Exam review document

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Contents

1 Exam 01

1.1 Summary

Exam 01 is based on Chapters 1, 2, 4, and 5.

In Chapter 1, we talked about how to solve systems of linear equations using Gauss-Jordan elimination. This technique is crucial for most of the computational problems in this course. Gaussian elimination is the process of finding a row echelon form of the matrix. Jordan elimination is the process of taking a row echelon form and finding the reduced row echelon form.

Chapter 2 focused on vectors in Euclidean space. We discussed addition and scalar multiplication of vectors, and linear combinations. In particular, our Quiz 01 was about determining whether a given vector was a linear combination of some other vectors. We also briefly mentioned length, and the dot product.

Chapter 4 was about matrices. We defined addition of matrices, scalar multiplication of matrices, and matrix products. We also defined the identity matrix, which serves as a multiplicative identity for matrices. This brought with it the topic of inverse matrices. We also defined the transpose of a matrix.

Chapter 5 was mainly about vector subspaces of $\mathbb{R}^n$. Every span of a sequence of vectors is a vector subspace, and conversely, every vector subspace is the span of some sequence of vectors. Now, among all the different sequences of vectors that could span a vector subspace, there are some that are "nice" in the sense of being linearly independent, while still corresponding to the entire vector subspace. These sequences are called bases of a vector subspace. We also introduced the columnspace and nullspace of a matrix, as examples of vector subspaces of $\mathbb{R}^n$.

1.2 Practice problems

Section 1.3: Exercises 1.3.1-4, 1.3.6

Section 1.4: Exercises 1.4.5-13, 1.4.16-25, 1.4.27, 1.4.28

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Section 2.1: Exercises 2.1.2, 2.1.3

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Section 4.3: Exercise 4.3.2

Section 4.4: Exercises 4.4.1, 4.4.2, 4.4.4-7, 4.4.9-12, 4.4.14

Section 4.5: Exercises 4.5.1-5, 4.5.7, 4.5.8, 4.5.18

Section 4.7: Exercises 4.7.1-3

Section 5.1: Exercises 5.1.1-5

Section 5.2: Exercises 5.2.3-11

Section 5.3: Exercises 5.3.1, 5.3.2

Section 5.4: Exercises 5.4.1-5, 5.4.8, 5.4.14, 5.4.15

Section 5.5: Exercises 5.5.1, 5.5.2

1.3 Test content

The directions of each problem are given here:

1. [20] Describe the set of solutions of the following linear system.

[linear system]

2. [20] Express the vector

$$\overrightarrow{u} = [\text{ vector }]$$

as a linear combination of the vectors

$$\overrightarrow{v_1} = [\text{ vector }], \quad \overrightarrow{v_2} = [\text{ vector }], \quad \overrightarrow{v_3} = [\text{ vector }],$$

or show that no such expression is possible.

3. [20] Find the inverse of the following matrix, or show that the matrix is not invertible.

$$A = [\text{ matrix }]$$

4. [20] Find a basis for the nullspace of the following matrix.

$$A = [\text{ matrix }]$$

5. [20] For each of the following sets, write "Y" if the set is a vector subspace of [some Euclidean space] and "N" if it is not. (It is not necessary to show your work or reasoning for this problem.)

[Four different sets]

2 Exam 02

2.1 Summary

Exam 02 is based on Chapters 6, 9, 10, 7, and 8.

Chapter 6 was our first look at linear transformations. Specifically, Chapter 6 only mentioned linear transformations between Euclidean spaces. We discussed the definition and various examples. Every linear transformation can be represented by a matrix. We opened up the discussion of how to find the matrix representation of a linear transformation, but didn't go into great detail until Chapter 10.

Chapter 9 was about extending the definitions and concepts that we learned about Euclidean spaces to more general settings. We defined the term "vector space" in a way that was general enough to include many interesting sets, such as vector spaces of polynomials, matrices, and general functions. We also discussed some concepts from Euclidean space which are valid in general vector spaces. In particular, linear combinations, vector subspaces, and bases (and all their consequences) can be discussed in general vector spaces.

In Chapter 10, we discussed the properties of linear transformations in the context of general vector spaces. Particular emphasis was placed on finding matrix representations of linear transformations with respect to given bases on the domain and codomain. This led us back to the context of matrices over the real numbers.

Chapter 7 was about determinants. With some amount of effort, we defined the determinant of a general square matrix in a computationally-minded way. We also discussed some shortcuts that assist in computing the determinant of a square matrix. The main reason that we concern ourselves with determinants is that it gives a straightforward criterion for figuring out whether a square matrix is invertible; a square matrix is invertible if and only if its determinant is nonzero.

Once we knew how to compute the determinant of a square matrix, we were then able to move into Chapter 8, which used these results to find eigenvalues. Eigenvalues and eigenvectors serve many important purposes in applied math, but we were mostly interested in them because they are crucial to the notion of diagonalization. Diagonalization allows for simplified computations involving matrices and exponents.

2.2 Practice problems

Section 6.1: Exercises 6.1.1, 6.1.2, 6.1.5
Section 6.2: Exercise 6.2.5

Section 9.1: Exercises 9.1.1-4, 9.17
Section 9.2: Exercises 9.2.2-4, 9.26, 9.2.7
Section 9.3: Exercises 9.3.2-4, 9.3.8, 9.3.9, 9.3.11, 9.3.16
Section 9.4: Exercises 9.4.1-4, 9.4.8, 9.4.9

Section 10.1: Exercises 10.1.1, 10.1.2, 10.1.4, 10.1.5,
Section 10.3: Exercise 10.3.1
Section 10.4: Exercises 10.4.1-6,

Section 7.1: Exercises 7.1.1, 7.1.2
Section 7.2: Exercises 7.2.1-7
Section 7.3: Exercise 7.3.1
Section 7.4: Exercise 7.4.1
Section 7.5: Exercises 7.5.1, 7.5.3, 7.5.6, 7.5.7, 7.5.10, 7.5.13

Section 8.1: Exercises 8.1.1-5
Section 8.2: Exercises 8.2.1-7
Section 8.4: Exercises 8.4.5, 8.4.6
Section 8.9: Exercises 8.9.1, 8.9.2

2.3 Test content

The directions of each problem are given here:

1. [20] For each of the following functions, write "Y" if the function is a linear transformation and "N" if it is not. (It is not necessary to show your work or reasoning for this problem.)

[four different functions]

2. [20] Determine whether the following sequence of vectors in [vector space] is linearly independent.

[finite sequence of vectors]

3. [20] Find the matrix representation of the linear transformation

$$T : [\text{vector space 1}] \to [\text{vector space 2}]$$
$$T\left([\text{generic input vector}]\right) = [\text{specific output vector}]$$

with respect to the bases

$$B = [\text{basis for vector space 1}]$$
$$C = [\text{basis for vector space 2}]$$

on [vector space 1] and [vector space 2] respectively.

4. [20] Compute the determinant of the following matrix.

$$A = [5 \times 5 \text{ matrix}]$$

5. [20] Diagonalize the following matrix.

$$A = [2 \times 2 \text{ matrix}]\,.$$

(To clarify: you are being asked to produce matrices P and D satisfying the equation $P^{-1}AP = D$, such that D is a diagonal matrix.)

3 Final exam

3.1 Summary

The final exam is cumulative, based on Chapters 1, 2, 4, 5, 6, 7, 8, 9, 10, and 11. Of these, only Chapter 11 is new.

In Chapter 11, we introduced inner products. An inner product is an operation that takes two input vectors and associates an output scalar, satisfying some particular properties. We began with real-valued inner products.

Defining an inner product allows us to define a notion of magnitude, which, in turn, allows us to consider distances. Not only that, but inner products also give a notion of orthogonality. With an orthogonal basis, it is much easier to find coordinates of a given vector. These coordinates are called "Fourier coefficients."

This brings up the question of how one can find an orthogonal basis for a given vector space. The theorem that answers this is the Gram-Schmidt procedure. This procedure takes a basis for an inner product space and finds an orthogonal basis. We also briefly mentioned orthogonal projections of a given vector onto a given vector subspace. In an infinite-dimensional vector space, this allows one to define what is called a "generalized Fourier series."

We then considered linear transformations between inner product spaces. We defined isometries and orthogonal transformations. This brought us, once again, to the topic of matrices. In particular, a linear transformation between inner product spaces is an isometry if and only if the matrix representation is an orthogonal matrix.

On the topic of matrices, we discussed that every symmetric matrix A with real entries can be "orthogonally diagonalized," meaning that we can find a diagonal matrix D and an orthogonal matrix P such that $P^{-1}AP = D$. We also talked about positive definite matrices, which can be used to define an inner product on a Euclidean space.

Finally, we briefly mentioned complex inner product spaces, saying that they follow all of the same theorems as real inner product space. We used the Gram-Schmidt procedure to find orthogonal bases in complex inner product spaces. (We also defined unitary and Hermitian matrices, featured in subsection 11.11.)

3.2 Practice problems

Section 11.1: Exercises 11.1.2, 11.1.3

Section 11.2: Exercises 11.2.1, 11.2.2, 11.2.4, 11.2.5

Section 11.3: Exercises 11.3.1, 11.3.2

Section 11.4: Exercises 11.4.1-4

Section 11.6: Exercises 11.6.1-3

Section 11.7: Exercise 11.7.1

Section 11.8: Exercise 11.8.2, 11.8.3

Section 11.10: Exercises 11.10.1-10

3.3 Test content

The directions of each problem are given here:

1. [10] Describe the set of solutions of the following linear system.

[linear system]

2. [10] Find the inverse of the following matrix, or show that the matrix is not invertible.

$$A = [\text{ matrix }]$$

3. [10] Find a basis for the nullspace of the following matrix.

$$A = [\text{ matrix }]$$

4. [10] Find the matrix representation of the linear transformation

$$T : [\text{vector space 1}] \to [\text{vector space 2}]$$
$$T\left([\text{generic input vector}]\right) = [\text{specific output vector}]$$

with respect to the bases

$$B = [\text{basis for vector space 1}]$$
$$C = [\text{basis for vector space 2}]$$

on [vector space 1] and [vector space 2] respectively.

5. [10] Determine whether the following matrix is diagonalizable.

$$A = [\text{square matrix}]\,.$$

6. [10] Consider the set $S = $ [set of vectors] in $V = $ [inner product space]. Find $S^{\perp}$, the orthogonal complement of S in V.

7. [10] Orthogonally diagonalize the following matrix.

$$A = [2 \times 2 \text{ symmetric matrix}]\,.$$

(To clarify: you are being asked to produce matrices P and D satisfying the equation $P^{-1}AP = D$, such that D is a diagonal matrix and P is an orthogonal matrix.)

8. [10] In the complex inner product space $\mathbb{C}^3$ with the (complex) dot product, let

$$\overrightarrow{v_1} = [\text{vector}]\,, \quad \overrightarrow{v_2} = [\text{vector}]\,, \quad \overrightarrow{v_3} = [\text{vector}].$$

Use the Gram-Schmidt procedure to find an orthogonal basis for the vector subspace $W = \operatorname{span}(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})$.

9. [10] For each of the following matrices, consider the operation $\langle , \rangle$ on $\mathbb{R}^2$ defined by the relationship

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = \overrightarrow{u}^T A \overrightarrow{v}.$$

For each matrix, write "Y" if the matrix defines an inner product on $\mathbb{R}^2$ and "N" if it does not. (It is not necessary to show your work or reasoning for this problem.)

[four different matrices]

10. [10] For each of the following matrices, write "Y" if the matrix is Hermitian and "N" if it is not. (It is not necessary to show your work or reasoning for this problem.)

[four different matrices]