Computational ontologies of parthood, componenthood, and containment

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Abstract

Parthood, componenthood, and containment relations are commonly assumed in biomedical ontologies and terminology systems, but are not usually clearly distinguished from another. This paper contributes towards a unified theory of parthood, componenthood, and containment relations. Our goal in this is to clarify distinctions between these relations as well as principles governing their interrelations. We first develop a theory of these relations in first order predicate logic and then discuss how description logics can be used to capture some important aspects of the first order theory.

1 Introduction

My car has components, for example, its engine, its oil pump, its wheels, etc. (See Figure 1.) Roughly, a *component* of an object is a proper part of that object which has a complete bona fide boundary (i.e., boundary that correspond discontinuities in reality) and a distinct function. Thus all components of my car are parts of my car, but my car has also parts that are not components. For example, the left side my car has neither a complete bona fide boundary nor a distinct function. My car is also a *container*. It contains the driver in the seat area and a tool box and a spare-tire in its trunk. Containment is here understood as a relation which holds between disjoint material objects when one object (the containee) is located within a space partly or wholly enclosed by the container. In this paper, we study formal properties of proper parthood, componenthood, and containment relations and demonstrate how they can be represented and distinguished from one other in formal ontologies expressed in languages of different expressive power.

At first sight, these three relations seem to have quite similar properties. All three are transitive and asymmetric. The screw-driver is contained in my tool box and the tool box is contained in the trunk of my car, therefore the screw-driver is contained in the trunk of my car. And if an object (e.g., a tool box) is contained in the trunk of my car, then the trunk of my car is *not* contained in that object. It is easy to see that the componenthood (See Figure 1) and proper parthood relations are also asymmetric and transitive. Due to their similarities these relations are not always clearly distinguished in ontologies such as, e.g., GALEN [6] or SNOMED [12].

However, there are important differences between these relations. There can be a container with a single containee (e.g., the screw-driver is the only tool in my tool box) but no object can have single proper part. Also the components of complex artifacts form tree-structures. Thus, two components share a component only when one is a sub-component of the other. (It is because components form tree structures that tree graphs of component structures can be given in assembly manuals.) The parthood relation does not have this property: The left half of my car and the bottom half of my car share the bottom left part of my car but they are not proper parts of each other.

Ontologies are tools for making explicit the semantics of terminology systems [2]. In this paper we develop ontologies which explicate the distinct properties of proper parthood, componenthood and containment relations. These ontologies can be used to specify the meaning of terms such as ‘proper-part-of’, ‘component-of’, and ‘contained-in’. We start by characterising important properties of binary relations and then study how these properties can be expressed both in ontological theories formulated in first order logic and in ontologies formulated in a description logic.

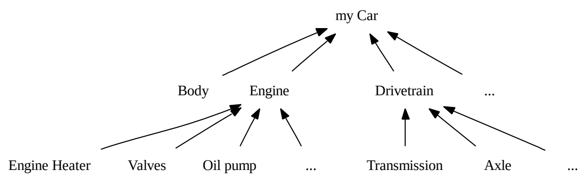

Figure 1: Car components

2 Binary relations

In this section, we define properties of binary relation structures that will be useful for distinguishing proper parthood, component-of, and containment relations.

2.1 R-structures

A R-structure is a pair, (Δ, R), that consists of a non-empty domain Δ and a binary relation $\emptyset \neq R \subseteq \Delta \times \Delta$. We write $R(x, y)$ to say that the binary relation R holds between the individuals $x, y \in \Delta$, i.e., $(x, y) \in R$. We can define the following relations on Δ in terms of R:

$$
\begin{array}{ll}
D_{R_=} & R_=(x,y) =_{df} R(x,y) \textit{ or } x = y \\
D_{R_O} & R_O(x,y) =_{df} \exists z \in \Delta :\ R_=(z,x)\ \&\ R_=(z,y) \\
D_{R_i} & R_i(x,y) =_{df} R(x,y) \& (\neg \exists z \in \Delta :\ R(x,z)\ \&\ R(z,y))
\end{array}
$$

For a given R-structure, the defined relations $R_=$, R_O, or R_i may be empty or identical to R. For example, if R is the identity relation on Δ, i.e., $R = \{(x,x) \mid x \in \Delta\}$, then $R_= = R = R_O$ and $R_i = \emptyset$.

2.2 Properties of binary relations

An R-structure (Δ, R) may have or lack the properties listed in Table 1. For example, for any Δ the identity relation on Δ is reflexive, symmetric and transitive. Moreover, for any (Δ, R), R_O is symmetric, R_i is intransitive, and $R_=$ is reflexive. As pointed out above, on their respective domains proper parthood, componenthood, and containment are asymmetric and transitive.

property	description
reflexive	$\forall x \in \Delta : R(x,x)$
irreflexive	$\forall x \in \Delta$: not $R(x,x)$
symmetric	$\forall x, y \in \Delta$: if $R(x,y)$ then $R(y,x)$
asymmetric	$\forall x, y \in \Delta$: if $R(x,y)$ then not $R(y,x)$
transitive	$\forall x, y, z \in \Delta$: if $R(x,y)$ and $R(y,z)$ then $R(x,z)$
intransitive	$\forall x, y \in \Delta$: if $R(x,y)$ and $R(y,z)$ then not $R(x,z)$
up-discrete	$\forall x, y \in \Delta$: if $R(x,y)$ then $R_i(x,y)$ or $\exists z \in \Delta$: $R(x,z)$ and $R_i(z,y)$
dn-discrete	$\forall x, y \in \Delta$: if $R(x,y)$ then $R_i(x,y)$ or $\exists z \in \Delta$: $R_i(x,z)$ and $R(z,y)$
discrete	up-discrete & dn-discrete
dense	$\forall x, y \in \Delta$: if $R(x,y)$ then $\exists z \in \Delta$: $R(x,z)$ and $R(z,y)$
WSP	$\forall x, y \in \Delta$: if $R(x,y)$ then $\exists z \in \Delta$: $R(z,y)$ & not $R_O(z,x)$
NPO	$\forall x, y \in \Delta$: if $R_O(x,y)$ then $x = y$ or $R(x,y)$ or $R(y,x)$
NSIP	$\forall x, y \in \Delta$: if $R_i(x,y)$ then $\exists z \in \Delta$: $R_i(z,y)$ & not $x = z$
SIS	$\forall x, y, z \in \Delta$: if $R_i(x,y)$ and $R_i(x,z)$ then $y = z$

Table 1: Properties of binary relations

We say (Δ, R) has the *weak supplementation property* (WSP) if and only if for all $x, y \in \Delta$ if $R(x,y)$ then there is a $z \in \Delta$ such that $R(z,y)$ but NOT $R_O(z,x)$. As an example of a relation that has the weak supplementation property, consider the proper parthood relation on the domain Δ_S of spatial objects, $(\Delta_S, \textsf{proper-part-of})$. In this structure $\textsf{proper-part-of}_O$ is the overlap relation. WSP tells us that if x is a proper part of y then there exists a proper part z of y that does not overlap x. For example, since the left side of my car is a proper part of my car there is some proper part of my car (e.g., the right side of my car) which is discrete from the left side of my car.

Another example of a structure that has the weak supplementation property is the componenthood relation on the domain of artifacts, $(\Delta_A, \textsf{component-of})$. Here $\textsf{component-of}_O$ is the relation of sharing a component. WSP tells us that if x is a component of y then there exists a component z of y such that z and x do not have a common component. For example, since the engine of my car is a component of my car there is some component of my car (e.g., the body of my car) which does not have a component in common with the engine. (See Figure 1.)

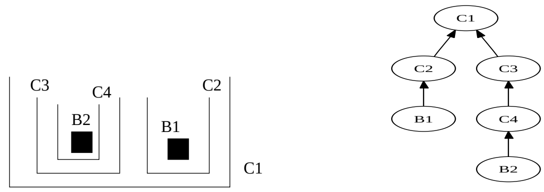

Figure 2: Nested containers

Consider the structure $(\Delta_C, \textsf{contained-in})$ with $\Delta_C = \{C_1, C_2, C_3, C_4, B_1, B_2\}$ as depicted in Figure 2. The block B_1 is immediately contained in the container C_2 which in turn is immediately contained in the container C_1. B_1 is contained, but not immediately contained, in C_1. Note that contained-in does NOT have the weak-supplementation property: B_1 is the only entity contained in C_2. Thus, every entity contained in C_2 stands in the $\textsf{contained-in}_O$ relation to B_1.

We say (R, Δ) has the *no-partial-overlap* property (NPO) if and only if for all $x, y \in \Delta$: if $R_O(x,y)$ then $x = y$ or $R(x,y)$ or $R(y,x)$. The structure $(\Delta_A, \texttt{component-of})$ has the NPO property. As a representative example consider the substructure of $(\Delta_A, \textsf{component-of})$ depicted in Figure 1: Two distinct car components share a component only if one is a subcomponent of the other.

The structure $(\Delta_S, \texttt{proper-part-of})$, on the other hand, does not have the no-partial-overlap property. As pointed out earlier, the left half of my car and the lower half of my car overlap partially. Note also that containment structures (domains with a containment relation) often do not have the NPO property: Consider the tool box in the trunk of my car. It is also contained in my car. My car and the trunk of my car share a containee (the tool box), i.e., $\textsf{contained-in}_O$ holds, but my car is not contained in the trunk of my car nor is the trunk contained in the car.

Containment structures are *discrete*. For example $(\Delta_C, \textsf{contained-in})$ is up- and dn-discrete: if x is contained in y then either x is an immediately contained in y or (a) there exists a z such that x is an immediately contained in z and z is contained in y , and (b) there exists a z such that x is contained in z and z is immediately contained in y. Similarly, the structure $(\Delta_A, \texttt{component-of})$ is discrete. If x is a component of y then either x is a immediate component of y or (a) there exists a z such that x is a immediate component of z and z is a component of y , and (b) there exists a z such that x is a component z and z is an immediate component of y. Again, Figure 1 is a representative example.

The structure $(\Delta_S, \texttt{proper-part-of})$, is *dense* due to the existence of *fiat* parts (parts which lack a complete bona fide boundary) [11]. Consider my car and its proper parts. My car does not have an immediate proper part – Whatever proper part x we chose, there exists another slightly bigger proper part of my car that has x as a proper part.

(R, Δ) has the *single-immediate-successor* property (SIS) if and only if no $x \in \Delta$ can stand in the R_i relation to two distinct members of Δ. Again, a representative example is the component-of structure depicted in Figure 1. In the structure $(\Delta_S, \texttt{proper-part-of})$ SIS trivially holds since this structure has the density property and no immediate proper parts exist. But note that containment structures often do not have the SIS property: Consider again the tool box in the trunk of my car. It is also contained in my car. My car and the trunk of my car are distinct immediate containers for my tool box.

(R, Δ) has the *no-single-immediate-predecessor* property (NSIP) if and only if for all $x, y \in \Delta$: if $R_i(x, y)$ then there exists a $z \in \Delta$ such that $R_i(z, y)$ and not $x = z$. Again, the componenthood structure depicted in Figure 1 is a representative example for a structure that has the NSIP property. Again, in the structure $(\Delta_S, \texttt{proper-part-of})$ NSIP is trivially true since no immediate proper parts exist. But containment structures like $(\Delta_C, \texttt{contained-in})$ lack the NSIP property.

Given the properties in Table 1 we can classify R-structures according to the properties of the relation R. In Table 2 we list classes of R-structures that will be useful for modelling proper parthood, componenthood, and containment relations.

R-structure	properties
partial ordering (PO)	asymmetric, transitive
discrete PO	PO + discrete
parthood structure	PO + WSP + dense
component-of structure	PO + WSP, NPO, discrete

Table 2: Classes of R-structures

Finally, note the following facts about R structures: (F1) If (Δ, R) has the no-partial-overlap property then it has the single-immediate-successor property; (F2) If (Δ, R) is finite and has the single-immediate-successor property then it has the no-partial-overlap property; (F3) If (Δ, R) is up-discrete and has also the no-partial-overlap property, then (Δ, R) has the weak-supplementation property if and only if it has the no-single-immediate-predecessor property; (F4) If (Δ, R) is reflexive, then $R_i = \emptyset$.

2.3 Parthood-containment-component structures

The relations that we are interested in do not exist in separation but form complex structures involving more than one relation. The structure $(\Delta, \mathbf{PP}, \mathbf{CntIn}, \mathbf{CmpOf})$ is a *parthood-containment-component structure* if and only if: (i) the substructure $(\Delta, \mathbf{PP})$ is a parthood structure; (ii) $(\Delta, \mathbf{CntIn})$ is a discrete partial ordering; (iii) $(\Delta, \mathbf{CmpOf})$ is a component-of structure; and addition the following conditions hold:

(iv) If $\mathbf{CntIn}(x, y)$ and $\mathbf{PP}(y, z)$ then $\mathbf{CntIn}(x, z)$;

(v) If $\mathbf{PP}(x, y)$ and $\mathbf{CntIn}(y, z)$ then $\mathbf{CntIn}(x, z)$;

(vi) If $\mathbf{CmpOf}(x, y)$ then $\mathbf{PP}(x, y)$;

As an example of a parthood-containment-component structure consider the set Δ formed by all parts of my car and everything that is contained in my car. The substructure $(\Delta, \mathbf{CmpOf})$ is depicted partly in Figure 1.

(iv) ensures that parts are contained in the container of the whole, e.g., my head is part of my body and my body is contained in my car, so my head must also be contained in my car. (v) ensures that if a part of some whole contains something then so does the whole, e.g., since my tool box is contained in the trunk of my car and the trunk is part of my car, my tool box is also contained in my car. (vi) tells us that componenthood is a special case of parthood, e.g., since the engine is a component of my car, it is also a proper part of my car.

3 A formal ontology of parthood, containment, and componenthood

The formal theory developed in this section is presented in standard first-order predicate logic with identity. We use x, y, and z for variables. Leading universal quantifiers are generally omitted. Names of axioms begin with the capital letter 'A', names of definitions begin with the capital letter 'D', and names of theorems begin with the capital letter 'T'.

We include the primitive relation symbols *PP*, *CntIn*, and *CmpOf* in the language of our theory. The intended interpretations are the relations $\mathbf{PP}$, $\mathbf{CntIn}$, and $\mathbf{CmpOf}$ respectively of parthood-containment-component structures.

3.1 Axioms for *PP*

We introduce the symbols $PP_=$, PP_O, and define that $PP_=\ xy$ holds if and only if either $PP\ xy$ or x and y are identical ($D_{PP_=}$); $PP_O\ xy$ holds iff x and y share a common part or are identical (D_{PP_O}).

$$
\begin{array}{ll}
D_{PP_=} & PP_=\ xy \equiv PP\ xy \lor x = y \\
D_{PP_O} & PP_O\ xy \equiv (\exists z)(PP_=\ zx \land PP_=\ zy)
\end{array}
$$

We then include the axioms of asymmetry and transitivity (APP1-APP2) as well as an axiom (APP3) that ensures that interpretations of *PP* have the weak supplementation property (WSP).

$$
\begin{array}{lll}
APP1 & PP\ xy \rightarrow \neg PP\ yx & \\
APP2 & (PP\ xy \land PP\ yz) \rightarrow PP\ xz & \\
APP3 & PP\ xy \rightarrow (\exists z)(PP\ zy \land \neg PP_O\ zx) & \textit{(WSP)}
\end{array}
$$

The theory that includes APP1-3 as axioms is known as basic mereology [10]. Finally we add a density axiom to include fiat parts into our domain (APP4).

$$APP4 \quad PP\ xy \rightarrow (\exists z)(PP\ xz \land PP\ zy)$$

Models the the theory that includes APP1-4 as axioms are parthood structures as defined in Table 2.

3.2 Axioms for *CmpOf*

We introduce the symbols $CmpOf_=$ and $CmpOf_O$ and add the respective definitions ($D_{CmpOf_=}$ and D_{CmpOf_O}).

$$
\begin{array}{ll}
D_{CmpOf_=} & CmpOf_=\ xy \equiv CmpOf\ xy \lor x = y \\
D_{CmpOf_O} & CmpOf_O\ xy \equiv (\exists z)(CmpOf_=\ zx \land CmpOf_=\ zy)
\end{array}
$$

We then include an axiom of transitivity (ACP1).

$$ACP1 \quad (\textit{CmpOf}\, xy \wedge \textit{CmpOf}\, yz) \rightarrow \textit{CmpOf}\, xz$$

Corresponding to (vi) we add an axiom that ensures that *CmpOf* xy implies *PP* xy (ACP2) and can then prove that *CmpOf* is asymmetric (TCP1).

$$\begin{array}{ll} ACP2 & \textit{CmpOf}\, xy \rightarrow \textit{PP}\, xy \\ TCP1 & \textit{CmpOf}\, xy \rightarrow \neg \textit{CmpOf}\, yx \end{array}$$

We introduce the symbol $\textit{CmpOf}_i$ and define $\textit{CmpOf}_i\, xy$ to hold iff $\textit{CmpOf}_i\, xy$ and there is no z such that $\textit{CmpOf}_i\, xz$ and $\textit{CmpOf}_i\, zy$ ($D_{\textit{CmpOf}_i}$). We then add an axiom that enforces that interpretations of *CmpOf* have the discreteness property (ACP3).

$$\begin{array}{ll} D_{\textit{CmpOf}_i} & \textit{CmpOf}_i\, xy \equiv \textit{CmpOf}\, xy \wedge \\ & \qquad \neg(\exists z)(\textit{CmpOf}\, xz \wedge \textit{CmpOf}\, zy) \\ ACP3 & \textit{CmpOf}\, xy \rightarrow (\textit{CmpOf}_i\, xy \vee \\ & \quad ((\exists z)(\textit{CmpOf}_i\, xz \wedge \textit{CmpOf}\, zy) \\ & \quad \wedge (\exists z)(\textit{CmpOf}\, xz \wedge \textit{CmpOf}_i\, zy))) \end{array}$$

From $D_{\textit{CmpOf}_i}$ we can prove immediately that $\textit{CmpOf}_i$ is intransitive (TCP2).

$$TCP2 \quad \textit{CmpOf}_i\, xy \wedge \textit{CmpOf}_i\, yz \rightarrow \neg \textit{CmpOf}_i\, xz$$

We then add axioms that require that *CmpOf* has the no-partial-overlap property (ACP4) and that *CmpOf* has the no-single-immediate-predecessor property (ACP5).

$$\begin{array}{ll} ACP4 & \textit{CmpOf}_O xy \rightarrow (\textit{CmpOf}_=\, xy \vee \textit{CmpOf}\, zx) \\ ACP5 & \textit{CmpOf}_i\, xy \rightarrow (\exists z)(\textit{CmpOf}_i\, zy \wedge \neg z = x) \end{array}$$

We now can prove that the the weak-supplementation principle holds (TCP3) and that nothing has two distinct immediate successors (TCP4).

$$\begin{array}{ll} TCP3 & \textit{CmpOf}\, xy \rightarrow (\exists z)(\textit{CmpOf}\, zy \wedge \neg \textit{CmpOf}_O\, zx) \\ TCP4 & \textit{CmpOf}_i\, xz_1 \wedge \textit{CmpOf}_i\, xz_2 \rightarrow z_1 = z_2 \end{array}$$

3.3 Axioms for *CntIn*

We introduce the symbols $\textit{CntIn}_=$, $\textit{CntIn}_O$, and $\textit{CntIn}_i$ and add the respective definitions ($D_{\textit{CntIn}_=}$, $D_{\textit{CntIn}_O}$, and $D_{\textit{CntIn}_i}$).

$$\begin{array}{ll} D_{\textit{CntIn}_=} & \textit{CntIn}_=\, xy \equiv \textit{CntIn}\, xy \vee x = y \\ D_{\textit{CntIn}_O} & \textit{CntIn}_O\, xy \equiv (\exists z)(\textit{CntIn}_=\, zx \wedge \textit{CntIn}_=\, zy) \\ D_{\textit{CntIn}_i} & \textit{CntIn}_i\, xy \equiv \textit{CntIn}\, xy \wedge \\ & \qquad \neg(\exists z)(\textit{CntIn}\, xz \wedge \textit{CntIn}\, zy) \end{array}$$

We then include axioms of asymmetry, transitivity, and discreteness (ACT1-3).

$$\begin{array}{ll} ACT1 & \textit{CntIn}\, xy \rightarrow \neg \textit{CntIn}\, yx \\ ACT2 & (\textit{CntIn}\, xy \wedge \textit{CntIn}\, yz) \rightarrow \textit{CntIn}\, xz \\ ACT3 & \textit{CntIn}\, xy \rightarrow (\textit{CntIn}_i\, xy \vee \\ & \quad ((\exists z)(\textit{CntIn}_i\, xz \wedge \textit{CntIn}\, zy) \\ & \quad \wedge (\exists z)(\textit{CntIn}\, xz \wedge \textit{CntIn}_i\, zy))) \end{array}$$

We add axioms, corresponding to (iv) and (v), parts are contained in the container of the whole (ACT4) and that if a part contains something then so does the whole (ACT5).

$$\begin{array}{ll} ACT4 & \textit{PP}\, xy \wedge \textit{CntIn}\, yz \rightarrow \textit{CntIn}\, xz \\ ACT5 & \textit{CntIn}\, xy \wedge \textit{PP}\, yz \rightarrow \textit{CntIn}\, xz \end{array}$$

We call the theory consisting of the axioms APP1-4, ACP1-5 and ACT1-5 *FO-PCC*. Parthood-composition-containment structures are models of this theory.

4 Representation in a description logic

Description Logics (DLs) are a family of logical formalisms which are significantly less powerful than first order logic but which are (relatively) easily implemented on the computer [1]. The task of this section is to investigate to what extent and how *FO-PCC* can be approximated by a theory expressed in a description logic. For this task, we consider DLs with different expressive capabilities, some of which are better suited than others for formulating properties of parthood, componenthood and containment relations. Notice, that it is not the purpose of this paper to provide a complexity analysis for these DLs.

4.1 The syntax and semantics of description logics

Basic expressions in description logics are *concept* and *role descriptions*. Concepts are interpreted as sets. Roles are interpreted as binary relations. General rules for forming concept and role descriptions (based on [1]) are given below. Note, however, that specific DLs typically allow for the formulation of some, but not all, of the complex concept and role descriptions listed.

Every concept name is a concept description (atomic concept), $\top$ is the *top-concept*. $\bot$ is the *bottom-concept*. If $\mathtt{C}$ and $\mathtt{D}$ are concept descriptions then $\mathtt{C} \sqcap \mathtt{D}$ (concept-intersection), $\mathtt{C} \sqcup \mathtt{D}$ (concept-union), $\sim \mathtt{C}$ (concept-complement) are also concept descriptions. Every role name, $\mathtt{R}$, is a role description (an atomic role). If $\mathtt{S}$ and $\mathtt{T}$ are role descriptions, then $\mathtt{S} \sqcap \mathtt{T}$ (role-intersection), $\mathtt{S} \sqcup \mathtt{T}$ (role-union), $\sim \mathtt{S}$ (role-complement), $\mathtt{S} \circ \mathtt{T}$ (role-composition), and $\mathtt{R}^-$ (role-inverse) are also role descriptions. $\mathtt{Id}$ is the name of the identity role. If $\mathtt{C}$ is a concept description and $\mathtt{R}$ is a role name then $(\exists \mathtt{R}.\mathtt{C})$, $(\forall \mathtt{R}.\mathtt{C})$, and $(= 1\mathtt{R})$ are concept descriptions.[1] The semantics of the various constructors is given in Table 3.

A *terminology* is a set of terminological axioms of the form $\mathtt{C} \doteq \mathtt{D}$ and $\mathtt{S} \doteq \mathtt{T}$ (called equalities) or $\mathtt{C} \sqsubseteq \mathtt{D}$ and $\mathtt{S} \sqsubseteq \mathtt{T}$ (called inclusions), where $\mathtt{C}$ and $\mathtt{D}$ are concept descriptions and $\mathtt{S}$ and $\mathtt{T}$ are role descriptions. An interpretation $\mathcal{I}$ satisfies an inclusion $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and $S \sqsubseteq T$ iff $S^{\mathcal{I}} \subseteq T^{\mathcal{I}}$. (See [1].) It satisfies an equality $C \doteq D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$ and $S \doteq T$ iff $S^{\mathcal{I}} = T^{\mathcal{I}}$.

4.2 Stating ontological principles

Let $\mathcal{L}_{WSP}$ be a language that includes at least the constructors (ia, iia, iii, via-c, vii, viii, ix). In this language we can state a DL-version of *FO-PCC*. In particular, if $\mathtt{R}$ is the name of a relation R then we are able to state in this language that R has the WSP property, we are able to define the relation R_i in terms of $\mathtt{R}$, and we are able to state that R is a discrete (or dense) relation:

(WSP)	$\mathtt{R}^- \sqsubseteq \mathtt{R}^- \circ \sim ((\mathtt{R}^- \sqcup \mathtt{Id}) \circ (\mathtt{R} \sqcup \mathtt{Id}))$
(def-i)	$\mathtt{R_i} \doteq \mathtt{R} \sqcap \sim (\mathtt{R} \circ \mathtt{R})$
(discrete)	$\mathtt{R} \sqsubseteq \mathtt{R_i} \sqcup (\mathtt{R} \circ \mathtt{R_i} \sqcap \mathtt{R_i} \circ \mathtt{R})$
(dense)	$\mathtt{R} \sqsubseteq \mathtt{R} \circ \mathtt{R}$

But since $\mathcal{L}_{WSP}$ is undecidable [9], it is important to identify less complex sub-languages of $\mathcal{L}_{WSP}$ that are still sufficient to

[1] $(= 1\mathtt{R})$ is a weak form of number restrictions. Usually stronger forms are used, e.g., [7, 4].

$(ia-b)$	$\top^{\mathcal{I}} = \Delta, \bot^{\mathcal{I}} = \emptyset$;
$(iia-c)$	$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\sim C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$;
(iii)	$(\exists R.C)^{\mathcal{I}} = \{a \in \Delta \mid (\exists b)((a,b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}})\}$
(iv)	$(\forall R.C)^{\mathcal{I}} = \{a \in \Delta \mid (b)((a,b) \in R^{\mathcal{I}} \rightarrow b \in C^{\mathcal{I}})\}$
(v)	$(=1R)^{\mathcal{I}} = \{a \in \Delta \mid \lvert\{b \mid (a,b) \in R^{\mathcal{I}}\}\rvert = 1\}$
$(via-c)$	$(S \sqcap T)^{\mathcal{I}} = S^{\mathcal{I}} \cap T^{\mathcal{I}}, (S \sqcup T)^{\mathcal{I}} = S^{\mathcal{I}} \cup T^{\mathcal{I}}$, $(\sim S)^{\mathcal{I}} = \Delta \times \Delta \setminus S^{\mathcal{I}}$;
(vii)	$(S \circ T)^{\mathcal{I}} = \{(a,c) \in \Delta \times \Delta \mid (\exists b)((a,b) \in S^{\mathcal{I}} \wedge (b,c) \in T^{\mathcal{I}})\}$
$(viii)$	$\mathtt{Id}^{\mathcal{I}} = \{(a,a) \mid a \in \Delta\}$
(ix)	$(\mathsf{R}^{-})^{\mathcal{I}} = \{(b,a) \in \Delta \times \Delta \mid (a,b) \in R^{\mathcal{I}}\}$

Table 3: Concept and role constructors with their semantics. $\mathcal{I}$ is the interpretation function and Δ is the domain of individuals.

state axioms distinguishing parthood, componenthood, and containment relations. Otherwise the DL version of *FO-PCC* would have no computational advantages over the first order theory.

Let $\mathcal{L}$ be the DL which includes only the constructors (ia-b, iia, iii, vii, ix, and v) and in which the role composition operator (vii) only occurs in acyclic role terminologies with inclusion axioms of the form $\mathsf{R} \circ \mathsf{R} \sqsubseteq \mathsf{R}$, $\mathsf{S} \circ \mathsf{R} \sqsubseteq \mathsf{R}$, and $\mathsf{R} \circ \mathsf{S} \sqsubseteq \mathsf{R}$. Unlike $\mathcal{L}_{WSP}$ the DL $\mathcal{L}$ is decidable [3].

If R is the name of the relation R then we are able to state in $\mathcal{L}$ that R is transitive ($\mathsf{R} \circ \mathsf{R} \sqsubseteq \mathsf{R}$). Moreover, in $\mathcal{L}$ we can very naturally represent DL-versions of the axioms ACP2 and ACT4-5. Unfortunately, in $\mathcal{L}$ we are not able to state either that R asymmetric, that R has the WSP property, or that R has the NPO property. Also we cannot state a DL-version of the definition of $\mathtt{R_i}$ in terms of R (as in def-i).

Let $\mathtt{R_i}$ be an undefined relation name interpreted as R_i in the R-structure (Δ, R) (e.g., as $\textsf{contained-in}_i$ in a containment structure). In $\mathcal{L}$ we are able to use this additional primitive to say that R has the no-single-immediate-predecessor property (NSIP) and the single-immediate-successor property (SIS).

(SIS)	$\exists \mathtt{R_i}.\top \sqsubseteq (=1)\mathtt{R_i}.\top$
(NSIP)	$(=1)\mathtt{R_i^{-}}.\top \sqsubseteq \bot$

Notice however that, since we introduced $\mathtt{R_i}$ as an undefined relation name we do not know that the interpretation of $\mathtt{R_i}$ is an intransitive subrelation of R unless additional axioms are included in the theory. In $\mathcal{L}$ we can state that $\mathtt{R_i}$ is a subrelation of R but we cannot not say that $\mathtt{R_i}$ is intransitive. Notice also, that in $\mathcal{L}$, we cannot say that R is irreflexive ($\mathsf{R} \circ \mathtt{Id} \sqsubseteq \bot$) since $\mathcal{L}$ does not include a constructor for the identity relation.

Let $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ be the DL obtained by extending $\mathcal{L}$ with the identity relation (viii), negation restricted to relation names (a restricted version of vic), and role union (vib). In this DL we can say that R_i is intransitive, that R is asymmetric, and that R has the NPO property.

(intrans)	$\mathtt{R_i} \circ \mathtt{R_i} \sqsubseteq (\sim \mathtt{R_i})$
(asym)	$\mathsf{R}^{-} \sqsubseteq (\sim \mathsf{R})$
(NPO)	$(\mathsf{R}^{-} \circ \mathsf{R}) \sqsubseteq \mathsf{R} \sqcup \mathtt{Id} \sqcup \mathsf{R}^{-}$

Unfortunately, including role negation into a DL-language significantly increases the complexity of the underlying reasoning [5]. Though $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ is less expressive than $\mathcal{L}_{WSP}$ (we cannot state WSP or discreteness axioms or define $\mathtt{R_i}$) it is an open question whether $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ is decidable. (It is known though that $\mathcal{ALC}$-DLs that include axioms of the form $\mathsf{R} \circ \mathsf{S} \sqsubseteq \mathtt{T_1} \sqcup \ldots \sqcup \mathsf{T_n}$ are undecidable [13].)

4.3 Describing parthood-composition-containment structures in $\mathcal{L}$

We chose $\mathcal{L}$ as the DL to formulate an *approximation* of *FO-PCC* because $\mathcal{L}$ is decidable and does include the composition operator which is important for expressing interrelations between relation and for reasoning (particularly in biomedical ontologies) [12, 6, 3].

We add the symbols $\mathtt{CP}$, $\mathtt{PP}$ and $\mathtt{CT}$ as well as $\mathtt{CP_i}$, $\mathtt{PP_i}$ and $\mathtt{CT_i}$ to $\mathcal{L}$. The intended interpretations of these symbols are the relations $\mathbf{CmpOf}$, $\mathbf{PP}$, $\mathbf{CntIn}$, $\mathbf{CmpOf_i}$, $\mathbf{PP_i}$, and $\mathbf{CntIn_i}$ of parthood-composition-containment structures. We then include the following axioms for $\mathtt{CP}$ and $\mathtt{PP}$:

component-of		proper-part-of	
(A1)	$\mathtt{CP_i} \sqsubseteq \mathtt{CP}$	(A5)	$\mathtt{PP_i} \sqsubseteq \mathtt{PP}$
(A2)	$\mathtt{CP} \circ \mathtt{CP} \sqsubseteq \mathtt{CP}$	(A6)	$\mathtt{PP} \circ \mathtt{PP} \sqsubseteq \mathtt{PP}$
(A3)	$(=1)\mathtt{CP_i^{-}}.\top \sqsubseteq \bot$	(A7)	$(=1)\mathtt{PP_i^{-}}.\top \sqsubseteq \bot$
(A4)	$\exists \mathtt{CP_i}.\top \sqsubseteq (=1)\mathtt{CP_i}.\top$		$-$

For $\mathtt{CT}$ we include a subrelation axiom and a transitivity axiom:

$$\text{A8 } \mathtt{CT_i} \sqsubseteq \mathtt{CT} \qquad \text{A9 } \mathtt{CT} \circ \mathtt{CT} \sqsubseteq \mathtt{CT}$$

We include also axioms A10-12 corresponding to (iv-vi) in Section 2.3.

$$\text{A10 } \mathtt{CP} \sqsubseteq \mathtt{PP} \quad \text{A11 } \mathtt{PP} \circ \mathtt{CT} \sqsubseteq \mathtt{CT} \quad \text{A12 } \mathtt{CT} \circ \mathtt{PP} \sqsubseteq \mathtt{CT}$$

We call the theory formed by A1-12 *DL-PCC*. The sub-theory formed by A1-4 is similar to the theories proposed by Sattler [7] and Lambrix and Padgham [4].

But, as discussed in the previous subsection, we are not able to add to *DL-PCC* the following axioms and definitions that are needed to constrain the models to parthood-composition-containment structures: (1) We are not able to state that $\mathtt{CP}$, $\mathtt{PP}$, and $\mathtt{CT}$ are asymmetric and irreflexive; (2) We are not able to state a discreteness axiom for $\mathtt{CP}$ or $\mathtt{CT}$ or a density axiom for $\mathtt{PP}$; (3) We are not able to define $\mathtt{CP_i}$, $\mathtt{PP_i}$, and $\mathtt{CT_i}$ in terms of $\mathtt{CP}$, $\mathtt{PP}$, and $\mathtt{CT}$ respectively; (4) We are not able to state the weak supplementation principle (WSP) for interpretations of $\mathtt{PP}$.

Consider (1). Since *DL-PCC* lacks asymmetry axioms it admits models in which $\mathtt{CP}$, $\mathtt{PP}$, and $\mathtt{CT}$ are interpreted as reflexive relations. In those models $\mathtt{CP_i}$, $\mathtt{PP_i}$, and $\mathtt{CT_i}$ are all interpreted as the empty relation (making the axioms A3, A4, and A7 trivially true). (See also F4 in Section 2.2.) For example the structure $(\Delta_C, \textsf{identical-to})$ is a model of *DL-PCC* (but not of *FO-PCC*) if we interpret $\mathtt{CP}$, $\mathtt{PP}$, and $\mathtt{CT}$ as $\textsf{identical-to}$ and $\mathtt{CP_i}$, $\mathtt{PP_i}$, and $\mathtt{CT_i}$ as $\emptyset$. Clearly, this model is not a parthood-component-containment structure.

Consider (3). We included $\mathtt{CP_i}$, $\mathtt{PP_i}$, and $\mathtt{CT_i}$ as undefined primitives in *DL-PCC* and added axioms (A1, A5, and A8) that require their interpretations to be sub-relations of the interpretations of $\mathtt{CP}$, $\mathtt{PP}$, and $\mathtt{CT}$. Unfortunately, *DL-PCC* admits models in which $\mathtt{PP_i}$ and $\mathtt{PP}$ are the same relation (similarly for $\mathtt{CP}$ and $\mathtt{CP_i}$ or $\mathtt{CT}$ and $\mathtt{CT_i}$). Consider Figure 2 and interpret $\mathtt{CP}$ and $\mathtt{CP_i}$ as the relation $\mathsf{icr} = \{(C_2, C_1), (C_3, C_1)\}$ (*immediately-contained-in-the-root-container*), and $\mathtt{PP}$, $\mathtt{PP_i}$, $\mathtt{CT}$, $\mathtt{CT_i}$ all as $\mathsf{contained\text{-}in}$. Then $(\Delta_C, \mathsf{contained\text{-}in}, \mathsf{icr})$ is a model of *DL-PCC* (but not of *FO-PCC*). This particular kind of unintended interpretations of $\mathtt{PP_i}$ and $\mathtt{CT_i}$ can be avoided by requiring that the interpretation of these relations are intransitive. However in $\mathcal{L}$ we are not able to require that a given relation is intransitive.

Consider (4). The closest we can get to requiring that the interpretation of $\mathtt{PP}$ has the WSP property is to require that the NSIP property holds (axiom 7). However the NSIP property is strictly weaker than the WSP property.[2] Consequently, *DL-PCC* admits models that would have been rejected by a theory including an axiom that requires WSP for interpretations of $\mathtt{PP}$ (e.g. *FO-PCC*). Similar comments apply to (2).

These are strong limitations if the purpose of the presented theory is to serve as an *ontology* that specifies the meaning of the terms ‘proper part of’, ‘component of’ and ‘contained in’ rather than to support automatic reasoning in some specific and possibly finite domain.[3] If the DL $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ is decidable we can get a better DL approximation of *FO-PCC* that is computationally tractable. But even a $\mathcal{L}^{\sim \mathtt{Id} \sqcup}$ version of *FO-PCC* will fall short of *FO-PCC* in expressivity since we cannot state WSP for $\mathtt{PP}$ or weaker versions of WSP that are useful in dense domains like $\mathtt{PP}^- \sqsubseteq \mathtt{PP}^- \circ \sim \mathtt{PP}$ and $\mathtt{PP}^- \sqsubseteq \mathtt{PP}^- \circ \sim \mathtt{Id}$.

5 Conclusions

We studied formal properties of parthood, componenthood and containment relations. Since it is the purpose of an ontology to make explicit the semantics of terminology systems, it is important to explicitly distinguish relations such as proper parthood, componenthood, and containment. We demonstrated that first order logic has the expressive power required to distinguish important properties of these relations. In description logics like $\mathcal{L}$ several important properties of these relations cannot be specified.

DLs are best used as reasoning tools for specific tasks in specific domains (as suggested in [8, 7, 4]). DLs are not appropriate for formulating complex interrelations between relations. Thus we need to understand a computational ontology as consisting of two complementary components: (1) a DL based ontology that enables automatic reasoning and constrains meaning as much as possible and (2) a first order ontology that serves as meta-data and makes explicit properties of relations that cannot be expressed in computationally efficient description logics. The first order theory then can be used by a human being to decide whether or not the DL-ontology in question is applicable to her domain. Moreover, meta-data can also be used to write special-purpose programs that phrase knowledge bases and enforce the usage of relations in accordance to the meta-data.

[2] NSIP entails WSP only in conjunction with discreteness and NPO. But we cannot require that interpretations of $\mathtt{PP}$ are discrete or have the NPO property since then, for example, the proper parthood relation on the domain of spatial objects could not be an interpretation for $\mathtt{PP}$. (See Section 2.2).

[3] If we constrain our models to finite domains then, for example, it is indeed sufficient to include (A3) and (A4) as axioms to require the WSP and the NPO properties for $\mathtt{CP}$ (F1-3).

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