INVENTORY CONTROL (KNOWN DEMAND)
Introduction

"3rd quarter of ’91 total US inventories = $1.03 trillion”

1. Types of Inventories
   • Raw Material - direct & indirect
   • Components
   • Work in Process (WIP)
   • Finished Good

2. Reasons for Inventory Holding
   • Set-up costs / Economies of scale
   • Uncertainties - internal & external
   • Smoothing
   • Speculation, Quantity discounts, Logistics
   • Transportation, Control costs
3. Relevant Costs
   - Holding (carrying): space, tax/insurance, breakage, interest
   - Order: bookkeeping, receiving, handling
   - Shortage (stock-out): lost sales, loss-of-goodwill

4. Characteristics of Inventory Systems
   - Demand: constant / variable; known / random
   - Lead time
   - Review time: continuous / periodic
   - Excess demand
   - Changing Inventory
THE EOQ MODEL
Assumptions

- Demand rate, known, constant = $\lambda$ units/time
- No shortages
- Zero Lead Time (will be relaxed later)
- Costs include
  1. Order Setup cost = $K$
  2. Order per unit cost = $c$
  3. Holding per unit cost per period = $h$

Constant Quantity to be ordered = $Q$
Min. average cost per unit time, $G(Q)$: Find $Q^*$
Ordering cost for each cycle = $K + cQ$
Ordering cost per unit time = $(K + cQ)/T; \ T = Q/\lambda$
Average inventory holding cost = $hQ/2$

$$G(Q) = \frac{K + cQ}{T} + \frac{hQ}{2} = \frac{K + cQ}{Q/\lambda} + \frac{hQ}{2}$$  \hspace{1cm} (18)$$

$$= \frac{K\lambda}{Q} + \lambda c + \frac{hQ}{2}$$  \hspace{1cm} (19)$$

To minimize $G(Q)$ set 1st derivative = 0

$$G'(Q) = -\frac{K\lambda}{Q^2} + \frac{h}{2} = 0$$  \hspace{1cm} (20)$$

$$Q^* = \sqrt[3]{\frac{2K\lambda}{h}}$$

Check $G''(Q)$

$$G''(Q) = \frac{2K\lambda}{Q^3} > 0 \ for \ Q > 0$$  \hspace{1cm} (21)$$
Inclusion of Order Lead Time

Order Lead time, $\tau, \tau < T$
Reorder point, $R$: on-hand inventory at order instant
$R = \lambda \tau$
If $\tau > T$, subtract from $\tau$ whole cycles: $R = \lambda$ remainder

- Suggested Reading: Sensitivity, p. 208.
EXTENSIONS TO THE EOQ MODEL
Finite Production Rate

- Finite and constant production rate $= P$ units/time
- Demand rate, known, constant $= \lambda$ units/time
- $P > \lambda$ for feasibility
- No shortages
- Costs include
  1. Order Setup cost $= K$
  2. Ignore per unit cost $= c$
  3. Holding per unit cost per period $= h$
Notation & Relations:

- Cycle length $= T$
- $T = T_1 + T_2; \; T_1$ is the uptime, $T_2$ is the downtime
- Maximal inventory level $= H \; (H \neq Q)$
- Per cycle: \# ordered $(Q) = \#$ produced $(PT_1) = \#$ consumed $(\lambda T)$

\[
T_1 = \frac{Q}{P} \quad (22)
\]
\[
\frac{H}{T_1} = P - \lambda \quad (23)
\]
\[
H = Q(1 - \frac{\lambda}{P}) \quad (24)
\]
Average annual cost function:

\[ G(Q) = \frac{K}{T} + \frac{hH}{2} \]

\[ = \frac{K\lambda}{Q} + \frac{hQ}{2}(1 - \lambda/P) \]  \hspace{1cm} (25)

\[ h' = h(1 - \lambda/P) \]  \hspace{1cm} (26)

\[ Q^* = \sqrt{\frac{2K\lambda}{h'}} \]  \hspace{1cm} (27)
Back-ordering model: shortages permitted

- Shortage cost per unit per period $= p$

Notation & Relations:

- $T = T_1 + T_2$; $T_1$ is +ve inv period, $T_2$ is -ve inv period
- Maximal inventory level $= M \ (M \neq Q)$

$$T_1 = \frac{M}{\lambda} \quad (28)$$
$$T_2 = \frac{(Q - M)}{\lambda} \quad (29)$$
Ordering cost per unit time = \( (K + cQ)/T \)
Average inventory holding cost = \( (h \frac{M^2}{2\lambda}) / T \)
Average inventory backlogging cost = \( (p \frac{(Q-M)^2}{2\lambda}) / T \)

\[
Q^* = \sqrt{\frac{2K\lambda(h+p)}{hp}} = Q\, EOQ\sqrt{\frac{h+p}{p}}
\]

\[
M^* = \sqrt{\frac{2K\lambda p}{h(h+p)}} = Q\, EOQ\sqrt{\frac{p}{h+p}}
\]
Quantity Discount Models

- Cost per unit \( c \) is dependent on order size \( Q \)

Discount schedules
- All-Units
- Incremental

Example: all-unit

\[
C(Q) = \begin{cases} 
.30Q & \text{for } 0 \leq Q < 500 \\
.29Q & \text{for } 500 \leq Q < 1000 \\
.28Q & \text{for } 1000 \leq Q 
\end{cases}
\]
EOQ Extensions: Quantity Discounts...

ALGORITHM: all-unit

1. Compute the largest realizable EOQ value.
   1.1. compute EOQ for lowest price.
   1.2. if feasible: stop
   1.3. otherwise: continue with next higher price

2. Compare annual cost at largest EOQ and break-point prices
   2.1. for each break-point price, compute annual cost
   2.2. select lowest of above and EOQ

3. Optimal $Q$ is the one for which annual cost is minimal.
EOQ Extensions: Quantity Discounts...

ALGORITHM: incremental
Example

\[ C(Q) = \begin{cases} 
0.30Q & \text{for } 0 \leq Q \leq 500 \\
150 + 0.29(Q - 500) & \text{for } 500 \leq Q \leq 1000 \\
295 + 0.28(Q - 1000) & \text{for } 1000 \leq Q 
\end{cases} \]

so that

\[ C(Q)/Q = \begin{cases} 
0.30 & \text{for } 0 \leq Q \leq 500 \\
0.29 + 5/Q & \text{for } 500 \leq Q \leq 1000 \\
0.28 + 15/Q & \text{for } 1000 \leq Q 
\end{cases} \]

Average annual cost function:

\[ G(Q) = \lambda C(Q)/Q + K\lambda/Q + I[C(Q)/Q]Q/2 \]
ALGORITHM: incremental

1. Determine the expression for $C(Q)/Q$
2. Define equation of $G(Q)$ using $C(Q)/Q$
3. Compute $Q^*$ for each price interval
4. From realizable $Q^*$'s above pick EOQ corresponding to the lowest annual cost
Resource Constrained Multi-Product Systems

- Multiple products
- Resource constraints: total $ value, space

Downscaling method:
Assumption: \[
\frac{\text{per unit cost or space consumed}}{\text{holding cost}} = \text{constant}
\]
- Compute unconstrained EOQ’s for all products
- If EOQ’s satisfy constraints, stop
- Otherwise, compute smallest downscaling ratio (available/required)
- Downscale every EOQ by the above ratio
Resource Constrained Multi-Product Systems

If assumption does not hold:

- Formulate a Lagrangean Function
- Find partial derivatives of $LF$ and set $= 0$

\[ Q^*_i = \sqrt{\frac{2K_i \lambda_i}{h_i+2\theta w_i}} \]

- Determine Lagrange multiplier $\theta$ by interval bisection

See Appendix 4-A for details