Change of Variables

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1 Introduction

The change of variables theorem is as follows.

Theorem 1.1 *Let $U, V \subseteq \mathbb{R}^n$ be open in $\mathbb{R}^n$, and let $T : U \to V$ be a diffeomorphism. Given a continuous function $f : V \to \mathbb{R}$,*

$$\int_V f = \int_U (f \circ T) \left| \det DT \right|. \tag{1}$$

There are some immediate questions that come with this. First, what is a diffeomorphism?

Definition 1.2 *Let $U, V \subseteq \mathbb{R}^n$ be open in $\mathbb{R}^n$. Given a bijection $T : U \to V$, we say that T is a diffeomorphism provided that T is C^1 and T^{-1} is C^1.*

In other words, a diffeomorphism is a C^1 function that has a C^1 inverse.

Next, and more importantly, how is this theorem used? In general, the change of variables theorem is used whenever we want to express an integral of f over a complicated region as an integral over a less complicated region. For example, suppose we want to integrate a function over the interior of the following curve:

$$\left(x^2 + y^2\right)^{\frac{5}{2}} = x^4 + 2\left(x^2 + y^2\right)\left(2x^2 + y^2\right). \tag{2}$$

This is positively horrific. Trying to express the interior would be even more of a nightmare. However, by expressing the curve in polar coordinates, this same curve can be written as

$$r = \left(\cos^2\theta + 1\right)^2. \tag{3}$$

This makes the region in question much easier to deal with. Change of variables is the theoretical foundation of such techniques.

2 Calculus, revisited

In this section, we'll show how to use the change of variables theorem to explain the techniques for simplifying integrals that you learned in Calculus III.

2.1 Polar coordinates

Let's say you want to find the integral of a function $f(x,y)$ over over the region bounded by the x-axis and the upper half of the unit circle. In Calculus III, you learned that you can express this integral in two different ways:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_{0}^{\pi}\int_{0}^{1} f\left(r\cos\theta, r\sin\theta\right) r\,\mathrm{d}r\mathrm{d}\theta. \tag{4}$$

Using the change of variables theorem, we can explain how this formula comes about.

First of all, let's understand that the integral on the right side of Equation 4 is actually an integral over a *rectangle*; it is a rectangle of width π and height 1 in the θr-plane (as opposed to the xy-plane). In order to use the change of variables theorem, we need a diffeomorphism that maps the interior of this rectangle to the interior of the upper unit semicircle. This diffeomorphism is exactly the transformation from polar to rectangular coordinates: $T:\mathbb{R}^2\to\mathbb{R}^2$ via

$$T\left(r,\theta\right) = \left(r\cos\theta, r\sin\theta\right). \tag{5}$$

The derivative DT can be represented by the Jacobi matrix:

$$J\left(T\right) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}. \tag{6}$$

The determinant of DT is now just the determinant of the Jacobi matrix:

$$\det DT = r\cos^2\theta + r\sin^2\theta = r. \tag{7}$$

Thus, the change of variables theorem states that

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{0}^{\pi}\int_{0}^{1} f\circ T\,(r,\theta)\left|\det DT\right|\mathrm{d}r\mathrm{d}\theta$$
$$= \int_{0}^{\pi}\int_{0}^{1} f\,(r\cos\theta, r\sin\theta)\,r\;\mathrm{d}r\mathrm{d}\theta, \quad (8)$$

as expected.

2.2 Cylindrical coordinates

Similarly, suppose we are given a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ and an integral

$$\iiint_R f(x,y,z)\,\mathrm{d}V, \quad (9)$$

where R is the region of xyz-space bounded by the planes $z=-1$, $z=1$ and the cylinder $x^2+y^2=4$. If we want to solve it by switching to cylindrical coordinates, then we can accomplish this by using the change of variables theorem. We seek some function $T : \mathbb{R}^3 \to \mathbb{R}^3$ that maps a nice region in $r\theta z$-space smoothly to R. In this case, the nice region that we want to integrate over is a rectangular prism in $r\theta z$-space:

$$S = \left\{(r,\theta,z)\in\mathbb{R}^3 \middle| 0\le r\le 2,\ 0\le\theta\le 2\pi \text{ and } -1\le z\le 1\right\}. \quad (10)$$

Let's take

$$T\,(r,\theta,z) = (r\cos\theta, r\sin\theta, z)\,. \quad (11)$$

Once again, we need the determinant of the Jacobi matrix:

$$\det DT = \begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix} = r. \quad (12)$$

Now the change of variables theorem dictates that

$$\iiint_R f(x,y,z)\,\mathrm{d}V = \iiint_S f\circ T\,(r,\theta,z)\,|\det DT|\,\mathrm{d}V$$
$$= \int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2} f\left(r\cos\theta, r\sin\theta, z\right) r\,\mathrm{d}r\mathrm{d}\theta\mathrm{d}z. \quad (13)$$

2.3 Spherical coordinates

Now, let's say we're given $f:\mathbb{R}^3\to\mathbb{R}^3$ and an integral

$$\iiint_R f(x,y,z)\,\mathrm{d}V, \quad (14)$$

where

$$R = \left\{(x,y,z)\in\mathbb{R}^3 \middle| x\geq 0, y\geq 0, \sqrt{x^2+y^2}\leq z\leq\sqrt{9-x^2-y^2}\right\}. \quad (15)$$

If we use spherical coordinates, we could instead integrate over the rectangle

$$S = \left\{(\rho,\phi,\theta)\in\mathbb{R}^3 \middle| 0\leq\rho\leq 3,\ 0\leq\phi\leq\frac{\pi}{4},\text{ and } 0\leq\theta\leq\frac{\pi}{2}\right\}. \quad (16)$$

To use the change of variables theorem to convert this integral to spherical coordinates, we need a function $T:\mathbb{R}^3\to\mathbb{R}^3$ that maps S smoothly to R. This will do:

$$T\left(\rho,\phi,\theta\right) = \left(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\right). \quad (17)$$

Again, we need the determinant of the Jacobi matrix of T:

$$\det DT = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + 0$$

$$= \cos\phi\left(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta\right) + \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi = \rho^2\sin\phi. \quad (18)$$

Therefore,

$$\iiint_R f(x,y,z)\,\mathrm{d}V = \iiint_S f\circ T\,(\rho,\phi,\theta)\,|\det DT|\,\mathrm{d}V$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^3 f\,(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\phi^2\sin\phi\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta. \quad (19)$$

3 Solutions to the final homework

From Section 4.19, page 167

$\boxed{1.}$ Let R be the region of $\mathbb{R}^2$ bounded by the curve $x^2 - xy + 2y^2 = 1$. We seek a diffeomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $B(0,1)$ to R. We define

$$T(u,v) = \left(u + \frac{1}{\sqrt{7}}v, \frac{2}{\sqrt{7}}v\right). \tag{20}$$

One can show that, for $x = u + \frac{1}{\sqrt{7}}v$ and $y = \frac{2}{\sqrt{7}}v$, if $u^2 + v^2 = 1$, then $x^2 - xy + 2y^2 = 1$. (Thus, T maps the boundary of $B(0,1)$ to the boundary of R.) We compute $\det DT$:

$$\det DT = \begin{vmatrix} 1 & \frac{1}{\sqrt{7}} \\ 0 & \frac{2}{\sqrt{7}} \end{vmatrix}. \tag{21}$$

By the change of variables theorem, this integral is given by

$$\iint_{B(0,1)} \left(u + \frac{1}{\sqrt{7}}v\right)\left(\frac{2}{\sqrt{7}}v\right)\left(\frac{2}{\sqrt{7}}\right) \mathrm{d}A. \tag{22}$$

□

$\boxed{2.}$ (a) Let $E \subseteq \mathbb{R}^3$ be the region in question, and let R be the image of E under projection onto the xy-plane. First, we can use Fubini's theorem to say that

$$V = \iiint_E 1 \,\mathrm{d}V = \iint_R \int_{x^2+2y^2}^{2x+6y+1} 1 \,\mathrm{d}z\mathrm{d}A = \iint_R 2x + 6y + 1 - x^2 - 2y^2 \,\mathrm{d}A. \tag{23}$$

Now we seek a diffeomorphism $T : \mathbb{R}^2 \to \mathbb{R}^2$ mapping $B(0,1)$ to R. In order to ensure that T maps the boundary of $B(0,1)$ to the boundary of R, we seek $T(u,v) = (x,y)$ so that if $u^2 + v^2 = 1$, then $x^2 + 2y^2 = 2x + 6y + 1$. This is possible if

$$T(u,v) = \left(\sqrt{20}u + 1, \sqrt{10}v + 3\right). \tag{24}$$

We evaluate $detDT$:

$$\det DT = \begin{vmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{10} \end{vmatrix} = 10\sqrt{2}. \tag{25}$$

Therefore, the volume can be expressed as

$$\iint_{B(0,1)} \left(2x + 6y + 1 - x^2 - 2y^2\right) 10\sqrt{2}\, \mathrm{d}A. \tag{26}$$

(b)

3.

4.

5.