{k\in K}(\Sigma_{(i,j)\in A} f_{ij}x_{ijk}^n - \Sigma_{l\in L} c_{kl}y_{kl}^n)^2}, \qquad (16)$$

where β is a scalar satisfying $0 < \beta^n < 2$.

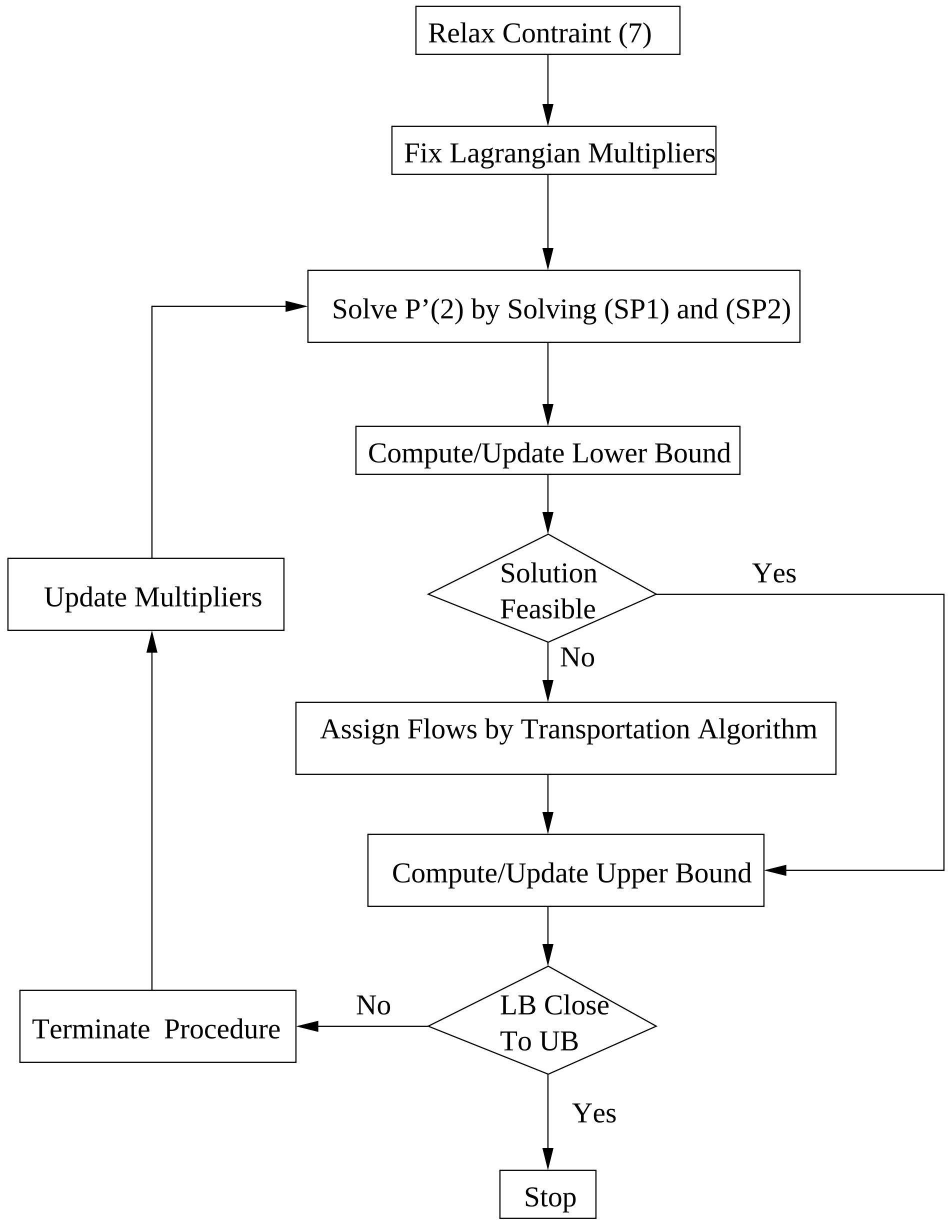

Figure 1: Flowchart of the Algorithm

Table 1: Comparison of optimal objective values

Data set	# of capacity options in discrete case			Continuous case
	Three	Five	Nine	
1	1224200	1219160	1218920	1218230
2	1859440	1854250	1852065	1850490

Table 2: Parameter values for test problems

	parameter	small	medium	large
1	number of departments	10-30	55-65	70-80
2	number of connections	15-45	110-160	180-200
3	number of capacity options	3 or 6	3 or 6	3 or 6
4	number of non-zero flows	45-435	1485-2080	2415-3160

Summary and Future Work

For the continuous capacity case, we prove that it can be reduced to the uncapacitated fixed charge facility location problem.

For the discrete capacity case, a Lagrangian relaxation based approach has been developed. ANOVA test shows that the capacity size and the number of options effect are significant to the algorithm

Table 3: Computational result for small sized problems

	dept #	option #	heuristic gap (%)	heuristic time (s)	CPLEX time (s)
Large capacity	10	3	3.74	0.07	0.24
	10	6	1.88	0.07	0.31
	20	3	3.40	0.38	1.48
	20	6	2.52	0.65	15.72
	30	3	2.63	7.86	219.23
	30	6	1.81	8.37	74.57
Small capacity	10	3	1.39	0.07	0.16
	10	6	2.33	0.07	0.37
	20	3	1.87	0.83	3.58
	20	6	1.15	1.03	39.30
	30	3	1.58	5.86	77.05
	30	6	1.33	6.87	449.83

performance.

Future works include:

- Stochastic version of this problem;
- Planar version of this problem.

Table 4: Computational result for medium sized problems

	dept #	option #	heuristic gap (%)	heuristic time (s)	CPLEX time (s)
Large capacity	55	3	1.96	255.60	545
	55	6	1.27	305.27	790
	60	3	2.45	698.42	3482
	60	6	1.44	694.49	3390
	65	3	1.86	779.64	2460
	65	6	1.36	1066.45	>3600
Small capacity	55	3	1.70	265.20	306
	55	6	1.15	295.43	324
	60	3	1.27	643.21	3572
	60	6	1.24	650.32	>3600
	65	3	1.23	767.24	>3600
	65	6	1.12	891.29	>3600

Table 5: Computational result for large sized problems

	dept #	option #	heuristic gap (%)	heuristic time (s)
Large capacity	70	3	2.10	2183.88
	70	6	1.67	1570.23
	75	3	2.38	2259.78
	75	6	1.35	2423.09
	80	3	1.82	4169.62
	80	6	1.51	2966.22
Small capacity	70	3	1.28	1899.93
	70	6	1.92	643.88
	75	3	1.66	2101.88
	75	6	0.99	1996.69
	80	3	1.84	2403.31
	80	6	1.61	1405.79

Table 6: Average gap for different problem sizes and parameter values

	Capacity size		Capacity options		
size	Large Capacity	Small Capacity	Three	Six	Average
small	2.66	1.61	2.44	1.84	2.14
medium	1.77	1.28	1.75	1.30	1.50
large	1.81	1.55	1.85	1.51	1.68