Types of statements

First, here are some general facts about mathematical statements.

1. If a statement is true, then its negation is false.

2. If a statement is false, then its negation is true.

3. Every statement can be phrased in terms of "for all" or "there exists." These are abbreviated by $\forall$ and $\exists$, respectively. Don't take my word for it; here are a few examples to illustrate this fact.

$A \subseteq B.$	is the same as	$\forall\, x \in A,\ x \in B.$
$\sqrt{2}$ is irrational.	is the same as	$\forall\, x \in \mathbb{Q},\ x \neq \sqrt{2}.$
$x^2 = 1$ has a real solution.	is the same as	$\exists\, x \in \mathbb{R}$ such that $x^2 = 1.$
Integers have non-negative squares.	is the same as	$\forall\, n \in \mathbb{Z},\ n^2 \geq 0.$
$\sqrt{-1}$ is a complex number.	is the same as	$\exists\, i \in \mathbb{C}$ such that $i^2 = -1.$
No real number has a negative square.	is the same as	$\forall\, x \in \mathbb{R},\ x^2 \nless 0.$
$2 + 2 = 4.$	is the same as	$\forall\, x \in \mathbb{R},$ if $x = 2 + 2$, then $x = 4.$

As we can see from the last example, this kind of rephrasing is not always natural or enlightening, but it can be done.

Often, the presence of "for all" or "there exists" is implicit, so those exact words do not appear. Let's look at a common example.

"Given sets X and Y, $X \subseteq X \cup Y$."

This can be written as

"For all sets X and Y, $X \subseteq X \cup Y$."

The negation of a "for all" statement is always a "there exists" statement. Observe:

Statement	Negation
"All sparrows are birds."	"There exists a sparrow that is not a bird."
"All vertebrates are fish."	"There exists a vertebrate that is not a fish."
"For any dog, the dog is a good boy."	"There exists a dog that is not a good boy."
"No insect can swim."	"There exists an insect that can swim."
$\forall\, x \in \mathbb{R},\ x^4 > 0.$	$\exists\, x \in \mathbb{R}$ such that $x^4 \leq 0.$
$\forall\, a \in A,\ a \in B$	$\exists\, a \in A$ such that $a \notin B.$
$\forall\, a \in A,\ a \notin B.$	$\exists\, a \in A$ such that $a \in B.$

(1)

By the same token, the negation of a "there exists" statement is always a "for all" statement.

Statement	Negation
"There exists a cat with red eyes."	"For any cat, the cat does not have red eyes."
"There exists a black swan."	"For any swan, the swan is not black."
"There exists an animal with no backbone."	"For any animal, the animal has a backbone."
$\exists\, x \in \mathbb{R}$ such that $x^2 = -1$	$\forall\, x \in \mathbb{R}, x^2 \neq -1.$
$\exists\, \omega \in \mathbb{C}$ such that $\omega \neq 1$ and $\omega^3 = 1$	$\forall\, \omega \in \mathbb{C},$ either $\omega = 1$ or $\omega^3 \neq 1.$
$\exists\, a \in A$ such that $a \in B$	$\forall\, a \in A,\ a \notin B.$
$\exists\, a \in A$ such that $a \notin B$	$\forall\, a \in A,\ a \in B.$

(2)

The basic strategy to prove a statement of the form "$\forall\, x \in S,\ p(x)$" is to begin with the sentence "Let $x \in S$ be arbitrary." The rest of the proof is showing that $p(x)$ is true.

The basic strategy to prove a statement of the form "$\exists\, x \in S$ such that $p(x)''$ is to produce some $x \in S$ such that $p(x)$ is true.

Example: we can understand the statement "$A \subseteq B$" as "$\forall\ x \in A,\ x \in B$." Therefore, to prove such a statement, begin by saying "Let $x \in A$ be arbitrary." For the rest of the proof, show that $x \in B$.