IE 680 – Special Topics in Production Systems: Networks, Routing and Logistics*

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**Lecture notes from Bruce Golden, U of MD and Introduction to Operations Research by Hiller and Lieberman (2001)*

Transportation Problem

- Special case of Min Cost Flow Problem
- Shipping and Distribution Applications
- From plants to warehouses or warehouses to retailers, respectively
- Let there be *m* plants with known supplies (S_i corresponding to supply point *i*)
- And *n* warehouses with know demands (D_j corresponding to demand point *j*)
- Identify flows at min cost to satisfy demand from given supply (c_{ij} is unit shipping cost point *i*)

Transportation Problem

- Minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

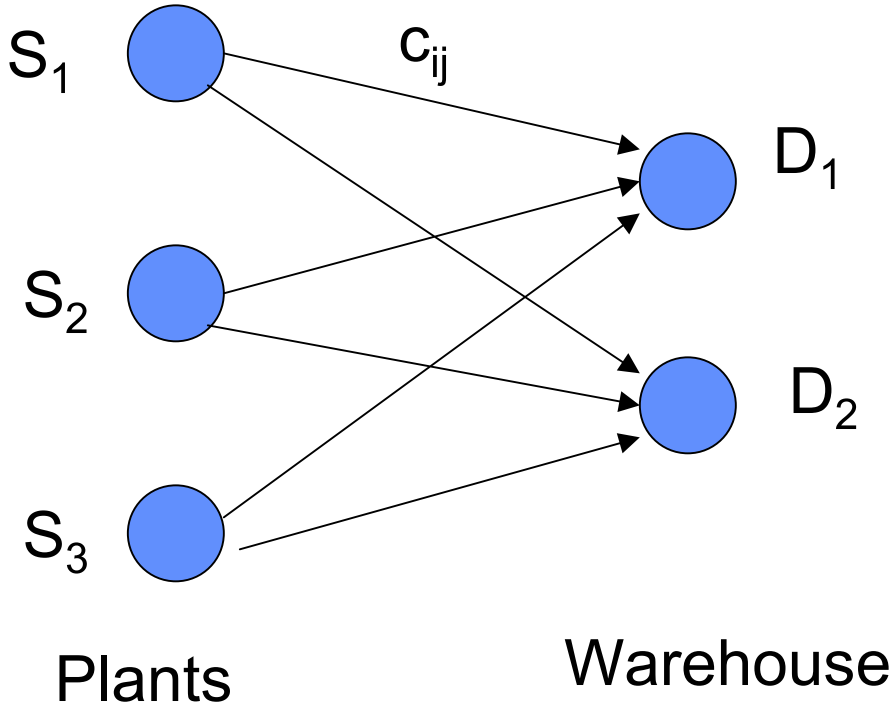
- S.t.

$$\sum_{j=1}^{n} x_{ij} \le S_i \qquad for \qquad i = 1,2,\ldots m$$

$$\sum_{i=1}^{m} x_{ij} \ge D_i \qquad for \qquad j = 1,2,\ldots n$$

$$x_{ij} \ge 0 \qquad for\, all \quad i, j$$

Transportation Network



Transportation Problem

- For the model to possess a feasible solution, total supply should be at least as large as total demand

$$\sum_{i=1}^{m} S_i \geq \sum_{j=1}^{n} D_j$$

- It is convenient to assume equality by adding a dummy destination with remaining supply

Transportation Problem

- Simplex solution to an LP problem has at most k positive variables, where k is the number of independent constraints
- Convince yourself that there are m+n-1 independent constraints in the transportation problem
- Also note that a key result in network theory is that at least one optimal solution to the TP has integer-valued x_{ij} provided S_i and D_j are integers.

Standard Simplex

- Step 1: Select a set of m+n-1 routes that provide an initial basic feasible solution
- Step 2: Check whether the solution is improved by introducing a non basic variable. If so, go to Step 3; otherwise, Stop.
- Step 3: Determine which routes leave the basis when the variable that you selected in Step 2 enters.
- Step 4: Adjust the flows of the other basic routes. Return to Step 2.

Initial Solution

- Northwest corner rule
- Algorithm: Begin by selecting X_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if X_{ij} was the last basic variable selected, then next select $X_{i,j+1}$ (that is, move one column to the right) if source i has any supply remaining. Otherwise, next select $X_{i+1,j}$ (that is, move one row down).

	1		2		3		4		Supply
	8		6		10		9		
A		35							35/0
	9		12		13		7		
B		10		20		20			50/40/20/0
	14		9		16		5		
C						10		30	40/30/0
Demand		45/10/0		20/0		30/10/0		30/0	

Initial Solution

- Northwest corner rule: Example

TABLE 8.16 Initial BF solution from the Northwest Corner Rule

		Destination						
		1	2	3	4	5	Supply	u_i
Source	1	16 (30)	16 (20)	13	22	17	50	
	2	14	14 (0)	13 (60)	19	15	60	
	3	19	19	20 (10)	23 (30)	*M* (10)	50	
	4(*D*)	*M*	0	*M*	0	0 (50)	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

Initial Solution

- Minimum Matrix Method
- Algorithm: Find the minimum C_{ij} and assign to X_{ij} the higher of S_i or D_j. Reduce the demand and supply (S_i- X_{ij}; D_j- X_{ij}). Repeat until done. Note that the last arcs chosen could be very expensive.

	1		2		3		4		Supply
	8		6		10		9		
A		3 (15)		2 (20)					35/15/0
	9		12		13		7		
B		(30)				(20)			50/20/0
	14		9		16		5		
C						(10)		1 (30)	40/10/0
Demand		45/30/0		20/0		30/10/0		30/0	

Initial Solution

- Vogel's Approximation Method
- Algorithm: For each row and column remaining under consideration, calculate its difference, which is defined as the arithmetic difference between the smallest and next-to-the-smallest unit cost C_{ij} still remaining in that row or column. (If two unit costs tie for being the smallest remaining in a row or column, then the difference is 0.) In that row or column having the largest difference, select the variable having the smallest remaining unit cost. (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

TABLE 8.17 Initial BF solution from Vogel's approximation method

		Destination 1	2	3	4	5	Supply	Row Difference
Source	1	16	16	13	22	17	50	3
	2	14	14	13	19	15	60	1
	3	19	19	20	23	M	50	0
	4(D)	M	0	M	[0]	0	50	0
Demand		30	20	70	30	60	Select $x_{44} = 30$	
Column difference		2	14	0	(19)	15	Eliminate column 4	

		Destination 1	2	3	5	Supply	Row Difference
Source	1	16	16	13	17	50	3
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
	4(D)	M	0	M	[0]	20	0
Demand		30	20	70	60	Select $x_{45} = 20$	
Column difference		2	14	0	(15)	Eliminate row 4(D)	

		Destination 1	2	3	5	Supply	Row Difference
Source	1	16	16	[13]	17	50	(3)
	2	14	14	13	15	60	1
	3	19	19	20	M	50	0
Demand		30	20	70	40	Select $x_{13} = 50$	
Column difference		2	2	0	2	Eliminate row 1	

		Destination 1	2	3	5	Supply	Row Difference
Source	2	14	14	13	[15]	60	1
	3	19	19	20	M	50	0
Demand		30	20	20	40	Select $x_{25} = 40$	
Column difference		5	5	7	($M - 15$)	Eliminate column 5	

		Destination 1	2	3	Supply	Row Difference
Source	2	14	14	[13]	20	1
	3	19	19	20	50	0
Demand		30	20	20	Select $x_{23} = 20$	
Column difference		5	5	(7)	Eliminate row 2	

		Destination 1	2	3	Supply
Source	3	19	19	20	50
Demand		30	20	0	Select $x_{31} = 30$, $x_{32} = 20$, $x_{33} = 0$

Z=2,460

Initial Solution

- Vogel's approximation method

	1		2		3		4		Supply	Row Diff
	8		6		10		9			2
A									35	
	9		12		13		7			2
B									50	
	14		9		16		5			4
C								30	40/10	
Demand		45		20		30		30/0		
Col Diff	1		3		3		2			

	1		2		3		4		Supply	Row Diff
	8		6		10		9			2
A									35	
	9		12		13		7			3
B									50	
	14		9		16		5			5
C				10				30	40/10/0	
Demand		45		20/10		30		30/0		
Col Diff	1		3		3					

Initial Solution

- Vogel's approximation method

	1		2		3		4		Supply	Row Diff
	8		6		10		9			2
A				10					35/25	
	9		12		13		7			3
B									50	
	14		9		16		5			
C				10				30	40/10/0	
Demand		45		20/10/0		30		30/0		
Col Diff	1		6		3					

	1		2		3		4		Supply	Row Diff
	8		6		10		9			2
A				10		25			35/25/0	
	9		12		13		7			4
B		45				5			50/0	
	14		9		16		5			
C				10				30	40/10/0	
Demand		45/0		20/10/0		30/0		30/0		
Col Diff	1		3							

Initial Solution

For ABC-> 1234 example

- Northwest corner rule $1,180
- Minimum Matrix Method $1,080
- Vogel's Approximation $1,020

Initial Solution

- Russell's approximation method
- Algorithm: For each source row i remaining under consideration, determine its $\bar{u}_i$ which is the largest unit cost C_{ij} still remaining in that row. For each destination column j remaining under consideration, determine its $\bar{v}_j$ which is the largest unit cost C_{ij} still remaining in that column. For each variable X_{ij} not previously selected in these rows and columns, calculate $\Delta_{ij} = C_{ij} - \bar{u}_i - \bar{v}_j$. Select the variable having the largest (in absolute terms) negative value of Δ_{ij} (Ties may be broken arbitrarily.)

TABLE 8.18 Initial BF solution from Russell's approximation method

Iteration	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{v}_1$	$\bar{v}_2$	$\bar{v}_3$	$\bar{v}_4$	$\bar{v}_5$	Largest Negative Δ_{ij}	Allocation
1	22	19	M	M	M	19	M	23	M	$\Delta_{45} = -2M$	$x_{45} = 50$
2	22	19	M		19	19	20	23	M	$\Delta_{15} = -5 - M$	$x_{15} = 10$
3	22	19	23		19	19	20	23		$\Delta_{13} = -29$	$x_{13} = 40$
4		19	23		19	19	20	23		$\Delta_{23} = -26$	$x_{23} = 30$
5		19	23		19	19		23		$\Delta_{21} = -24$*	$x_{21} = 30$
6										Irrelevant	$x_{31} = 0$
											$x_{32} = 20$
											$x_{34} = 30$
											$Z = 2{,}570$

*Tie with $\Delta_{22} = -24$ broken arbitrarily.

Step 2: Optimality Test

- Dual Linear Program is
- Maximize $\sum_{i=1}^{m} S_i v_i + \sum_{j=1}^{n} D_j w_j$
- S.t. $v_i + w_j \leq C_{ij} \quad \textit{for all} \quad i = 1,2,\ldots m \quad j = 1,2,\ldots n$

$$\textit{where} \quad v_i, \quad w_j \quad \textit{are unrestricted in sign}$$

- Theorem of complementary slackness gives:

$$\overline{C}_{ij} = v_i + w_j - C_{ij} \leq 0 \quad \textit{if} \quad x_{ij} = 0$$

$$\overline{C}_{ij} = v_i + w_j - C_{ij} = 0 \quad \textit{if} \quad x_{ij} > 0$$

Format for Transportation Simplex

TABLE 8.15 Format of a transportation simplex tableau

		Destination					
		1	**2**	...	***n***	**Supply**	V_i / u_i
Source	1	c_{11}	c_{12}	...	c_{1n}	s_1	
	2	c_{21}	c_{22}	...	c_{2n}	s_2	
	⋮	...	...	...	...	⋮	
	m	c_{m1}	c_{m2}	...	c_{mn}	s_m	
Demar ⁻ᵈ		d_1	d_2	...	d_n	$Z =$	
W_j	v_j						

Additional information to be added to each cell:

If x_{ij} is a basic variable	*If x_{ij} is a nonbasic variable*
c_{ij} ; (x_{ij})	c_{ij} ; $c_{ij} - u_i - v_j$
0	$v_i + w_j - C_{ij}$

Example

	1	2	3	4	Supply
	2	3	11	7	
A					6
	1	0	6	1	
B					1
	5	8	15	9	
C					10
Demand	7	5	3	2	

Initial Basic Feasible Solution using Vogel's Approximation = 112

	1	2	3	4	Supply
	2	3	11	7	
A	6				6
	1	0	6	1	
B		1			1
	5	8	15	9	
C	1	4	3	2	10
Demand	7	5	3	2	

Example

- Finding v_i and w_j
- Set $v_A = 0$ arbitrarily to determine all other duals
- $v_A + w_2 - 6 = 0$; so $w_2 = 6$
- $v_C + w_2 - 9 = 0$; so $v_C = 3$
- $v_C + w_4 - 5 = 0$; so $w_4 = 2$
- $v_A + w_3 - 10 = 0$; so $w_3 = 0$
- $v_C + w_1 - 9 = 0$; so $w_1 = 6$, etc.

Example

- Step 3: Identify incoming variable (A2)

	1		2		3		4		v_I
	2		3		11		7		
A		0		2		1		-1	0
	1		0		6		1		
B		-4		0		1		0	-5
	5		8		15		9		
C		0		0		0		0	3
w_j		2		5		12		6	

- Identify which variable leaves (C2)

	1		2		3		4		Supply
	2		3		11		7		
A		6-z		z					6
	1		0		6		1		
B				1					1
	5		8		15		9		
C		1+z		4-z		3		2	10
Demand		7		5		3		2	

Example

- Step 4: Adjust the flows – find new basis (cost = 104)

	1		2		3		4		Supply
	2		3		11		7		
A		2		4					6
	1		0		6		1		
B				1					1
	5		8		15		9		
C		5		0		3		2	10
Demand		7		5		3		2	

- Step 3: Identify incoming variable (B3)

	1		2		3		4		v_I
	2		3		11		7		
A		0		0		1		-1	0
	1		0		6		1		
B		-2		0		3		2	-5
	5		8		15		9		
C		0		-2		0		0	3
w_j		2		3		12		6	

Example

- Step 3: Identify which variable leaves (B2)

	1		2		3		4		Supply
	2		3		11		7		
A		2-z		4+z					6
	1		0		6		1		
B				1-z		z			1
	5		8		15		9		
C		5+z				3-z		2	10
Demand		7		5		3		2	

- Step 4: Adjust the flows – find new basis (cost = 101)

	1		2		3		4		Supply
	2		3		11		7		
A		1		5					6
	1		0		6		1		
B				0		1			1
	5		8		15		9		
C		6		0		2		2	10
Demand		7		5		3		2	

Example

- Step 3: Identify incoming variable (A3)

	1		2		3		4		v_I
	2		3		11		7		
A		0		0		1		-1	0
	1		0		6		1		
B		-5		-3		0		-1	-5
	5		8		15		9		
C		0		-2		0		0	3
w_j		2		3		12		6	

- Identify which variable leaves (A1)

	1		2		3		4		Supply
	2		3		11		7		
A		1-z		5		z			6
	1		0		6		1		
B				0		1			1
	5		8		15		9		
C		6+z		0		2-z		2	10
Demand		7		5		3		2	

Example

- Step 4: Adjust the flows – find new basis (cost = 100)

	1		2		3		4		Supply
	2		3		11		7		
A		0		5		1			6
	1		0		6		1		
B						1			1
	5		8		15		9		
C		7				1		2	10
Demand		7		5		3		2	

- Step 3: Test for optimality again

	1		2		3		4		v_I
	2		3		11		7		
A		-1		0		0		-2	0
	1		0		6		1		
B		-5		-2		0		2	-5
	5		8		15		9		
C		0		-1		0		0	4
w_j		1		3		11		5	