RESEARCH STATEMENT

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1. INTRODUCTION

My research involves studying the modular representations of symmetric groups. In particular, I wish to understand the relationship between simple extensions and the representation theory.

Specht modules play a central role in the modular representation theory of symmetric groups; however, still little is understood about their structures in prime characteristic (e.g., the composition factors and homomorphisms between Specht modules). In 2005 the problem of determining the Specht modules that remain irreducible over prime characteristic was finally solved in [Fay05]. Also, it is was once thought that the Hom-space between two Specht modules might always be at most one-dimensional. However, recently Dodge [Dog11] showed that the dimension of a Hom-space can be made arbitrarily large.

Describing the decomposition numbers of the Specht modules over an arbitrary field of prime characteristic is widely regarded as the most important open problem in the modular representation theory of symmetric groups. The problem has been solved in certain cases; however, we are still far from a solution. It is known that the decomposition numbers are at most one when a Specht module lies in a block of a symmetric group of defect at most 3. Closed form descriptions of the decomposition numbers in defects at most 2 have been established; however, there is no such description in the defect 3 case.

Currently I am interested in understanding the structures of Specht modules that lie in defect 3 blocks of symmetric groups (over characteristic at least five). More generally, the goal is the understand the structures of Specht modules in blocks of small defect.

It is fascinating to see the role cohomology plays in determining the composition factors of Specht modules. In 2014 it was determined [DE14] that each row sum of the decomposition matrix of a defect 2 block of a symmetric group algebra is at most 5. This upper bound was deduced from properties of the Ext$^1$-quiver of the block [CT00]. So cohomology was able to detect the composition factors of the Specht modules in defect 2. The question I wish to answer is: To what extent does cohomology determine the decomposition matrix for a defect 3 block of a symmetric group algebra?

2. NOTATION AND PRELIMINARIES

Let $F$ be a field of characteristic $p$ and let $\Sigma_n$ denote the symmetric group on the letters $1, \ldots, n$. A partition of a positive integer $n$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-negative integers such that $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for all $i$. We write $\lambda \vdash n$ to mean $\lambda$ is a partition of $n$. We denote the transpose of $\lambda$ by $\lambda'$. 
A partition $\lambda$ is called $p$-regular if there does not exist an $i$ so that $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+p-1} > 0$. Otherwise, $\lambda$ is called $p$-singular. Finally $\lambda$ is $p$-restricted if $\lambda'$ is $p$-regular.

If the characteristic of $F$ is zero, the simple $F\Sigma_n$-modules are the Specht modules, denoted $S^\lambda$. If the characteristic of $F$ is $p > 0$, the simple $F\Sigma_n$ modules are of the form $D^\lambda = S^\lambda/\text{rad}S^\lambda$ where $\lambda$ is a $p$-regular partition of $n$ [Jam78].

Let $\text{sgn}$ denote the one-dimensional signature representation of $F\Sigma_n$. There is a bijection $m$ from the set of $p$-regular partitions of $n$ onto itself given by $D^\lambda \otimes \text{sgn} \cong D^m(\lambda)$. This bijection is known as the Mullineux map. See [FK97] for a proof of an algorithm to compute $m$.

The Ext$^1$-quiver of a finite dimensional $F$-algebra $A$ is a directed graph with vertices the isomorphism classes of the simple $A$-modules where one draws an arrow from the simple $A$-module $S$ to the simple $A$-module $T$ if $\text{Ext}^1_A(S,T)$ is nonzero. The number of such arrows is $\dim(\text{Ext}^1_A(S,T))$. If the simple $A$-modules are self-dual, as is the case with the symmetric group algebra, we draw a line segment from $S$ to $T$ instead of an arrow since $\text{Ext}^1_A(S,T) \cong \text{Ext}^1_A(T,S)$. The Ext$^1$-quiver of $A$ tells us how to build modules of the algebra. In other words, the first cohomology of the simples modules builds the $A$-modules.

Suppose $\tau \vdash n - 3p$ is the $p$-core of the defect 3 block $B$ of $F\Sigma_n$, and suppose that an abacus display for $\tau$ has $k$ more beads on runner $i$ than on runner $i - 1$. Interchanging runners $i$ and $i - 1$ gives an abacus display for a $p$-core $\tau' \vdash n - k - 3p$ associated to a defect 3 block $B$ of $F\Sigma_{n-k}$. In this case we say the blocks $B$ and $\tilde{B}$ form a $[3 : k]$-pair. Scopes [Sco91] proved that if $k \geq 3$ then the blocks $B$ and $\tilde{B}$ are Morita equivalent. If $k \leq 2$ the blocks $B$ and $\tilde{B}$ need not be Morita equivalent; however, they share many properties via restriction and induction. In particular, there is an algorithm to compute the Ext$^1$-quiver of $B$ from the Ext$^1$-quiver for $\tilde{B}$ when $B$ and $\tilde{B}$ form a $[3 : 1]$-pair (see Proposition 4.25 of [MT01]).

3. Cohomology and Specht modules

In [Ros15] I build on the work of Martin, Russell, and Tan found in [MR96] and [MT01] to study the Loewy structures of Specht modules that lie in the principal block of $F\Sigma_{3p}$, where $F$ is a field of characteristic $p$ at least five. Henceforth, $F$ is a field of characteristic $p \geq 5$ and $B$ is the principal block of $F\Sigma_{3p}$.

The Ext$^1$-quiver of $B$ was constructed by Martin and Russell in [MR96]. The authors were able to construct most of the quiver by restricting to blocks of defect 1 and 2 of $F\Sigma_{3p-1}$. They developed rules of restriction to these blocks and rules for inducing back up to $B$. Later in 2000 Chaung and Tan [CT00] described the Loewy structures of Specht modules that lie in defect two blocks of symmetric groups in odd characteristic. They showed that a Specht module in a defect 2 block has Loewy length at most 3, and the only Specht modules in a defect 2 that can have Loewy length 3 must correspond to a partitions that are both $p$-regular and $p$-restricted. Furthermore, it was shown that the Ext$^1$-quiver of the block determines second radical layer of a reducible Specht module.

Since the Ext$^1$-quiver of $B$ is closely related to the Ext$^1$-quivers of defect one and defect two blocks of $F\Sigma_{3p-1}$ via restriction, one is left wondering: How do the Loewy lengths of Specht modules in $B$ relate to the Loewy lengths of Specht modules for $F\Sigma_{3p-1}$? Furthermore, how do the radical series and socles relate?
3.1. Loewy lengths of Specht modules in $\mathcal{B}$. In [Ros15] I show that if the Specht module $S^\lambda$ lies in the principal block of $F\Sigma_3p$ then $S^\lambda$ has Loewy length at most 4. Furthermore, $S^\lambda$ has Loewy length 4 if and only if the partition $\lambda$ is both $p$-regular and $p$-restricted. In fact, a more general statement is given.

**Theorem 3.1** ([Ros15]). Suppose $S^\lambda$ lies in a defect 3 block of $F\Sigma_n$. If the partition $\lambda$ is $p$-regular and $p$-restricted then $S^\lambda$ has Loewy length 4.

**Theorem 3.2** ([Ros15]). Suppose $\lambda$ is a partition in $\mathcal{B}$. Then $S^\lambda$ has Loewy length 4 if and only if $\lambda$ is $p$-regular and $p$-restricted.

The proof of Theorem 3.1 uses the facts that the projective indecomposable modules in defect 3 blocks of symmetric groups have a common Loewy length of 7 and the Young module has a Specht module filtration where $S^\lambda$ is seen only once and all other Specht modules in the filtration have the form $S^\mu_i$ with $\mu_i \triangleright \lambda$ [HN04].

Martin’s conjecture states that if $B$ is a defect $w < p$ block of $F\Sigma_n$ then the projective indecomposable modules of $B$ have a common Loewy length of $2w + 1$. The conjecture has been proved for $w \leq 3$. If Martin’s conjecture holds for the block $B$ then we see: if $S^\lambda$ lies in $B$ and $\lambda$ is $p$-regular and $p$-restricted then $S^\lambda$ has Loewy length $w + 1$.

3.2. The second radical layer of $S^\lambda$ and $\operatorname{Ext}^1$-quivers. Kleshchev and Sheth [KS99] prove a result which tells us that the second radical layer of a reducible Specht module $S^\lambda$ is partially determined by the $\operatorname{Ext}^1$-quiver of $F\Sigma_n$ whenever $\lambda$ is a $p$-regular partition of $n$ with at most $p - 1$ nonzero parts.

**Theorem 3.3** ([KS99]). Let $p > 2$ and let $\lambda$ and $\mu$ be $p$-regular partitions of $n$ with at most $p - 1$ nonzero parts. If $\lambda$ does not strictly dominate $\mu$ then

$$\operatorname{Hom}_{F\Sigma_n}(\operatorname{rad}S^\lambda, D^\mu) \cong \operatorname{Ext}^1_{F\Sigma_n}(D^\lambda, D^\mu).$$

It remains open whether Theorem 3.3 holds without the restriction of at most $p - 1$ nonzero parts. In [Ros15] I show that the restriction of at most $p - 1$ parts may be removed for $p$-regular partitions in the principal block of $F\Sigma_3p$. Hence the $\operatorname{Ext}^1$-quiver of the principal block determines the second radical layer of $S^\lambda$ whenever $\lambda$ is $p$-regular. Theorem 3.1 tells us that the composition factors in the heart of $S^\lambda$ are determined by the $\operatorname{Ext}^1$-quiver of $B$ when $\lambda$ is $p$-regular and $p$-restricted.

**Theorem 3.4** ([Ros15]). Suppose $\lambda$ and $\mu$ are $p$-regular partitions in the principal block of $F\Sigma_{3p}$. If $\lambda$ does not strictly dominate $\mu$ then

$$\operatorname{Hom}_{F\Sigma_{3p}}(\operatorname{rad}S^\lambda, D^\mu) \cong \operatorname{Ext}^1_{F\Sigma_{3p}}(D^\lambda, D^\mu).$$

Fayers and Tan [FT07] showed the following: If $B$ is a block of $F\Sigma_n$ of defect 3 and $\lambda$ and $\mu$ are $p$-regular partitions in $B$ with the same parity then $\operatorname{Ext}^1_B(D^\lambda, D^\mu) = 0$. They also proved that $\lambda$ and $m(\lambda)$ have different parity. Hence if $\lambda$ is also $p$-restricted then $\lambda$ and $m(\lambda')$ have different parity. It is unknown whether there is a non-split extension of $D^\lambda$ by $D^{m(\lambda')}$. It is shown that no such extension exists in $B$.

**Theorem 3.5** ([Ros15]). Suppose $\lambda$ is a partition in $\mathcal{B}$ that is $p$-regular and $p$-restricted. Then $\operatorname{Ext}^1_{F\Sigma_{3p}}(D^\lambda, D^{m(\lambda')}) = 0$.

Interestingly, if $\lambda$ is a weight 2 partition of $n$ that is both $p$-regular and $p$-restricted then $\operatorname{Ext}^1_{F\Sigma_n}(D^\lambda, D^{m(\lambda')}) = 0$ since $\lambda$ and $m(\lambda')$ have the same parity.
4. Future Research

Let $B$ be a block of $F\Sigma_n$ of defect 3. We list what is known about the block $B$.

1. The principal indecomposable modules of $B$ have a common Loewy length of 7 [Tan08].
2. The decomposition numbers of the Specht modules in $B$ are at most 1 [Fay08].
3. The Ext$^1$-quiver of $B$ is bipartite [FT07].
4. If $D^\lambda$ and $D^\mu$ are simple modules in $B$ then Ext$^1_{F\Sigma_n}(D^\lambda, D^\mu)$ is at most one-dimensional [MR99].
5. If $S^\lambda$ lies in $B$ and $\lambda$ is $p$-regular and $p$-restricted then $S^\lambda$ has Loewy length 3 [Ros15].

Statement (1) was first announced in [MR99]; however, gaps in the proof were discovered. A complete proof of (1) was given in [Fay08].

As $B$ is the unique defect 3 block of $F\Sigma_3p$, we are left with the following problem.

**Problem 4.1.** Do Theorems 3.2, 3.4, 3.5 generalize to an arbitrary defect 3 block of $F\Sigma_n$?

Martin and Russell [MR99] showed that the Ext$^1$-quivers of defect 3 blocks all have the same basic shape. So the Ext$^1$-quiver of $B$ is in a sense a prototype for the Ext$^1$-quiver of an arbitrary defect 3 block of $F\Sigma_n$. Based on this evidence, we believe that the answer to Problem 4.1 is “yes.” We are left to study relationships between Specht modules lying in blocks that form $[3:1]$ or $[3:2]$-pairs.

**Conjecture 4.2.** Let $B$ be a block $F\Sigma_n$ of defect 3. If the Specht module $S^\lambda$ lies in $B$ then it has Loewy length at most 4. Furthermore, $S^\lambda$ has Loewy length 4 if and only if $\lambda$ is $p$-regular and $p$-restricted.

In [Ros15] I showed 14 to be the largest row sum in the decomposition matrix for the block $B$. I list the largest row sum for a block of small defect $w \leq 4$ in Table 1.

<table>
<thead>
<tr>
<th>Defect $w &lt; p$</th>
<th>Largest row sum in decomposition matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>14 (for the block $B$)</td>
</tr>
<tr>
<td>4†</td>
<td>33 (principal block of $F\Sigma_{20}$, $p = 5$)</td>
</tr>
<tr>
<td></td>
<td>42 (principal block of $F\Sigma_{28}$, $p = 7$)</td>
</tr>
</tbody>
</table>

**Table 1.** Row sums

The defect 4 cases (†) in Table 1 were computed using GAP. We see that the $(w + 1)$th Catalan number is bounding the row sums of decomposition matrix for the block of defect $w \leq 4$. We consider two problems.

**Problem 4.3.** Let $B$ be a defect 3 block of $F\Sigma_n$. Is each row sum in the decomposition matrix for $B$ bounded above by 14?

**Problem 4.4.** Does the pattern in Table 1 continue for the principal block of $F\Sigma_{np}$, $5 \leq n \leq p - 1$?
References


[Ros15] M. Rosas, On the Loewy structures of Specht modules in the principal block of $F^{\Sigma_{3}}$. Accepted to The Journal of Algebra for publication (paper accepted in 2015).
