Mathematical induction

Mathematical induction is a technique of proving that a statement is true for all natural numbers. Here are some examples of such statements:

"$\forall\, n \in \mathbb{N}$, $1 + 2 + ... + n = \frac{n(n+1)}{2}$."

"$\forall\, n \in \mathbb{N}$, $1 + 3 + ... + (2n - 1) = n^2$."

"$\forall\, n \in \mathbb{N}$, $9^n - 5^n$ is divisible by 4."

"$\forall\, n \in \mathbb{N}$, if X is a set and $|X| = n$, then $\big|P(X)\big| = 2^n$."

The principal of mathematical induction is as follows:

Axiom 5 For each $n \in \mathbb{N}$, let $P(n)$ be a statement. If both of the following conditions are true:
(1) $P(1)$ is true
(2) $\forall\, k \in \mathbb{N}$, $P(k) \Rightarrow P\,(k + 1)$,
then $\forall\, n \in \mathbb{N}$, $P(n)$ is true.

From this, one can prove the principal of strong (or "complete") mathematical induction:

Theorem 9.5 For each $n \in \mathbb{N}$, let $P(n)$ be a statement. If both of the following conditions are true:
(1) $P(1)$ is true
(2) $\forall\, k \in \mathbb{N}$, $(P(1), P(2), ..., P(k-1),$ and $P(k)) \Rightarrow P(k + 1)$,
then $\forall\, n \in \mathbb{N}$, $P(n)$ is true.

In order to use this fact to prove statements, we need to follow three steps:

1. Prove that $P(1)$ is true. This is called the basis for induction.

2. For weak induction, assume that for some $k \in \mathbb{N}$, $P(k)$ is true. For strong induction, assume that for some $k \in \mathbb{N}$, $P(1), P(2), ..., P(k - 1)$ and $P(k)$ are all true. This is called the induction hypothesis.

3. Prove that $P(k + 1)$ is true. This proof is called the induction step.

Therefore, proofs by induction will *always* take one of these two forms:

Weak mathematical induction: "We proceed by mathematical induction on n. We notice that [...proving $P(1)$...]. This establishes a basis for induction. As the induction hypothesis, assume that for some $k \in \mathbb{N}$, [...$P(k)$...]. Now, [...proving $P(k+1)$...]."

Strong mathematical induction: "We proceed by strong mathematical induction on n. We notice that [...proving $P(1)$...]. This establishes a basis for induction. As the induction hypothesis, assume that for some $k \in \mathbb{N}$, $\forall\ n \in \{1, 2, ..., k\}$, [...$P(n)$...]. Now, [...proving $P(k+1)$...]."

Following are two examples of proofs by induction: one by weak induction and one by strong induction.

Theorem Given $n \in \mathbb{N}$, $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.
Proof We proceed by mathematical induction on n. We notice that

$$\frac{\mathrm{d}}{\mathrm{d}x}x^1 = \lim_{h \to 0} \frac{(x+h)^1 - x^1}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = \lim_{h \to 0} \frac{h}{h} = 1. \tag{1}$$

This establishes a basis for induction. As the induction hypothesis, assume that for some $k \in \mathbb{N}$, $\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}$. Now,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}x^{k+1} &= \lim_{h \to 0} \frac{(x+h)^{k+1} - x^{k+1}}{h} = \lim_{h \to 0} \frac{(x+h)^k (x+h) - x^{k+1}}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^k x + (x+h)^k h - x^{k+1}}{h} = \lim_{h \to 0} \frac{(x+h)^k x - x^k x + (x+h)^k h}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^k x - x^k x}{h} + \lim_{h \to 0} \frac{(x+h)^k h}{h} \\ &= x \left(\lim_{h \to 0} \frac{(x+h)^k - x^k}{h} \right) + \lim_{h \to 0} (x+h)^k. \end{aligned} \tag{2}$$

However, by our induction hypothesis, we know that

$$\lim_{h\to 0}\frac{(x+h)^k - x^k}{h} = \frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}. \tag{3}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{k+1} = x\left(kx^{k-1}\right) + \lim_{h\to 0}(x+h)^k = kx^k + x^k = (k+1)\,x^k. \tag{4}$$

This completes the induction. □

Theorem Define a sequence (s_n) such that $s_1 = 1$, $s_2 = 5$ and $s_{n+1} = s_n + 2s_{n-1}$. For each $n \in \mathbb{N}$, $s_n = (-1)^n + 2^n$.
Proof We proceed by strong mathematical induction on n. We notice that

$$\begin{array}{c}(-1)^1 + 2^1 = 1 = s_1 \\ (-1)^2 + 2^2 = 5 = s_2\end{array}. \tag{5}$$

This establishes a basis for induction. As the induction hypothesis, assume that for some $k \in \mathbb{N}$, $\forall\, n \in \{1, 2, ..., k\}$, $s_n = (-1)^n + 2^n$. Now, if $k = 1$, then $s_{k+1} = s_2$, and we have already shown that that $s_2 = (-1)^2 + 2^2$. Suppose, then, that $k > 1$. By definition, $s_{k+1} = s_k + 2s_{k-1}$. By the induction hypothesis, $s_k = (-1)^k + 2^k$ and $s_{k-1} = (-1)^{k-1} + 2^{k-1}$. Therefore,

$$\begin{aligned} s_{k+1} &= s_k + 2s_{k-1} \\ &= \left((-1)^k + 2^k\right) + 2\left((-1)^{k-1} + 2^{k-1}\right) \\ &= (-1)^k + 2(-1)^{k-1} + 2^k + 2\left(2^{k-1}\right) \\ &= (-1)\,(-1)^{k-1} + 2(-1)^{k-1} + 2^k + 2^k \\ &= (-1+2)\,(-1)^{k-1} + 2\left(2^k\right) \\ &= (-1)^{k-1} + 2^{k+1} \\ &= (-1)^{k+1} + 2^{k+1}. \end{aligned} \tag{6}$$

This completes the induction. □