Relative Places

Maureen Donnelly
Department of Philosophy
New York State Center of Excellence in Bioinformatics and Life Sciences
University at Buffalo
135 Park Hall
Buffalo, NY 14260
Telephone: 49-(0)681-982-2821 (until 20/5)
1-716-645-2444 (after 20/5)
Fax: 49-(0)681--302-64-772 (until 20/5)
1-716-645-6139 (after 20/5)
email: maureen.donnelly@ifomis.uni-saarland.de

Abstract. Newton distinguishes between *absolute* and *relative* places. Both types of places endure through time and may be occupied by various objects at various times. But unlike absolute places, each relative place stands in fixed spatial relations with one or more reference objects. Relative places with independent reference objects (e.g. a ship and the earth) may move relative to one another.

Relative places, not absolute places, are used to locate objects and track their movements in common-sense reasoning and in disciplines such as biology, engineering, and geology. The purpose of this paper is to develop a formal theory for reasoning about relative places and their changing relations to both other places and to material objects.

1 Introduction

In the Scholium to the Definitions of the *Principia*, Newton distinguishes between *absolute* and *relative* places. For Newton, both absolute places and relative places endure through time and may be occupied by various material objects at various times. But absolute places are parts of absolute space that are independent of material objects and remain forever in an unchanging arrangement[1]. A relative place, by contrast, stands in fixed spatial relations with one or more material objects, which I will call its *reference objects*. Newton gives as examples of relative places: places in and around a ship whose reference object is the ship, and places in and around the earth whose reference object is the earth. Unlike absolute places, relative places may move relative to one another. This happens when the reference objects for the places move relative to one another. For example, when a ship moves relative to the earth, places with the ship as their reference object (e.g. the ship's hold) move relative to places with the earth as their reference object.

Because absolute places do not stand in fixed relations to objects, we cannot track them over any time interval. Thus, absolute places are not of much use for locating things in the world. Newton himself points out that we use relative places, never absolute places, in ordinary spatial reasoning[2].

I take it that, among other things, relative places include:

- interiors of artifacts such as ovens, cups, rooms, buildings, ships, and subways
- neighborhoods, cities, countries, deserts, and other geographical entities, and
- "organic spaces" such as body cavities and the niches of organisms [15].

Relative places are used to locate objects and track their movements, not only in common-sense reasoning, but also in biology, engineering, meteorology, and other disciplines. In the life sciences, relative places are particularly important since they function as the loci of the specific types of environments which are necessary for organic processes. Relative places are also important in legal contexts. Individuals and institutions may have (or lack) rights to enter or perform specific types of actions within places such as military zones, air traffic corridors, parks, and interiors of airplanes.

Natural language offers imprecise tools for describing spatial structure generally and, in particular, for talking about relations involving places [10][3]. The purpose of this paper is to propose a step in the direction of a more systematic understanding of relative places. I develop here a formal theory for reasoning about relative places and their changing relations to both other places and to material objects. The theory is useful for reasoning about spatial relations among places and the locations and movements of material objects, as well as for distinguishing the different kinds of places (e.g. sealed body cavities vs. open geographic places) which are assumed in various disciplines.

My work is in part inspired by the analyses of spatial relations presented in [4, 5]. Like Casati and Varzi, I am interested in spatial structures that involve entities of various types, including objects, places[4], and holes. But Casati and Varzi distinguish holes from places. For them, both are immaterial but, unlike places, holes can move.[5] By contrast, I assume that all

---

[1] Newton says, for example, that absolute places retain "from infinity to infinity,...the same given position one to another" and that it is possible that there is no body which remains over any interval in the same absolute place (Scholium, IV).

[2] For example, Newton says: "instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs" (Scholium, IV).

[3] Herskovits points out, for example, that the preposition "in" is used in the same grammatical constructions to describe several different spatial relations, some of which involve specific kinds of relative places.

[4] [4, 5], as well as [2, 3, 6, 7, 9], tend to use the term "region" instead of "place".

[5] A similar approach is taken in [3].

places can move (relative to other places) and treat holes as special kinds of places. My approach makes explicit the need for understanding changing relations among places. It also allows for a more economical spatial framework which treats places as the only kind of immaterial entity and does justice to the common nature of the interior of a room and an air traffic corridor. They are both places in which we can move ourselves and other objects, but only the first is a hole.

Formal theories intended for describing changing spatial relations among objects and places are also presented in [2, 6, 7]. As in [3, 4, 5, 9], these treatments assume that relations among places do not change. But unlike [3, 4, 5], holes are not included as special members of the domains. The result is that these theories give us no way of relating immaterial entities that move relative to one another, such an organism's cranial cavity (a hole) and the geographic places through which the organism moves.

The remainder of this paper is organized as follows. §2 is an informal discussion of the role of relative places in spatial reasoning. §3-§5 present a formal theory, Basic Place Theory (BPT), in which we can describe changing relations among places, form sums of mutually fixed places, specify the reference object of a place, and track objects' movements through sums of mutually-fixed places. §3 presents the core time-dependent spatial relations of BPT. §4 introduces a relation for linking places to their reference objects and time-independent relations among places with the same reference objects. §5 develops some formal tools for locating objects and tracking their movements. Finally, in §6, I demonstrate how some important properties of places might be described in terms of their relations to objects.

2 Location-Complexes

Spatial relations between places such as my chest cavity and the interior of my kitchen can change sharply and unpredictably. This morning my chest cavity was inside the interior of my kitchen. I have driven to Berlin during the afternoon and the two places are now separated by 200 kilometers. The distance between the places may increase or decrease tomorrow depending on how far and in which direction I travel.

Fortunately, some places stand in much more stable relations than do my chest cavity and the interior of my kitchen. My abdominal cavity remains next to my chest cavity throughout my life. The interior of my bedroom will remain one meter from the interior of my kitchen until my apartment is destroyed or remodeled.

In practical reasoning, we group together places, like the interior of my bedroom and the interior of my kitchen or my chest cavity and my abdominal cavity, whose spatial relations remain fixed. Such places share a reference object. My apartment is a reference object for both the interior of my bedroom and the interior of my kitchen. My body is a reference object for both my chest cavity and my abdominal cavity. I will call a maximal collection of places with the same reference object a *location-complex*. Examples of location-complexes are: the collection of cavities and pathways in a given body, the collection of places fixed relative to a given ship, and the collection of all places fixed relative to the earth.

I thus assume that reference objects include not only (more or less) rigid objects such a ship and the earth, but also non-rigid objects such as an organism's body. This assumption fits the way in which body cavities and pathways are treated in bio-medical contexts—as enduring immaterial entities whose boundaries are fixed by the organism's material parts. However, it would seem that the location-complexes determined by such non-rigid reference objects must have a different type of spatial structure than those determined by rigid objects. For example, the interior of a rigid box is divided into parts of determinate shapes and sizes which remain at fixed distances from one another. By contrast, the places within an organism do not have precisely fixed metrical properties — their shapes, sizes, and distances from one another are in general merely constrained to given ranges. I will not, however, attempt in this paper to characterize such differences in the geometrical structure of location-complexes. This is an important issue for future work. Throughout this paper, I focus on only the topological properties of location-complexes and I assume that each location-complex (whether or not its reference object is rigid) has fixed topological properties.

A location-complex functions as a stable array within which we can locate objects and across which we can track their movements. In many contexts, we limit possible places to those of a single location-complex. For example, when tracking the movements of an object on a ship, we normally consider only places on the ship. We might say that yesterday the harpoon was in the galley but today it is in the forehold. In certain circumstances (e.g. when we are nowhere near the ship), we may instead use geographic places. But we do not usually mix the two groups of places together. We would not describe the harpoon’s movements by saying that, yesterday it was in the galley, this morning it was fifty miles south of Hawaii, and now it is in the forehold.

Time-independent relations usually suffice for describing the arrangement of places within a location-complex. To describe the layout of a ship, we say "the forehold is below the forecastle, the main hold is behind the forehold", and so on, not "the main hold is today at noon behind the forehold".

Certain situations, however, require reasoning about changing relations among places with different reference objects. The purpose of a vehicle is that it provides a collection of human-inhabitable places that can move with respect to certain other places (usually places with the earth as a reference object). When we take a subway between two stations, we need to know not only the fixed positions of the stations in their location-complex (which includes other stations, neighborhoods, parks etc). We must also know that our subway car and the places within it will be at one time contained in the first station and at a later time will be contained in the second station. It is because of the changes in the spatial relations holding between places in the two location-complexes that we can remain in the same place on the subway car while moving from station to station. Reasoning about changing relations among places is also important when the places are the loci of specific types of environments. For example, the environmental features (temperature, air pressure, radiation levels) of the places through which an organism moves may affect the specialized environmental features of the places within the organism.

We see then two foundational tasks for a theory of places. One is to provide a mechanism for dividing places into separate location-complexes. Another is to introduce time-dependent relations that may hold between places in different location-complexes. Both tasks are addressed in the formal theory, Basic Place Theory, presented in the next three sections.

3 Basic Place Theory--Time-Dependent Relations

Basic Place Theory should be seen as a first step toward a comprehensive theory of changing relations among places and objects over time. There are several issues which are not addressed in BPT but which may be treated in a more comprehensive theory. In particular, since my focus here is on places, objects are dealt with only insofar as they relate to places, either as their reference objects or as individuals which are located in or move through places. Thus, although I introduce mereological relations for places, I do not introduce mereological relations for objects. This is in part so that I can leave open issues of whether distinct material objects may coincide or more generally whether, when two material objects are partially co-located, they must share a part. However, questions of structural relations among material objects will need to be addressed, e.g., for developing a more precise theory of reference objects.

Basic Place Theory is also simplified by the assumption that neither objects nor places come into or go out of existence throughout the time interval under consideration. We are thus spared the complication of having to explicitly state whether a given object or place exists at a given time -- it is assumed that all objects and places in the domain of the theory endure throughout all times in the domain of the theory. This assumption seems appropriate for most spatial reasoning contexts. We do not usually need to refer to objects or places that do not exist a given time when we are describing the spatial arrangements of objects or places that do exist at that time. A somewhat more complicated theory is required for the rare occasions in which the distinction between the durations of different places or objects plays a role in our reasoning[6].

The domains of Basic Place Theory are partitioned into four nonempty sorts:

time instants, for which the variables s, t are used
time intervals, for which the variables I, J are used and
places, for which the variables w, x, y, z are used.
material objects, for which the variables m, o are used.

All quantification is restricted to a single sort. However, all axioms, definitions, and theorems given in this section for the time-dependent spatial relations apply to both places and objects. To simply the presentation of these formulae, I will use the Greek letters α, β, χ as meta-variables which can stand for either place variables or object variables. Restrictions on quantification will be understood from these conventions on (meta-)variable usage.

I assume that BPT includes a temporal sub-theory, but I leave open the specific form of that sub-theory. In what follows, I make use of a partial ordering, «, on intervals and a binary relation, $\propto$, between instants and intervals where

$$t \propto I$$

is interpreted as:

instant t is in interval I.

I assume that I is a subinterval of J if and only if all instants in I are also in J:

$$I \ll J \leftrightarrow \forall t\, (t \propto I \rightarrow t \propto J)^7$$

that every interval has some proper sub-interval:

$$\exists I (I \ll J \;\&\; I \neq J)$$

and that there are no empty intervals:

$$\exists t\; t \propto I.$$

The time-dependent relations of BPT are introduced in terms of a single primitive -- the ternary relation MT, which holds between an instant and either two places, two objects, or a place and an object. On the intended interpretation,

$$MT_t\alpha\beta$$

means

$$\alpha \text{ meets } \beta \text{ at instant } t$$

[6] One way of constructing such a theory is, roughly, to define that a place or object exists whenever it meets itself. Appropriate existence-at-given-time assumptions should then be added to axioms and definitions.

[7] Throughout this paper, initial universal quantifiers are suppressed.

Places or objects meet when the distance them is zero (where distance is understood as the greatest lower bound of the distance between any point of the first extended entity and any point of the second extended entity). For example, a subway car meets the interior of the station when it is partially or completely within the station or when it first reaches the station. Also, my esophagus meets my stomach and the interior of my coffee cup meets the exterior of my coffee cup.

The first two axioms require that, at a fixed instant, MT is reflexive and symmetric.

(A1)[8] $MT_t\alpha\alpha$ (any place or object meets itself at all times)
(A2) $MT_t\alpha\beta \rightarrow MT_t\beta\alpha$ (if α meets β at time t, then β meets α at time t)

Relations defined in terms of MT include the following.

(D1) $COV_t\alpha\beta =: \forall\chi\ (MT_t\chi\alpha \rightarrow MT_t\chi\beta)$ (α is covered by β at t)
(D2) $ECOIN_t\alpha\beta =: \forall\chi\ (MT_t\chi\alpha \leftrightarrow MT_t\chi\beta)$ (α and β exactly coincide at t)
(D3) $PCOIN_t\alpha\beta =: \exists\chi\ (COV_t\chi\alpha\ \&\ COV_t\chi\beta)$ (α and β partially coincide at t)
(D4) $ABUT_t\alpha\beta =: MT_t\alpha\beta\ \&\ \sim PCOIN_t\alpha\beta$ (α and β abut at t)

It is assumed that all places and objects are three-dimensional, regular, and subdivided into arbitrarily small parts. Thus, the covering relation holds between α and β only when α is located within β. For example, an organism's brain is covered by its cranial cavity and its left heart ventricle is covered by its heart. While the organism occupies a spaceship, the organism and all of its cavities and material parts are covered by the interior of the spaceship. For domains which include lower dimensional boundaries, atomic places, or atomic objects, a slightly different theory is required, since BPT would on these domains conflate the covering relation with the *surrounds* relation[9]. However, for reasons of simplicity, I do not consider such domains in this paper.

An object o exactly coincides with place x when o exactly occupies x. Also, two places can exactly coincide -- when a subway car stops in a station, the interior of the subway car exactly coincides with a part of the interior of the station.

α and β partially coincide when α and β are partially co-located. My esophagus partially coincides with my chest cavity, but also partially coincides with the space of the anterior compartment of my neck. As another example, the right half of my car partially coincides with the front half of my car.

α and β abut when they meet but do not partially coincide. The interior of my hallway abuts the interior of my kitchen. The right half of my car abuts the left half of my car.

At a fixed instant, COV is reflexive and transitive, ECOIN is an equivalence relation, PCOIN is symmetric and reflexive, and ABUT is symmetric and irreflexive.

We can define interval versions of all instant-indexed relations. I will use **bold** text for all interval versions of the relations above. For example:

(D5) $\mathbf{MT}_I\alpha\beta =: \forall t(t \propto I \rightarrow MT_t\alpha\beta)$ (α and β meet throughout interval I)
(D6) $\mathbf{COV}_I\alpha\beta =: \forall t(t \propto I \rightarrow COV_t\alpha\beta)$ (β covers α throughout interval I)

[8] To be precise, (A1) should be read as an abbreviation for two distinct axioms: $MT_t xx$ (every place meets itself) and $MT_t oo$ (every object meets itself). But I will treat (A1) throughout this paper as a single axiom. Similar points apply to the other expressions formulated in terms of the meta-variables.

[9] One way of handling these kinds of domains would be to treat MT and COV as separate primitives. See Chapter 4 of [5] and [12] for discussions of these issues.

The following axiom tells us that when α is not covered by β at t, there is a place or object χ that is covered by α at t, but does not partially coincide with β at t.

(A3) $\sim COV_t\alpha\beta \rightarrow \exists\chi(COV_t\chi\alpha \;\&\; \sim PCOIN_t\chi\beta)$[10]

For example, when the interior of the subway car is not yet fully covered by the interior of the train station, there is some place (e.g. the space in the back of the car) that is covered by the interior of the subway car, but does not partially coincide with the interior of the station.

COV and MT behave much like time-dependent versions of the mereotopological parthood (P) and connection (C) relations axiomatized in [1, 5, 6] in the sense that, at a fixed instant, COV and MT have many of the same logical properties as the parthood and connection relations of these theories.[11] However, the following time-dependent analogue of the antisymmetry axiom for the parthood relation is NOT appropriate for COV:

$COV_t\alpha\beta \;\&\; COV_t\beta\alpha \rightarrow \alpha = \beta$

We want to allow *distinct* places or objects to exactly coincide at an instant. An object may, at a given time, exactly coincide with a given place, but the object is never identical to the place. Also, distinct places can exactly coincide. The interior of the subway car exactly coincides with a part of the interior of the station when the car stops in the station, but these places cannot be identical since they have different reference objects. On the other hand, one might plausibly hold that distinct material objects cannot exactly coincide[12]. If so, BPT could be strengthened through the addition of an axiom requiring that if object o covers object m at time t and m also covers o at t, then o and m are identical.

Note that, more generally, BPT allows a place x to cover either other places or objects which are not part of x. For example, an organism's brain is covered by its cranial cavity, but the brain is not part of the cranial cavity. While the organism is within a spaceship, the organism, its cranial cavity, and its brain are all covered by the interior of the spaceship, but none of these are parts of the interior of the spaceship. Also, objects or places may partially coincide without sharing parts. My esophagus partially coincides with my chest cavity, but nothing is part of both my esophagus and my chest cavity. In the terminology of [8], MT, COV, ECOIN, PCOIN, and ABUT are intended as time-dependent *relative location* relations -- relations which depend only on the spatial entities' locations, not on their mereotopological structure. In BPT, the stronger mereotopological relations (parthood, connection, and so on) are introduced only for places and hold only between places with the same reference object (see Section 4.2).

4 Basic Place Theory--Time-Independent Relations

4.1 Relatively Fixed Places

To divide places into separate location-complexes, I introduce the binary relation RO which holds between an object and a place, where on the intended interpretation

ROox

means

object o is a reference object for place x.

Though the focus of this paper is on relative places, I do not require that every place has a reference object. I thus leave open the possibility that a domain of BPT includes some abso-

---

[10] Compare (A3) to the Strong Supplementation Principle of [14].

[11] See also Simons' treatment of temporary parts in [14].

[12] But see, for example [16], for an argument that distinct material objects may exactly coincide.

lute places. I also do not require that every object is a reference object for some place. For example, completely amorphous objects (e.g. a lump of jelly) are perhaps not reference objects for places. I do require that at least some place has a reference object (A4) and that if any two places share a reference object, then they share all reference objects (A5).

(A4) $\exists o \exists x ROox$
(A5) $ROox \ \& \ ROoy \rightarrow \forall m(ROmx \leftrightarrow ROmy)$

Places x and y are relatively fixed if and only if either x and y have a common reference object or neither x nor y has a reference object.

(D7) $RFxy =: \exists o(ROox \ \& \ ROoy) \vee \forall o(\sim ROox \ \& \ \sim ROoy)$
(place x and place y are relatively fixed)

Thus, the forehold and main hold of a ship are relatively fixed -- both places have the ship as a reference object. Also, my chest cavity and abdominal cavity are relatively fixed (both have my body as their reference object), Hawaii and Acapulco are relatively fixed (both have the earth as their reference object), and, if there are absolute places, any two absolute places are relatively fixed (since neither has a reference object).

It follows from (A5) that RF is an equivalence relation on the sub-domain of places.

(T1) $RFxx$ (every place is fixed relative to itself)
(T2) $RFxy \rightarrow RFyx$ (if x is fixed relative to y then y is fixed relative to x)
(T3) $RFxy \ \& \ RFyz \rightarrow RFxz$
(if x is fixed relative to y and y is fixed relative to z, then x is fixed relative to z)

Each RF equivalence class corresponds to a location-complex. One of these classes might include all places with a particular ship as their reference object and another might contain all places with the earth as their reference object.

The next axioms tie RO and RF to the spatial relations defined in the previous subsection.

(A6) $ROox \rightarrow \exists y(RFxy \ \& \ \forall t \ ECOIN_t oy)$ (if o is a reference object for x, then there is some place y such that x and y are relatively fixed and o exactly coincides with y at all instants)

(A7) $COV_t \alpha y \rightarrow \exists z(RFzy \ \& \ ECOIN_t z\alpha)$ (if place y covers α at t, then there is some place z such that z and y are relatively fixed and z exactly coincides with α at t)

(A8) $RFxy \rightarrow (\exists t \ MT_t xy \rightarrow \forall t \ MT_t xy)$ (if x and y are relatively fixed, then x and y meet at some instant only if x and y meet at all instants)

(A9) $RFxy \rightarrow (\exists t \ COV_t xy \rightarrow \forall t \ COV_t xy)$ (if x and y are relatively fixed, then y covers x at some instant only if y covers x at all instants)

(A6) requires that the reference object for place y always occupies a fixed place in y's location-complex. For example, the location-complex of places with a ship as their reference object includes the place which the ship itself always occupies.

(A7) ensures that places are divided into smaller places within a location-complex in a way that matches the divisions imposed by objects or by places outside of the location-complex. It tells us, for example, that if a subway car is within a station at t, then there is a part of the interior of the station which exactly coincides with the subway car at t and also a part of the interior of the station which exactly coincides with the interior of the subway car at t.

(A8) and (A9) require that MT and COV relations among members of the same location-complex never change. For example, the interior of my kitchen will never move away from the interior of the hallway which it meets (though it is possible to destroy one or both of these places and create new places in the apartment which are some distance apart). Also, the left half of the interior of my kitchen is always be covered by the interior of my kitchen.

Using (A8) and (A9), we can derive the following theorems, requiring that ECOIN, PCOIN, and ABUT relations among places in the same location-complex never change.

(T4) $RFxy \rightarrow (\exists t\ ECOIN_t xy \rightarrow \forall t\ ECOIN_t xy)$ (if x and y are relatively fixed, then if x and y exactly coincide at some instant, x and y exactly coincide at all instants)

(T5) $RFxy \rightarrow (\exists t\ PCOIN_t xy \rightarrow \forall t\ PCOIN_t xy)$ (if x and y are relatively fixed, then if x and y partially coincide at some instant, x and y partially coincide at all instants)

(T6) $RFxy \rightarrow (\exists t\ ABUT_t xy \rightarrow \forall t\ ABUT_t xy)$ (if x and y are relatively fixed, then if x and y abut at some instant, x and y abut at all instants)

4.2 Mereotopological Relations among Places

The converses of (A8)-(A9) and (T4)-(T6) do not in general hold. My cranial cavity may always be covered by, but not fixed relative to, a geographic place such as Europe. In this case, I never leave Europe, though I may move around extensively within it. A much stronger relation holds between a geographic place and another geographic place that it covers. Unlike my cranial cavity, Scandinavia is a part of Europe.

Mereotopological relations among places are in BPT defined in terms of RF and the time-dependent spatial relations. Note that these mereotopological relations are time-independent -- unlike objects, places do not gain and lose parts.

(D8) $Pxy =: RFxy\ \&\ \forall t\ COV_t xy$	(place x is part of place y)
(D9) $Oxy =: RFxy\ \&\ \forall t\ PCOIN_t xy$	(place x overlaps place y)
(D10) $Cxy =: RFxy\ \&\ \forall t\ MT_t xy$	(place x and place y are connected)
(D11) $ECxy =: RFxy\ \&\ \forall t\ ABUT_t xy$	(place x and place y are externally connected)

The relations defined in (D8) - (D11) behave very much like the mereotopological relations of [8]. P is reflexive and transitive, O and C are reflexive and symmetric, and EC is irreflexive and symmetric. We can also derive the following theorems, which are common to most mereotopologies.

(T7) $Oxy \leftrightarrow \exists z\ (Pzx\ \&\ Pzy)$	(x and y overlap iff there is some place z that is part of both x and y)
(T8) $ECxy \leftrightarrow Cxy\ \&\ {\sim}Oxy$	(x and y are externally connected if and only if x and y are connected but do not overlap)
(T9) $Pxy \rightarrow \forall z\ (Czx \rightarrow Czy)$	(if x is part of y then every place connected to x is also connected to y)

(T10) $\sim Pxy \rightarrow \exists z\ (Pzx\ \&\ \sim Ozy)$ (if x is not part of y, then there is a part z of x that does not overlap y)

To obtain a stronger mereotopology, I add to BPT axiom (A10) and axiom schema (A11) (where ϕx represents any formula in which only x occurs free and ϕy is the same formula with x replaced by y but with variable substitution performed as necessary so that y is free in ϕy exactly where x is free in ϕx).

(A10) $Pxy\ \&\ Pyx \rightarrow x = y$
(if x is part of y and y is part of x, then x and y are identical)

(A11) $\exists x\ \phi x\ \&\ \forall x\ \forall y(\phi x\ \&\ \phi y \rightarrow RFxy) \rightarrow \exists z\ \forall w(Owz \leftrightarrow \exists x\ (\phi x\ \&\ Oxw))$
(if some place satisfies ϕ and all places satisfying ϕ are mutually fixed, then there is a sum of all ϕ-ers)

Using (A10) and (T10), we can prove that O is extensional:

(T11) $x = y \leftrightarrow \forall z\ (Ozx \leftrightarrow Ozy)$

Thus, whenever the antecedent of (A11) is satisfied, there is a *unique* sum of all ϕ-ers. For example, there is a unique (disconnected) place which is the sum of the interior of my kitchen and the interior of my bedroom.

4.3 The Formal Definition of a Location-Complex

Location-complexes are introduced in BPT as the sums of all places fixed relative to a given place.

(D12) $LCXxz =: \forall w(Owz \leftrightarrow \exists y\ (RFxy\ \&\ Oyw))$ (z is x's location-complex)[13]

The location-complex of a geographic place is the sum of all places with the earth as a reference object. The location-complex of a place on a ship is the sum of all places with the ship as a reference object. If there are any absolute places (i.e. places without a reference object), then the location-complex of any absolute place is the sum of all absolute places.

It follows from (A11) and (T11) that every place has a unique location-complex.

(T12) $\exists z\ LCXxz$
(T13) $LCXxy\ \&\ LCXxz \rightarrow y = z$

Thus, LCX is a function on the sub-domain of places. For convenience, I will introduce the function term lcx(x) to stand for place x's location-complex.

Additional theorems concerning location-complexes can be derived.

(T14) $RFxy \leftrightarrow lcx(x) = lcx(y)$ (place x and place y are relatively-fixed iff they have the same location-complex)
(T15) $RFxy \leftrightarrow P(x, lcx(y))$ (place x and place y are relatively-fixed iff x is part of y's location-complex)

[13] Compare with the definition of the layer-of relation in [8]. It is easy to verify that all axioms of Layered Mereology are theorems of BPT and that location-complexes are the layers of BPT.

(T16) $RFxy \leftrightarrow \exists z\,(Pxz \,\&\, Pyz)$ (place x and place y are relatively-fixed iff they are both parts of some place z)

A location-complex is any place that is the location-complex for some place.

(D13) $LCz =: \exists x\, LCXxz$ (z is a location-complex)

All location-complexes are maximal places in the sense that they overlap only their own parts.

(T17) $LCz \rightarrow \forall y(Oyz \rightarrow Pyz)$ (if z is a location-complex, then any place y overlaps z only if y is part of z)

However, there is still a sense in which we might distinguish larger and smaller location-complexes. For example, the location-complex of geographic places should always cover the location-complex of places which have my body as a reference object, but it does not seem that the converse holds. We will call a place *comprehensive* if it always covers every place (and thus also every location-complex) and every object.

(D14) $CMPy =: \forall t \forall \alpha\, COV_t\, \alpha y$ (y is a comprehensive place)

Any comprehensive place is a location-complex

(T18) $CMPy \rightarrow LCy$ (if y is a comprehensive place, then y is a location-complex)

Comprehensive location-complexes mirror the universe in the sense that, if y is comprehensive, then for every instant t and every place or object α, there is a part of y that exactly coincides with α at t.

(T19) $CMPy \rightarrow \forall t \forall \alpha \exists z(Pzy \,\&\, ECOIN_t \alpha z)$

Comprehensive location-complexes are useful because they function as a background space in terms of which we can relate non-coincident objects or places. In many contexts, the location-complex of geographic places is assumed to be comprehensive. We can use it to relate places on two separate moving ships or within two separate organisms. Alternatively, if we allow any absolute places at all, then we would probably hold that the location-complex of absolute places is comprehensive.

It does not follow from the axioms given so far that there is any comprehensive place. If desired, this could be required by an additional axiom.

4.4 Mathematical Models of BPT

To show that BPT is consistent and to better illustrate the intended interpretations of BPT's relations, I present in this subsection one class of models for BPT. However, I do not here attempt to construct a class of models whose generality matches that of BPT. It will be clear that all models described below share specific mathematical properties which are not required by BPT. For example, all object and place movements in the models are continuous. But nothing in BPT requires object or place movements to be continuous. Where appropriate, more complex extensions of BPT can be constructed which capture these kinds of properties.

But, in fact, the particular models presented here do exclude some types of location-complexes that we would like to represent if we allow for flexible reference objects, such as organisms' bodies. These location-complexes do not (like the location-complexes in the models presented here) preserve all of their geometrical properties over time, but may shrink, grow, and so on. Models which can adequately represent such location-complexes are more complicated than the models presented here, but can be developed along the same lines.

For the presentation of the models, I will make use of $\Re$ (the real numbers) and $\Re^3$. I assume that both $\Re$ and $\Re^3$ are endowed with the usual Euclidean metric.

Throughout this subsection, all variables are used only for the mathematical objects under discussion and should not be confused with the variables belonging to the formal language in which BPT is stated.

The domains of the models are divided into four disjoint sets -- *T*, *I*, *Pl*, *Ob* -- where time instant variables of BPT are interpreted as elements of *T*, time interval variables are interpreted as elements of *I*, place variables are interpreted as elements of *Pl*, and object variables are interpreted as elements of *Ob*.

T may be any non-degenerative interval of $\Re$.[14] In particular, *T* may be $\Re$ itself. Given *T*, *I* is fixed as the set of all non-degenerative intervals which are included in *T*. In other words, $I =: \{I : I$ is a non-degenerative interval of $\Re$ and $I \subseteq T\}$.

The members of *Pl* and *Ob* are considerably more complicated than those of *T* and *I*. To construct them, I make use of functions from $T \times \Re^3$ to $\Re^3$. I will also refer frequently to the set of all closed, regular, non-empty subsets of $\Re^3$ which I call *CR*.

$CR =: \{X : \varnothing \neq X \subseteq \Re^3$ and X is closed, regular$\}$

Let f: $T \times \Re^3 \to \Re^3$, let $t \in T$, and let $X \subseteq \Re^3$. Then by f_t, I mean the function f_t: $\Re^3 \to \Re^3$ such that for all $x \in \Re^3$,

$$f_t(x) = f(\langle t, x\rangle).$$

By f^X, I mean the function f^X: $T \to \wp(\Re^3)$ (where $\wp(\Re^3)$ is the power set of $\Re^3$) such that for all $t \in T$,

$$f^X(t) = f_t[X] = \{y \in \Re^3 : \text{for some } x \in X, y = f_t(x)\}.$$

Let

$F =: \{f: T \times \Re^3 \to \Re^3$: f is continuous and for all $t \in T$, f_t: $\Re^3 \to \Re^3$ is an isometry$\}$.

Notice that for any $f \in F$, $t \in T$, and X, $Y \subseteq \Re^3$:

- $f^X(t) \cap f^Y(t) \neq \varnothing$ if and only if $X \cap Y \neq \varnothing$
- $f^X(t) \subseteq f^Y(t)$ if and only if $X \subseteq Y$
- $f^X(t) \in CR$ if and only if $X \in CR$.

Let *LC* be any non-empty set of ordered triples $\langle X, f, 1\rangle$, with $X \in CR$ and $f \in F$, satisfying the condition:

if $\langle X, f, 1\rangle, \langle Y, g, 1\rangle \in LC$, and f = g, then X = Y.

LC is the set of location-complexes in the model.

Let $Pl =: \{\langle X, f, 1\rangle : X \in CR$ and there is some $\langle Y, f, 1\rangle \in LC$ such that $X \subseteq Y\}$. Notice that $LC \subseteq Pl$.

Let *Ob* be any set of ordered triples $\langle X, f, 0\rangle$, with $X \in CR$ and $f \in F$, satisfying the following conditions:

i) if $\langle Y, f, 0\rangle \in Ob$, $X \subseteq Y$, and $X \in CR$, then $\langle X, f, 0\rangle \in Ob$.

ii) if $\langle X, f, 0\rangle \in Ob$ and $\langle Y, f, 1\rangle \in LC$, then $X \subseteq Y$.

[14] An *interval* I of $\Re$ is any subset of $\Re$ with the following property: if x, y $\in$ I and x < z < y, then z $\in$ I. In other words, an interval of $\Re$ is any convex subset of $\Re$. A *non-degenerative* interval of $\Re$ is one that includes at least two (and thus uncountably many) real numbers.

iii) for all but at most one $<Y, f, 1> \in LC$, there is some $X \in CR$ such that $<X, f, 0> \in Ob$ and there is at least one $<Y, f, 1> \in LC$, such that $<X, f, 0> \in Ob$.

As interpretations of the primitive relations in BPT, I introduce the following two relations. **MT** is a ternary relation on $T \times (Pl \cup Ob) \times (Pl \cup Ob)$ and **RO** is a binary relation on $Ob \times Pl$.

$$(t, <X, f, i>, <Y, g, j>) \in \textbf{MT} \text{ if and only if } f^X(t) \cap g^Y(t) \neq \varnothing.$$

$$(<X, f, 0>, <Y, g, 1>) \in \textbf{RO} \text{ if and only if } f = g.$$

It is straightforward to check that, given these interpretations of the BPT primitives MT and RO, BPT's axioms are satisfied and defined relations have the following interpretations:

- COV is interpreted as: **COV** on $T \times (Pl \cup Ob) \times (Pl \cup Ob)$, where $(t, <X, f, i>, <Y, g, j>) \in \textbf{COV}$ iff $f^X(t) \subseteq g^Y(t)$.
- ECOIN is interpreted as: **ECOIN** on $T \times (Pl \cup Ob) \times (Pl \cup Ob)$, where $(t, <X, f, i>, <Y, g, j>) \in \textbf{ECOIN}$ iff $f^X(t) = g^Y(t)$.
- PCOIN is interpreted as: **PCOIN** on $T \times (Pl \cup Ob) \times (Pl \cup Ob)$, where $(t, <X, f, i>, <Y, g, j>) \in \textbf{PCOIN}$ iff there is some $Z \in CR$ such that $Z \subseteq f^X(t) \cap g^Y(t)$.
- ABUT is interpreted as: **ABUT** on $T \times (Pl \cup Ob) \times (Pl \cup Ob)$, where $(t, <X, f, i>, <Y, g, j>) \in \textbf{ABUT}$ iff $f^X(t) \cap g^Y(t) \neq \varnothing$ and there is no $Z \in CR$ such that $Z \subseteq f^X(t) \cap g^Y(t)$.
- RF is interpreted as: **RF** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{RF}$ iff $f = g$.
- P is interpreted as: **P** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{P}$ iff $f = g$ and $X \subseteq Y$.
- O is interpreted as: **O** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{O}$ iff $f = g$ and there is some $Z \in CR$ such that $Z \subseteq X \cap Y$.
- C is interpreted as: **C** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{C}$ iff $f = g$ and $X \cap Y \neq \varnothing$.
- EC is interpreted as: **EC** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{EC}$ iff $f = g$, $X \cap Y \neq \varnothing$, and there is no $Z \in CR$ such that $Z \subseteq X \cap Y$.
- LCX is interpreted as: **LCX** on $Pl \times Pl$, where $(<X, f, 1>, <Y, g, 1>) \in \textbf{LCX}$ iff $f = g$ and $<Y, g, 1> \in LC$.
- LC is interpreted as: LC.
- CMP is interpreted as: $\textbf{CMP} \subseteq LC$, where $<X, f, 1> \in \textbf{CMP}$ iff for all $t \in T$ and all $<Y, g, 1> \in Pl$, $g^Y(t) \subseteq f^X(t)$.

As an illustration of what one of these models might look like, I will give a very simple example. In presenting the example, I will make use of the notation

$$B^r_{xyz}$$

for the closed ball in $\Re^3$ centered at $<x, y, z>$ with radius r. For example, B^1_{000} is the closed ball centered at the origin (i.e., $< 0, 0, 0>$) with radius 1.

In the example model, $T = \Re$ and I is the set of all non-empty intervals of $\Re$.

For Pl and Ob, I make use of the following functions from $\Re \times \Re^3$ to $\Re^3$:

$$Id(<t, <x, y, z>>) = <x, y, z>$$
$$f(<t, <x, y, z>>) = <x + t, y, z>$$
$$g(<t, <x, y, z>>) = <x, y + t, z>$$

Location-complexes, places, and objects are as follows:

$LC = \{< \Re^3, Id, 1>, <B^2_{000}, f, 1>, <B^2_{030}, g, 1>\}$

$Pl = \{\langle X, Id, 1\rangle: X \in CR\} \cup \{\langle X, f, 1\rangle: X \in CR \text{ and } X \subseteq B^2{}_{000}\} \cup \{\langle X, g, 1\rangle: X \in CR$ and $X \subseteq B^2{}_{030}\}$
$Ob = \{\langle X, f, 0\rangle: X \in CR \text{ and } X \subseteq B^1{}_{000}\} \cup \{\langle X, g, 0\rangle: X \in CR \text{ and } X \subseteq B^1{}_{030}\}$

In this model, $\langle \mathfrak{R}^3, Id, 1\rangle$ is the only member of **CMP**. Note also that none of the members of $\{\langle X, Id, 1\rangle: X \in CR\}$, including $\langle \mathfrak{R}^3, Id, 1\rangle$, has a reference object. Thus, they represent absolute places.

Examples of changing relations among members of the domain are:
1. $(0, \langle B^2{}_{000}, f, 1\rangle, \langle B^1{}_{030}, g, 0\rangle) \in$ **ABUT**, but for $t > 0$, $(t, \langle B^2{}_{000}, f, 1\rangle, \langle B^1{}_{030}, g, 0\rangle) \notin$ **ABUT**. (In fact, for $t > 0$, $(t, \langle B^2{}_{000}, f, 1\rangle, \langle B^1{}_{030}, g, 0\rangle) \notin$ **MT**.)

2. $(0, \langle B^2{}_{000}, f, 1\rangle, \langle B^2{}_{030}, g, 1\rangle) \in$ **PCOIN**, but $(1, \langle B^2{}_{000}, f, 1\rangle, \langle B^2{}_{030}, g, 1\rangle) \notin$ **PCOIN**.

3. $(0, \langle B^1{}_{000}, Id, 1\rangle, \langle B^1{}_{000}, f, 0\rangle)$, $(1, \langle B^1{}_{100}, Id, 1\rangle, \langle B^1{}_{000}, f, 0\rangle)$, $(2, \langle B^1{}_{200}, Id, 1\rangle, \langle B^1{}_{000}, f, 0\rangle) \in$ **ECOIN**. More generally, $(t, \langle B^r{}_{xyz}, Id, 1\rangle, \langle B^1{}_{000}, f, 0\rangle) \in$ **ECOIN** if and only if $r = 1$, $x = t$, and $y = 0 = z$.

Examples of unchanging relations among places are among places and objects are:

4. $(\langle B^1{}_{000}, f, 0\rangle, \langle B^2{}_{000}, f, 1\rangle)$, $(\langle B^1{}_{030}, g, 0\rangle, \langle B^2{}_{030}, g, 1\rangle) \in$ **RO**.

5. $(\langle B^1{}_{000}, f, 1\rangle, \langle B^2{}_{000}, f, 1\rangle) \in$ **P**.

6. $(\langle B^1{}_{-100}, f, 1\rangle, \langle B^1{}_{100}, f, 1\rangle) \in$ **EC**.

Figure 1 presents two-dimensional views of places and objects in the model at times 0 and 1.

5 Locating and Tracking the Movements of Objects

The purpose of this section is to demonstrate how BPT can be used to represent and reason about the locations of objects in places or the movements of objects through places. Though I focus here on object-place relations, all of the relations defined in this section can also be used to describe the movements of places through location-complexes.

We locate an object within a location-complex by determining which places in the complex cover the object. For example, a doctor may locate a parasite in a patient's body by determining which cavities or passageways it is contained in. Or, someone may locate my keys by determining which (appropriately small) parts of the interior of my apartment cover them.

An object is *exactly located* at a place with which it exactly coincides. It follows from (A7) that an object is exactly located at more than one place when it is covered by more than one location-complex. When my train stops in a station, I exactly coincide with both a part of the interior of the train and a part of interior of the station.

We can, however, prove that an object or place exactly coincides with at most one place at a time in a given location-complex.

(T20) $RFxy \;\&\; ECOIN_t\alpha x \;\&\; ECOIN_t\alpha y \rightarrow x = y$
(if place x and place y are relatively fixed and α exactly coincides with both x and y at t, then x and y are identical)

An object need not occupy any place in a given location-complex. I do not coincide with any place in a location-complex that is limited to places within your body. But an object must at all times occupy a place in a comprehensive location-complex.

(T21) $CMPz \rightarrow \exists x(Pxz \,\&\, ECOIN_t\alpha x)$
(if z is comprehensive, then there is some part x of z with which α exactly coincides at t)

All movement is relative to a location-complex. An object may at the same time move through one location-complex and rest within another location-complex. I move with respect to the location-complex of geographic places while I stay in the same place on my train.

I will say that α *moves in* location-complex z during interval I if α is covered by z throughout I and α is exactly located at more than one place in z during I. (But note that this relation covers movement only in the sense of re-location. A slightly more complex relation is needed to capture also cases such as the rotation of a ball in a fixed place.)

(D15) $MOV\text{-}IN_I\alpha z =: LCz \,\&\, \mathbf{COV}_I\alpha z \,\&\, \exists s, t, x, y\ (s, t \propto I \,\&\, Pxz \,\&\, Pyz \,\&\, x \neq y \,\&\, ECOIN_s\alpha x \,\&\, ECOIN_t\alpha y)$ (α moves within z during I)

By contrast α *rests within* location-complex z throughout I if there is a part of z which exactly coincides with α throughout I.

(D16) $RT\text{-}IN_I\alpha z =: LCz \,\&\, \exists x\ (Pxz \,\&\, \mathbf{ECOIN}_I\alpha x)$ (α rests within z throughout I)

For example, a harpoon rests within the complex of places on a ship throughout I when it occupies the same place on the ship throughout I. It moves in this complex during I when it remains on the ship throughout I but occupies different places on the ship during I (for example, the harpoon may be in the galley during one sub-interval of I and on the deck during another sub-interval of I).

Note that $\sim MOV\text{-}IN_I\alpha z$ does not imply $RT\text{-}IN_I\alpha z$. $MOV\text{-}IN_I\alpha z$ may fail to hold either because z is not a location-complex or because α is not covered by z throughout I. In both of these cases, $RT\text{-}IN_I\alpha z$ also fails to hold. But we can prove:

(T22) $LCz \,\&\, \mathbf{COV}_I\alpha z \,\&\, \sim MOV\text{-}IN_I\alpha z \rightarrow RT\text{-}IN_I\alpha z$
(if z is a location-complex that covers α throughout I and α does not move within z during I, then α rests within z throughout I)

Note also that it follows from (A6) that any reference object for a place rests within that place's location-complex throughout every interval (T23). In particular, if z is a location-complex and o is a reference object for z, then o rests within z throughout every interval (T24).

(T23) $ROox \rightarrow RT\text{-}IN_I(o, lcx(x))$
(T24) $ROoz \,\&\, LCz \rightarrow RT\text{-}IN_Ioz$

If desired, other types of moving or resting relations can be introduced. For example, we could say that α moves within location-complex z *throughout* I if and only if α moves within z during every sub-interval of I. Also, with a parthood relation on objects, we could define a weaker movement relation which holds between α and z on I if and only if some part of α moves within z during I. The corresponding stronger rest relation would hold between α and z if and only if all parts of α rest within z throughout I.

In many contexts, it is useful to know the path that an object has followed through a given location-complex. For example, a doctor may want to know the path that a parasite has followed through a patient's body or an ecologist may want to know the path that an organism has taken between two geographic places. We can define the *path* y of α through location-complex z over interval I as the sum of all places in z with which α exactly coincides at some instant in I.

(D17) $\mathrm{PATH_I}\alpha zy$ =: $\mathrm{LC}z$ & $\forall w(\mathrm{O}wy \leftrightarrow \exists x\, \exists t\, (\mathrm{P}xz\ \&\ t \propto \mathrm{I}\ \&\ \mathrm{ECOIN}_t\alpha x\ \&\ \mathrm{O}wx))$
(y is the path of α through location-complex z over interval I)

When it exists, the path of α through z over I is a unique place in z.

α does not have any path through location-complex z over I if α is not covered by z at any instant in I. But every object or place has a path through a comprehensive location-complex over every interval.

(T25) $\mathrm{CMP}z \rightarrow \exists y\ \mathrm{PATH_I}\alpha zy$

We can use paths to give alternative characterizations of the *rests within* and *moves within* relations. α rests within location-complex z throughout I if and only if α has a path y in z over I and α exactly coincides with y throughout I.

(T26) $\mathrm{RT\text{-}IN_I}\alpha z \leftrightarrow \exists y(\mathrm{PATH_I}\alpha zy\ \&\ \mathbf{ECOIN}_\mathrm{I}\alpha y)$

α moves within location-complex z during I if and only if α has a path y in z over I and y covers α throughout I but does not exactly coincide with α throughout I.

(T27) $\mathrm{MV\text{-}IN_I}\alpha z \leftrightarrow \exists y(\mathrm{PATH_I}\alpha zy\ \&\ \mathbf{COV}_\mathrm{I}\alpha y\ \&\ {\sim}\mathbf{ECOIN}_\mathrm{I}\alpha y)$

6 Properties of Places

Leibniz' objection to Newton's theory of absolute space was based in part on the assumption that "space being uniform, there can be neither any external nor internal reason by which to distinguish its parts and make any choice among them"[11]. But clearly relative places can have important properties that make them more or less appropriate targets for our actions. I go into my bedroom (not onto the street) when I want to sleep. I put milk in my refrigerator (not in my oven) so that it will keep.

Obviously the purely geometrical properties of a place, such as its size and shape, are important in decisions about where we should locate ourselves or other objects. I cannot put my sewing machine inside my trunk if its diameter is larger than that of the trunk's interior. Other important properties of places depend on relations between places and objects. The purpose of this section is to briefly indicate how a few of these properties might be defined in BPT or an extension of BPT.

We can say that place x is *filled throughout* interval I if there is some object that covers x throughout I.

(D18) $\mathrm{filled_I}(x)$ =: $\exists o\ \mathbf{COV}_\mathrm{I}xo$ (x is filled throughout interval I)

If we assume that distinct objects cannot occupy the same place at the same time, then no object (other than a part of the filler) can move into or out of a filled place while it is filled. For example, I cannot move my desk into a place occupied by my sofa while my sofa remains in this place.

It follows immediately from (D18), that if x is filled throughout I, then every part of x is filled throughout every subinterval of I.

(T28) $Pyx \,\&\, filled_I(x) \,\&\, J \ll I \rightarrow filled_J(x)$

But we cannot infer from $filled_I(x)$ that any part of x is filled on intervals before or after I. Although a given part of the interior of my living room is currently covered by my sofa, I can always move the sofa to a different place. However, in the special case of a place which is covered by one of its reference objects, we cannot move the filler away since the place stands always in a fixed relation to its reference objects. For example, I cannot move my house away from the fixed place it occupies relative to its interior.

We can introduce a stronger predicate for places that are filled by their reference objects.

(D19) $N\text{-}filled(x) =: \exists o \exists t(ROox \,\&\, ECOIN_t ox)$ (x is necessarily filled)

It follows from the axioms of BPT that if x is a necessarily filled place, then x is filled on every interval.

(T29) $N\text{-}filled(x) \rightarrow filled_I(x)$

If desired, additional filling relations can be defined. For example, we can say that x is *partly filled throughout* I if and only if some part of x is filled throughout I or that x is *filled during* I if and only if x is filled throughout some subinterval of I.

Place x is *free throughout* I if no part of x is filled throughout any subinterval of I.

(D20) $free_I(x) =: \forall J \, \forall y(J \ll I \,\&\, Pyx \rightarrow \sim filled_J(y))$ (x is free throughout I)

Objects might move into or through any part of a free place. The interior of a room is free while it contains no furniture or other stationary objects. Note that x may be free throughout I, but never empty. The interior of the room always contains air, but the air moves continuously through it. On the other hand, if the interior of the room contains an object, such as a sofa, which rests within it on any subinterval of I, then the interior of the room is not free throughout I. It may, however, be free on subintervals of I (shorter periods when all of its contents are moved or removed) and it will almost certainly have proper parts (e.g. the space above, in front of, or underneath the sofa) that are free throughout I.

Additional relations can be defined. We can say that x is *partly free throughout* I if and only if some part of x is free throughout I, that x is *free during* I if and only if x is free throughout some subinterval of I, or that x is *empty* at instant t if and only if no object partially coincides with x at t[15].

Further properties of places can be distinguished in an extension of BPT which includes more complex spatial relations. A *convex* place x is one which covers any line segment connecting two parts of x. For example, a ball-shaped place is convex and a doughnut-shaped place is not convex. The *convex hull* of a place x is the smallest convex place including x. For example, the convex hull of a doughnut-shaped place x includes both x and the space in the middle of x.

[15]Also, with additional predicates distinguishing different kinds of objects (organisms, portions of toxic substances, vehicles, etc), we could define relations that distinguish places which contain no objects of a given type over a given interval.

A convex hull operator has been added to a formal theory somewhat similar to BPT in [6, 7][16]. I will not here attempt to add such an operator to BPT, but will briefly describe some of the properties of places that could be defined in a stronger theory which includes this operator. Some formal definitions along these lines (but without time-dependent relations, a distinction between places and objects, or a distinction between places in separate location-complexes) are given in [7].

In the appropriate extension of BPT, the convex hull operator is restricted to places and I assume for now that every location-complex is convex and every place has a convex hull. For any place, x, ConHull(x) is interpreted as the smallest convex part of x's location-complex which covers x. (In fact though, it is not clear exactly how we should handle a convex hull operator in the case of places such as an abdominal cavity or stomach cavity whose shape can change significantly. I will not consider these issues here, but leave open the possibility that the convex hull operator may be restricted to a sub-domain of places with appropriately rigid reference objects.)

We can say that place y is *sheltered* throughout I if and only if there is some object o and place x such that o exactly coincides with x throughout I, x is self-connected, and y is covered throughout I by ConHull(x). Here, a place z is *self-connected* (SC) if and only if it is not the sum of two disconnected places. (This predicate can be defined in BPT.)

$\text{sheltered}_I(y) =: \exists o \exists x\ (\mathbf{ECOIN}_I(o, x)\ \&\ \text{SC}x\ \&\ \mathbf{COV}_I(y, \text{ConHull}(x)))$

For example, the interior of a house, a village in a valley, and the interior of a tunnel are sheltered places as long as the relevant objects surround them[17]. Notice that all parts of these places are also sheltered places.

A place is *open* throughout I if and only if it is not sheltered on any sub-interval of I.

$\text{open}_I(y) =: \forall J(\ J \ll I \rightarrow {\sim}\text{sheltered}_J(y))$

For example, an air traffic corridor and a city on a plain are open places.

A place y is *sealed* throughout I if and only if there are object o and place x such that i) o exactly coincides with x throughout I, ii) y is covered throughout I by ConHull(x), iii) y does not partially coincide with o at any instant in I, and iv) any self-connected place z that partially coincides during I both with y and the exterior of ConHull(x) also partially coincides with o.

$\text{sealed}_I(y) =: \exists o \exists x\ (\mathbf{ECOIN}_I(o, x)\ \&\ \mathbf{COV}_I(y, \text{ConHull}(x))\ \&\ \forall t(t \propto I \rightarrow {\sim}\text{PCOIN}_t yo\ \&\ \forall z(\text{SC}z\ \&\ \text{PCOIN}_t zy\ \&\ \text{PCOIN}_t zw\ \&\ {\sim}\text{PCOIN}_t w\text{ConHull}(x) \rightarrow \text{PCOIN}_t zo))$

The purpose of conditions iii) and iv) is to prohibit sealed places from coinciding with part of the surrounding object o (condition iii) and to require that the only continuous paths from y out of ConHull(x) are through o (condition iv). Thus, sealed places are places which are completely surrounded (throughout a given interval) by a fixed object. For example, the interior of an unbroken eggshell is sealed and the interior of a spacecraft is sealed while its doors are shut. Sealed places are possible sites of controlled environments. Until the eggshell is broken, its interior is free of environmental substances which might interfere with

[16] See also [3] which defines a convexity predicate using a congruence relation.

[17] Note that the interior of the house and the interior of the tunnel are surrounded by their reference objects. Thus, they are sheltered as long as they exist. The village, by contrast, might continue to exist even if the mountains were leveled. If this happened, it would no longer be a sheltered place.

the developing organism. While the doors of the spacecraft are shut, a specific temperature and air pressure can be maintained in its interior.[18]

Additional properties of places can be introduced in terms of ConHull. For example, [7] proposes a relation intended to distinguish what they call the "containable inside" of a place. Very roughly, place y lies within the containable inside of place x if y is surrounded by x on all but one side. If, in addition, x is the location of an object o then we will say that y is a *container place*. Notice that, if o is a solid object, then y can contain liquids when the openings in o are orientated away from the direction of gravity. Examples of container places are the interior of a coffee cup and the interior of a bottle. Examples of sheltered places which are NOT container places are the interior of a tunnel, the interior of a colander, and the space in between the base and the handle of a coffee cup.

Holes are special kinds of places. All holes are holes in some object, called in [4] the *host* of the hole. My glass is the host of its interior. My skull is the host of my cranial cavity. Using a convex hull operator in an extension of BPT, we can require i) that the host is a reference object for its holes and ii) that all holes are maximal self-connected parts of what is left over when we subtract a place filled by the host from the convex hull of that place. More precisely, let o be a reference object and let x be the unique place which has o as a reference object and which exactly coincides with o. o has a hole only if x is not convex. In other words, o has a hole only if x ≠ ConHull(x). In this case, there is some place, ConHull(x)/x, which is the difference of x in ConHull(x). Any hole in o is a maximal self-connected part of ConHull(x)/x.

However, as is pointed out in [4], not every such place is a hole in o (or in any other object). For example, the space surrounding the stem of a wine glass meets this criterion but is not a hole in the wine glass. The condition stated in the previous paragraph is only a necessary condition, not a sufficient condition, for place x being a hole in object o. To distinguish holes in an extension of BPT, it seems that we have to add a separate primitive relation which states that place x is a hole in object o and which is tied to the convex hull operator and the relations of BPT by a condition such as that stated informally above. But we can at least construe more precise criteria for certain types of holes by drawing upon additional properties and relations. For example, we might require that a *container-hole* in o meets both the condition stated in the previous paragraph and is surrounded on all but one side by o. This more specialized hole relation would hold between the wine glass and its interior but not between the wine glass and the space surrounding its stem.

Other kinds of spatial relations can be introduced to distinguish properties of places which are important in practical contexts. With qualitative distance relations, we could distinguish places as being closer to or further from a distinguished object, such as the earth, the top of a person's skull (for places in a body), or the front of a ship (for places on the ship). With orientation relations determined by the intrinsic orientation of a reference object, we could order geographic places in terms of the cardinal directions or order places in an organism's body along axes determined by the body.

6 Conclusion

[18] It might be useful to introduce place-relative versions of some of the relations above. For example, while the doors and windows of my apartment are shut, the space below my sofa is a sealed place. But this place is not (like parts of the interior of my refrigerator) sealed relative to the interior of my apartment.

Other axiomatic theories of qualitative spatial relations, such as parthood and connection, can be found in [1, 2, 3, 5, 6, 7, 8, 9, 12]. In all of this work, the domains of the spatial theories include (and in [2, 3, 6, 9] are limited to) regions. However, the authors of these theories rarely say whether their regions are supposed to be absolute places, relative places, or something else. The distinction is important -- absolute places cannot be identified over time and thus are not used in practical reasoning. One would expect theories intended for practical applications to deal explicitly with relative places. But in the work cited above, no formal tools are provided for describing changing relations among places and there is no indication of how we might, within the theory, characterize places (as having a certain reference object or as filled, free, open, etc) in terms of their relations to objects.

The goal of this paper has been to construct a theory for describing spatial relations between objects and relative places. Basic Place Theory allows us to identify the reference object of a place, to relate places in different location-complexes, and to describe some important non-geometric properties of a place. Basic Place Theory can be refined, for example, by the addition of axioms requiring that all movement is continuous. It can also be expanded, for example, through the addition of a convex hull operator or orientation relations determined by the intrinsic orientations of reference objects.

Acknowledgements

I am very grateful for the helpful comments of Antony Galton, Barry Smith, and two anonymous reviewers of a previous version of this paper. My work on this paper has been supported by the Wolfgang Paul Program of the Alexander von Humboldt Foundation, the Network of Excellence in Semantic Interoperability and Data Mining in Biomedicine of the European Union, and the project Forms of Life sponsored by the Volkswagen Foundation.

References

[1] N. Asher and L. Vieu (1995), Towards a Geometry of Commonsense: A Semantics and a Complete Axiomatization of Mereotopology, in: *Proceedings of IJCAI´95*, Morgan Kaufmann, San Mateo, CA, 846-852.

[2] B. Bennett (2001), Space, Time, Matter, and Things, in: *Proceedings of FOIS'01*, ACM Press, New York, 105-116.

[3] S. Borgo, N. Guarino, and C. Masalo (1997), An Ontological Theory of Physical Objects, in: *Proceedings of QR´97*, Cortona, Italy, 223-231.

[4] R. Casati and A.C. Varzi (1994), *Holes and Other Superficialities*, MIT Press, Cambridge.

[5] R. Casati and A.C. Varzi (1999), *Parts and Places*, MIT Press, Cambridge.

[6] A.G. Cohn, B. Bennett, J. Gooday, and N.M. Gotts (1997), Qualitative Spatial Representation and Reasoning with the Region Connection Calculus, *Geoinformatica*, 1, 1-44.

[7] A.G. Cohn, D.A. Randell, and Z. Cui (1995), Taxonomies of logically defined qualitative spatial relations, *International Journal of Human Computer Studies* 43, 831-846.

[8] M. Donnelly (2004) A Formal Theory for Reasoning about Parthood, Connection, and Location, *Artificial Intelligence* 160, 145-172.

[9] A. Galton (1995) Towards a qualitative theory of movement, in: *Proceedings of COSIT'95*, Springer-Verlag, New York, 377-396.

[10] A. Herskovits (1986) *Language and Spatial Cognition. An Interdisciplinary Study of the Prepositions in English*, Cambridge University Press, Cambridge.

[11] G.W. Leibniz (1989) *Philosophical Papers and Letters*. Vol. 2, L. Loemker, ed., Kluwer, Dordrecht.

[12] C. Masalo and L. Vieu (1999) Atomicity vs. Infinite Divisibility of Space, in: *Proceedings of COSIT'99*, 235-250.

[13] I. Newton (1966) The Scholium to the Definitions in: *The Mathematical Principles of Natural Philosophy*, A. Motte and F. Cajori, trans., University of California Press, Los Angeles.

[14] P. Simons (1987) *Parts: A Study in Ontology*, Clarendon Press, Oxford.

[15] B. Smith and A.C. Varzi (2000) Surrounding Space: The Ontology of Organism-Environment Relations, *Theory in Biosciences* 121, 139-162.

[16] D. Wiggens (1968) On Being in the Same Place at the Same Time, *Philosophical Review* 77: 90-95.

Figure 1: Two-dimensional views of the location-complexes $< \mathfrak{R}^3, Id, 1>$, $<\mathbb{B}^2_{000}, f, 1>$, and $<\mathbb{B}^2_{030}, g, 1>$ and objects $<\mathbb{B}^1_{000}, f, 0>$ and $<\mathbb{B}^1_{030}, g, 0>$ at times t = 0 and t = 1.