Rudimentary Matrix Algebra

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Contents

1 Preliminaries

In this document, we introduce the basic principles of using matrices in mathematics and applied mathematics to solve systems of equations. In order to do this, we'll need to explain how to do algebra on matrices, just as one would do algebra on the numbers.

The practice of redefining and generalizing operations on numbers to work for different mathematical contexts is a field of mathematics known as "abstract algebra." The particular case of vectors and matrices is the sub-field known as "linear algebra." These two fields are a primary research interest for mathematicians of the past, present and future. There are entire (enormous) books devoted solely to these subjects, and the author admits to a proclivity to ramble for hours on end about them.

With such an abundance of literature in mind, you may be asking:

1.1 Why does this document exist?

Unfortunately, many students do not take a course in abstract or linear algebra until late in their undergraduate education (if ever). This leaves them mostly in the dark concerning matrices. I wrote this document for the sake of those who want to learn how to use matrices despite not having that background.

So:

1.2 Why does anyone care about matrices?

Matrix algebra is a skill that can significantly reduce the effort required to do a computation. As a visual strategy, it can highlight ways of reducing the amount of steps in a computation. Additionally, computers think of systems of equations primarily in terms of matrices. Beyond this, matrices can provide a link between solving systems of equations and deeper concepts in linear algebra. In general, *just about every person ever doing mathematical work can benefit from learning about matrices*.

1.3 What is a matrix?

Definition 1.1 *An* <u>$m \times n$ *matrix*</u> *(pronounced "m by n matrix") is an assignment of some mathematical objects to each ordered pair* (i, j)*, where* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*. We say that an* $m \times n$ *matrix* <u>*has* m *rows*</u> *and* <u>*has* n *columns*</u>*.*

Matrices are generally visualized as rectangles made up of squares, each one labeled by (i, j), with larger values for i below smaller ones, and larger values for j to the right of smaller ones. In each square, the value corresponding to (i, j) is placed. (In other words, the value assigned to (i, j) is placed in the cell contained by the ith row and the jth column.)

For example, a 3×4 matrix of numbers making the following assignments:

$$\begin{array}{ll} (1,1) & 1 \\ (1,2) & 5 \\ (1,3) & -\frac{4}{3} \\ (1,4) & 4 \\ (2,1) & 0 \\ (2,2) & 4 \\ (2,3) & -2 \\ (2,4) & 3 \\ (3,1) & \sqrt{2} \\ (3,2) & 6 \\ (3,3) & e^2 \\ (3,4) & \pi \end{array} \tag{1}$$

would be visualized like this:

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & -\frac{4}{3} & 4 \\ \hline 0 & 4 & -2 & 3 \\ \hline \sqrt{2} & 6 & e^2 & \pi \\ \hline \end{array}. \tag{2}$$

However, we need not bother ourselves with drawing lines dividing the cells. Instead, we'll just write the assignments like this:

$$\begin{pmatrix} 1 & 5 & -\frac{4}{3} & 4 \\ 0 & 4 & -2 & 3 \\ \sqrt{2} & 6 & e^2 & \pi \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 5 & -\frac{4}{3} & 4 \\ 0 & 4 & -2 & 3 \\ \sqrt{2} & 6 & e^2 & \pi \end{bmatrix}. \tag{3}$$

We define two matrices as being equal if they are the same size and for each ordered pair (i, j), the two matrices associate (i, j) to equal mathematical objects. Thus, two matrices are equal only if *all* of their entries agree:

$$\begin{pmatrix} 1 & 0 & 9 \\ 0 & 6 & 7 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 9 \\ 0 & 6 & 6 \end{pmatrix}. \tag{4}$$

There are plenty generalizations of matrices. For one thing, we've only dealt with matrices that have two coordinates for each cell, creating a 2-dimensional diagram. What if we had triplets, (i, j, k), creating a 3-dimensional diagram? What if we had quadruplets, (i, j, k, l), creating a 4-dimensional diagram? What if we had more than that? These generalizations are known in mathematics as "tensors," although some applied mathematicians refer to them as "n-dimensional matrices." For our purposes, though, we will only be need with "2-dimensional" matrices, such as those diagrammed above.

2 Matrix operations

It will be convenient to be able to add, subtract, multiply and (sort of) divide matrices, in the same way that one would add, subtract, multiply and (sometimes) divide real numbers. So first, we'll have to explain what these operations mean.

2.1 Addition of matrices

Addition of matrices is unnervingly simple. Given two $m \times n$ matrices, one adds them by simply adding the entries in corresponding positions:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}. \tag{5}$$

For example:

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 9 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 0+4 & 1+9 \\ 1+(-1) & 2+8 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 0 & 10 \end{pmatrix}$$
$$\begin{pmatrix} -12 & 7 \\ 6 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 6 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -12+4 & 7+6 \\ 6+(-1) & 2+5 \end{pmatrix} = \begin{pmatrix} -8 & 13 \\ 5 & 7 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1+2 & 0+(-2) & 1+2 \\ -1+0 & 1+2 & 0+(-2) \\ 0+0 & -1+2 & -1+0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 3 \\ -1 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}.$$
$$\begin{pmatrix} -6 & 1 & 0 & 0 \\ 1 & 2 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 8 & 9 & -1 & -4 \\ 4 & 6 & -12 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 10 & -1 & -4 \\ 5 & 8 & -14 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 4 \\ -2 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 9 \\ -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 13 \\ -3 \\ -2 \\ 5 \end{pmatrix} \tag{6}$$

The sum of two $m \times n$ matrices will always be an $m \times n$ matrix. Addition of matrices makes sense only when the matrices have the same numbers of rows and columns; one cannot add a 2×3 matrix to a 4×3 matrix:

$$\begin{pmatrix} 3 & 1 & 7 \\ 0 & -2 & 8 \end{pmatrix} + \begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 12 \\ 8 & -9 & 1 \end{pmatrix} \text{ (is meaningless)} \tag{7}$$

2.2 Multiplication of matrices

There are two sorts of multiplications for matrices, depending on *what* kind of mathematical object you want to multiply by a matrix.

2.2.1 Scalar multiplication

"Scalar multiplication" is the multiplication of a "scalar," (that is, a number) by a matrix. This, too, is very simple; just multiply each entry by the scalar:

$$k \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}. \tag{8}$$

For example:

$$\begin{aligned} 12 \begin{pmatrix} 1 & 5 & 0 \\ 3 & -2 & 1 \end{pmatrix} &= \begin{pmatrix} 12 & 60 & 0 \\ 36 & -24 & 12 \end{pmatrix} \\ 0 \begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 5 & 2 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{9}$$

$$-1\begin{pmatrix}0 & -2 & 8\end{pmatrix} = \begin{pmatrix}0 & 2 & -8\end{pmatrix}$$

$$6\begin{pmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\9 & 10 & 11 & 12\\13 & 14 & 15 & 16\end{pmatrix} = \begin{pmatrix}6 & 12 & 18 & 24\\30 & 36 & 42 & 48\\54 & 60 & 66 & 72\\78 & 84 & 90 & 96\end{pmatrix} \tag{10}$$

The product of a scalar and a matrix will *always* be a matrix of the same size.

2.2.2 Multiplication of matrices by matrices

Multiplying two matrices together can be significantly more complicated.

$$\begin{pmatrix}a_{11} & a_{12} & \dots & a_{1n}\\a_{21} & a_{22} & \dots & a_{2n}\\\vdots & \vdots & \ddots & \vdots\\a_{m1} & a_{m2} & \dots & a_{mn}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12} & \dots & b_{1p}\\b_{21} & b_{22} & \dots & b_{2p}\\\vdots & \vdots & \ddots & \vdots\\b_{n1} & b_{n2} & \dots & b_{np}\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^n a_{1i}b_{i1} & \sum_{i=1}^n a_{1i}b_{i2} & \dots & \sum_{i=1}^n a_{1i}b_{ip}\\\sum_{i=1}^n a_{2i}b_{i2} & \sum_{i=1}^n a_{2i}b_{i2} & \dots & \sum_{i=1}^n a_{2i}b_{ip}\\\vdots & \vdots & \ddots & \vdots\\\sum_{i=1}^n a_{mi}b_{i1} & \sum_{i=1}^n a_{mi}b_{i2} & \dots & \sum_{i=1}^n a_{mi}b_{ip}\end{pmatrix}. \tag{11}$$

To summarize, the (i, j)th entry of the product of two matrices will be the sum of the products of the entries in the ith row of the first factor and the jth column of the second factor. For example:

$$\begin{pmatrix}1\\0\\-1\\2\\4\end{pmatrix}\begin{pmatrix}3 & -2 & 1\end{pmatrix} = \begin{pmatrix}3 & -2 & 1\\0 & 0 & 0\\-3 & 2 & -1\\6 & -4 & 2\\12 & -8 & 4\end{pmatrix} \tag{12}$$

$$\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}11\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}\begin{pmatrix}2 & -1 & 0 & 1 & 2\\1 & 1 & -1 & 0 & 0\end{pmatrix} = \begin{pmatrix}4 & 1 & -2 & 1 & 2\\-1 & -1 & 1 & 0 & 0\end{pmatrix}$$

$$
\begin{pmatrix}1 & 0\\ 1 & 2\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 1 & 2\\ 2 & -1 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 & 1 & 2\\ 4 & -1 & 1 & 4\\ -2 & 1 & 0 & -1\end{pmatrix}
$$

$$
\begin{pmatrix}1 & 2 & 1\\ 0 & -1 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 3 & 2\\ 2 & -1 & -1 & 2\\ 3 & -2 & 1 & 2\end{pmatrix} = \begin{pmatrix}7 & -3 & 2 & 8\\ -5 & -1 & 0 & -4\end{pmatrix}. \tag{13}
$$

$$
\begin{pmatrix}1 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 3 & 0 & 3\\ 2 & 0 & 2 & 0\\ 0 & 3 & 0 & 3\end{pmatrix}
$$

Matrix multiplication makes sense only when the number of columns in the first factor is equal to the number of rows in the second factor. *Id est*, given matrices A and B, in order to make sense of the product AB, the number of columns of A must equal the number of rows of B. If this condition is not satisfied, then the two matrices simply cannot be multiplied. Additionally, the product of an $m \times n$ matrix and an $n \times p$ matrix will always be an $m \times p$ matrix.

An interesting feature of matrix multiplication is that it is not *commutative*; this means that, given matrices A and B, AB might not be the same as BA. First of all, it's possible that one of those multiplications makes sense, but not the other. (For example, if A is a 2×3 matrix and B is a 3×5 matrix, then AB is a 2×5 matrix, but BA is not defined.) Besides, even if the dimensions *do* match in both cases, the two products might still be different, as in the following example:

$$
\begin{matrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}\\ \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\end{matrix}. \tag{14}
$$

Thankfully, however, matrix multiplication is *associative* (which means that, for matrices A, B and C, $A(BC) = (AB)C$) and *distributive* (which means that, for matrices A, B and C, $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$).

2.3 Division of matrices?

Given two numbers, a and b, what exactly do we mean by $\frac{b}{a}$? We'll need to introduce some terminology first.

Definition 2.1 *Let a be a nonzero real (or complex) number. The multiplicative inverse of a is a number x such that $ax = 1$ and $xa = 1$.*

We typically denote "the multiplicative inverse of a" by a^{-1}, or $\frac{1}{a}$. When we refer to $\frac{b}{a}$, what we really mean is ba^{-1}. Therefore, as long as one can define a multiplicative inverse, one can define division. In order to explain what we mean by "division of matrices," we'll need to do a similar investigation.

First, it's necessary to come up with a matrix that can serve the same purpose as 1 does for the real (and complex) numbers:

Definition 2.2 *The $n \times n$ identity matrix is a matrix I_n such that if A is any $m \times n$ matrix, then $AI_n = A$, and if B is any $n \times m$ matrix, then $I_nB = B$.*

The $n \times n$ identity matrix is always just a matrix of 1's along the diagonal and 0's elsewhere:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{15}$$

For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Now we're in a position to define a multiplicative inverse of a matrix.

Definition 2.3 *Let A be an $m \times n$ matrix. An inverse matrix of A is an $n \times m$ matrix B such that $AB = I_m$ and $BA = I_n$.*

We often denote "the inverse matrix of A" by A^{-1}. (No one ever uses the notation $\frac{1}{A}$.)

Of particular interest are the square matrices:

Definition 2.4 *A square matrix is an $m \times n$ matrix such that $m = n$.*

In other words, a square matrix has as many rows as columns. As it turns out, only square matrices can have inverse matrices. This fact is **not** easy to prove, but using it, we can restate the definition:

Definition 2.5 *Let A be an $n \times n$ matrix. An inverse matrix of A is an $n \times n$ matrix B such that $AB = I_n = BA$.*

Do all square matrices have inverses? No, in the same way that some numbers (namely, 0) do not have multiplicative inverses. The following are some examples of square matrices that do not have inverses:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

(There are infinitely many for each size $n \times n$.)

How would one compute the inverse matrix of a given matrix? In order to answer this, we'll need to understand the procedure of row reduction of matrices, which is the topic of the next section.

3 Matrix equations

A matrix equation is exactly what the name implies; an equation with matrices in it. In particular, we'll be interested in equations of the form $Ax = B$, where A is an $m \times n$ matrix, x is an $n \times 1$ matrix, and B is an $m \times 1$ matrix:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{B}. \tag{18}$$

The goal here will be to solve for the $n \times 1$ matrix x. Here are some examples, and their solutions (in the following, c, c_1, c_2 and c_3 refer to arbitrary scalars):

$$\begin{gathered}
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 2-c \end{pmatrix} \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\
\begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Solution: } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 1-c_1 \\ -c_2 \\ c_3 \end{pmatrix} \\
\begin{pmatrix} 4 & 2 \\ 1 & -1 \\ 3 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 \\ 3 \\ 5 \\ -7 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}
\end{gathered} \tag{19}$$

$$\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\qquad \text{Solution: }\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad \text{Solution: }\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\c\\-c\end{pmatrix}$$

$$\begin{pmatrix}1&1\\0&2\\-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\2\\2\\2\end{pmatrix}\qquad \text{No solution} \tag{20}$$

$$\begin{pmatrix}1&1\\0&1\\1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\qquad \text{No solution}$$

$$\begin{pmatrix}-1&-3&-9\\2&6&1\\3&9&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-10\\3\\7\end{pmatrix}\qquad \text{No solution}$$

As with any other kind of equation, some matrix equations have one solution, some have no solution, and some have multiple solutions. A theorem from linear algebra indicates that a matrix equation that has multiple solutions will always have infinitely many solutions. This is why some of the systems above involve arbitrary scalars in their solutions; by choosing any (yes, any) value for the constant(s), one can create a new vector x that is a solution of the system.

A system of linear equations in n variables x_1, x_2,..., x_n corresponds exactly to a matrix equation. To see this, consider the following example:

$$\begin{pmatrix}1&2\\0&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}-4\\5\end{pmatrix} \tag{21}$$

By computing the matrix multiplication, we get:

$$\begin{pmatrix} 1x_1 + 2x_2 \\ 0x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}, \tag{22}$$

which exactly means that

$$\begin{aligned} 1x_1 + 2x_2 &= -4 \\ 0x_1 + 3x_2 &= 5 \end{aligned}. \tag{23}$$

On the other hand, by just taking the coefficients on each variable, we can revert back to the original matrix equation:

$$\begin{aligned} \boxed{1}x_1 + \boxed{2}x_2 &= \boxed{\text{-4}} \\ \boxed{0}x_1 + \boxed{3}x_2 &= \boxed{5} \end{aligned} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}. \tag{24}$$

The result is that matrix equations can be used as a shorthand for linear systems. In fact, we can go even further. If we know that we're dealing in the variables x_1 and x_2 (which, by the way, are just names; calling them something different would have changed nothing), then we can write the matrix equation above in an even simpler form:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \longrightarrow \left(\begin{array}{cc:c} 1 & 2 & -4 \\ 0 & 3 & 5 \end{array}\right). \tag{25}$$

This notation is known as an augmented matrix.

Since systems of linear equations (and matrix equations) correspond exactly to augmented matrices, we can solve them (and/or matrix equations) by dealing only with augmented matrices. This process is called Gauss-Jordan elimination.

3.1 Elementary row operations

We'd like to discuss a procedure of solving matrix equations commonly referred to as "row reduction." First, we'll have to define the moves that are legal in the procedure.

Definition 3.1 *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{26}$$

be an $m \times n$ matrix. The following are the *elementary row operations* *of A.*
(i) Multiplication of all of the entries in a single row (say, the ith row) by a nonzero constant k:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_i \to kR_i} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{i1} & ka_{i2} & \dots & ka_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{27}$$

(ii) Swapping two rows (say, the ith row and the jth row):

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_i \leftrightarrow R_j} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{28}$$

(iii) Adding a nonzero constant k times all of the entries in a row (say, the ith row)

to all of the corresponding entries in another row (say, the jth row):

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \xrightarrow{R_j \to R_j + kR_i} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} + ka_{i1} & a_{j2} + ka_{i2} & \dots & a_{jn} + ka_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}. \tag{29}$$

For example:

$$\begin{matrix} \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_2 \to 3R_2} \begin{pmatrix} 1 & 5 \\ -3 & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -1 & 2 \\ 1 & 5 \end{pmatrix} \\ \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} -1 & 9 \\ -1 & 2 \end{pmatrix} \end{matrix}. \tag{30}$$

We have a name for the relationship between two matrices when one can be gotten from the other by doing only elementary row operations.

Definition 3.2 *Let A and B be $m \times n$ matrices. We say that A and B are <u>row equivalent matrices</u> provided that there exists a sequence*

$$A = M_1 \to M_2 \to M_3 \to \dots \to M_{k-1} \to M_k = B \tag{31}$$

of $m \times n$ matrices M_1, M_2, ..., M_k such that $M_1 = A$, $M_k = B$ and for each i (where $1 \leq i < k$), M_{i+1} can be produced from an elementary row operation on M_i.

In other words, two matrices are equivalent if you can get one from the other by a sequence of elementary row operations. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 9 & 18 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \tag{32}$$

are row equivalent matrices, because you can produce B by adding -9 times the first row of A to the second row of A:

$$\begin{pmatrix} 1 & 2 \\ 9 & 18 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 9R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

As another example,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

are row equivalent matrices, because

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 4R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_2}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

Note that row equivalent matrices are *not* necessarily equal (unless, of course, all of the elementary row operations done to get from one to the other cancel each other out). As we mentioned in the first section, two matrices are equal only if all of their corresponding entries are equal. What, then, is so great about this relationship of row equivalence? The following sections answer this question.

3.2 Gauss-Jordan elimination

Here is the main interest of row equivalence:

Theorem 3.3 *If two linear systems have row equivalent augmented matrices, then the two linear systems have the same solution(s).*

This means that for any augmented matrix, we can do elementary row operations without worrying about whether the solutions will be altered. Let's look at the following example:

$$\begin{aligned} 1x_1 + 0x_2 + 1x_3 &= 1 \\ 1x_1 + 1x_2 + 2x_3 &= 1 \\ 1x_1 + 0x_2 + 2x_3 &= 1 \\ 1x_1 - 1x_2 + 2x_3 &= 0 \end{aligned}. \tag{36}$$

Even when we look at the augmented matrix for this, it's not really clear how to proceed:

$$\left(\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{array}\right). \tag{37}$$

However, this augmented matrix is row equivalent to:

$$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right). \tag{38}$$

Based on the theorem, this augmented matrix can then be thought of as a restatement of the original system:

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + 1x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 1x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}. \tag{39}$$

It's now clear what the solutions of this system are. (In this case, there aren't any, since $0x_1 + 0x_2 + 0x_3$ cannot be anything other than 0, but the last equation requires that it be 1.)

This example illustrates that if we can use elementary row operations to find a sufficiently simple augmented matrix, then solving the matrix equation will be easy. But how simple is "sufficiently simple?" The following definitions will formalize this concept.

Definition 3.4 *Let A be an $m \times n$ matrix. A row echelon form of A is a matrix B that is row equivalent to A such that:*
(i) The left-most nonzero entry of each row of B is 1.
(ii) The left-most nonzero entry of each row of B contains only zeros below it in its column.
(iii) If (i, j) and (k, l) are two positions of left-most nonzero entries in B and $i < k$, then $j < l$.
(iv) Any rows that do not contain nonzero entries are at the bottom of the matrix.

Here are some examples of matrices that meet the criteria of row echelon form, and some that don't:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is row echelon.} \quad \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not; it doesn't satisfy criterion (i).}$$
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is row echelon.} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ is not; it doesn't satisfy criterion (ii).}$$
$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \text{ is row echelon.} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \text{ is not; it doesn't satisfy criterion (iii).}$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is row echelon.} \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is not; it doesn't satisfy criterion (iv).} \tag{40}$$

To summarize: row echelon form matrices have only 1's as the left-most nonzero entries of each row, below which are only 0's; further, the left-most nonzero 1's must move to the right as one moves down the rows.

Row echelon form is simple, but it can get even simpler:

Definition 3.5 *Let A be an $m \times n$ matrix. A* <u>*reduced row echelon form of A*</u> *is a matrix B that is row equivalent to A such that:*
(i) The left-most nonzero entry of each row of B is 1.
(ii) The left-most nonzero entry of each row of B contains only zeros above and below it in its column.
(iii) If (i, j) and (k, l) are two positions of left-most nonzero entries in B and $i < k$, then $j < l$.
(iv) Any rows that do not contain nonzero entries are at the bottom of the matrix.

To summarize: reduced row echelon matrices are row echelon matrices with the additional condition that each left-most nonzero entry in each row is the only nonzero entry in its entire column. Here are some examples of reduced row echelon matrices, and some examples which are row echelon but not reduced row echelon:

$$\begin{array}{ll}
\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \text{ is reduced row echelon.} & \begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \text{ is not.}\\
\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix} \text{ is reduced row echelon.} & \begin{pmatrix}1&0&3&0\\0&0&1&0\end{pmatrix} \text{ is not.}\\
\begin{pmatrix}1&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix} \text{ is reduced row echelon.} & \begin{pmatrix}1&-1\\0&1\\0&0\\0&0\\0&0\end{pmatrix} \text{ is not.}\\
\begin{pmatrix}0&1&4&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \text{ is reduced row echelon.} & \begin{pmatrix}0&1&4&1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix} \text{ is not.}
\end{array} \tag{41}$$

As long as we can find a reduced row echelon form of an augmented matrix, we can solve the system. So, which matrices have reduced row echelon forms? All of them:

Theorem 3.6 *Every $m \times n$ matrix has a unique reduced row echelon form.*

Thus, whatever matrix we are given, we can perform some sequence of elementary row operations to arrive at a reduced row echelon matrix. The process of using elementary row operations to find the reduced row echelon form of an augmented matrix is commonly called "row reduction," or "Gauss-Jordan elimination."

Here are some examples of augmented matrices, and their reduced row echelon forms:

$$\begin{array}{cc}
\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) & \text{RREF: } \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \\
\left(\begin{array}{ccc|c} 5 & 0 & 2 & -1 \\ 0 & 0 & 2 & 4 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right) & \text{RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\left(\begin{array}{cccc|c} 1 & 2 & -2 & 2 & 3 \\ -1 & 4 & -4 & -2 & -15 \end{array}\right) & \text{RREF: } \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 & 2 \end{array}\right) \\
\left(\begin{array}{ccc|c} 6 & -6 & 0 & 24 \\ 2 & -1 & 0 & 5 \\ 0 & 2 & 0 & -6 \end{array}\right) & \text{RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\left(\begin{array}{ccc|c} 2 & 0 & 3 & -1 \\ 0 & 1 & 3 & -3 \\ 1 & 0 & 3 & -5 \\ 1 & 1 & 2 & 4 \end{array}\right) & \text{RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 3 & 2 & 3 \\ 0 & 4 & 1 & 21 & 6 & -4 \\ 0 & 6 & 3 & 27 & 6 & -6 \end{array}\right) & \text{RREF: } \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 6 & 0 & -1 \\ 0 & 0 & 1 & -3 & 0 & 0 \end{array}\right) \\
\left(\begin{array}{ccc|c} 4 & 2 & 3 & 28 \\ -6 & -2 & -1 & -28 \\ 9 & 3 & 1 & 40 \end{array}\right) & \text{RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right)
\end{array} \tag{42}$$

$$\left(\begin{array}{cccc:c} 2 & 0 & 4 & 2 & -2 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -2 \\ -1 & 1 & -3 & -1 & 0 \end{array}\right) \quad \text{RREF:} \left(\begin{array}{cccc:c} 1 & 0 & 2 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{43}$$

When dealing with a linear system, it is important to understand how the reduced row echelon form of the corresponding matrix equation should be interpreted. Let's look at the following example:

$$\begin{gathered} -3w + 1x - 7y - 3z = 2 \\ -2w + 1x - 5y + 0z = -5 \\ -w + 1x - 1y + 1z = 0 \\ 2w + 0x + 4y + 0z = -4 \end{gathered} \tag{44}$$

First, convert this system into an augmented matrix:

$$\left(\begin{array}{cccc:c} -3 & 1 & -7 & -3 & 2 \\ -2 & 1 & -5 & 0 & -5 \\ 0 & 1 & -1 & 1 & 0 \\ 2 & 0 & 4 & 0 & -4 \end{array}\right), \tag{45}$$

and find its reduced row echelon form:

$$\left(\begin{array}{cccc:c} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{46}$$

What does this mean? In order to interpret this, let's revert from the augmented matrix to the system notation:

$$\begin{gathered} 1w + 0x + 2y + 0z = -1 \\ 0w + 1x - 1y + 0z = -1 \\ 0w + 0x + 0y + 1z = 1 \\ 0w + 0x + 0y + 0z = 0 \end{gathered} \Rightarrow \begin{gathered} w + 2y = -1 \\ x - y = -1 \\ z = 1 \end{gathered}. \tag{47}$$

At first, this does not seem helpful. It appears that there's still more to solve. However, that is not the case; these are the *only* restrictions on the values of w, x, y and z. In other words, their values can be *anything*, as long as they satisfy that $w + 2y = -1$, $x - y = -1$, and $z = 1$. Select a value c for y. This completely determines the values for w, x, y and z:

$$\begin{aligned} w + 2c &= -1 \\ x - c &= -1 \\ y &= c \\ z &= 1 \end{aligned} \Rightarrow \begin{aligned} w &= -2c - 1 \\ x &= c - 1 \\ y &= c \\ z &= 1 \end{aligned}, \text{ or } \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2c - 1 \\ c - 1 \\ c \\ 1 \end{pmatrix}. \tag{48}$$

In linear algebra, we call y a *free variable*, since any chosen value for y will produce a solution to the system.

Some systems have more than one free variable. For example,

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \Rightarrow \begin{aligned} 1x_1 + 0x_2 + 0x_3 - 1x_4 + 1x_5 &= 1 \\ 0x_1 + 1x_2 + 0x_3 + 2x_4 + 2x_5 &= 0 \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 + 0x_5 &= -1 \end{aligned} \tag{49}$$

which gives the equations

$$\begin{aligned} x_1 - x_4 + x_5 &= 1 \\ x_2 + 2x_4 + 2x_5 &= 0. \\ x_3 &= -1 \end{aligned} \tag{50}$$

The situation is similar here. Again, *any* values for x_1, x_2, x_3, x_4 and x_5 which satisfy the above relationships will be a solution. By choosing values $x_4 = c_1$ and $x_5 = c_2$, we can completely determine the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 + c_1 - c_2 \\ -2c_1 - 2c_2 \\ -1 \\ c_1 \\ c_2 \end{pmatrix}. \tag{51}$$

3.3 Computing inverse matrices

4 Determinants

A common theme in mathematics is drawing information from expressions by associating simpler expressions to them.

For example, given a quadratic equation $ax^2 + bx + c = 0$, how can we tell whether the solutions will be real, non-real, or repeated? In this case, we look at the associated "discriminant," the expression $b^2 - 4ac$, and study it in order to learn about the original equation. Just from this single number, we can tell whether the equation has two real solutions (in the case that the discriminant is positive), and whether they are rational (if the discriminant is a nonzero perfect square) or irrational (if the discriminant is not a perfect square), one real solution (in the case that the discriminant is zero), or two non-real solutions (in the case that the discriminant is negative).

In this section, we'll see what we can learn about a square matrix from an associated expression known as its "determinant:"

Definition 4.1 *Let A be an $n \times n$ (square) matrix. The determinant of A is defined recursively as follows.*
(i) If $n = 2$, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{52}$$

then the determinant $\det(A) = ad - bc$.
(ii) If $n > 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \tag{53}$$

then the determinant

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(M_{1i}), \tag{54}$$

where M_{1i} is the submatrix defined via

$$M_{1i} = \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2(i-1)} & a_{2(i+1)} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3(i-1)} & a_{3(i+1)} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(i-1)} & a_{n(i+1)} & \dots & a_{nn} \end{pmatrix}. \tag{55}$$

This formula may seem somewhat complicated, and that's because it is. Thankfully, for this document, we will only consider determinants of 2×2 or 3×3 matrices, since computing determinants for larger matrices is easy, but tiresome.

As the definition says, for a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{56}$$

the determinant is just

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{57}$$

On the other hand, for a 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{58}$$

the determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{59}$$

To summarize: one takes each entry along the first row, with alternating $\pm$ signs, multiplies them by the determinant of the submatrix obtained by deleting the row and the column of the entry in question, and then adding it all up. With some practice, you'll see that the process is easy, but it can get very tedious.

Here are some examples:

$$
\begin{gathered}
\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = (1)(5) - (0)(2) = 5 \\
\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \\
\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = -7 \\
\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2 \\
\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1 \\
\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -2 \\
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 \\
\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 24 \\
\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 18 \\
\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 30 \\
\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0
\end{gathered} \quad . \tag{60}
$$

4.1 Tricks for computing determinants

It didn't take long for people do realize that the determinant is useful, but tiresome to compute. Therefore, some mathematicians started to look for ways to simplify the process. The following theorem from linear algebra doesn't provide any new theory, and it's not particularly important for doing this kind of work, but it can help to reduce the pain of figuring out the determinant of a matrix by hand.

Theorem 4.2 *The following statements are true.*
(i) $\det(I_n) = 1$.
(ii) If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.
(iii) If A is an $n \times n$ matrix, then $\det(A) = \det\left(A^T\right)$.
(iv) If A is an $n \times n$ matrix and B is obtained from A by multiplying a nonzero constant r by a single row (or column) of A, then $\det(B) = r\det(A)$.
(v) If A is an $n \times n$ matrix and B is obtained from A by swapping two different rows (or columns) of A, then $\det(B) = -\det(A)$.
(vi) If A is an $n \times n$ matrix and B is obtained from A by adding a nonzero scalar multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

Here A^T refers to the transpose matrix, which is the matrix you get by exchanging all of the rows and columns of A:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \quad \to \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{pmatrix}. \tag{61}$$

Among the implications of this theorem is the fact that one can do elementary row (and column) operations in order to simplify the calculation of a determinant. For example, given the matrix

$$A = \begin{pmatrix} 87 & 37 & 9 \\ -3 & -3 & -1 \\ 19 & 8 & 2 \end{pmatrix}, \tag{62}$$

it would be rather annoying to compute the determinant of A directly. However, using the theorem above,

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 87 & 37 & 9 \\ -3 & -3 & -1 \\ 19 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -50 & -20 \\ -3 & -3 & -1 \\ 19 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -50 & -20 \\ -3 & -3 & -1 \\ 1 & -10 & -4 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 50 & 20 \\ 0 & -33 & -13 \\ 1 & -10 & -4 \end{vmatrix} = 10 \begin{vmatrix} 0 & 5 & 2 \\ 0 & -33 & -13 \\ 1 & -10 & -4 \end{vmatrix} = 10 \begin{vmatrix} 0 & 3 & 2 \\ 0 & -20 & -13 \\ 1 & -6 & -4 \end{vmatrix} \\
&= 10 \begin{vmatrix} 0 & 3 & 2 \\ 0 & -20 & -13 \\ 1 & 0 & 0 \end{vmatrix} = 10 \begin{vmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -10 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{vmatrix} \\
&= -10 \left(1 \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} \right) = 10. \qquad (63)
\end{aligned}$$

Another implication of this theorem is that the first row is not special; one can compute the determinant by using any kth row (or column), just by multiplying $(-1)^k$ by each term:

$$\begin{vmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = - \left(0 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \right). \qquad (64)$$

All of these sorts of approaches are common: simplify the matrix or look at a particular row or column of it until the determinant becomes easy. (This usually involves a lot of entries being zero.)

4.2 The meaning of the determinant

What is the point of all this? The point is the following theorem, which forms the core of every first course in linear algebra.

Theorem 4.3 *Let A be an $n \times n$ matrix. The following statements are equivalent.*
(i) $\det(A) = 0$.
(ii) There exists a vector $\overrightarrow{v} \neq \overrightarrow{0}$ such that $A\overrightarrow{v} = \overrightarrow{0}$.
(iii) The matrix A is not invertible.

By a "vector," in this context, we simply mean an $n \times 1$ matrix:

$$\overrightarrow{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}. \tag{65}$$

In particular, the "zero vector" is the vector whose entries are all zero:

$$\overrightarrow{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{66}$$

Let's take a moment to examine the implications of this theorem. Suppose that we're doing a calculation that involves finding all the solutions of this system of equations:

$$\begin{aligned} 2x - y + z &= 0 \\ 0x + 1y + 1z &= 0. \\ 1x + 2y + 2z &= 0 \end{aligned} \tag{67}$$

This is the same as the matrix equation

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{68}$$

So now, in order to find the solutions, we have no choice but to do Gauss-Jordan elimination to find the reduced row echelon form of the augmented matrix, right? Wrong.

Using finding the reduced row echelon form of the augmented matrix would work, but alternatively, we could just find the determinant of the matrix:

$$\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -2. \tag{69}$$

By the theorem, since the determinant is nonzero, there cannot be a vector $\overrightarrow{v}$ which is not $\overrightarrow{0}$ that satisfies the equation. As a result, our only solution is the zero vector:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{70}$$

and so $x = 0$, $y = 0$, and $z = 0$.

This sort of use of the theorem is of fundamental importance to the theory of linear transformations, but that is a topic best left for a course in linear algebra. In the next section, we will use this theorem to determine another important piece of data concerning matrices: eigenvalues.

5 Eigenvalues, eigenvectors, generalized eigenvectors

An eigenvalue of a square matrix is a scalar that can simplify the multiplication of the matrix by certain vectors.

Definition 5.1 *Let A be an $n \times n$ matrix. Given a scalar λ, we say that $\underline{\lambda \text{ is an eigenvalue of } A}$ provided that there exists a vector $\overrightarrow{v} \neq \overrightarrow{0}$ such that $A\overrightarrow{v} = \lambda \overrightarrow{v}$. Any such vector $\overrightarrow{v}$ is called an $\underline{\text{eigenvector of } A \text{ corresponding to } \lambda}$.*

Let's discuss the process of finding eigenvalues. First, we know that if A is any $n \times n$ matrix and λ is an eigenvalue of A, then there has to be a vector $\overrightarrow{v} \neq \overrightarrow{0}$ such that:

$$A\overrightarrow{v} = \lambda \overrightarrow{v}. \tag{71}$$

We can write this as:

$$A\overrightarrow{v} - \lambda \overrightarrow{v} = \overrightarrow{0}. \tag{72}$$

Now, given any matrix X with n rows, we know that $I_n X = X$, by definition of I_n as the identity matrix. Therefore, this equation can be written as

$$A\overrightarrow{v} - \lambda I_n \overrightarrow{v} = \overrightarrow{0}. \tag{73}$$

As we briefly mentioned earlier, matrix multiplication can distribute over addition, so we can factor out $\overrightarrow{v}$ to write this as:

$$\left(A - \lambda I_n\right) \overrightarrow{v} = \overrightarrow{0}. \tag{74}$$

By Theorem 4.3, this equations will have solutions with $\overrightarrow{v} \neq \overrightarrow{0}$ if and only if the determinant $\det\left(A - \lambda I_n\right) = 0$. Therefore, **a scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if** $\det\left(A - \lambda I_n\right) = 0$. In order to find the eigenvalues of a matrix, we'll assume this equation and then solve for λ.

We demonstrate the procedure in the case of

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \tag{75}$$

1. Set $0 = \det(A - \lambda I_2)$ and solve as follows:

$$0 = \det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{vmatrix} 1-\lambda & 3-0 \\ 3-0 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)(1-\lambda) - (3)(3) = 1 - 2\lambda + \lambda^2 - 9 = \lambda^2 - 2\lambda - 8. \quad (76)$$

The expression $\lambda^2 - 2\lambda - 8$ is known as the characteristic polynomial of A, and the equation $\lambda^2 - 2\lambda - 8 = 0$ is known as the characteristic equation of A.

2. Solve the characteristic equation:

$$0 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2), \quad (77)$$

and so A has two eigenvalues: $\lambda_1 = 4$ and $\lambda_2 = -2$.

Now, how can one find the eigenvectors associated to an eigenvalue? We will again refer to the equation

$$(A - \lambda I_n)\overrightarrow{v} = \overrightarrow{0}. \quad (78)$$

Once the eigenvalues have been determined, one can substitute them into this equation to solve for the eigenvectors. We demonstrate the procedure for the case of

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \quad (79)$$

1. Find the eigenvalues of A. (By the example above, these are $\lambda_1 = 4$ and $\lambda_2 = 2$.)

2. For each eigenvalue, find the solutions of the equation $(A - \lambda I_n)\overrightarrow{v} = \overrightarrow{0}$. In our case, for the eigenvalue $\lambda_1 = 4$:

$$(A - 4I_2)\overrightarrow{v_1} = \left(\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\overrightarrow{v_1}. \quad (80)$$

Thus, finding the eigenvector(s) $\overrightarrow{v_1}$ corresponding to $\lambda_1 = 4$ is reduced to the problem of solving the matrix equation

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{81}$$

By using Gauss-Jordan elimination, we find that this equation has the same solutions as the equation

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{82}$$

(In the case of 2×2 matrices, such as this one, the solutions are very easy to see even without Gauss-Jordan elimination, but for larger matrices, things could be more opaque at first.) From this, we get the relationship

$$a - b = 0, \tag{83}$$

which cannot be solved for an explicit value of a and b. This is normal. In general, any nonzero scalar times an eigenvector will also be an eigenvector corresponding to the same eigenvalue, so we should expect to have infinitely many solutions. In this case, the eigenvectors corresponding to $\lambda_1 = 4$ are

$$\overrightarrow{v_1} = \begin{pmatrix} a \\ a \end{pmatrix}, \tag{84}$$

where a is any value whatsoever (except 0), known as a free variable. Similarly, for the eigenvalue $\lambda_2 = 2$, we have the matrix equation

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \overrightarrow{v_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{85}$$

whose solutions are of the form

$$\overrightarrow{v_2} = \begin{pmatrix} 3b \\ b \end{pmatrix}, \tag{86}$$

where b is any value whatsoever (except 0).

5.1 Distinct, real eigenvalues

Here are some examples of matrices with eigenvalues that are real and distinct:

$$
\begin{aligned}
\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 1, \lambda_2 = -1 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} -b \\ b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 3, \lambda_2 = -1 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} -3b \\ b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 2, \lambda_2 = -1 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -2b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 3, \lambda_2 = -2 \\
& \quad \text{with eigenvectors } \begin{pmatrix} 2a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -2b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -1 & -5 \\ -1 & 3 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 4, \lambda_2 = -2 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} 5b \\ b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 2 & 2 \\ 9 & -1 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 5, \lambda_2 = -4 \\
& \quad \text{with eigenvectors } \begin{pmatrix} 2a \\ 3a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -3b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 0 & -2 \\ 5 & -7 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -5, \lambda_2 = -2 \\
& \quad \text{with eigenvectors } \begin{pmatrix} 2a \\ 5a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ b \end{pmatrix} \text{ for } \lambda_2
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
\begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -6, \lambda_2 = -1 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ 4b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -12 & -2 \\ 10 & 0 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -10, \lambda_2 = -2 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -5b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -2 & 12 \\ 4 & -10 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 2, \lambda_2 = -14 \\
& \quad \text{with eigenvectors } \begin{pmatrix} 3a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} -b \\ b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -1 & 8 \\ 11 & 2 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 10, \lambda_2 = -9 \\
& \quad \text{with eigenvectors } \begin{pmatrix} 8a \\ 11a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -8 & -1 \\ 7 & 0 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -7, \lambda_2 = -1 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -7b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -7 & 8 \\ 2 & -1 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = 1, \lambda_2 = -9 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} -4b \\ b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} 1 & 7 \\ -3 & -9 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -6, \lambda_2 = -2 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} 7b \\ -3b \end{pmatrix} \text{ for } \lambda_2 \\
\begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} & \quad \text{has eigenvalues } \lambda_1 = -2, \lambda_2 = -3 \\
& \quad \text{with eigenvectors } \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} b \\ -b \end{pmatrix} \text{ for } \lambda_2
\end{aligned}
\tag{88}
$$

$$\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = -2, \lambda_2 = -4$$

$$\text{with eigenvectors } \begin{pmatrix}a\\ -a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ b\end{pmatrix} \text{ for } \lambda_2$$

$$\begin{pmatrix}7 & 5\\ 5 & 7\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = 12, \lambda_2 = 2$$

$$\text{with eigenvectors } \begin{pmatrix}a\\ a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ -b\end{pmatrix} \text{ for } \lambda_2$$

$$\begin{pmatrix}-4 & -3\\ -7 & -8\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = -11, \lambda_2 = -1$$

$$\text{with eigenvectors } \begin{pmatrix}3a\\ 7a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ -b\end{pmatrix} \text{ for } \lambda_2$$

$$\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ -2 & -1 & 0\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$$

$$\text{with eigenvectors } \begin{pmatrix}0\\ 0\\ a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ -b\\ -b\end{pmatrix} \text{ for } \lambda_2, \begin{pmatrix}c\\ c\\ -3c\end{pmatrix} \text{ for } \lambda_3$$

$$\begin{pmatrix}1 & -1 & 0\\ -1 & 0 & -1\\ 0 & -1 & 1\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$$

$$\text{with eigenvectors } \begin{pmatrix}a\\ 2a\\ a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ 0\\ b\end{pmatrix} \text{ for } \lambda_2, \begin{pmatrix}c\\ -c\\ c\end{pmatrix} \text{ for } \lambda_3$$

$$\begin{pmatrix}1 & 0 & 0\\ -1 & 2 & 0\\ 3 & 1 & 0\end{pmatrix} \quad \text{has eigenvalues } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 0$$

$$\text{with eigenvectors } \begin{pmatrix}a\\ a\\ 4a\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}0\\ 2b\\ b\end{pmatrix} \text{ for } \lambda_2, \begin{pmatrix}0\\ 0\\ c\end{pmatrix} \text{ for } \lambda_3 \tag{89}$$

5.2 Repeated, real eigenvalues

The characteristic polynomial is, of course, a polynomial. Therefore, it may sometimes have repeated roots.

Definition 5.2 *Let A be an $n \times n$ matrix with characteristic polynomial*

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0. \quad (90)$$

Suppose that the characteristic polynomial factors as:

$$p(\lambda) = (\lambda - r_1)^{m_1}(\lambda - r_2)^{m_2}...(\lambda - r_k)^{m_k}, \quad (91)$$

where r_1, r_2, ..., r_{k-1} and r_k are all distinct. For each i, the multiplicity of the eigenvalue r_i is the corresponding exponent m_i.

In other words, the multiplicity of an eigenvalue a is the number of times the factor $\lambda - a$ appears in the characteristic polynomial. To have distinct roots literally means that the multiplicity of each eigenvalue is 1. If an eigenvalue has multiplicity greater than 1, it is called a repeated eigenvalue. (Some sources also refer to these as "multiple eigenvalues." The opinion of the author is that this term is extremely confusing.)

Here are some examples of matrices with repeated eigenvalues:

$$
\begin{aligned}
&\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} && \text{has eigenvalue } \lambda = 1 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} && \text{has eigenvalue } \lambda = 1 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} 0 \\ a \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} -1 & 3 \\ -3 & -7 \end{pmatrix} && \text{has eigenvalue } \lambda = -4 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ -a \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} && \text{has eigenvalue } \lambda = -1 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ -2a \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} -5 & 2 \\ -2 & -1 \end{pmatrix} && \text{has eigenvalue } \lambda = -3 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix} && \text{has eigenvalue } \lambda = -12 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ b \end{pmatrix} \text{ for } \lambda \\
&\begin{pmatrix} 3 & 6 \\ -6 & 15 \end{pmatrix} && \text{has eigenvalue } \lambda = 9 \text{ with multiplicity 2} \\
& && \text{with eigenvectors } \begin{pmatrix} a \\ a \end{pmatrix} \text{ for } \lambda
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
&\begin{pmatrix}6 & 4\\ -4 & -2\end{pmatrix} && \text{has eigenvalue } \lambda = 2 \text{ with multiplicity } 2\\
& && \text{with eigenvectors } \begin{pmatrix}a\\ -a\end{pmatrix} \text{ for } \lambda\\
&\begin{pmatrix}3 & 6\\ -6 & 15\end{pmatrix} && \text{has eigenvalue } \lambda = 9 \text{ with multiplicity } 2\\
& && \text{with eigenvectors } \begin{pmatrix}a\\ a\end{pmatrix} \text{ for } \lambda\\
&\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 3\\ 2 & 0 & -1\end{pmatrix} && \text{has eigenvalues } \begin{matrix}\lambda_1 = 1 \text{ with multiplicity } 2,\\ \lambda_2 = -2 \text{ with multiplicity } 1\end{matrix}\\
& && \text{with eigenvectors } \begin{pmatrix}0\\ a\\ 0\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}b\\ 2b\\ -2b\end{pmatrix} \text{ for } \lambda_2\\
&\begin{pmatrix}1 & -1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix} && \text{has eigenvalue } \lambda = 1 \text{ with multiplicity } 3\\
& && \text{with eigenvectors } \begin{pmatrix}a\\ 0\\ 0\end{pmatrix} \text{ for } \lambda\\
&\begin{pmatrix}9 & 4 & 0\\ -6 & -1 & 0\\ 6 & 4 & 3\end{pmatrix} && \text{has eigenvalues } \begin{matrix}\lambda_1 = 3 \text{ with multiplicity } 2,\\ \lambda_2 = 5 \text{ with multiplicity } 1\end{matrix}\\
& && \text{with eigenvectors } \begin{pmatrix}2a\\ -3a\\ b\end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix}c\\ -c\\ c\end{pmatrix} \text{ for } \lambda_2\\
&)(
\end{aligned}
\tag{93}
$$

$$\begin{pmatrix} 2 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{r l} & \lambda_1 = 1 \text{ with multiplicity } 2, \\ \text{has eigenvalues} & \lambda_2 = -1 \text{ with multiplicity } 1, \\ & \lambda_3 = 2 \text{ with mulitiplicity } 1 \end{array}$$

$$\text{with eigenvectors } \begin{pmatrix} a \\ b \\ 0 \\ a+b \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} c \\ c \\ -2c \\ 0 \end{pmatrix} \text{ for } \lambda_2, \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix} \text{ for } \lambda_3$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & 2 \end{pmatrix} \quad \text{has eigenvalues } \begin{array}{l} \lambda_1 = 2 \text{ with multiplicity } 2, \\ \lambda_2 = -1 \text{ with multiplicity } 2 \end{array}$$

$$\text{with eigenvectors } \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} 0 \\ 0 \\ 3b \\ b \end{pmatrix} \text{ for } \lambda_2$$

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \text{has eigenvalues } \begin{array}{l} \lambda_1 = 1 \text{ with multiplicity } 3, \\ \lambda_2 = 4 \text{ with multiplicity } 1 \end{array}$$

$$\text{with eigenvectors } \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix} \text{ for } \lambda_2$$

$$\begin{pmatrix} -4 & -4 & -4 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \quad \text{has eigenvalue } \lambda = -4 \text{ with multiplicity } 4$$

$$\text{with eigenvectors } \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} \text{ for } \lambda \tag{94}$$

There is a strangeness to some repeated eigenvalues. Some eigenvalues have a property known as "defectiveness."

Definition 5.3 *Let A be a matrix with an eigenvalue λ. The dimension of the eigenspace corresponding to λ is the number of free variables in the eigenvectors of λ.*

For example, the matrix

$$A = \begin{pmatrix} -5 & -6 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & -2 \end{pmatrix} \tag{95}$$

has the eigenvalue $\lambda_1 = -2$, with multiplicity 2, and the eigenvalue $\lambda_2 = 1$, with multiplicity 1. The eigenvectors corresponding to these take the form:

$$\begin{pmatrix} a \\ b \\ a+2b \end{pmatrix} \text{ for } \lambda_1, \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix} \text{ for } \lambda_2, \tag{96}$$

where a, b and c are any real numbers. We say that λ_1 has an eigenspace of dimension 2, since any of its eigenvectors can be completely determined by specifying the values of at least 2 free variables: a and b. As for λ_3, there is only one free variable involved in its eigenvectors: c. Therefore, λ_3 has an eigenspace of dimension 1.

Definition 5.4 *Let A be a matrix with an eigenvalue λ. The defect of λ is the multiplicity of λ minus the dimension of the eigenspace corresponding to λ.*

In the example above, λ_1 had multiplicity 2 and an eigenspace of dimension 2, so its defect was 0. Similarly, λ_2 had multiplicity 1 and an eigenspace of dimension 1, so its defect was also 0.

On the other hand, the matrix

$$A = \begin{pmatrix} 10 & 0 \\ -10 & 10 \end{pmatrix} \tag{97}$$

has the eigenvalue $\lambda = 10$ with multiplicity 2, but the eigenvectors corresponding to it have the form

$$\begin{pmatrix} 0 \\ a \end{pmatrix}. \tag{98}$$

With only one free variable (that is, a), λ has an eigenspace of dimension 1. Therefore the defect of λ is $2 - 1 = 1$.

An eigenvalue with a defect of 0 is called a complete eigenvalue, while one with a defect greater than 0 is called a defective eigenvalue.

5.3 Complex eigenvalues