A theory of granular parthood based on qualitative cardinality and size measures

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Abstract. We present a theory of granular parthood based on qualitative cardinality and size measures. Using standard mereological relations and qualitative, context-dependent relations such as *roughly the same size*, we define a granular parthood relation and distinguish different ways in which a collection of smaller objects may sum to a larger object. At one extreme, an object x may be a mereological sum of a large collection p where the members of p are all negligible in size with respect to x (e.g., x is a human body and p is the collection of its molecules). At the other extreme, x may be a mereological sum of a collection q none of whose members are negligible in size with respect to x (e.g., x is again a human body and p is the collection consisting of its head, neck, torso, and limbs).

We cannot give precise quantitative definitions for relations such as *roughly the same size* or *negligible in size with respect to* since these are, even within a fixed context, vague relations. The primary focus in the formal theory presented in this paper is on the context-independent logical properties of these qualitative cardinality and size relations and their interaction with mereological relations. In developing our formal theory, we draw upon work on order of magnitude reasoning.

Keywords. Qualitative reasoning, vagueness, context, formal ontology

1. Introduction

There have been some interesting recent proposals for developing theories of parthood which take into account aspects of granularity, scale, and context [**?**,**?**,**?**]. The importance of taking into account granularity and scale in bio-medical ontologies has been emphasized, for example, in [**?**,**?**,**?**,**?**]. It is the aim of this paper to contribute to this work by presenting an axiomatic theory of granular parthood and scale based on qualitative cardinality and size relations, such as *roughly the same size*. For the development of the axiomatic theory we draw on work on order of magnitude reasoning by Raiman, Mavrovouniotis *et al*, and Dague [**?**,**?**,**?**,**?**].

That the interpretation of expressions like 'roughly the same size' is context dependent is widely acknowledged [**?**,**?**,**?**,**?**]. However, there are different strategies for dealing with this context-dependence. Van Deemter [**?**], for example, explicitly represents context in the object-language of his theory. In this paper, we deal with context-dependence in a more indirect way: context is represented abstractly in numerical parameters which determine the canonical interpretations of the qualitative size and cardinality relations of the formal theory. This allows us to focus in the theory only on context-independent logical properties of the qualitative relations.

Obviously we cannot, even in a given context, specify precisely what is meant by, e.g, *roughly the same size* since this is, even once the context is fixed, a vague relation. Although the canonical models use precise numerical parameters for fixing the interpretation of the qualitative size and relations, it is not expected that precise numerical parameters can be fixed in an actual practical context. At best, we associate contexts demanding high precision with a different range of numerical parameters than contexts requiring only loose precision. Since the logical properties of the relations of our theory are valid over a range of numerical parameters, the formal theory can be used for reasoning even where relations such as *roughly the same size* lack precise numerical definitions.

The remainder of this paper is structured as follows: we start by presenting an axiomatic theory of finite collections and relative cardinality. We then extend the theory by introducing parthood and relative size relations among the objects in the collections.

We present the formal theory in a sorted first-order predicate logic with identity. We use the letters w, x, y, z as variables ranging over objects and p, q, r as variables ranging over collections of objects. All quantification is restricted to a single sort. Leading universal quantifiers are generally omitted and restrictions on quantification are to be understood by the conventions on variable usage.

2. Collections

We use $\in$ for the member-of relation between objects and collections. Collections are finite sets of two or more objects.

We require: every collection has two or more members (AC1); two collections are identical if and only if they have the same members (AC2); if x and y are distinct objects, there is a collection consisting of just x and y (AC3).

$$AC1\ (\exists x)(\exists y)(x \in p \wedge y \in p \wedge x \neq y)$$
$$AC2\ p = q \leftrightarrow (x)(x \in p \leftrightarrow x \in q)$$
$$AC3\ x \neq y \rightarrow (\exists p)(x \in p \wedge y \in p \wedge (z)(z \in p \rightarrow z = x \vee z = y))$$

We define union, intersection, and difference relations between collections. It follows from AC2 that unions, intersections, and differences of collections are unique whenever they exist. r is the union of p and q if and only if x is a member and p or x is a member of q ($D_\cup$). r is the intersection of p and q if and only if x is a member of r if and only if x is a member of p and x is a member of q ($D_\cup$). r is the difference of q in p if and only if x is a member of r if and only if x is a member of p and x is not a member of q ($D_\setminus$).

$$D_\cup \quad \cup pqr \equiv (x)(x \in r \leftrightarrow (x \in p \vee x \in q))$$
$$D_\cap \quad \cap pqr \equiv (x)(x \in r \leftrightarrow (x \in p \wedge x \in q))$$
$$D_\setminus \quad \setminus pqr \equiv (x)(x \in r \leftrightarrow (x \in p \wedge x \notin q))$$

We require: the union of two collections always exists (AC4); if p and q share at least two members, then the intersection of p and q exists (AC5); if p has at least two members that are not members of q, then the difference of q in p exists (AC6).

$AC4\ (\exists r) \cup pqr$
$AC5\ (\exists x)(\exists y)(x \neq y \wedge x \in p \wedge y \in p \wedge x \in q \wedge y \in q) \rightarrow (\exists r) \cap pqr$
$AC6\ (\exists x)(\exists y)(x \neq y \wedge x \in p \wedge y \in p \wedge x \notin q \wedge y \notin q)) \rightarrow (\exists r) \setminus pqr$

Axioms AC1-AC6 ensure that collections behave roughly like sets with at least two members. We introduce the term $p \cup q$ for the union of p and q.

p is a *sub-collection* of q ($p \subseteq q$) if and only if every member of p is also a member of q ($D_{\subseteq}$). p is a *proper sub-collection* of q ($p \subset q$) if and only if p is a sub-collection of q and p and q are not identical ($D_{\subset}$).

$$D_{\subseteq}\ p \subseteq q \equiv (x)(x \in p \rightarrow x \in q) \qquad D_{\subset}\ p \subset q \equiv p \subseteq q \wedge p \neq q$$

We can prove that $\subseteq$ is reflexive, antisymmetric, and transitive.

Collection r is *symmetric with respect to* collections p and q if and only if any member of r is member of p if and only if it is a member of q (D_{Sym_C}).

$$D_{Sym_C}\ Sym_C\ rpq \equiv (x)(x \in r \rightarrow (x \in p \leftrightarrow x \in q))$$

On the intended interpretation, collection r is symmetric with respect to p and q whenever the standard set-theoretic intersection of r and p is identical to the standard set-theoretic intersection of r and q. For example, the collection $C_1 = \{1, 2, 3, 4, 5\}$ is symmetric with respect to $C_2 = \{4, 5, 10, 20, 30\}$ and $C_3 = \{-5, -4, 4, 5, 10\}$. But C_2 is not symmetric with respect to C_1 and C_2.

We use $[p]$ in the meta-language to refer to the number of members of p. Notice that if, as intended, p ranges over finite sets with at least two members, $[p]$ must be a natural number greater than one. In the formal theory, we introduce an equivalence relation $\asymp$ between collections where the intended interpretation of $p \asymp q$ is: p and q have the same cardinality ($[p] = [q]$). We require that: $\asymp$ is reflexive, symmetric, and transitive (AC7-9); if p is a sub-collection of q and p and q have the same cardinality then p and q are identical (AC10); if r is symmetric with respect to p and q then p and q have the same cardinality if and only if the union of p and r has the same cardinality as the union of q and r (AC11) ; for all collections p and q there is a collection r such that either (i) r and p have the same cardinality and r is a sub-collection of q or (ii) r and q have the same cardinality and r is a sub-collection of p (AC12); if there is a sub-collection of q that has the same cardinality as p and there is a sub-collection of p that has the same cardinality as q then p and q have the same cardinality (AC13).

$AC7\ \ p \asymp p$
$AC8\ \ p \asymp q \rightarrow q \asymp p$
$AC9\ \ p \asymp q \wedge q \asymp r \rightarrow p \asymp r$
$AC10\ p \subseteq q \wedge p \asymp q \rightarrow p = q$
$AC11\ Sym_C\ rpq \rightarrow (p \asymp q \leftrightarrow (p \cup r) \asymp (q \cup r))$
$AC12\ (\exists r)[(r \asymp p \wedge r \subseteq q) \vee (r \asymp q \wedge r \subseteq p)]$
$AC13\ (\exists r_1)(r_1 \asymp p \wedge r_1 \subseteq q) \wedge (\exists r_2)(r_2 \asymp q \wedge r_2 \subseteq p) \rightarrow p \asymp q$

We can prove: if p is a proper sub-collection of q and q has the same cardinality as r then p and r have different cardinalities (TC1); if s_1 is the difference of r in p and s_2 is the difference of r in q and r is symmetric with respect to p and q then p and q have the same cardinality if and only if s_1 and s_2 have the same cardinality (TC2).

$$TC1\ (p \subset q \wedge q \asymp r) \rightarrow \neg p \asymp r$$
$$TC2\ \backslash prs_1 \wedge\ \backslash qrs_2 \wedge \mathit{Sym}_C\ rpq \rightarrow (p \asymp q \leftrightarrow s_1 \asymp s_2)$$

The cardinality of p is *less than or equal* to the cardinality of q if and only if there is a sub-collection r of q that has the same cardinality as p ($D_{\leq}$). On the intended interpretation, $p \leq q$ holds if and only if $[p]$ is less than or equal to to $[q]$. The cardinality of p is *less than* the cardinality of q if and only if the cardinality of p is less than or equal to the cardinality of q and p and q do not have the same cardinality($D_{<}$).

$$D_{\leq}\ p \leq q \equiv (\exists r)(r \asymp p \wedge r \subseteq q) \qquad D_{<}\ p < q \equiv p \leq q \wedge \neg p \asymp q$$

We can prove: if p is a sub-collection of q, then the cardinality of p is less than or equal to the cardinality of q (TC3); if p is a proper sub-collection of q, then the cardinality of p is less than the cardinality of q (TC4); for any collections p and q, the cardinality of p is less than or equal to the cardinality of q or the cardinality of q is less than or equal to the cardinality of p (TC5); $\leq$ is reflexive (TC6); if the cardinality of p is less than or equal to the cardinality of q and cardinality of q is less than or equal to the cardinality of p, then p and q have the same cardinality (TC7); $\leq$ is transitive (TC8); $<$ is transitive (TC9); $<$ is asymmetric (TC10); if the cardinality of p is less than or equal to the cardinality of q and q and r have the same cardinality, then the cardinality of p is less than or equal to the cardinality of r (TC11); if r and p have the same cardinality and the cardinality of p is less than or equal to the cardinality of q then the cardinality of r is less than or equal to the cardinality of p (TC12).

$$TC3\ p \subseteq q \rightarrow p \leq q \qquad TC8\ \ p \leq q \wedge q \leq r \rightarrow p \leq r$$
$$TC4\ p \subset q \rightarrow p < q \qquad TC9\ \ p < q \wedge q < r \rightarrow p < r$$
$$TC5\ p \leq q \vee q \leq p \qquad TC10\ p < q \rightarrow \neg q < p$$
$$TC6\ p \leq p \qquad TC11\ p \leq q \wedge q \asymp r \rightarrow p \leq r$$
$$TC7\ p \leq q \wedge q \leq p \rightarrow p \asymp q \qquad TC12\ r \asymp p \wedge p \leq q \rightarrow r \leq q$$

3. Close and negligible cardinalites of collections

In this section we formalize the binary relations between collections: *close-to* (in cardinality) and *negligible with respect to*. Let ϵ be a parameter such that $0 < \epsilon < 0.5$. On the intended interpretation, p is close to q if and only if $1/(1+\epsilon) \leq [p]/[q] \leq 1+\epsilon$. p is negligible with respect to q if and only if $[p]/[q]$ is smaller than $\epsilon/(1+\epsilon)$.

Consider Figure **??**. Values for the cardinality of p range along the positive horizontal axis and values for the cardinality of q range along the positive vertical axis. If p and q have the same cardinality then $([p], [q])$ represents a point on the dotted line. If $1/(1+\epsilon) \leq [p]/[q] \leq 1+\epsilon$ (i.e., p is close to q), then $([p], [q])$ represents a point lying within the area delimited by the dashed lines. If $[p]/[q]$ is smaller than $\epsilon/(1+\epsilon)$ (i.e., p is negligible with respect to q), then $([p], [q])$ represents a point lying between the positive vertical axis and the solid diagonal line.

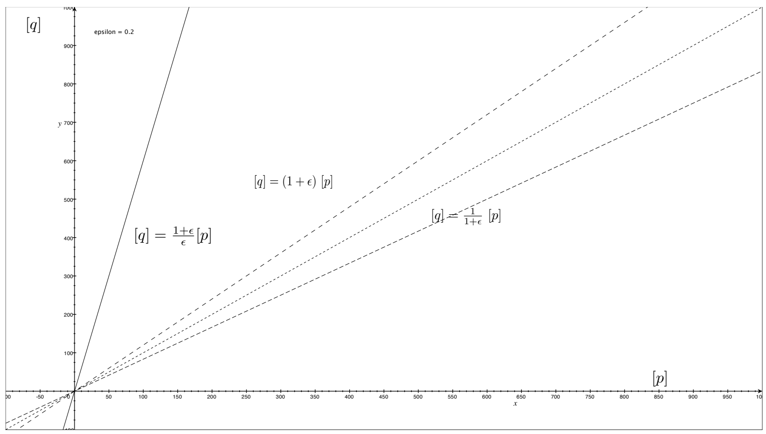

Figure 1. Graph for $\epsilon = 0.2$

Now consider a fixed collection q and imagine that different values of ϵ are appropriate for different contexts. The smaller the value of ϵ, the smaller the value of $|[p] - [q]|$ must be for p to count as close to q and the smaller $[p]$ must be for p to count as negligible with respect to q. To picture this situation graphically: the smaller the value of ϵ, the narrower the corridor between the dashed diagonal lines in Figure **??** and also the narrower the corridor between the solid diagonal line and the positive vertical axis. Consider Table **??**. If $\epsilon = 0.2$ and q has cardinality 100, then collections with cardinalities between 84 and 120 count as close to q and collections with less than 17 members count as negligible with respect to q. By contrast, if $\epsilon = 0.01$ and q has cardinality 100, then $[p]$ must equal 100 or 101 for p to count as close to q and no collection has a cardinality small enough to count as negligible with respect to q.

ϵ	$[q]$	$p \simeq q$	$p \ll q$
0.7	100	$58.8 \leq [p] \leq 170$	$[p] < 41.146$
0.2	100	$83.3 \leq [p] \leq 120$	$[p] < 16.666$
0.1	100	$9.9 \leq [p] \leq 110$	$[p] < 9.0909$
0.01	100	$99.009 \leq [p] \leq 101$	$[p] < 0.99$

Table 1. The parameter ϵ determines which collections are close and which collections are negligible with respect to other collections.

The choice of a value of ϵ between 0 and 0.5 is determined by the level of precision assumed in a particular context. For example, one would chose a larger value of ϵ in a context where the goal is to represent the general functions of the human organ systems than in contexts where the goal is to represent precise analyses of particular blood samples. An important advantage of the presented theory is that the axioms are valid for all choices of ϵ between 0 and 0.5.

Axioms for 'close to'. In the axiomatic theory, we represent *close to* as a relation $\simeq$ between collections, where $p \simeq q$ is interpreted as: $1/(1+\epsilon) \leq [p]/[q] \leq 1+\epsilon$. We require: $\simeq$ is reflexive (AC14) and symmetric (AC15); if r is symmetric with respect to p and q and p is close to q, then $p \cup r$ is close to $q \cup r$ (AC16); if p is close to q and the cardinality of r is greater than or equal to that of p and less than or equal to that of q, then p is close to r and q is close to r (AC17).

$$AC14\; p \simeq p$$
$$AC15\; p \simeq q \rightarrow q \simeq p$$
$$AC16\; \mathit{Sym}_C\; rpq \wedge p \simeq q \rightarrow (p \cup r) \simeq (q \cup r)$$
$$AC17\; p \simeq q \wedge p \leq r \wedge r \leq q \rightarrow (p \simeq r \wedge q \simeq r)$$

Notice that unlike [**?**] and [**?**] we do not require $\simeq$ to be transitive. In many of the intended models of our theory, it is possible to find collections $r_1, \ldots, r_n$ such that $p \simeq r_1$, $r_1 \simeq r_2$, ... and $r_n \simeq q$ and but NOT $p \simeq q$. Hence, adding a transitivity axiom for $\simeq$ would give rise to a version of the Sorites paradox [**?**,**?**].

If the cardinalities of p and q are the same and q is close to r, then p is close to r (TC13); if p is close to q and the cardinalities of q and r are the same, then p is close to r (TC14); if the cardinalities of p and q are the same, then p is close to q (TC15).

$$TC13\; p \asymp q \wedge q \simeq r \rightarrow p \simeq r$$
$$TC14\; p \simeq q \wedge q \asymp r \rightarrow p \simeq r$$
$$TC15\; p \asymp q \rightarrow p \simeq q$$

Notice that the axioms for $\simeq$ are significantly weaker than the axioms for $\asymp$. $\simeq$ is not an equivalence relation; a collection may be close to one of its proper sub-collections; for disjoint collections p and q, there may be some collection r such that the union of p and r is close to the union of q and r even though p is not close to q.

Definition of 'negligible'. Let p and q be collections. p is *negligible with respect to* q if and only if there exist r and s such that (i) p and r have the same cardinality, (ii) r is a sub-collection of q, (iii) s is the difference of r in q and (iii) s is close to q ($D_{\ll}$).

$$D_{\ll}\; p \ll q \equiv (\exists r)(\exists s)(r \asymp p \wedge r \subseteq q \wedge \setminus qrs \wedge s \simeq q)$$

When $\simeq$ is interpreted so that $s \simeq q$ holds if and only if $1/(1+\epsilon) \leq [s]/[q] \leq 1+\epsilon$, then $p \ll q$ holds if and only if $[p]/[q]$ is smaller than $\epsilon/(1+\epsilon)$. We require that if p is negligible with respect to q and the cardinality of q is less than or equal to the cardinality of r, then p is negligible with respect to r (AC18).

$$AC18\; p \ll q \wedge q \leq r \rightarrow p \ll r$$

We can prove: if p is negligible with respect to q, then the cardinality of p is smaller than the cardinality of q (TC16); if the cardinality of p is less than or equal to the cardinality of q and q is negligible with respect to r, then p is negligible with respect to r (TC17); if p is a sub-collection of q and q is negligible with respect to r, then p is negligible with respect to r (TC18); if p is negligible with respect to q and q a sub-collection of r, then p is negligible with respect to r (TC19); $\ll$ is transitive (TC20).

$$TC16\; p \ll q \rightarrow p < q$$
$$TC17\; p \leq q \wedge q \ll r \rightarrow p \ll r$$
$$TC18\; p \subseteq q \wedge q \ll r \rightarrow p \ll r$$
$$TC19\; p \ll q \wedge q \subseteq r \rightarrow p \ll r$$
$$TC20\; p \ll q \wedge q \ll r \rightarrow p \ll r$$

Definition of 'large'. p is *large* if and only if some other collection is negligible with respect to p (D_{Lg}). When $\simeq$ is interpreted so that $p \simeq q$ holds if and only if $1/(1+\epsilon) \leq [p]/[q] \leq 1+\epsilon$, p is large if and only if $[p] > (2+2\epsilon)/\epsilon$. For example, if $\epsilon = 0.01$, then collections of cardinality greater than 202 are large.

$$D_{Lg}\ \mathit{Lg}\ p \equiv (\exists q)(q \ll p)$$

We can prove: super-collections of large collections are large (TC21); sub-collections of non-large collections are non-large (TC22).

$$TC21\ \mathit{Lg}\ p \wedge p \subseteq q \rightarrow \mathit{Lg}\ q \qquad TC22\ p \subseteq q \wedge \neg \mathit{Lg}\ q \rightarrow \neg \mathit{Lg}\ p$$

4. The mereology of objects

We introduce the primitive binary relation P, where Pxy is interpreted as: object x is part of object y.

We define: x *overlaps* y if and only if there is an object z such that z is part of both x and y (D_O); x is a *proper part* of y if and only if x is part of y and y is not part of x (D_{PP}); z is a *difference* of y in x if and only if any object w overlaps z if and only if w overlaps some part of x and that does not overlap y (D_-); z is a *sum* of x and y if and only if any object w overlaps z if and only if w overlaps x or y (D_+); z is a *sum* of collection p, $z\sigma p$, if and only if any object overlaps z just in case it overlaps a member of p (D_σ). We also say in this case that z is a p-*sum*.

$$\begin{array}{ll} D_O & O\ xy \equiv (\exists z)(P\ zx \wedge P\ zy) \\ D_{PP} & \mathit{PP}\ xy \equiv P\ xy \wedge \neg P\ yx \\ D_- & -\ xyz \equiv (w)(O\ wz \leftrightarrow (\exists w_1)(P\ w_1x \wedge \neg O\ w_1y \wedge O\ w_1w)) \\ D_+ & +\ xyz \equiv (w)(O\ wz \leftrightarrow (O\ wx \vee O\ wy)) \\ D_\sigma & z\sigma p \equiv (w)(O\ wz \leftrightarrow (\exists x)(x \in p \wedge O\ xw)) \end{array}$$

We have the usual axioms of reflexivity (AP1) and transitivity (AP2). We also require that if x is not a part of y then there is a difference of y in x (AP3) and that there is a binary sum of any two objects (AP4).

$$\begin{array}{ll} AP1\ P\ xx & AP3\ \neg P\ xy \rightarrow (\exists z)(-\ xyz) \\ AP2\ P\ xy \wedge P\ yz \rightarrow P\ xz & AP4\ (\exists z)(+\ xyz) \end{array}$$

We can prove: if everything that overlaps x overlaps y then x is part of y (TP1); if x is a p-sum, then every member of p is part of x (TP2); if x is a p-sum, y is a q-sum, and p is a sub-collection of q then x is part of y (TP3).

$$\begin{array}{ll} TP1\ (z)(O\ zx \rightarrow O\ zy) \rightarrow P\ xy & \\ TP2\ x \in p \wedge y\sigma p \rightarrow P\ xy & TP3\ x\sigma p \wedge y\sigma q \wedge p \subseteq q \rightarrow P\ xy \end{array}$$

A collection p is *discrete* if and only if distinct members of p do not overlap (D_D).

$$D_D\ D\ p \equiv (x)(y)(x \in p \wedge y \in p \wedge O\ xy \rightarrow x = y)$$

We say that object z is a *discrete sum* of the collection p , $z\Delta p$, if and only if p is discrete and z is a p sum (D_Δ). We can prove that if x is a discrete p-sum then the members of p are proper parts of x (TP4).

$$D_\Delta\ z\Delta p \equiv D\ p \wedge z\sigma p \qquad TP4\ x\Delta p \wedge y \in p \rightarrow \mathit{PP}\ yx$$

We define that z is *mereologically symmetric* with respect to x and y if and only if for every object w that is part of z: w is part of x if and only if w is part of y (D_{Sym_P}).

$$D_{\mathit{Sym}_P}\ \mathit{Sym}_P\ zxy \equiv (w)(P\ wz \rightarrow (P\ wx \leftrightarrow P\ wy))$$

5. Relative size of objects and granular parthood

Exactly the same size. We use $\|x\|$ in the meta-language to refer to the exact volume size of object x. x and y have *exactly the same size* if and only if $\|x\| = \|y\|$. In the formal theory we introduce the *same size* relation $\sim$ where, on the intended interpretation, $x \sim y$ holds if and only if $\|x\| = \|y\|$. We require: if x is part of y and y is part of x, then x and y are the same size (AP5); $\sim$ is symmetric (AP6); $\sim$ is transitive (AP7); if x is part of y and x and y have the same size then y is part of x (AP8); if w_1 is a sum of x and z and w_2 is a sum of y and z and z is symmetric with respect to x and y then: x and y have the same size if and only if w_1 has the same size as w_2 (AP9).

$$AP5\ P\ xy \wedge P\ yx \rightarrow x \sim y$$
$$AP6\ x \sim y \rightarrow y \sim x$$
$$AP7\ x \sim y \wedge y \sim z \rightarrow x \sim z$$
$$AP8\ P\ xy \wedge x \sim y \rightarrow P\ yx$$
$$AP9\ +xzw_1 \wedge +yzw_2 \wedge \mathit{Sym}_P\ zxy \rightarrow (x \sim y \leftrightarrow w_1 \sim w_2)$$

We can prove: $\sim$ is reflexive (TP5); if x is a proper part of y and y has the same size as z or if x has the same size as y and y is a proper part of z, then x and z are different sizes (TP6); if w_1 is a difference of z in x and w_2 is a difference of z in y and z is symmetric with respect to x and y, then x and y have the same size if and only if w_1 and w_2 have the same size (TP7).[1]

$$TP5\ x \sim x$$
$$TP6\ [(\mathit{PP}\ xy \wedge y \sim z) \vee (x \sim y \wedge \mathit{PP}\ yz)] \rightarrow \neg x \sim z$$
$$TP7\ -xzw_1 \wedge -yzw_2 \wedge \mathit{Sym}_P\ zxy \rightarrow (x \sim y \leftrightarrow w_1 \sim w_2))$$

[1]Notice that we do not introduce a total size ordering on objects analogous to the $\leq$ ordering on collections. This is because we do not want to commit to the assumption that for any two objects x and y, either x has a part of exactly the same size as y or y has a part of exactly the same size as x.

Roughly the same size and granular parthood. We introduce the relations *roughly the same size* ($\approx$) and *granular parthood* ($\lll$) between objects, which are roughly analogous to the relations *close to* and *negligible with respect to* on collections. Let ω be a parameter such that $0 < \omega < 0.5$. On the intended interpretation, x is *roughly same size as* y if and only if $1/(1+\omega) \leq \|x\|/\|y\| \leq 1+\omega$. x is a *granular part* of y (i.e., a part of y of negligible size) if and only if x is part of y and $\|x\|/\|y\|$ is less than $\omega/(1+\omega)$.

The parameter ω determines which objects are roughly the same size and which of an object's parts are negligible in size with respect to it. This corresponds to the way in which the parameter ϵ determines which cardinalities are close and which cardinalities negligible with respect to others. As with ϵ, the value of ω can vary according to context. The axioms of our theory are valid for all choices of ω between 0 and 0.5.

Consider Table **??**. If HB is a human body of average volume 70 liter and HH is HB's heart of average volume 0.3 liter, then HH is a granular part of HB for choices of ω larger than 0.0043. HB's cells (average size $400 * 10^{-15}$) are granular parts of HB for all choices of ω listed in the table.

ω	$HB \simeq y$	$y \preceq HB$	$y \lll HB$
0.2	$58.333 \leq \|y\| \leq 84$	$11.666 \leq \|y\| \leq 70$	$\|y\| < 11.666$
0.1	$63.636 \leq \|y\| \leq 77$	$6.363 \leq \|y\| \leq 70$	$\|y\| < 6.363$
0.01	$69.307 \leq \|y\| \leq 70.7$	$0.693 \leq \|y\| \leq 70$	$\|y\| < 0.693$
0.001	$69.93 \leq \|y\| \leq 70.07$	$0.0699 \leq \|y\| \leq 70$	$\|y\| < 0.0699$

Table 2. The parameter ω determines which objects are roughly the same size and which of an object's parts are granular parts. Average volume in liters: human body (HB) = 70 liter, human heart (HH) = 0.3 liter, average cell (HC) = $400 * 10^{-15}$ liter.

Axioms for $\approx$. We require: $\approx$ is reflexive (AP10); $\approx$ is symmetric (AP11); if w_1 is a sum of x and z and w_2 is a sum of y and z and z is symmetric with respect to x and y and x and y are roughly the same size, then w_1 and w_2 are the roughly the same size (AP12); if x and y are roughly the same size and y and z are the same size, then x and z are roughly the same size (AP13); if x and y are roughly the same size and x is a part of z and z is a part of y, then z and x, as well as z and y, are roughly the same size (AP14).

$$AP10\; x \approx x$$
$$AP11\; x \approx y \rightarrow y \approx x$$
$$AP12\; +xzw_1 \wedge +yzw_2 \wedge \mathit{Sym}_P\, zxy \wedge x \approx y \rightarrow w_1 \approx w_2$$
$$AP13\; x \approx y \wedge y \sim z \rightarrow x \approx z$$
$$AP14\; x \approx y \wedge P\, xz \wedge P\, zy \rightarrow (z \approx x \wedge z \approx y)$$

We can prove: x and y are the same size and y and z are roughly the same size, then x and z are roughly the same size (TP8); if x and y are the same size, then x and y are roughly the same size (TP9).

$$TP8\; x \sim y \wedge y \approx z \rightarrow x \approx z \qquad\qquad TP9\; x \sim y \rightarrow x \approx y$$

For reasons analogous to those discussed in the context of $\simeq$ we do not require $\approx$ to be transitive.

Granular and non-granular parthood. x is a *granular part* of y (i.e., x is a part of y whose size is negligible with respect to y) if and only if x is a proper part of y and any difference of x in y has roughly the same size as y ($D_{\lll}$).[2]

$$D_{\lll}\ x \lll y \equiv \mathit{PP}\ xy \wedge (z)(-\ yxz \rightarrow z \approx y)$$

As discussed above, on the intended interpretation $x \lll y$ holds if and only if $\|x\|/\|y\| < \omega/(1+\omega)$. Consider Table **??**. For $\omega = 0.01$, if x is a human body of size 70 liter, then any part y of x with $\|y\| < 0.693$ liter is a granular part of x.

We can prove: $\lll$ is asymmetric (TP10) and transitive (TP11); if x is part of y and y is a granular part of z then x is granular part of z (TP12); if x is a granular part of y and y is part of z then x is granular part of z (TP13).

$$TP10\ x \lll y \rightarrow \neg y \lll x \qquad TP12\ P\ xy \wedge y \lll z \rightarrow x \lll z$$
$$TP11\ x \lll y \wedge y \lll z \rightarrow x \lll z \qquad TP13\ x \lll y \wedge P\ yz \rightarrow x \lll z$$

x is a *non-granular part* of y if and only if x is part of y and x is not a granular part of y ($D_{\preceq}$). It follows immediately that non-granular parthood is reflexive.

$$D_{\preceq}\ x \preceq y \equiv P\ xy \wedge \neg x \lll y$$

On the intended interpretation, $x \preceq y$ holds if and only if x is part of y and $\|x\|/\|y\| \geq \omega/(1+\omega)$.

x and y are of the same *scale with respect to* z if and only if x and y are both non-granular parts of z ($D_{\cong}$)

$$D_{\cong}\ x \cong_z y \equiv x \preceq z \wedge y \preceq z$$

On the intended interpretation, $x \cong_z y$ holds if and only if x and y are parts of z, $\|x\|/\|z\| \geq \omega/(1+\omega)$, and $\|y\|/\|z\| \geq \omega/(1+\omega)$. Consider Table **??**. For $\omega = 0.001$, an average-sized human heart and an average sized human leg are of the same scale with respect to the 70 liter human body of which both are parts.

6. Aggregates and scale

We require: if x is a p-sum and all members of p are granular parts of x, then p is large (AA1); if x is a discrete p-sum and all members of p are of non-granular parts of x, then p is not large (AA2).

$$AA1\ x\sigma p \wedge (y)(y \in p \rightarrow y \lll x) \rightarrow \mathit{Lg}\ p$$
$$AA2\ x\Delta p \wedge (y)(y \in p \rightarrow y \preceq x) \rightarrow \neg \mathit{Lg}\ p$$

It follows from (AA1) that if x is part of y and x is roughly the same size as y, then x is a non-granular part of y (TA1).

[2]Notice that we do not define a relation 'of negligible size with respect to' for arbitrary, possibly disjoint objects analogous to $\ll$ on collections. This because we do not want to commit to the general thesis that any object x has a part of that is roughly the same size as any smaller object.

$$TA1\ P\,xy \wedge x \approx y \rightarrow x \preceq y$$

Object x is a p-*assembly* if and only if x is a discrete p-sum and all members of p are non-granular parts of x (D_{Ass}). Object x is a p-*aggregate* if and only if x is a discrete p-sum and all members of p are granular parts of x(D_{Ag}).

$$D_{Ass}\ \mathit{Ass}\ xp \equiv x\Delta p \wedge (y)(y \in p \rightarrow y \preceq x)$$
$$D_{Ag}\ \mathit{Ag}\ xp \equiv x\Delta p \wedge (y)(y \in p \rightarrow y \lll x)$$

For example, my liver is an aggregate of liver cells in contexts with ω larger than $5.7143 * 10^{-13}$ and ϵ larger than $1.143 * 10^{-12}$ ($\|$my liver$\| = 0.7$ liter, $\|$an average cell$\| = 400 * 10^{-15}$ liter). My body is an assembly of the collection of my major body parts (my torso, my head, my neck, my left arm, my left leg, $\ldots$) in contexts with $\omega < 0.01$ and $\epsilon < 0.02$ ($\|$my neck$\| = 0.7$ liter and $\|$my body$\| = 70$ liter).

We can prove: if x is a p-assembly then p is not large (TA2); if x is a p-aggregate, then p is large (TA3); if x is a p-assembly and y and z are members of p, then y and z are of the same x-scale (TA4).

$$TA2\ \mathit{Ass}\ xp \rightarrow \neg \mathit{Lg}\ p$$
$$TA3\ \mathit{Ag}\ xp \rightarrow \mathit{Lg}\ p$$
$$TA4\ \mathit{Ass}\ xp \wedge y \in p \wedge z \in p \rightarrow y \cong_x z$$

7. Conclusions

We have presented an axiomatic theory of size and granular parthood. The theory is based on the formal characterization of the primitive relations: member of ($\in$) (between objects and collections); same-cardinality-as ($\asymp$) and close-to-in-cardinality ($\simeq$) (between collections); part-of (P), exactly-the-same-size ($\sim$) and roughly-the-same-size ($\approx$) (between objects). In our theory, we are able to formally distinguish between: i) large and non-large collections, ii) the granular and non-granular parts of a given object, and iii) assemblies and aggregates. We thereby extend existing work on mereology, context, and order of magnitude reasoning.

Our theory has a number of limitations: (1) It does not take into account time. Hence we cannot do justice to the fact that most objects most objects gain and lose parts over times. Moreover, there is a critical distinction between gaining or losing granular parts and gaining or losing non-granular parts. Only in rare contexts does it matter whether a human body loses cells, but the loss of a limb or an organ is always a significant event. In [**?**], we develop a time-dependent mereology. We are currently working on a combined theory of parthood, change, and scale.

(2) We focus in this paper exclusively on similarity in cardinality and size, leaving aside similarity in type. However, there are critical distinctions between homogeneous aggregates (p-aggregates where all members of p are of the same type) and heterogeneous aggregates (p-aggregates where members of p are of different types) [**?**]. By combining the work in this paper with a theory of types or universals, we can distinguish between different sorts of homogeneous and heterogenous aggregates.

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