Handbook of Ordinary Differential Equations

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1 Preliminaries

Classifying differential equations means coming up with a term for each type of differential equation, and (if possible) a strategy for finding the solution. The key here is that the term should be applied unambiguously. That is, if two mathematicians look at the same differential equation (perhaps simplified or written in a different way), then they ought to come up with the same terms to describe it.

At this point, a very natural question may come to mind:

1.1 Why bother?

Indeed, when dealing with regular algebraic equations that don't involve derivatives, there is very little, if any need for classification. The reason for this is simple: the essential strategy of solving an algebraic equation is almost always the same: get the variable by itself on one side of the equation.

However, for differential equations, the situation is not so simple. There is no single method or strategy for solving differential equations. This is reminiscent of something you've learned before. Consider the following integrals:

$$\int \frac{x}{x+1}\,\mathrm{d}x \qquad \int x^2 e^x\,\mathrm{d}x \qquad \int \frac{1}{x^2+5x+6}\,\mathrm{d}x \tag{1}$$

A different method is necessary for evaluating each of these. Therefore, we've come up with terms like "u-substitution," "integration by parts," and "integration by partial fractions," in order to quickly refer to these strategies. This necessity is exactly the reason that classification of differential equations is so important; each classification needs to be solved using a different method.

1.2 What's so ordinary about ordinary differential equations?

First, let's address the broadest classification of differential equations that we'll be dealing with: ordinary ones.

Definition 1.1 *An ordinary differential equation (ODE) is a differential equation in which the solution has only one independent variable.*

The solution to a differential equation will always be a function. Therefore, it will have at least one independent variable and at least one dependent variable. Here are some ODEs and their (general) solutions:

$$\begin{array}{cc} y' = 2x & y(x) = x^2 + C \\ y' = x^2 y & y(x) = Ce^{\frac{1}{3}x^3} \\ x'' + 9x = 10\cos(2t) & x(t) = 2\cos(2t) + c_1\cos(3t) + c_2\sin(3t) \end{array} . \tag{2}$$

The attribute to notice about these differential equations is that they all have solutions depending only on *one* independent variable (and some undetermined constants): in the first two, this variable is x, and in the third, it is t. We will also study systems of ODEs:

$$\begin{array}{cc} x' = x + 2y & x(t) = c_1 e^{3t} + c_2 e^{-t} \\ y' = 2x + y & y(t) = c_1 e^{3t} - c_2 e^{-t} \end{array} . \tag{3}$$

In these cases, there is more than one dependent variable. However, the number of independent variables *is still just one*: in the case above, x and y are dependent variables, while t is the only independent variable.

So, if ordinary differential equations have solutions with only one independent variable, then surely an "extraordinary" differential equation would have more than one, right? Indeed, but the term is "partial differential equations," due to the presence of partial derivatives in the equation:

$$\frac{\partial^2 u}{\partial t^2} + 2x^2 \frac{\partial u}{\partial x} - 3x \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0. \tag{4}$$

In general, partial differential equations are **much** more difficult to solve than ordinary differential equations.

In Math 306, we will only consider problems involving ordinary differential equations.

2 First Order Ordinary Differential Equations

After specifying whether a differential equation is ordinary or partial, the next most important classification is the "order:"

Definition 2.1 *The* <u>*order*</u> *of a differential equation is the highest order of all of the derivatives upon which it depends.*

Here are some examples:

$$\begin{array}{cc} y' = \sqrt{x+y} & \text{first order} \\ y'' = \frac{x}{y} & \text{second order} \\ y'y = \sin\left(x^2+y\right) & \text{first order} \\ y''y + \left(y'\right)^2 = 0 & \text{second order} \\ y^{\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime\prime} = x & \text{nineteenth order} \end{array} . \tag{5}$$

As the last example above demonstrates, it is an eyesore to write many primes for a high-order derivative. In general, for dealing with higher order derivatives than the second derivative, we will write $y^{(3)}$ instead of y''', $y^{(4)}$ instead of y'''', and so on.

Here are some more examples:

$$\begin{array}{cc} y^{(4)} + y'' + y = e^x & \text{fourth order} \\ y^{(3)} + x^2y'' + xy' + y = 0 & \text{third order} \\ y' + e^{\sin x}y = \frac{1}{x - \ln x} & \text{first order} \\ \frac{1}{y''} + \frac{1}{y'} + \frac{1}{y} = \tan^{-1}x & \text{second order} \\ y^{(100)} + xy^{(75)} + x^2y^{(50)} + x^3y^{(25)} + y' = y^{200} & \text{one hundredth order} \\ x^3y' - e^{\left(y^{(5)}\right)} = \frac{1}{x} & \text{fifth order} \\ \left(y' + y\right)^2 = 1 - x & \text{first order} \\ x^2y'' + 2xy' + y = 16y^2 & \text{second order} \\ y^{(3)} + \left(x^2 + xy^{(3)}\right)^4 = \sec(x) & \text{third order} \\ \cos\left(xy'' + \pi y\right) = y^{(4)} & \text{fourth order} \end{array} \tag{6}$$

True to this section's name, we will consider only first-order ordinary differential equations in this section.

2.1 Separable equations

Definition 2.2 *We say that a first order differential equation with solution $y(x)$ is* <u>*separable*</u> *provided that it can be written in the form $y' = f(x)g(y)$ for some functions $f(x)$ and $g(y)$.*

Here are some examples, and their general solutions:

$$\begin{array}{cc} y' = xy^4 & y(x) = \frac{8}{(C-3x^2)} \\ (x^2+1)\,y' = 1 & y(x) = \tan^{-1}x + C \\ y'\tan(x) = 2(y-1) & y(x) = 1 + C\sin^2 x \\ y' = \frac{2xy}{y+1} & \text{implicit: } y + \ln|y| = x^2 + C \end{array} . \tag{7}$$

In general, separable equations are among the easiest types of differential equations to solve. They are solved by the method of "separation of variables," demonstrated here in finding the solution of $(1+x)^2 y' = (1+y)^2$:

1. Write the differential equation in the form described by Definition 2.2:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1+x)^{-2}(1+y)^2. \tag{8}$$

2. Re-write the differential equation so that every part that involves y is **multiplied** by y':

$$(1+y)^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} = (1+x)^{-2}. \tag{9}$$

3. Integrate both sides of the differential equation:

$$\int (1+y)^{-2}\mathrm{d}y = \int (1+x)^{-2}\mathrm{d}x \tag{10}$$

4. Solve for y (as much as possible):

$$-(1+y)^{-1} = -(1+x)^{-1} + C_1 \tag{11}$$

$$1+y = \frac{1}{\frac{1}{1+x} + C_2} = \frac{x+1}{1+C_2(x+1)} \tag{12}$$

$$\boxed{y(x) = \frac{x+1}{C_2(x+1)+1} - 1} \tag{13}$$

2.2 Linear first order equations

Definition 2.3 *We say that a differential equation with solution $y(x)$ is* <u>*linear*</u> *provided that it can be written in the standard form*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x), \tag{14}$$

for some functions $p_{n-1}(x)$, $p_{n-2}(x)$, ..., $p_2(x)$, $p_1(x)$, $p_0(x)$, and $q(x)$.

This definition has greater significance in dealing with differential equations of second order and higher. In the case of first order differential equations, a linear equation is one that can be written in the form

$$y' + p(x)y = q(x), \tag{15}$$

for some functions $p(x)$ and $q(x)$. (The functions p and q must depend *only* on x, not on y.)

Here are some examples, and their general solutions:

$$\begin{array}{rl} y' + y = 0 & y(x) = Ce^{-x} \\ y' - \tan(x)y = 2\sin(x) & y(x) = C\sec(x) - \cos(x). \\ xy' + 2\ln(x)y = x^{2-\ln(x)} & y(x) = \left(\frac{1}{2}x^2 + C\right)x^{-\ln(x)} \end{array} \tag{16}$$

The technique here is a clever use of the product rule, and multiplication by an "integration factor," as described below in solving $xy' + y = xe^x$:

1. Put the differential equation in the standard form described in Definition 2.3:

$$y' + \frac{1}{x}y = e^x \tag{17}$$

2. The integration factor is defined as the exponential of the antiderivative of the coefficient of y:

$$\rho(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln|x|} = x. \qquad \text{(for } x > 0) \tag{18}$$

3. Multiply the integration factor by the differential equation:

$$xy' + x\frac{1}{x}y = xe^x \tag{19}$$

4. By design, the side of the equation containing y' is now the derivative of $\rho(x)y$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = xe^x. \tag{20}$$

5. Integrate both sides of the equation:

$$xy = \int xe^x \, \mathrm{d}x = (x-1)e^x + C. \tag{21}$$

6. Solve for $y(x)$:

$$\boxed{y(x) = \tfrac{x-1}{x}e^x + \tfrac{C}{x}}. \tag{22}$$

2.3 Exact equations

Definition 2.4 *Let $M(x,y)+N(x,y)y'=0$ be a differential equation with solution $y(x)$. We say that this differential equation is exact provided that there exists a function $F(x,y)$ such that $M=\frac{\partial F}{\partial x}$ and $N=\frac{\partial F}{\partial y}$.*

Here are some examples, and their general solutions:

$$\begin{array}{cc} y+(x+y)\,y'=0 & y(x)=-x\pm\sqrt{x^2+C} \\ (\cos x+\ln y)+\left(\frac{x}{y}+e^y\right)y'=0 & \text{implicit: } \sin x+x\ln y+e^y+C \\ (x+\tan^{-1}y)+\frac{x+y}{1+y^2}y'=0 & \text{implicit: } \frac{1}{2}x^2+x\tan^{-1}y+\frac{1}{2}\ln\left(1+y^2\right)=C \\ (e^x\sin y+\tan y)+(e^x\cos y+x\sec^2 y)\,y'=0 & \text{implicit: } e^x\sin y+x\tan y=C \end{array} \tag{23}$$

Exact equations are somewhat different from the other types of ODEs discussed so far. With most ODEs, you can determine the classification by just a glance. On the other hand, part (and sometimes, most) of the challenge of exact equations is that it's not so easy to tell whether an equation is exact or not. Therefore, we need to use the following theorem.

Theorem 2.5 *A differential equation $M(x,y)+N(x,y)y'=0$ with solution $y(x)$ is exact if and only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.*

The procedure of determining exactness and finding the solution is outlined below, for the case of $(2xy^2+3x^2)+(2x^2y+4y^3)\,y'=0$:

1. Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$:

$$\begin{array}{c} \frac{\partial}{\partial y}\left(2xy^2+3x^2\right)=4xy \\ \frac{\partial}{\partial x}\left(2x^2y+4y^3\right)=4xy \end{array}. \tag{24}$$

If these two partial derivatives are equal, then the equation is exact; if the two partial derivatives are not equal, then the equation is not exact, and should be solved by a different method.

2. If the equation is exact, then set $M=\frac{\partial F}{\partial x}$ and $N=\frac{\partial F}{\partial y}$, to get a system of

equations:

$$\begin{aligned} 2xy^2 + 3x^2 &= \tfrac{\partial F}{\partial x} \\ 2x^2y + 4y^3 &= \tfrac{\partial F}{\partial y} \end{aligned}. \tag{25}$$

3. To find $F(x, y)$, integrate both sides of one of the equations (it makes no difference which one is chosen; here we will use the first one):

$$\begin{aligned} \int 2xy^2 + 3x^2 \, \mathrm{d}x &= F(x, y) \\ F(x, y) &= x^2y^2 + x^3 + g(y) \end{aligned} \tag{26}$$

Here our constant of integration (which we have called $g(y)$) should be a function of y, only.

4. Now we must find $g(y)$. First, take the other partial derivative of this expression for $F(x, y)$:

$$\tfrac{\partial F}{\partial y} = \tfrac{\partial}{\partial y} \left(x^2y^2 + x^3 + g(y)\right) = 2x^2y + g'(y). \tag{27}$$

5. Set this equal to the other equation:

$$2x^2y + 4y^3 = \frac{\partial F}{\partial y} = 2x^2y + g'(y). \tag{28}$$

6. Based on this, integrate to find $g(y)$:

$$\begin{aligned} g'(y) &= 4y^3 \\ g(y) = \int 4y^3 \, \mathrm{d}y &= y^4 + C. \end{aligned} \tag{29}$$

7. Write $F(x, y)$ in full:

$$F(x, y) = x^2y^2 + x^3 + y^4 + C. \tag{30}$$

8. Set $F(x, y)$ to a constant and solve for y (as much as possible):

$$\boxed{C = x^2y^2 + x^3 + y^4} \tag{31}$$

2.4 Substitution methods

As you may have already noticed, differential equations can be quite complicated. So, just as with integrals, sometimes it is helpful to define new variables and substitute them into a differential equation. With some luck, this can result in a new differential equation which is a bit simpler.

However, also similar to integration by u-substitution, there is no procedure of determining *when* a substitution is necessary, or *how* it should be done. As with integrals, substitution methods involve making a guess, and then seeing if it's correct.

What follows are several examples of first order ODEs which cannot be solved directly by the usual methods, a suitable substitution, and the eventual general solution.

$$
\begin{array}{cc}
y' = \sqrt{x+y+1} & \\
v = x+y+1 & y(x) = \left(\frac{1}{2}x + C\right)^2 - x - 1 \\
(\ln y)^2 + \left(\frac{y'}{y}\right) = 1 & \\
v = \ln y & y(x) = e^{\sin(C \pm x)} \\
y^2 y' + 2xy^3 = 6x & \\
v = y^3 & y(x) = \sqrt[3]{3 + Ce^{-3(x^2)}} \\
y' = y + y^3 & \\
v = y^2 + 1 & \text{implicit: } y^2 = \frac{Ce^{2x}}{1-Ce^{2x}} \\
(2x\sin(y)\cos(y))\, y' = 4x^2 + \sin^2 y & \\
v = \sin^2 y & \text{implicit: } \sin^2 y = 4x^2 + Cx \\
(x^2+1)\sec(y)y' + \sin(y) = 2\cos(y) & \\
v = \tan y & \text{implicit: } \tan(y) = Ce^{-\tan^{-1}(x)} + 2 \\
y'\tan(y) + 2xe^{\left(x^2\right)} = xe^{\left(x^2\right)}\csc(y) & \\
v = \sec(y) & \text{implicit: } \sec(y) = e^{\left(e^{\left(x^2\right)}\right)}\left(C + \frac{1}{2}e^{\left(x^2\right)}\right) \\
2yy'\ln\left(y^2-4\right) = \left(y^2-4\right)\sqrt{e^{2x}+e^{-2x}+2} & \\
v = y^2 - 4 & \text{implicit: } y^2 = e^{\pm\sqrt{e^x+e^{-x}+C}} + 4
\end{array}
\tag{32}
$$

Substitution problems can be the most difficult kinds of ODE problems to solve.

It can be very far from obvious what the appropriate substitution should be, and even when (if) it is found, it can be very far from obvious what to do next. It is the opinion of the author that, hypothetically, if he wanted to create the most unfair problem in the history of first courses in differential equations, he would first look to the bizarre world of substitution problems.

However, there is good news: besides the obvious good news that I just saved a bunch of money on my car insurance by switching to Geico, there are three cases of substitution problems that can be solved through more predictable approaches. These are the homogeneous substitutions, the Bernoulli substitutions, and the order-reducing substitutions.

2.4.1 Homogeneous substitutions

Definition 2.6 *A first order ordinary differential equation with solution $y(x)$ is called <u>homogeneous of first order</u> provided that it can be written in the form $y' = f\left(\frac{y}{x}\right)$ for some function f.*

Here are some examples, and their general solutions:

$$\begin{array}{cc} xy^2y' = x^3 + y^3 & y(x) = x\sqrt[3]{C + 3\ln|x|} \\ xy' = y + 2\sqrt{xy} & y(x) = x(C + \ln|x|)^2 \\ (x+y)y' = x - y & \text{implicit: } y^2 + 2xy - x^2 = C \\ (x-y)y' = x + y & \text{implicit: } \tan^{-1}\left(\frac{y}{x}\right) = \ln\left(\sqrt{x^2+y^2}\right) + C \\ \left(x^2 - y^2\right)y' = 2xy & \text{implicit: } y = C\left(x^2 + y^2\right) \\ xy' = y + \sqrt{x^2+y^2} & \text{implicit: } y + \sqrt{x^2 - y^2} = Cx^2 \end{array} \tag{33}$$

A pattern to notice in these examples is that many homogeneous equations involve x and y raised to some rational powers, with no other functions involved.

Homogeneous equations can be solved via the substitution $v = \frac{y}{x}$. We demonstrate the procedure of homogeneous substitution in the case of the differential equation $yy' + x = \sqrt{x^2 + y^2}$:

1. Rewrite the equation so that y and x only appear in fractions of the form $\frac{y}{x}$ or $\frac{x}{y}$:

$$\begin{aligned} y' + \tfrac{x}{y} &= \sqrt{\left(\tfrac{x}{y}\right)^2 + 1} \\ y' &= -\tfrac{x}{y} + \sqrt{\left(\tfrac{x}{y}\right)^2 + 1} \end{aligned}. \tag{34}$$

2. Write $v = \frac{y}{x}$, and replace y' with $v'x + v$:

$$v'x + v = -\frac{1}{v} + \sqrt{\left(\frac{1}{v}\right)^2 + 1}. \tag{35}$$

3. Solve as normally for $v(x)$. (Typically this means either separate variables or write the new equation as a linear first order ODE):

$$\begin{aligned} v'xv + v^2 &= -1 + \sqrt{1+v^2} \\ v'xv &= -\left(1+v^2\right) + \sqrt{1+v^2} \\ \int \tfrac{v}{\sqrt{1+v^2} - (1+v^2)} \, \mathrm{d}v &= \int \tfrac{1}{x} \, \mathrm{d}x \\ \int \tfrac{v}{\sqrt{1+v^2}\left(1-\sqrt{1+v^2}\right)} \, \mathrm{d}v &= \ln|x| + C_1. \end{aligned} \tag{36}$$

Let $u = 1 - \sqrt{1+v^2}$. (Author's note: this happens to be the most difficult u-substitution I have ever seen.) In that case, $\mathrm{d}u = \frac{-v}{\sqrt{1+v^2}} \, \mathrm{d}v$, and so

$$\begin{aligned} -\int \tfrac{1}{u} \, \mathrm{d}u &= \ln|x| + C_1 \\ \ln|u| &= -\ln|x| + C_2 \\ |u| &= \tfrac{e^{C_2}}{|x|} \\ u &= \pm\tfrac{e^{C_2}}{x} \\ 1 - \sqrt{1+v^2} &= \tfrac{C_3}{x} \\ 1 + v^2 = \left(1 - \tfrac{C_3}{x}\right)^2 &= 1 - \tfrac{2C_3}{x} + \tfrac{{C_3}^2}{x^2} \\ v^2 &= \tfrac{{C_3}^2}{x^2} - \tfrac{2C_3}{x}. \end{aligned} \tag{37}$$

4. Reverse the substitution and solve for y (as much as possible):

$$\boxed{y^2 = {C_3}^2 - 2C_3 x}. \tag{38}$$

2.4.2 Bernoulli substitutions

Definition 2.7 *A Bernoulli equation is a differential equation with solution $y(x)$ which can be written in the standard form $y' + p(x)y = q(x)y^n$ for some functions $p(x)$ and $q(x)$ and some real number $n \neq 1$.*

Here are some examples, and their general solutions:

$$\begin{array}{cc} 3y^2y' + y^3 = e^{-x} & y(x) = e^{-3x}(x+C)^3 \\ xy' + 6y = 3xy^{\frac{4}{3}} & y(x) = \frac{1}{x^3(x+C)^3} \\ y^2\left(xy' + y\right)\sqrt{1+x^4} = x & \text{implicit: } 2x^3y^3 = 3\sqrt{1+x^4} + C \end{array} \tag{39}$$

Bernoulli equations can always be solved via the substitution $v = y^{1-n}$. We demonstrate the method for the case of $x^2y' + 2xy = 5y^4$:

1. Write the equation in the standard form to which the definition refers:

$$y' + \frac{2}{x}y = \frac{5}{x^2}y^4. \tag{40}$$

2. Identify the n to which the definition refers, and state the intended substitution $v = y^{1-n}$:

$$v = y^{1-4} = y^{-3}. \tag{41}$$

3. Solve for y and y' in terms of v and v':

$$\begin{array}{c} y = v^{-\frac{1}{3}} \\ y' = -\frac{1}{3}v^{-\frac{4}{3}}v' \end{array}. \tag{42}$$

4. Rewrite the equation with this information:

$$-\frac{1}{3}v^{-\frac{4}{3}}v' + \frac{2}{x}v^{-\frac{1}{3}} = \frac{5}{x^2}v^{-\frac{4}{3}}. \tag{43}$$

5. Cancel the coefficient of v' to find a first order linear ODE:

$$v' - \frac{6}{x}v = -\frac{15}{x^2} \tag{44}$$

6. Solve as usual for v:

$$\begin{aligned} \rho(x) = e^{\int -\frac{6}{x}\,\mathrm{d}x} = x^{-6} \\ x^{-6}v' - 6x^{-7}v = -15x^{-8} \\ \tfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-6}v\right) = -15x^{-8} \\ x^{-6}v = \tfrac{15}{7}x^{-7} + C_1 \\ v = \tfrac{15}{7}x^{-1} + C_1x^6 \end{aligned} \tag{45}$$

7. Revert to y and solve for $y(x)$ (as much as possible):

$$\begin{aligned} y^{-3} = \tfrac{15}{7x} + C_1x^6 \\ y^3 = \tfrac{1}{\frac{15}{7x}+C_1x^6} = \tfrac{7x}{15+C_2x^7} \\ \boxed{y(x) = \sqrt[3]{\tfrac{7x}{15+C_2x^7}}.} \end{aligned} \tag{46}$$

2.4.3 Order-reducing substitutions

See Section 3.1 for methods of solving second order equations by first-order methods using substitutions.

3 Second Order Ordinary Differential Equations

3.1 Substitution methods

Some second order ODEs can be solved using methods from the theory of first order ODEs. If a second order ODE is of the form $F(x, y', y'') = 0$ or the form $F(y, y', y'') = 0$, then it may be possible to do a substitution and solve the second-order equation as a sequence of two first-order equations. If neither of these is the case (*id est*, the equation can only be written as $F(x, y, y', y'') = 0$), then no such thing is possible in general.

3.1.1 Non-autonomous reducible ODEs

Let's begin with the simpler case: $F(x, y', y'') = 0$. Here are some examples, and their general solutions.

$$\begin{array}{cc} xy'' - y' = 0 & y(x) = C_1 x^2 + C_2 \\ xy'' + y' = 4x & y(x) = x^2 + C_1 \ln(x) + C_2 \\ x^2 y'' + 3xy' = 2 & y(x) = \ln|x| + \frac{C_1}{x^2} + C_2 \\ y'' = (x + y')^2 & y(x) = \ln\big|\sec(x + C_1)\big| - x^2 + C_2 \end{array} \tag{47}$$

The strategy in this case is to make the substitution $v(x) = y'$. We demonstrate this procedure for the case of $y'' - \frac{x}{1-x^2} y' = \frac{1}{\sqrt{1-x^2}}$:

1. Substitute $v(x) = y'$, so that $v'(x) = y''$:

$$v' - \frac{x}{1 - x^2} v = \frac{1}{\sqrt{1 - x^2}}. \tag{48}$$

2. Solve as usual for $v(x)$:

$$\begin{aligned}
\rho(x) = e^{\int -\frac{x}{1-x^2}\,\mathrm{d}x} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1-x^2}.\\
\sqrt{1-x^2}v' - \tfrac{x}{\sqrt{1-x^2}}v = 1\\
\tfrac{\mathrm{d}}{\mathrm{d}x}\left(v\sqrt{1-x^2}\right) = 1\\
v\sqrt{1-x^2} = x + C_1\\
v = \tfrac{x}{\sqrt{1-x^2}} + \tfrac{C_1}{\sqrt{1-x^2}}.
\end{aligned} \tag{49}$$

3. Revert to y' and solve as usual for y (as much as possible):

$$\begin{aligned}
y' = \tfrac{x}{\sqrt{1-x^2}} + \tfrac{C}{\sqrt{1-x^2}}\\
y = \int \tfrac{x}{\sqrt{1-x^2}}\,\mathrm{d}x + C_1\int \tfrac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\\
\boxed{y(x) = -\sqrt{1-x^2} + C_1\sin^{-1}x + C_2.}
\end{aligned} \tag{50}$$

3.1.2 Autonomous reducible ODEs

The case of $F(y, y', y'') = 0$ is a bit more non-intuitive. Here are some examples, and their general solutions.

$$\begin{array}{cc}
yy'' + (y')^2 = yy' & y(x) = \pm\sqrt{C_1e^x + C_2}\\
y'' + 4y = 0 & y(x) = C_1\sin(2x) + C_2\cos(2x)\\
yy'' + (y')^2 = 0 & y(x) = \pm\sqrt{C_1x + C_2}\\
y'' = 2yy' & y(x) = C_1\tan(C_1x + C_2)
\end{array} \tag{51}$$

For this type of problem, the strategy is to substitute $v(y) = y'$, which is just subtly different from the previous strategy. We demonstrate for the case of the differential equation $yy'' = 3(y')^2$:

1. Write the intended substitution $v(y) = y'$ and solve for y'':

$$\begin{aligned} v(y) &= y' \\ \frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x}y' &= \frac{\mathrm{d}v}{\mathrm{d}y}y' \\ y'' &= v'v \end{aligned}. \tag{52}$$

2. Rewrite the original ODE with this information:

$$\begin{aligned} yvv' &= 3v^2 \\ yv' &= 3v \end{aligned}. \tag{53}$$

3. Solve as usual for $v(y)$:

$$\begin{aligned} \frac{1}{v}v' &= 3\frac{1}{y} \\ \int \frac{1}{v}\,\mathrm{d}v &= 3\int \frac{1}{y}\mathrm{d}y \\ \ln|v| &= 3\ln|y| + A_1 \\ v(y) &= A_2 y^3 \end{aligned}. \tag{54}$$

4. Revert to y' and solve for $y(x)$ (as much as possible):

$$\begin{aligned} y' &= A_2 y^3 \\ y^{-3}y' &= A_2 \\ \int y^{-3}\,\mathrm{d}y &= \int A_2\,\mathrm{d}x \\ -\frac{1}{2}y^{-2} &= A_2 x + B_1 \\ y^{-2} &= A_3 x + B_2 \\ y^2 &= \frac{1}{A_3 x + B_2} \\ &\boxed{y(x) = \pm\frac{1}{\sqrt{A_3 x + B_2}}.} \end{aligned} \tag{55}$$

3.2 Linear, homogeneous, with constant coefficients

In Section 2.2, we discussed first order equations that happened to be linear:

$$y' + p(x)y = q(x). \tag{56}$$

As we've discussed, the integration factor provides a simple and complete method of solution for any and every differential equation of such a form.

As with first order ODEs, we will take some time to restrict our attention second order equations that happen to be linear:

$$y'' + p_1(x)y' + p_2(x)y = q(x). \tag{57}$$

However, even in the linear case, the theory of second order ODEs can be much more difficult to handle than that of first order ODEs. In order to approach these problems, we will first consider only those linear second order ODEs in which $p_1(x)$ and $p_2(x)$ are both constants:

$$y'' + ay' + by = q(x). \tag{58}$$

Even this is not enough. We will need one more restriction to introduce the theory of second order ODEs.

Definition 3.1 *A linear differential equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = q(x) \tag{59}$$

with solution $y(x)$ is <u>*homogeneous*</u> *provided that $q(x) = 0$ for all x.*
(Note that this definition is completely unrelated to Definition 2.6. The author expresses his condolences.)

In particular, a second order linear ODE is homogeneous provided that it can be written in the form

$$y'' + p_1(x)y' + p_2(x)y = 0. \tag{60}$$

To reiterate, the characteristic that makes a linear ODE with solution $y(x)$ homogeneous or not is whether there is a nonzero term that contains neither y nor any of its derivatives.

For now, we will only be concerned with linear, homogeneous second order ODEs with constant coefficients:

$$y'' + ay' + by = 0. \tag{61}$$

Note that most of these ODEs cannot be solved by order-reducing substitutions. Here are some examples, and their general solutions:

$$\begin{array}{cc}
4y'' + 4y' + y = 0 & y(x) = (c_1 + c_2 x)\, e^{-\frac{1}{2}x} \\
y'' + 11y' = 0 & y(x) = c_1 + c_2 e^{-11x} \\
y'' - y = 0 & y(x) = c_1 e^{x} + c_2 e^{-x} \\
y'' + y = 0 & y(x) = c_1 \cos(x) + c_2 \sin(x) \\
y'' + y' + y = 0 & y(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \\
y'' + y' - y = 0 & y(x) = c_1 e^{\frac{-1+\sqrt{5}}{2}x} + c_2 e^{\frac{-1-\sqrt{5}}{2}x} \\
y'' - y' + y = 0 & y(x) = e^{\frac{1}{2}x}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) \\
y'' - y' - y = 0 & y(x) = c_1 e^{\frac{1+\sqrt{5}}{2}x} + c_2 e^{\frac{1-\sqrt{5}}{2}x} \\
y'' - y' - 2y = 0 & y(x) = c_1 e^{-x} + c_2 e^{2x} \\
y'' + y' - 2y = 0 & y(x) = c_1 e^{x} + c_2 e^{-2x} \\
y'' - y' + 2y = 0 & y(x) = e^{\frac{1}{2}x}\left(c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right) \\
y'' + y' + 2y = 0 & y(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right) \\
y'' + 2y' + y = 0 & y(x) = (c_1 + c_2 x)\, e^{-x} \\
y'' + 5y' + 6y = 0 & y(x) = c_1 e^{-3x} + c_2 e^{-2x} \\
y'' + y' - 6y = 0 & y(x) = c_1 e^{-3x} + c_2 e^{2x} \\
y'' - 5y' + 6y = 0 & y(x) = c_1 e^{3x} + c_2 e^{2x} \\
y'' - y' - 6y = 0 & y(x) = c_1 e^{3x} + c_2 e^{-2x}
\end{array} \tag{62}$$

Notice that very small differences between two ODEs can cause the solutions to be wildly different. (Conversely, notice that sometimes, large differences can cause the solutions to be eerily similar!) This is just one of many mathematical instances of the butterfly effect.

The method of solving linear ODEs with constant coefficients hinges on finding the "characteristic equation" of the ODE, which we demonstrate for the case of $y'' - 2y' - 8y = 0$:

1. Assume that $y = e^{rx}$ is a solution to the differential equation for some unknown value of r:

$$\begin{array}{c} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(e^{rx}\right) - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{rx}\right) - 8\left(e^{rx}\right) = 0 \\ r^2 e^{rx} - 2re^{rx} - 8e^{rx} = 0 \\ e^{rx}\left(r^2 - 2r - 8\right) = 0 \end{array}. \tag{63}$$

2. Divide both sides by e^{rx} to get the characteristic equation:

$$r^2 - 2r - 8 = 0. \tag{64}$$

3. Find the roots of the characteristic equation. (In this case, we can factor the characteristic equation, but this is not always possible.)

$$\begin{array}{c} (r-4)(r+2) = 0, \\ r_1 = 4,\ r_2 = -2 \end{array}. \tag{65}$$

4. If the roots are real and unequal, proceed to Section 3.2.1. If the roots are repeated (equal), proceed to Section 3.2.2. If the roots are complex, proceed to Section 3.2.3.

3.2.1 Real distinct roots

If the characteristic equation has real valued roots r_1 and r_2, where $r_1 \neq r_2$, then the general solution is simply

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{66}$$

For example, in the case of $y'' - 2y' - 8y = 0$, the roots are $r_1 = 4$ and $r_2 = -2$. Thus, the general solution of $y'' - 2y' - 8y = 0$ is

$$\boxed{y(x) = c_1 e^{4x} + c_2 e^{-2x}.} \tag{67}$$

3.2.2 Repeated roots

If the characteristic equation has exactly one root r, then the general solution is

$$y(x) = (c_1 + c_2 x)\, e^{rx}. \tag{68}$$

For example, in the case of $y'' + 6y + 9 = 0$, the only root is $r = 4$. Thus, the general solution of $y'' + 6y + 9 = 0$ is

$$\boxed{y(x) = (c_1 + c_2 x)\, e^{3x}.} \tag{69}$$

3.2.3 Complex roots

If the characteristic equation has two complex roots, then they take the form $r_1 = a + bi$ and $r_2 = a - bi$. In this case, the general solution is

$$y(x) = e^{ax} \left(c_1 \cos(bx) + c_2 \sin(bx)\right). \tag{70}$$

For example, in the case of $y'' - 2y' + 10 = 0$, the roots are $r_1 = 1 + 6i$ and $r_2 = 1 - 6i$. Thus, the general solution of $y'' - 2y' + 10 = 0$ is

$$\boxed{y(x) = e^{x} \left(c_1 \cos(6x) + c_2 \sin(6x)\right).} \tag{71}$$

3.3 Linear, non-homogeneous, with constant coefficients

The theory of second order linear ODEs with constant coefficients which are *not* homogeneous can be handled together with higher-order ODEs with constant coefficients. For these techniques, refer to Section 4.2.

3.4 Linear, homogeneous, with non-constant coefficients

3.4.1 Solutions by power series

The power series method of solving differential equations is as follows: assume that the solution y is a power series, and then attempt to determine its coefficients. We place this topic in a section on non-constant coefficients because that is the situation in which power series methods are most effective.

So, first of all, what is a power series?

Definition 3.2 *A power series in the variable x is a function g with independent variable x of the form*

$$g(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where a is a real number and, for each integer n, c_n is a real number.

If a function is equal to a particular power series, then we call that equality "a power series representation" of the function. Many of the functions that are important to us will have representations as power series. Here are the most essential examples:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \qquad \text{for } 0 \le |x| < 1 \end{aligned} \quad . \tag{72}$$

These equations, and many more, can be figured out through the use of Taylor series.

Definition 3.3 *Let f be a function defined on the real line, and let a be a real value. The Taylor series of f centered at $x = a$ is the power series*

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n.$$

If $a = 0$, we call this the <u>*Maclaurin series of f*</u>.

As you may know, with every series, there are issues of convergence that must be addressed. However, we will not concern ourselves much with the radii of convergence of the power series we deal with in this document; that would be a better topic for a textbook than for a handbook on ODEs. It will be enough for us to simply note that *some ODEs cannot be solved by power series.*

The method of power series hinges on the following theorem.

Theorem 3.4 *If*

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

for all real values of x, then for each integer $n \geq 0$, $a_n = b_n$.

In particular, if $b_0 = b_1 = ... = 0$, then $a_0 = a_1 = ... = 0$.

Here are some examples of linear, homogeneous ODEs with non-constant coefficients, and their general solutions:

$$\begin{array}{cc}
(x-3)y' + 2x = 0 & y(x) = c_0 \sum_{n=0}^{\infty} (n+1)\left(\frac{x}{3}\right)^n \\
(2x-1)y' + 2y = 0 & y(x) = \frac{c_0}{1-2x} \\
(x-2)y' + y = 0 & y(x) = \frac{2c_0}{2-x} \\
(x^2+1)\,y'' + 6xy' + 4y = 0 & y(x) = c_0 \sum_{n=0}^{\infty} (-1)^n (n+1)\, x^{2n} + \frac{c_1}{3} \sum_{n=0}^{\infty} (-1)^n (2n+3)\, x^{2n+1} \\
(x^2-3)\,y'' + 2xy' = 0 & y(x) = c_0 + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{3^n(2n+1)} \\
(x^2-1)\,y'' - 6xy' + 12y = 0 & y(x) = c_0\,(1+6x^2+x^4) + c_1\,(x+x^3)
\end{array} \tag{73}$$

We describe the procedure for solving such problems below for the cases of four different ODEs.

To solve $y'' + 4y = 0$ (note that this ODE can be solved by other methods as well):

1. Assume that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution to the ODE for some unknown values of the c_n. In that case, $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, so in this case, we get the equation

$$\sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} + 4\sum_{n=0}^{\infty} c_n x^n = 0. \tag{74}$$

2. Distribute.

$$\sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} + \sum_{n=0}^{\infty} 4c_n x^n = 0. \tag{75}$$

3. Shift the indices of the sums so that the exponents of x are the same. In this case, we need to shift the indices of the term

$$\sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} \tag{76}$$

so that the exponent of x will be n. First, we note that the 0 and 1 terms will both be zero, so

$$\sum_{n=0}^{\infty} n\,(n-1)\,c_n x^{n-2} = \sum_{n=2}^{\infty} n\,(n-1)\,c_n x^{n-2}. \tag{77}$$

We now make the substitution $m = n - 2$, the current exponent of x. When $n = 2$, $m = 0$, so

$$\sum_{n=2}^{\infty} n\,(n-1)\,c_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)\,(m+1)\,c_{m+2} x^m. \tag{78}$$

This same sum can be written as

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,c_{n+2} x^n, \tag{79}$$

since m and n are just names of the index counting the terms. Thus, the equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0. \tag{80}$$

4. Combine the sums and like terms.

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\, c_{n+2} + 4c_n\right) x^n = 0. \tag{81}$$

5. By the theorem, the coefficients must now be zero. In this case, for $n \geq 0$,

$$(n+2)(n+1)c_{n+2} + 4c_n = 0. \tag{82}$$

6. Find a relationship among the coefficients. (This is called a "recurrence relation.")

$$\begin{array}{c} (n+2)(n+1)c_{n+2} = -4c_n \\ c_{n+2} = \frac{-4}{(n+2)(n+1)} c_n \end{array}. \tag{83}$$

7. If the original ODE has order r, then regard the coefficients $c_0, c_1, ..., c_{r-1}$ as arbitrary constants, and determine the values of the other constants in terms of them using the recurrence relation. In this case, c_0 and c_1 are arbitrary. As for any other constants, the recurrence relation implies

$$\begin{array}{cc} c_0 & c_1 \\ c_2 = \frac{-4}{(2)(1)} c_0 & c_3 = \frac{-4}{(3)(2)} c_1 \\ c_4 = \frac{4^2}{(4)(3)(2)(1)} c_0 & c_5 = \frac{4^2}{(5)(4)(3)(2)} \\ c_6 = \frac{-4^3}{(6)(5)(4)(3)(2)} c_0 & c_7 = \frac{-4^3}{(7)(6)(5)(4)(3)(2)(1)} \\ c_8 = \frac{4^4}{(8)(7)(6)(5)(4)(3)(2)} c_0 & c_9 = \frac{4^4}{(9)(8)(7)(6)(5)(4)(3)(2)} \\ \vdots & \vdots \end{array}. \tag{84}$$

These equations can be summarized as follows:

$$c_{2k} = \frac{(-1)^k 4^k}{(2k)!} c_0, \quad c_{2k+1} = \frac{(-1)^k 4^k}{(2k+1)!} c_1. \tag{85}$$

8. Write $y = \sum_{n=0}^{\infty} c_n x^n$ with this insight about the c_n. In this case, the value of n will depend on whether n is even or odd. If n is even, then $n = 2k$ for some

integer k, and $c_n = c_0$. If n is odd, then $n = 2k+1$ for some integer k, and $c_n = c_1$. Thus,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1}$$
$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 4^k}{(2k+1)!} x^{2k+1}. \quad (86)$$

9. Determine whether any familiar power series occur in $y(x)$. In this case, we note that

$$y(x) = c_0 \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!} + \frac{1}{2} c_1 \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k+1}}{(2k+1)!}. \quad (87)$$

We know that $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ and $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so

$$\boxed{y(x) = c_0 \cos(2x) + \tfrac{1}{2} c_1 \sin(2x)}. \quad (88)$$

To solve $(x^2+2)\, y'' + 4xy' + 2y = 0$:

1. Assume that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution to the ODE for some unknown values of the c_n. In that case, $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, so in this case, we get the equation

$$\left(x^2+2\right) \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + 4x \sum_{n=0}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0. \quad (89)$$

2. Distribute.

$$\sum_{n=0}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0 \quad (90)$$

3. Shift the indices of the sums so that the exponents of x are the same. In this

case, we need to shift the indices of the term

$$\sum_{n=0}^{\infty} 2n\,(n-1)\,c_n x^{n-2} \tag{91}$$

so that the exponent of x will be n. First, we note that the 0 and 1 terms will both be zero, so

$$\sum_{n=0}^{\infty} 2n\,(n-1)\,c_n x^{n-2} = \sum_{n=2}^{\infty} 2n\,(n-1)\,c_n x^{n-2}. \tag{92}$$

We now make the substitution $m = n - 2$, the current exponent of x. When $n = 2$, $m = 0$, so

$$\sum_{n=2}^{\infty} 2n\,(n-1)\,c_n x^{n-2} = \sum_{m=0}^{\infty} 2\,(m+2)\,(m+1)\,c_{m+2} x^{m}. \tag{93}$$

This same sum can be written as

$$\sum_{n=0}^{\infty} 2\,(n+2)\,(n+1)\,c_{n+2} x^{n}, \tag{94}$$

since m and n are just names of the index counting the terms. Thus, the equation becomes

$$\sum_{n=0}^{\infty} n\,(n-1)\,c_n x^n + \sum_{n=0}^{\infty} 2\,(n+2)\,(n+1)\,c_{n+2} x^n + \sum_{n=0}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0. \tag{95}$$

4. Combine the sums and like terms.

$$\sum_{n=0}^{\infty} \left(n\,(n-1)\,c_n + 2\,(n+2)\,(n+1)\,c_{n+2} + 4nc_n + 2c_n\right) x^n = 0. \tag{96}$$

5. By the theorem, the coefficients must now be zero. In this case, for $n \geq 0$,

$$n\,(n-1)\,c_n + 2\,(n+2)\,(n+1)\,c_{n+2} + 4nc_n + 2c_n = 0. \tag{97}$$

6. Find a relationship among the coefficients. (This is called a “recurrence

relation.")

$$\begin{gathered}\left(n^2-n\right) c_n+2\,(n+2)\,(n+1)\, c_{n+2}+(4 n+2)\, c_n=0 \\ \left(n^2+3 n+2\right) c_n+2\,(n+2)\,(n+1)\, c_{n+2}=0 \\ 2\,(n+2)\,(n+1)\, c_{n+2}=-\left(n^2+3 n+2\right) c_n \\ c_{n+2}=-\tfrac{n^2+3 n+2}{2(n+2)(n+1)} c_n \\ c_{n+2}=-\tfrac{(n+2)(n+1)}{2(n+2)(n+1)} c_n=-\tfrac{1}{2} c_n .\end{gathered} \quad . \tag{98}$$

7. If the original ODE has order r, then regard the coefficients $c_0, c_1, ..., c_{r-1}$ as arbitrary constants, and determine the values of the other constants in terms of them using the recurrence relation. In this case, c_0 and c_1 are arbitrary. As for any other constants, the recurrence relation implies

$$\begin{array}{cc} c_0 & c_1 \\ c_2=\frac{-1}{2} c_0 & c_3=\frac{-1}{2} c_1 \\ c_4=\frac{1}{4} c_0 & c_5=\frac{1}{4} c_1 \\ c_6=\frac{-1}{8} c_0 & c_7=\frac{-1}{8} c_1 \\ c_8=\frac{1}{16} c_0 & c_9=\frac{1}{16} c_1 \\ \vdots & \vdots \end{array} . \tag{99}$$

These equations can be summarized as follows:

$$c_{2k}=\tfrac{(-1)^k}{2^k} c_0, \quad c_{2k+1}=\tfrac{(-1)^k}{2^k} c_1 . \tag{100}$$

8. Write $y=\sum_{n=0}^{\infty} c_n x^n$ with this insight about the c_n. In this case, the value of n will depend on whether n is even or odd. If n is even, then $n=2k$ for some integer k, and $c_n=c_0$. If n is odd, then $n=2k+1$ for some integer k, and $c_n=c_1$. Thus,

$$\begin{aligned} y(x)=\sum_{n=0}^{\infty} c_n x^n=\sum_{k=0}^{\infty} c_{2k} x^{2k}+\sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ =c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} x^{2k}+c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} x^{2k+1} .\end{aligned} \tag{101}$$

9. Determine whether any familiar power series occur in $y(x)$. In this case, we notice that

$$y(x) = c_0 \sum_{k=0}^{\infty} \left(-\frac{x^2}{2}\right)^k + c_1 \sum_{k=0}^{\infty} x \left(-\frac{x^2}{2}\right)^k. \tag{102}$$

We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so this is

$$\boxed{y(x) = \frac{c_0}{1-\left(-\frac{x^2}{2}\right)} + \frac{c_1 x}{1-\left(-\frac{x^2}{2}\right)} = \frac{2c_0+2c_1x}{2+x^2}}. \tag{103}$$

To solve $y'' + xy' + y = 0$:

1. Assume that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution to the ODE for some unknown values of the c_n. In that case, $y' = \sum_{n=0}^{\infty} nc_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$, so in this case, we get the equation

$$\sum_{n=0}^{\infty} n\left(n-1\right)c_n x^{n-2} + x\sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0. \tag{104}$$

2. Distribute.

$$\sum_{n=0}^{\infty} n\left(n-1\right)c_n x^{n-2} + \sum_{n=0}^{\infty} nc_n x^{n} + \sum_{n=0}^{\infty} c_n x^n = 0. \tag{105}$$

3. Shift the indices of the sums so that the exponents of x are the same. In this case, we need to shift the indices of the term

$$\sum_{n=0}^{\infty} n\left(n-1\right)c_n x^{n-2} \tag{106}$$

so that the exponent of x will be n. First, we note that the 0 and 1 terms will both be zero, so

$$\sum_{n=0}^{\infty} n\left(n-1\right)c_n x^{n-2} = \sum_{n=2}^{\infty} n\left(n-1\right)c_n x^{n-2}. \tag{107}$$

We now make the substitution $m = n-2$, the current exponent of x. When $n = 2$,

$m = 0$, so

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)\, c_{m+2} x^m. \tag{108}$$

This same sum can be written as

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n, \tag{109}$$

since m and n are just names of the index counting the terms. Thus, the equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0. \tag{110}$$

4. Combine the sums and like terms.

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\, c_{n+2} + n c_n + c_n\right) x^n = 0. \tag{111}$$

5. By the theorem, the coefficients must now be zero. In this case, for $n \geq 0$,

$$(n+2)(n+1)\, c_{n+2} + n c_n + c_n = 0. \tag{112}$$

6. Find a relationship among the coefficients. (This is called a "recurrence relation.")

$$\begin{array}{c} (n+2)(n+1)\, c_{n+2} + (n+1)\, c_n = 0 \\ (n+2)(n+1)\, c_{n+2} = -(n+1)\, c_n \\ c_{n+2} = \frac{-1}{n+2} c_n \end{array}. \tag{113}$$

7. If the original ODE has order r, then regard the coefficients $c_0, c_1, ..., c_{r-1}$ as arbitrary constants, and determine the values of the other constants in terms of them using the recurrence relation. In this case, c_0 and c_1 are arbitrary. As for any other

constants, the recurrence relation implies

$$\begin{array}{cc} c_0 & c_1 \\ c_2 = \frac{-1}{2}c_0 & c_3 = \frac{-1}{3}c_1 \\ c_4 = \frac{1}{(2)(4)}c_0 & c_5 = \frac{1}{(3)(5)}c_1 \\ c_6 = \frac{-1}{(2)(4)(6)}c_0 & c_7 = \frac{-1}{(3)(5)(7)}c_1 \\ c_8 = \frac{1}{(2)(4)(6)(8)}c_0 & c_9 = \frac{1}{(3)(5)(7)(9)}c_1 \\ \vdots & \vdots \end{array}. \tag{114}$$

These equations can be summarized as follows:

$$c_{2k} = \frac{(-1)^k}{2^k k!}c_0 \quad c_{2k+1} = \frac{(-1)^k 2^k k!}{(2k+1)!}c_1. \tag{115}$$

(Note: many authors refer to $\frac{(2k+1)}{2^k k!}$ as $(2k+1)!!$, the so-called "double factorial.")

8. Write $y = \sum_{n=0}^{\infty} c_n x^n$ with this insight about the c_n. In this case, the value of n will depend on whether n is even or odd. If n is even, then $n = 2k$ for some integer k, and $c_n = c_0$. If n is odd, then $n = 2k+1$ for some integer k, and $c_n = c_1$. Thus,

$$\begin{aligned} y(x) = \sum_{n=0}^{\infty} c_n x^n &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}. \end{aligned} \tag{116}$$

9. Determine whether any familiar power series occur in $y(x)$. In this case, we do not recognize either of the power series appearing in the solution, so there is nothing more to do:

$$\boxed{y(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}}. \tag{117}$$

To solve $(x^2 - 1)\, y'' + 4xy' + 2y = 0$:

1. Assume that $y = \sum_{n=0}^{\infty} c_n x^n$ is a solution to the ODE for some unknown values of the c_n. In that case, $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$, so in this case, we get the equation

$$\left(x^2 - 1\right) \sum_{n=0}^{\infty} n\left(n-1\right) c_n x^{n-2} + 4x \sum_{n=0}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0. \tag{118}$$

2. Distribute.

$$\sum_{n=0}^{\infty} n\left(n-1\right) c_n x^n - \sum_{n=0}^{\infty} n\left(n-1\right) c_n x^{n-2} + \sum_{n=0}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0. \tag{119}$$

3. Shift the indices of the sums so that the exponent on each x is n. In this case, we need to shift the indices of the term

$$\sum_{n=0}^{\infty} n\left(n-1\right) c_n x^{n-2} \tag{120}$$

so that the sum will be of the form $\sum_{n=0}^{\infty} a_n x^n$. First, we note that the 0 and 1 terms are both zero, so

$$\sum_{n=0}^{\infty} n\left(n-1\right) c_n x^{n-2} = \sum_{n=2}^{\infty} n\left(n-1\right) c_n x^{n-2}. \tag{121}$$

We now make the substitution $m = n - 2$, the exponent on x. When $n = 2$, $m = 0$, so

$$\sum_{n=2}^{\infty} n\left(n-1\right) c_n x^{n-2} = \sum_{m=0}^{\infty} \left(m+2\right)\left(m+1\right) c_{m+2} x^m. \tag{122}$$

This same sum can be written as

$$\sum_{n=0}^{\infty} \left(n+2\right)\left(n+1\right) c_{n+2} x^n, \tag{123}$$

since m and n are just names of the index. Thus, the equation becomes

$$\sum_{n=0}^{\infty} n\left(n-1\right)c_n x^n - \sum_{n=0}^{\infty}\left(n+2\right)\left(n+1\right)c_{n+2}x^n + \sum_{n=0}^{\infty} 4nc_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0. \tag{124}$$

4. Combine the sums and like terms.

$$\sum_{n=0}^{\infty}\left(n\left(n-1\right)c_n - \left(n+2\right)\left(n+1\right)c_{n+2} + 4nc_n + 2c_n\right)x^n = 0. \tag{125}$$

5. By the theorem, the coefficients must now be zero. In this case, for $n \geq 0$,

$$n\left(n-1\right)c_n - \left(n+2\right)\left(n+1\right)c_{n+2} + 4nc_n + 2c_n = 0. \tag{126}$$

6. Find a relationship among the coefficients. (This is called a "recurrence relation.")

$$\begin{array}{c} \left(n^2-n\right)c_n - \left(n+2\right)\left(n+1\right)c_{n+2} + 4nc_n + 2c_n = 0 \\ \left(n^2+3n+2\right)c_n - \left(n+2\right)\left(n+1\right)c_{n+2} = 0 \\ \left(n+2\right)\left(n+1\right)c_{n+2} = \left(n^2+3n+2\right)c_n \\ c_{n+2} = \frac{(n+2)(n+1)}{n^2+3n+2}c_n = \frac{(n+2)(n+1)}{(n+2)(n+1)}c_n = c_n \end{array}. \tag{127}$$

7. If the original ODE has order r, then regard the coefficients $c_0, c_1, ..., c_{r-1}$ as arbitrary constants, and determine the values of the other constants in terms of them using the recurrence relation. In this case, c_0 and c_1 are arbitrary, and

$$\begin{array}{c} c_0 = c_2 = c_4 = c_6 = c_8 = ... \\ c_1 = c_3 = c_5 = c_7 = c_9 = ... \end{array}. \tag{128}$$

8. Write $y = \sum_{n=0}^{\infty} c_n x^n$ with this insight about the c_n. In this case, the value of n will depend on whether n is even or odd. If n is even, then $n = 2k$ for some integer k, and $c_n = c_0$. If n is odd, then $n = 2k+1$ for some integer k, and $c_n = c_1$. Thus,

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} = \sum_{k=0}^{\infty} c_0 x^{2k} + \sum_{k=0}^{\infty} c_1 x^{2k+1}. \tag{129}$$

9. Determine whether any familiar power series occur in $y(x)$. In this case, we note that

$$y(x) = c_0 \sum_{k=0}^{\infty} \left(x^2\right)^k + c_1 x \sum_{k=0}^{\infty} \left(x^2\right)^k. \tag{130}$$

We recognize that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Therefore, $\sum_{k=0}^{\infty} \left(x^2\right)^k = \frac{1}{1-x^2}$. Thus,

$$\boxed{y(x) = c_0 \tfrac{1}{1-x^2} + c_1 x \tfrac{1}{1-x^2} = \tfrac{c_0 + c_1 x}{1-x^2}}. \tag{131}$$

4 Higher Order Ordinary Differential Equations

4.1 Linear, homogeneous, with constant coefficients

In section 2.2, we defined what is meant by an ODE being linear (this is Definition 2.3). So far, we've dealt with two orders of linear ODEs:

$$\begin{array}{rc} y' + p(x)y = q(x) & \text{first order} \\ y'' + p_1(x)y' + p_0(x)y = q(x) & \text{second order} \end{array} \tag{132}$$

In this section, we'll take the next logical steps forward:

$$\begin{array}{cc} y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x) & \text{third order} \\ y^{(4)} + p_3(x)y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x) & \text{fourth order} \\ y^{(5)} + p_4(x)y^{(4)} + p_3(x)y^{(3)} + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x) & \text{fifth order} \\ \vdots & \text{(you get the idea)} \end{array} \tag{133}$$

However, just as we needed to make the additional assumptions of homogeneity and constant coefficients in Section 3.2, we also need those assumptions here. Therefore, for now, we'll only deal with ODEs that look like:

$$\begin{array}{c} y^{(3)} + a_2y'' + a_1y' + a_0y = 0 \\ y^{(4)} + a_3y^{(3)} + a_2y'' + a_1y' + a_0y = 0 \\ y^{(5)} + a_4y^{(4)} + a_3y^{(3)} + a_2y^{(2)} + a_1y' + a_0y = 0\text{'} \\ \vdots \end{array} \tag{134}$$

where all of the coefficients a_i are constants.

On the following page are some examples, and their general solutions.

$$
\begin{array}{cr}
y^{(4)}-5y''+6y=0 & y(x)=c_1e^{2x}+c_2e^{-2x}+c_3e^{3x}+c_4e^{-3x} \\
y^{(3)}-y'=0 & y(x)=c_1+c_2e^{x}+c_3e^{-x} \\
y^{(3)}+y'=0 & y(x)=c_1+c_2\cos x+c_3\sin x \\
y^{(3)}-3y''+2y'=0 & y(x)=c_1+c_2e^{x}+c_3e^{2x} \\
y^{(5)}-15y^{(4)}+85y^{(3)}-225y''+274y'-120y=0 & y(x)=c_1e^{x}+c_2e^{2x}+c_3e^{3x}+c_4e^{4x}+c_5e^{5x} \\
y^{(100)}=0 & y(x)=c_0+c_1x+c_2x^2+...+c_{98}x^{98}+c_{99}x^{99} \\
y^{(3)}-3y''+3y'-y=0 & y(x)=\left(c_1+c_2x+c_3x^2\right)e^{x} \\
y^{(4)}-3y^{(3)}+3y''-y'=0 & y(x)=\left(c_1+c_2x+c_3x^2\right)e^{x}+c_4 \\
y^{(6)}-6y^{(5)}+25y^{(4)}=0 & y(x)=c_1+c_2x+c_3x^2+c_4x^3 \\
 & +e^{3x}\left(c_4\cos(4x)+c_5\sin(4x)\right) \\
y^{(5)}-2y^{(4)}+y^{(3)}=0 & y(x)=c_1+c_2x+c_3x^2+c_4e^{x}+c_5xe^{x} \\
y^{(4)}+2y''+y=0 & y(x)=\left(c_1+c_2x\right)\cos x+\left(c_3+c_4x\right)\sin x \\
y^{(5)}+3y^{(4)}+3y^{(3)}+y''-4y'+2y=0 & y(x)=e^{x}\left(c_1+c_2x+c_3\cos x+c_4\sin x\right)+c_5e^{-x} \\
y^{(4)}-y=0 & y(x)=c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x
\end{array}
$$

Thankfully, the methods of solving linear homogeneous ODEs with constant coefficients that have orders greater than two are almost identical to those for order two. We describe the procedure below for the cases of four different ODEs.

To solve $9y^{(3)}+12y''+4y'=0$:

1. Assume that $y=e^{rx}$ is a solution to the differential equation for some unknown value of r to find the characteristic equation:

$$
\begin{array}{c}
9\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(e^{rx}\right)+12\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(e^{rx}\right)+4\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{rx}\right)=0 \\
9r^3r^{rx}+12r^2e^{rx}+4re^{rx}=0 \\
e^{rx}\left(9r^3+12r^2+4r\right)=0 \\
9r^3+12r^2+4r=0
\end{array}
\quad . \tag{135}
$$

2. Find the roots of the characteristic equation:

$$\begin{aligned} r\left(9r^2+12r+4\right)&=0\\ r\left(r+\tfrac{2}{3}\right)\left(r+\tfrac{2}{3}\right)&=0 \end{aligned} \tag{136}$$

The number of times a root repeats is called the multiplicity of the root. (In other words, the multiplicity of a root k is the number of times the factor $r-k$ appears in the complete factorization of the equation.) In this example, the roots are:

$$\begin{array}{rl} r_1=0 & \text{with multiplicity } 1\\ r_2=-\frac{2}{3} & \text{with multiplicity } 2 \end{array}. \tag{137}$$

3. Write the linearly independent particular solutions that span the general solution (this may involve multiplying e^{rx} by x some number of times in order to compensate for multiplicities greater than 1). In general, there must be as many linearly independent solutions as the order of the equation. In this case, the order is 3, so we need 3 solutions based on the roots of the characteristic equation:

$$\begin{array}{rl} y_1(x)=e^{0x}=1 & \text{corresponding to } r_1=0\\ y_2(x)=e^{-\frac{2}{3}x} & \text{corresponding to } r_2=-\frac{2}{3}\\ y_3(x)=xe^{-\frac{2}{3}x} & \text{corresponding to } r_2=-\frac{2}{3} \end{array} \tag{138}$$

4. The general solution is formed by linear combinations of these solutions:

$$\begin{gathered} y(x)=c_1y_1(x)+c_2y_2(x)+c_3y_3(x)\\ \boxed{y(x)=c_1+c_2e^{-\frac{2}{3}x}+c_3xe^{-\frac{2}{3}x}} \end{gathered} \tag{139}$$

To solve $y^{(4)}=16y$:

1. Assume that $y=e^{rx}$ is a solution to the differential equation for some

unknown value of r to find the characteristic equation:

$$\begin{gathered}\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(e^{rx}\right) = 16\left(e^{rx}\right)\\ r^4 e^{rx} = 16 e^{rx}\\ e^{rx}\left(r^4 - 16\right) = 0\\ r^4 - 16 = 0\end{gathered}. \tag{140}$$

2. Find the roots of the characteristic equation:

$$\begin{gathered}\left(r^2 - 4\right)\left(r^2 + 4\right) = 0\\ \left(r - 2\right)\left(r + 2\right)\left(r - 2i\right)\left(r + 2i\right)\end{gathered}. \tag{141}$$

In this example, the roots are:

$$\begin{array}{cl} r_1 = 2 & \text{with multiplicity } 1\\ r_2 = -2 & \text{with multiplicity } 1\\ r_3 = 2i & \text{with multiplicity } 1\\ r_4 = -2i & \text{with multiplicity } 1\end{array}. \tag{142}$$

3. Write the linearly independent particular solutions that span the general solution. In this case, the order is 4, so we need 4 solutions based on the roots of the characteristic equation:

$$\begin{array}{cc} y_1(x) = e^{2x} & \text{corresponding to } r_1 = 2\\ y_2(x) = e^{-2x} & \text{corresponding to } r_2 = -2\\ y_3(x) = \cos\left(2x\right) & \text{corresponding to } r_3 = 2i \text{ and } r_4 = -2i\\ y_4(x) = \sin\left(2x\right) & \text{corresponding to } r_3 = 2i \text{ and } r_4 = -2i\end{array}. \tag{143}$$

4. The general solution is formed by linear combinations of these solutions:

$$\begin{gathered}y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y_4\\ \boxed{y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos\left(2x\right) + c_4 \sin\left(2x\right)}\end{gathered}. \tag{144}$$

To solve $y^{(4)} + 18y'' + 81y = 0$:

1. Assume that $y = e^{rx}$ is a solution to the differential equation for some unknown value of r to find the characteristic equation:

$$\begin{gathered} \frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(e^{rx}\right) + 18\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(e^{rx}\right) + 81\left(e^{rx}\right) = 0 \\ r^4 e^{rx} + 18r^2 e^{rx} + 81e^{rx} = 0 \\ e^{rx}\left(r^4 + 18r^2 + 81\right) = 0 \\ r^4 + 18r^2 + 81 = 0 \end{gathered}. \tag{145}$$

2. Find the roots of the characteristic equation:

$$\begin{gathered} \left(r^2+9\right)\left(r^2+9\right) = 0 \\ \left(r-3i\right)\left(r+3i\right)\left(r-3i\right)\left(r+3i\right) \end{gathered}. \tag{146}$$

In this example, the roots are:

$$\begin{aligned} r_1 &= 3i \quad \text{with multiplicity } 2 \\ r_2 &= -3i \quad \text{with multiplicity } 2 \end{aligned}. \tag{147}$$

3. Write the linearly independent particular solutions that span the general solution. In this case, the order is 4, so we need 4 solutions based on the roots of the characteristic equation:

$$\begin{aligned} y_1(x) &= \cos\left(3x\right) \quad \text{corresponding to } r_1 = 3i \text{ and } r_2 = -3i \\ y_2(x) &= \sin\left(3x\right) \quad \text{corresponding to } r_1 = 3i \text{ and } r_2 = -3i \\ y_3(x) &= x\cos\left(3x\right) \quad \text{corresponding to } r_1 = 3i \text{ and } r_2 = -3i \\ y_4(x) &= x\sin\left(3x\right) \quad \text{corresponding to } r_1 = 3i \text{ and } r_2 = -3i \end{aligned}. \tag{148}$$

4. The general solution is formed by linear combinations of these solutions:

$$\begin{gathered} y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y_4 \\ \boxed{y(x) = \left(c_1 + c_3 x\right)\cos\left(2x\right) + \left(c_2 + c_4 x\right)\sin\left(2x\right)} \end{gathered}. \tag{149}$$

To solve $y^{(6)} + 4y^{(5)} + 13y^{(4)}$:

1. Assume that $y = e^{rx}$ is a solution to the differential equation for some unknown value of r to find the characteristic equation:

$$\begin{gathered}\frac{\mathrm{d}^6}{\mathrm{d}x^6}\left(e^{rx}\right) + 4\frac{\mathrm{d}^5}{\mathrm{d}x^5}\left(e^{rx}\right) + 13\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(e^{rx}\right) = 0\\ r^6e^{rx} + 4r^5e^{rx} + 13r^4e^{rx} = 0\\ e^{rx}\left(r^6 + 4r^5 + 13r^4\right) = 0\\ r^6 + 4r^5 + 13r^4 = 0\end{gathered}. \tag{150}$$

2. Find the roots of the characteristic equation:

$$\begin{gathered}r^4\left(r^2 + 4r + 13\right) = 0\\ r^4\left(r + 2 - 3i\right)\left(r + 2 + 3i\right)\end{gathered}. \tag{151}$$

In this example, the roots are:

$$\begin{aligned} r_1 &= 0 && \text{with multiplicity } 4\\ r_2 &= -2 + 3i && \text{with multiplicity } 1.\\ r_3 &= -2 - 3i && \text{with multiplicity } 1\end{aligned} \tag{152}$$

3. Write the linearly independent particular solutions that span the general solution. In this case, the order is 6, so we need 6 solutions based on the roots of the characteristic equation:

$$\begin{aligned} y_1(x) &= e^{0x} = 1 && \text{corresponding to } r_1 = 0\\ y_2(x) &= xe^{0x} = x && \text{corresponding to } r_1 = 0\\ y_3(x) &= x^2e^{0x} = x^2 && \text{corresponding to } r_1 = 0\\ y_4(x) &= x^3e^{0x} = x^3 && \text{corresponding to } r_1 = 0\\ y_5(x) &= e^{-2x}\cos\left(3x\right) && \text{corresponding to } r_2 = -2 + 3i \text{ and } r_3 = -2 - 3i\\ y_6(x) &= e^{-2x}\sin\left(3x\right) && \text{corresponding to } r_2 = -2 + 3i \text{ and } r_3 = -2 - 3i\end{aligned}. \tag{153}$$

4. The general solution is formed by linear combinations of these solutions:

$$\begin{gathered} y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + c_4y_4 + c_5y_5 + c_6y_6\\ \boxed{y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + e^{-2x}\left(c_5\cos(3x) + c_6\sin(3x)\right)}\end{gathered}. \tag{154}$$

4.2 Linear, non-homogeneous, with constant coefficients

Now we begin to strip away some of the assumptions that we previously made in order to simplify our analysis of ODEs. We will now deal with linear ODEs with constant coefficients which are not necessarily homogeneous:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_2 y'' + a_1 y' + a_0 y = q(x), \tag{155}$$

where $q(x)$ is some function that depends only on the independent variable x. Here are some examples, and their general solutions:

$$\begin{array}{cc}
y'' - 2y' - 3y = e^{4x} & y(x) = c_1 e^{-x} + c_2 e^{3x} + \frac{1}{5} e^{4x} \\
y'' - 2y' - 3y = e^{-x} & y(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{1}{4} x e^{-x} \\
y^{(4)} - 2y'' + y = \cos(3x) & y(x) = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x) e^{-x} + \frac{1}{82} \cos(3x) \\
y^{(3)} + y'' - y' - y = e^{x} + e^{-x} & y(x) = c_1 e^{x} + (c_2 + c_3 x) e^{-x} + \frac{1}{4} x e^{x} - \frac{1}{4} x^2 e^{-x} \\
y'' + 5y' + 6y = (x+1)^3 & y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{6} x^3 + \frac{1}{12} x^2 + \frac{7}{36} x - \frac{5}{216} \\
y'' + 2y' + 5y = e^{x} \sin x & y(x) = e^{-x} (c_1 \cos(2x) + c_2 \sin(2x)) + e^{x} \left(-\frac{4}{65} \cos x + \frac{7}{65} \sin x\right)
\end{array} . \tag{156}$$

We first introduce some terminology that will make the discussion of the procedures of these ODEs a bit less cumbersome.

Definition 4.1 *Let*

$$p_n(x) y^{(n)} + p_{n-1}(x) y^{(n-1)} + ... + p_2(x) y'' + p_1(x) y' + p_0(x) y - q(x) = 0$$

be a linear ODE with solution $y(x)$. The <u>inhomogeneous term(s)</u> of this ODE is the function $q(x)$.

In other words, the inhomogeneous term of a linear ODE is the function of x which is not multiplied by y or any of the derivatives of y. Here are some examples of linear ODEs, with their inhomogeneous terms stated alongside them.

$$
\begin{array}{cc}
y'' + 6y' + 9y = \sin x & \text{Inhomogeneous term: } \sin x \\
x^2 y' + 3xy = 1 & \text{Inhomogeneous term: } 1 \\
\frac{1}{x}y^{(15)} + \frac{1}{x^2}y^{(10)} + \frac{1}{x^3}y^{(5)} + \frac{1}{x^4}y + \frac{1}{x^5} = 0 & \text{Inhomogeneous term: } \frac{1}{x^5} \\
y^{(4)} + y'' - 2y = 0 & \text{Inhomogeneous term: } 0 \\
y'' = 4y + 16 & \text{Inhomogeneous term: } 16 \\
y^{(3)} = x^2 \sin x + e^x & \text{Inhomogeneous terms: } x^2 \sin x + e^x \\
x^2 + y'' + xy = x^3 & \text{Inhomogeneous terms: } x^3 - x^2
\end{array}
\quad . \tag{157}
$$

To clarify: the statement "this ODE is homogeneous" is exactly equivalent to "the inhomogeneous term of this ODE is zero." In order to distinguish the inhomogeneous term(s), we often write linear ODEs with the inhomogeneous term(s) on the right side, and everything else on the left side.

We'll look at two main methods of solving non-homogeneous ODEs with constant coefficients: the method of undetermined coefficients, and the method of Laplace transforms.

4.2.1 Method of undetermined coefficients

Next, we need a term to specify the ODE that results when you set the inhomogeneous term(s) to zero:

Definition 4.2 *Let*

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x)$$

be a linear ODE with solution $y(x)$ and inhomogeneous term(s) $q(x)$. The associated homogeneous equation of this ODE is the homogeneous ODE

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

It will also be helpful to come up with a name for the general solution of the associated homogeneous equation:

Definition 4.3 *The complementary solution of a linear ODE is the general solution of the associated homogeneous equation of the ODE.*

Therefore, the complementary solution of an ODE is the solution you get by setting the inhomogeneous term(s) to zero. Note the unfortunate, but important point that *the complementary solution of an ODE is* ***not*** *a solution of the ODE itself, unless the ODE in question is homogeneous!*

The theory of linear ODEs revolves around the following theorem.

Theorem 4.4 *If $p_n(x)y^{(n)}+p_{n-1}(x)y^{(n-1)}+...+p_2(x)y''+p_1(x)y'+p_0(x)y = q(x)$ is a linear ODE, and $y_P(x)$ is any solution whatsoever of the ODE, then the general solution of the ODE is $y(x) = y_C(x) + y_P(x)$, where $y_C(x)$ is the complementary solution of the ODE.*

This powerful theorem tells us that the problem of solving an inhomogeneous linear ODE with constant coefficients can be broken into two parts: solving the associated homogeneous equation, and finding *any* solution. The problem of solving the associated homogeneous equation is exactly the topic of Section 4.1. Therefore, we will spend the remainder of this section on figuring out how to determine the particular solution of a non-homogeneous ODE.

The strategy of figuring out the particular solution of a non-homogeneous ODE is called the "method of undetermined coefficients." Its core is this: take a guess at the solution with arbitrary coefficients, based solely on the inhomogeneous term, and then solve for the coefficients.

How does one know what guess to make? In Math 306, we will deal only with inhomogeneous terms which involve exponentials, polynomials and/or sine and cosine functions. By restricting our attention to these, we can make very reliable guesses as to what the particular solution will be. The following is a summary of our strategy:

If the inhomogeneous term is:	then make this guess:
e^{kx}	$y_t(x) = Ae^{kx}$
a polynomial of degree m	$y_t(x) = A_m x^m + A_{m-1}x^{m-1} + ... + A_2 x^2 + A_1 x + A_0$
$\sin(kx)$ or $\cos(kx)$	$y_t(x) = A\cos(kx) + B\sin(kx)$
A product of the functions above	A product of the guesses above
A sum of the functions above	A sum of the guesses above

(158)

Here are some examples of non-homogeneous ODEs, and the initial guess that should be made to solve them:

$$\begin{array}{cc}
y'' - 2y' - 3y = e^{4x} & y_t(x) = Ae^{4x} \\
y^{(4)} - 2y'' + y = \cos(3x) & y_t(x) = A\cos(3x) + B\sin(3x) \\
y'' - y' - 2y = x^2 + 4 & y_t(x) = Ax^2 + Bx + C \\
y'' + 5y' + 6y = (x+1)^3 & y_t(x) = Ax^3 + Bx^2 + Cx + D \\
y^{(3)} - 16y'' + 64y' = e^x \cos x & y_t(x) = (A\cos x + B\sin x)\, e^x \\
y'' + 3y' + 2y = e^x\left(x^2 + \cos(2x)\right) & y_t(x) = \left(Ax^2 + Bx + C + D\cos(2x) + E\sin(2x)\right) e^x \\
y'' + 2y' + 5y = x^2 e^x & y_t(x) = \left(Ax^2 + Bx + C\right) e^x \\
y^{(4)} + 32y'' + 256y = x^2 \sin x & y_t(x) = \left(Ax^2 + Bx + C\right)(D\cos x + E\sin x) \\
y^{(4)} - y = \cos(2x) - x & y_t(x) = A\cos(2x) + B\sin(2x) + Cx + D
\end{array} \tag{159}$$

These guesses have varying effectiveness depending on whether the inhomogeneous terms share terms with the complementary solution, a phenomenon known as "duplication." We will describe the procedure for both cases below.

Case of non-duplication

In this case, the initial guess is the correct one, and so the method goes smoothly. We describe the procedure in the case of $y^{(4)} - 8y'' + 16y = x\cos x$:

1. Find the complementary solution of the ODE:

$$\begin{gathered} y^4 - 8y'' + 16y = 0 \\ r^4 - 8r^2 + 16 = 0 \\ \left(r^2 - 4\right)^2 = 0 \\ (r+2)^2(r-2)^2 = 0 \\ y_C(x) = (c_1 + c_2 x)\, e^{2x} + (c_3 + c_4 x)\, e^{-2x} \end{gathered} \quad . \tag{160}$$

2. Guess the solution based on the inhomogeneous term(s). In this case, the inhomogeneous term is $x\cos x$. The factor x, a polynomial of degree 1, corresponds to the initial guess $y_t(x) = Ax + B$, and the factor $\cos x$ corresponds to the initial guess $y_t(x) = C\cos x + D\sin x$. Therefore, the guess corresponding to the product $x\cos x$ will be the product of these two guesses:

$$\begin{aligned} & y_t(x) = (Ax+B)\,(C\cos x + D\sin x) \\ \text{(Equivalently,)} \quad & y_t(x) = A\cos x + Bx\cos x + C\sin x + Dx\sin x \end{aligned} \tag{161}$$

3. Assume that y_t is a solution and subsitute it into the original ODE:

$$\begin{aligned} {y_t}^{(4)} - 8y_t'' + 16y_t &= ((A-4D)\cos x + Bx\cos x + (4B+C)\sin x + Dx\sin x) \\ &\quad - 8\left((2D-A)\cos x - Bx\cos x - (2B+C)\sin x - Dx\sin x\right) \\ &\quad + 16\left(A\cos x + Bx\cos x + C\sin x + Dx\sin x\right) \\ &= (25A - 20D)\cos x + 9Bx\cos x + (20B + 25C)\sin x + 25Dx\sin x \end{aligned} \tag{162}$$

4. Set this equal to the inhomogeneous terms and use the linear independence of the functions to equate the coefficients:

$$\begin{aligned} (25A-20D)\cos x+9Bx\cos x+(20B+25C)\sin x+25Dx\sin x=x\cos x \\ 25A-20D=0 \\ 25B=1 \\ 20B+25C=0 \\ 25D=0 \end{aligned}. \tag{163}$$

5. Solve the system for the coefficients:

$$\begin{aligned} A&=0 \\ B&=\tfrac{1}{25} \\ C&=-\tfrac{4}{125} \\ D&=0 \end{aligned}. \tag{164}$$

6. Write the particular solution using the newly found coefficients:

$$y_P(x)=0\cos x+\frac{1}{25}x\cos x-\frac{4}{125}\sin x+0x\sin x. \tag{165}$$

7. Write the general solution using Theorem 4.4:

$$\boxed{y(x)=(c_1+c_2x)\,e^{2x}+(c_3+c_4x)\,e^{-2x}+\tfrac{1}{25}x\cos x-\tfrac{4}{125}\sin x.} \tag{166}$$

Case of duplication

When given a linear ODE with constant coefficients, it is necessary to find the complementary solution before beginning to choose a guess for a particular solution. For example, take $y''+y'-6y=e^{2x}$. The initial guess would be $y_t(x)=Ae^{2x}$. However, the complementary solution of this ODE is $y_C(x)=c_1e^{-3x}+c_2e^{2x}$. Thus, our guess y_t *will* be a solution of the associated homogeneous ODE, since $y_t=0e^{-3x}+Ae^{2x}$ for some real value A. Ergo, plugging this guess into the original ODE will always give us zero! Observe:

$$\begin{aligned} y_t'' + y_t' - 6y_t &= e^{2x} \\ \tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(Ae^{2x}\right) + \tfrac{\mathrm{d}}{\mathrm{d}x}\left(Ae^{2x}\right) - 6\left(Ae^{2x}\right) &= e^{2x} \\ 4Ae^{2x} + 2Ae^{2x} - 6Ae^{2x} &= e^{2x} \\ 0 &= e^{2x} \qquad\qquad\qquad \text{(WHAT?!)} \end{aligned} \tag{167}$$

In these situations, the initial guess cannot work, so we must modify the guess.

To clarify, the situation above, called "duplication," occurs whenever one can choose the arbitrary constants in the initial guess so that the initial guess is a solution of the associated homogeneous equation. For example, in the case of the non-homogeneous linear ODE $y^{(4)} - y = x^2 + \cos x$, the complementary solution is

$$y_C(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x. \tag{168}$$

The initial guess for the particular solution is

$$y_t(x) = Ax^2 + Bx + C + D\cos x + E\sin x. \tag{169}$$

Now, if one chooses $A = 0$, $B = 0$, $C = 0$, $D = 1$ and $E = 1$, then this becomes

$$y_t(x) = \cos x + \sin x, \tag{170}$$

which is described in the complementary solution $y_C(x)$.

Here are some examples of non-homogeneous ODEs which have duplication, and the appropriate refined guess that should be used to find a particular solution:

$$\begin{aligned} y^{(4)} + 2y^{(3)} + \left(\pi^2+1\right)y'' = e^{-x}\sin(\pi x) &\qquad y_t(x) = xe^{-x}\left(A\cos(\pi x) + B\sin(\pi x)\right) \\ y^{(4)} + 8y'' + 16y = \cos(2x) &\qquad y_t(x) = Ax^2\cos(2x) + Bx^2\sin(2x) \\ y^{(4)} + 9y'' = \left(x^2+1\right)\sin(3x) &\qquad y_t(x) = x\left(Ax^2 + Bx + C\right)\left(D\cos(3x) + E\sin(3x)\right) \\ y^{(5)} - y^{(3)} = e^{x} + 2x^2 - 5 &\qquad y_t(x) = Axe^{x} + x^3\left(Bx^2 + Cx + D\right) \end{aligned} \tag{171}$$

We describe the procedure of altering the initial guess in order to produce an appropriate guess in the case of three different ODEs.

To solve $y'' - 2y' - 3y = e^{-x}$:

1. Find the complementary solution:

$$\begin{aligned} r^2 - 2r - 3 &= 0 \\ (r-3)(r+1) &= 0 \\ y_C(x) &= c_1 e^{3x} + c_2 e^{-x} \end{aligned} \quad . \tag{172}$$

2. Guess the solution based on the inhomogeneous term(s):

$$y_t(x) = Ae^{-x} \tag{173}$$

3. Multiply each term of the guess y_t that appears in the complementary solution by x. Repeat this step until the guess is no longer a solution of the associated homogeneous equation:

$$\begin{aligned} &\cancel{y_t(x) = Ae^{-x}} \\ &y_t(x) = Axe^{-x} \end{aligned} . \tag{174}$$

4. As before, assume that this new y_t is a solution and substitute it into the original ODE:

$$y_t'' - 2y_t' - 3y_t = \left(A(x-2)e^{-x}\right) - 2\left(A(1-x)e^{-x}\right) - 3\left(Axe^{-x}\right) = (0x-4)Ae^{-x}. \tag{175}$$

5. As before, set this equal to the inhomogeneous term(s) and solve for the constants:

$$\begin{aligned} -4Ae^{-x} &= e^{-x} \\ A &= -\tfrac{1}{4} \\ y_P(x) &= -\tfrac{1}{4}e^{-x} \\ &\boxed{y(x) = c_1 e^{3x} + c_2 e^{-x} - \tfrac{1}{4}e^{-x}.} \end{aligned} \tag{176}$$

To solve $y^{(4)} - y = x^2 + \cos x$:

1. Find the complementary solution:

$$\begin{aligned} r^4 - 1 &= 0 \\ (r+1)(r-1)(r+i)(r-i) &= 0 \\ y_C(x) &= c_1 e^{-x} + c_2 e^{x} + c_3 \cos x + c_4 \sin x \end{aligned} \quad . \tag{177}$$

2. Guess the solution based on the inhomogeneous term(s):

$$y_t(x) = \left(Ax^2 + Bx + C\right) + (D\cos x + E\sin x) \tag{178}$$

3. Multiply each term of the guess y_t that appears in the complementary solution by x. Repeat this step until the guess is no longer a solution of the associated homogeneous equation:

$$\begin{aligned} &\cancel{y_t(x) = (Ax^2 + Bx + C) + (D\cos x + E\sin x)} \\ &y_t(x) = (Ax^2 + Bx + C) + x\,(D\cos x + E\sin x) \end{aligned} . \tag{179}$$

4. As before, assume that this new y_t is a solution and substitute it into the original ODE:

$$\begin{aligned} {y_t}^{(4)} - y_t &= (Dx\cos x + 4D\sin x + Ex\sin x - 4E\cos x) \\ &\qquad - \left(Ax^2 + Bx + C + Dx\cos x + Ex\sin x\right) \\ &= -Ax^2 - Bx - C - 4E\cos x + 4D\sin x + 0x\cos x + 0x\sin x \end{aligned} \tag{180}$$

5. As before, set this equal to the inhomogeneous term(s) and solve for the

constants:

$$\begin{gathered}
-Ax^2 - Bx - C - 4E\cos x + 4D\sin x + 0x\cos x + 0x\sin x = x^2 + \cos x \\
-A = 1 \\
-B = 0 \\
-C = 0 \\
-4E = 1 \\
4D = 0 \\
y_P(x) = (-x^2 + 0x + 0) + x\left(0\cos x - \tfrac{1}{4}\sin x\right) \\
\boxed{y_C(x) = c_1 e^{-x} + c_2 e^{x} + c_3\cos x + c_4\sin x - x^2 - \tfrac{1}{4}x\sin x.}
\end{gathered} \tag{181}$$

To solve $y^{(3)} + y'' - y' - y = x + e^{-x}$:

1. Find the complementary solution:

$$\begin{aligned}
r^3 + r^2 - r - 1 &= 0 \\
(r+1)^2(r-1) &= 0 \\
y_C(x) &= c_1 e^{x} + (c_2 + c_3 x)\, e^{-x}
\end{aligned} . \tag{182}$$

2. Guess the solution based on the inhomogeneous term(s):

$$y_t(x) = (Ax + B) + \left(Ce^{-x}\right) \tag{183}$$

3. Multiply each term of the guess y_t that appears in the complementary solution by x. Repeat this step until the guess is no longer a solution of the associated homogeneous equation:

$$\begin{aligned}
&\cancel{y_t(x) = (Ax + B) + (Ce^{-x})} \\
&\cancel{y_t(x) = (Ax + B) + x\,(Ce^{-x})} \\
&y_t(x) = (Ax + B) + x^2\,(Ce^{-x})
\end{aligned} . \tag{184}$$

4. As before, assume that this new y_t is a solution and substitute it into the

original ODE:

$$\begin{aligned} {y_t}^{(3)} + y_t'' - y_t' - y_t = &\left(Ce^{-x}\left(-x^2+6x-6\right)\right) + \left(Ce^{-x}\left(x^2-4x+2\right)\right) \\ &- \left(A + Ce^{-x}\left(-x^2+2x\right)\right) - \left(Ax + B + Ce^{-x}x^2\right) \\ &= -\left(A+B\right) - Ax + Ce^{-x}\left(0x^2+0x-4\right) \end{aligned} \tag{185}$$

5. As before, set this equal to the inhomogeneous term(s) and solve for the constants:

$$\begin{gathered} -(A+B) - Ax - 4Ce^{-x} = x + e^{-x} \\ -(A+B) = 0 \\ -A = 1 \\ -4C = 1 \\ y_P(x) = (-x+1) - \tfrac{1}{4}x^2e^{-x} \\ \boxed{y(x) = c_1e^x + \left(c_2 + c_3x\right)e^{-x} - x + 1 - \tfrac{1}{4}x^2e^{-x}.} \end{gathered} \tag{186}$$

4.2.2 Laplace transform methods

In solving linear ODEs, the characteristic equation carries immense power, because it allows us to turn a differential equation into an algebraic equation. By then solving the algebraic equation, we can solve the differential equation. In this section, we'll discuss another way of turning a linear ODE into an algebraic equation, using a concept known as the Laplace transform.

Finding Laplace transforms

First, of course, what is a Laplace transform?

Definition 4.5 *Let $f(t)$ be a function defined for $t \geq 0$. The Laplace transform of f is a function $F(s)$ defined as follows:*

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

Notice that the definition of the Laplace transform is dependent on an improper integral. Since improper integrals do not always converge, the Laplace transform does not necessarily exist for all values of s.

We will often denote the Laplace transform of a function $f(t)$ as $\mathcal{L}\left(f(t)\right)$. This notation has a slight disadvantage: the Laplace transform of a function $f(t)$ has a different independent variable than t. We call this new independent variable s, but "s" does not appear in the expression "$\mathcal{L}\left(f(t)\right)$."

One can easily verify the following theorem using the properties of integrals.

Theorem 4.6 *Given functions $f(t)$ and $g(t)$, and constants a and b,*

$$\mathcal{L}\left(af(t) + bg(t)\right) = a\mathcal{L}\left(f(t)\right) + b\mathcal{L}\left(g(t)\right),$$

for all s values such that $\mathcal{L}\left(f(t)\right)$ and $\mathcal{L}\left(g(t)\right)$ both exist.

We say that the Laplace transform is a "linear" operation because of this theorem.

Here are some examples of functions, and their Laplace transforms:

$$
\begin{array}{ll}
f(t) = 1 & F(s) = \frac{1}{s} \text{ for } s > 0 \\
f(t) = e^{at}, \text{ for a real value } a & F(s) = \frac{1}{s-a} \text{ for } s > a \\
f(t) = t^n, \text{ for an integer } n \geq 0 & F(s) = \frac{n!}{s^{n+1}} \text{ for } s > 0 \\
f(t) = \cos(kt), \text{ for a real value } k & F(s) = \frac{s}{s^2+k^2} \text{ for } s > 0 \\
f(t) = \sin(kt), \text{ for a real value } k & F(s) = \frac{k}{s^2+k^2} \text{ for } s > 0
\end{array}
\tag{187}
$$

Each of these can be derived from the definition and/or from the linearity of the Laplace transform.

We will now demonstrate how to find the Laplace transform of a given function by proving two of the above equations. First, we find $\mathcal{L}\left(e^{at}\right)$ for any real value a:

$$
\mathcal{L}\left(e^{at}\right) = \int_0^\infty e^{-st}e^{at} \, \mathrm{d}t = \int_0^\infty e^{(a-s)t} \, \mathrm{d}t = \frac{1}{a-s}e^{(a-s)t}\bigg|_0^\infty . \tag{188}
$$

By definition of the improper integral, this is

$$
\mathcal{L}\left(e^{at}\right) = \frac{1}{a-s}\left(\lim_{t\to\infty} e^{(a-s)t} - e^{(a-s)0}\right) = \frac{1}{a-s}\left(\lim_{t\to\infty} e^{(a-s)t} - 1\right). \tag{189}
$$

If $s < a$, then this limit is ∞, and so the integral diverges. If $s = a$, then $\frac{1}{a-s}$ is undefined. Yet if $s > a$, then

$$
\mathcal{L}\left(e^{at}\right) = \frac{1}{a-s}(0-1) = \frac{1}{s-a}. \tag{190}
$$

Next, we will find $\mathcal{L}(\cos(kt))$ for any real value k:

$$
\mathcal{L}(\cos(kt)) = \int_0^\infty e^{-st}\cos(kt) \, \mathrm{d}t. \tag{191}
$$

We proceed by integration by parts. We set $u = \cos(kt)$, so that $\mathrm{d}v = e^{-st} \, \mathrm{d}t$. Therefore $\mathrm{d}u = -k\sin(kt) \, \mathrm{d}t$ and $v = -\frac{1}{s}e^{-st}$, and so

$$
\int_0^\infty e^{-st}\cos(kt) \, \mathrm{d}t = -\frac{1}{s}e^{-st}\cos(kt)\bigg|_0^\infty - \frac{k}{s}\int_0^\infty e^{-st}\sin(kt) \, \mathrm{d}t. \tag{192}
$$

First, let's evaluate $-\frac{1}{s}e^{-st}\cos(kt)\Big|_0^\infty$. This is

$$-\frac{1}{s}e^{-st}\cos(kt)\Bigg|_0^\infty = \frac{1}{s}e^{-st}\cos(kt)\Bigg|_\infty^0 = \frac{1}{s} - \lim_{t\to\infty}\frac{1}{s}e^{-st}\cos(kt). \tag{193}$$

Now, given any t, $-1 \leq \cos(kt) \leq 1$. Thus, $-\frac{e^{-st}}{s} \leq \frac{1}{s}e^{-st}\cos(kt) \leq \frac{e^{-st}}{s}$. Given that $s > 0$, $\lim_{t\to\infty} -\frac{e^{-st}}{s} = 0 = \lim_{t\to\infty}\frac{e^{-st}}{s}$. Therefore, the squeeze theorem indicates that $\lim_{t\to\infty}\frac{1}{s}e^{-st}\cos(kt) = 0$. We deduce that

$$\int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t = \frac{1}{s} - \frac{k}{s}\int_0^\infty e^{-st}\sin(kt)\ \mathrm{d}t. \tag{194}$$

Now we use integration by parts on $\int_0^\infty e^{-st}\sin(kt)\ \mathrm{d}t$. Setting $u = \sin(kt)$, $\mathrm{d}v = e^{-st}\ \mathrm{d}t$, so $\mathrm{d}u = k\cos(kt)\ \mathrm{d}t$ and $v = -\frac{1}{s}e^{-st}$. Thus,

$$\int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t = \frac{1}{s} - \frac{k}{s}\left(-\frac{1}{s}e^{-st}\sin(kt)\Bigg|_0^\infty + \frac{k}{s}\int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t\right). \tag{195}$$

By applying the squeeze theorem again, we can find that $-\frac{1}{s}e^{-st}\sin(kt)\Big|_0^\infty = 0$. Ergo,

$$\int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t = \frac{1}{s} - \frac{k^2}{s^2}\int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t. \tag{196}$$

This implies that

$$\left(1 + \frac{k^2}{s^2}\right)\int_0^\infty e^{-st}\cos(kt) = \frac{1}{s}. \tag{197}$$

Thus,

$$\mathcal{L}(\cos(kt)) = \int_0^\infty e^{-st}\cos(kt)\ \mathrm{d}t = \frac{1}{s\left(1 + \frac{k^2}{s^2}\right)} = \frac{s}{s^2 + k^2}. \tag{198}$$

As you can see, in computing Laplace transforms from the definition, some functions are easier than others. It is important to note that finding the Laplace transform of $f(t) = t^n$ for $n \geq 0$ uses the principle of mathematical induction; if you are not familiar with mathematical induction, then the author does not suggest attempting to derive it.

There is one more type of function that will be of interest to us.

Definition 4.7 *The unit step function is the piecewise function*

$$u(t) = \begin{cases} 0 & \textit{for } t < 0 \\ 1 & \textit{for } t \geq 0 \end{cases}.$$

Notice that, for any real value a,

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}. \tag{199}$$

Let's try to figure out the Laplace transform of the function $u(t-a)$ for $a \geq 0$. By definition,

$$F(s) = \mathcal{L}\left(u\left(t-a\right)\right) = \int_0^\infty e^{-st} u\left(t-a\right) \, \mathrm{d}t. \tag{200}$$

We can split this into two integrals to evaluate:

$$F(s) = \int_0^a e^{-st} u\left(t-a\right) \, \mathrm{d}t + \int_a^\infty e^{-st} u\left(t-a\right) \, \mathrm{d}t \tag{201}$$

On the interval $[0,a]$, $u(t-a) = 0$, and on the interval $[a,\infty)$, $u(t-a) = 1$. Thus,

$$\begin{aligned} F(s) = \int_0^a e^{-st}(0) \, \mathrm{d}t + \int_a^\infty e^{-st}(1) \, \mathrm{d}t = 0 + \int_a^\infty e^{-st} \, \mathrm{d}t \\ = -\frac{1}{s} e^{-st}\bigg|_a^\infty = \frac{1}{s} e^{-st}\bigg|_\infty^a = \lim_{b\to\infty} \frac{1}{s} e^{-st}\bigg|_b^a \\ = \lim_{b\to\infty} \frac{1}{s}\left(e^{-sa} - e^{-sb}\right) = \frac{1}{s} e^{-sa} - \frac{1}{s} \lim_{b\to\infty} e^{-sb} \end{aligned} \tag{202}$$

The limit $\lim\limits_{b\to\infty} e^{-sb}$ has a finite value if and only if $s > 0$. In that case, the limit is zero, and so

$$\mathcal{L}\left(u\left(t-a\right)\right) = F(s) = \frac{1}{s} e^{-as} \text{ for } s > 0. \tag{203}$$

Existence of Laplace transforms

We've seen that the Laplace transform of a function won't be defined over all real numbers. Is it possible that the Laplace transform would fail to exist *everywhere*? Indeed, it is possible. For example, the Laplace transform of $e^{\left(t^2\right)}$ would not exist for any real number, because the integral $\int_0^\infty e^{-st}e^{\left(t^2\right)}\,\mathrm{d}t$ will always fail to converge. In this sort of case, we'd say that $e^{\left(t^2\right)}$ does not have a Laplace transform.

So, which functions can we count on to have a Laplace transform? The class of functions that do not grow faster than exponential functions would qualify. We give them a name.

Definition 4.8 *Let $f(t)$ be a function defined on the real line. We say that f is of exponential order provided that there exist non-negative constants M, c and T such that for all $t \geq T$, $|f(t)| \leq Me^{ct}$.*

In other words, f is of exponential order if $\lim\limits_{t\to\infty} \frac{f(t)}{e^{ct}}$ is finite for some non-negative constant c.

Theorem 4.9 *If a function $f(t)$ is piecewise continuous for $t \geq 0$ and f is of exponential order, then $\mathcal{L}\left(f(t)\right)$ exists.*

Of course, this means that we can only use Laplace transform methods to solve ODEs when we're dealing with piecewise continuous functions of exponential order.

Laplace transforms and initial value problems

Are Laplace transforms useful for solving ODEs? Well of course; I wouldn't be writing about it here if they weren't, you dullard. The Laplace transform techniques of solving ODEs revolve around the following theorem.

Theorem 4.10 *If a function $f(t)$ is piecewise smooth for $t \geq 0$ and f is of exponential order, then*

$$\mathcal{L}\left(f'(t)\right) = s\mathcal{L}\left(f(t)\right) - f(0). \tag{204}$$

Similarly,

$$\begin{aligned}\mathcal{L}\left(f''(t)\right) &= s\mathcal{L}\left(f'(t)\right) - f'(0) \\ &= s\left(s\mathcal{L}\left(f(t)\right) - f(0)\right) - f'(0) \\ &= s^2\mathcal{L}\left(f(t)\right) - sf(0) - f'(0), \end{aligned} \tag{205}$$

and so on.

Therefore, if we're given an intial value problem (that is, an nth order ODE in f where the values $f(0), f'(0), f''(0), ..., f^{(n)}(0)$ are given), then we can take the Laplace transform of both sides. For example, given the intial value problem

$$\begin{aligned} x'' + 3x' + 2x = t; \\ x(0) = 0 \qquad , \\ x'(0) = 2 \end{aligned} \tag{206}$$

We can take the Laplace transform of both sides of the ODE:

$$\mathcal{L}\left(x'' + 3x' + 2x\right) = \mathcal{L}\left(t\right). \tag{207}$$

By the linearity of the Laplace transform, this is

$$\mathcal{L}\left(x''\right) + 3\mathcal{L}\left(x'\right) + 2\mathcal{L}\left(x\right) = \mathcal{L}\left(t\right). \tag{208}$$

The theorem above indicates that this can be written as

$$\left(s^2\mathcal{L}(x) - sx(0) - x'(0)\right) + 3\left(s\mathcal{L}\left(x\right) - x(0)\right) + 2\mathcal{L}\left(x\right) = \mathcal{L}\left(t\right). \tag{209}$$

Let's denote $\mathcal{L}(x) = X(s)$. Since we have that $x(0) = 0$ and $x'(0) = 2$, this becomes

$$\left(s^2X(s) - 2\right) + 3\left(sX(s)\right) + 2X(s) = \mathcal{L}(t). \tag{210}$$

We know that $\mathcal{L}(t) = \frac{1}{s^2}$, so

$$s^2X - 2 + 3sX + 2X = \frac{1}{s^2}. \tag{211}$$

We isolate X:

$$\left(s^2+3s+2\right)X=\frac{1}{s^2}+2, \tag{212}$$

and so

$$X(s)=\frac{\frac{1}{s^2}+2}{s^2+3s+2}=\frac{1}{s^2\left(s+2\right)\left(s+1\right)}+\frac{2}{\left(s+2\right)\left(s+1\right)}. \tag{213}$$

Now, we have the Laplace transform of the solution. If we can just figure out which function has a Laplace transform of $\frac{1}{s^2(s+2)(s+1)}+\frac{2}{(s+2)(s+1)}$, then we'll be able to find x and solve the problem. Therefore, we seek the *inverse Laplace transform* of $\frac{1}{s^2(s+2)(s+1)}+\frac{2}{(s+2)(s+1)}$.

Here are some examples of initial value problems involving linear ODEs with constant coefficients, and the Laplace transforms of their solutions:

$$\begin{array}{cc} x''+4x=0 & X(s)=\frac{5s}{s^2+4} \\ x(0)=5,\ x'(0)=0 & \\ x''-x'-2x=0 & X(s)=\frac{2}{s^2-s-2} \\ x(0)=0,\ x'(0)=2 & \\ x''+x=\sin\left(2t\right) & X(s)=\frac{2}{(s^2+1)(s^2+4)} \\ x(0)=0,\ x'(0)=0 & \\ x''+x=\cos\left(3t\right) & X(s)=\frac{s}{(s^2+9)(s^2+1)}+\frac{s}{s^2+1} \\ x(0)=1,\ x'(0)=0 & \end{array} \tag{214}$$

In the same way that finding an antiderivative is much harder than finding a derivative, finding the inverse Laplace transform is much harder than finding the Laplace transform. Ultimately, every method of doing so boils down to dissolving the function in question into parts that resemble the Laplace transforms of familiar functions.

Here are some examples of functions, and their inverse Laplace transforms:

$$
\begin{array}{cc}
F(s)=\frac{-5}{s} & f(t)=-5 \\
F(s)=\frac{s+k}{s^2+4} & f(t)=\cos(2t)+\sin(2t) \\
F(s)=\frac{1}{s^5} & f(t)=\frac{1}{24}t^4 \\
F(s)=\frac{1}{5-s} & f(t)=-e^{5t} \\
F(s)=\frac{e^{-5s}+1}{s} & f(t)=u(t-5)+1 \\
F(s)=\frac{k}{(s-2)^2+\pi^2} & f(t)=e^{2t}\sin(\pi t) \\
F(s)=\frac{s^2+1}{s^3-2s^2-8s} & f(t)=-\frac{1}{8}+\frac{17}{24}e^{4t}+\frac{5}{12}e^{-2t} \\
F(s)=\frac{1}{s(s-3)} & f(t)=\frac{1}{3}\left(e^{3t}-1\right) \\
F(s)=\frac{1}{s(s^2+4)} & f(t)=\frac{1}{4}\left(1-\cos(2t)\right) \\
F(s)=\frac{1}{s^2(s^2+1)} & f(t)=t-\sin t \\
F(s)=\frac{1}{s^2(s^2-1)} & f(t)=\frac{e^t-e^{-t}}{2}-t \\
F(s)=\frac{3}{2s-4} & f(t)=\frac{3}{2}e^{2t} \\
F(s)=\frac{1}{s^2+4s+4} & f(t)=te^{-2t} \\
F(s)=\frac{1}{s^3-5s^2} & f(t)=\frac{1}{25}\left(e^{5t}-5t-1\right) \\
F(s)=\frac{5s-6}{s^2-3s} & f(t)=2+3e^{3t} \\
F(s)=\frac{s}{(s-3)(s^2+1)} & f(t)=\frac{1}{10}\left(3e^{2t}+\sin t-3\cos t\right)
\end{array}
\tag{215}
$$

Notice that for each of these examples, $\lim\limits_{s\to\infty} F(s)=0$. Indeed, this is how it must be. If $f(t)$ is of exponential order, then $\lim\limits_{s\to\infty}\mathcal{L}\left(f\left(t\right)\right)=0$.

Theorems on Laplace transforms

Before going further, let's discuss a few important theorems that can assist us in finding inverse Laplace transforms. Using the fundamental theorem of calculus, it is possible to prove the following theorem.

Theorem 4.11 *If $f(t)$ is a piecewise continuous function for $t\geq 0$ and f is of exponential order, then*

$$
\mathcal{L}\left(\int_0^t f\left(r\right)\,\mathrm{d}r\right)=\frac{\mathcal{L}\left(f(t)\right)}{s}
$$

for $s>0$.

The theorem implies that the inverse Laplace transform of $\frac{\mathcal{L}(f(t))}{s}$ is $\int_0^t f(u)\,\mathrm{d}u$. For example, $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = \int_0^t 1\,\mathrm{d}u = t$.

The next theorem can be proven directly from the definition of the Laplace transform.

Theorem 4.12 *Let f be a function defined on the real line. If $F(s) = \mathcal{L}(f(t))$ exists, then*

$$\mathcal{L}\left(e^{at}f(t)\right) = F(s-a).$$

This theorem implies that if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}^{-1}(F(s-a)) = e^{at}f(t)$. For example, $\mathcal{L}^{-1}\left(\frac{6}{(s-10)^4}\right) = e^{10t}\mathcal{L}^{-1}\left(\frac{6}{s^4}\right) = e^{10t}t^3$. On the other hand, we also have the following theorem.

Theorem 4.13 *Let f be a function defined on the real line, and let a be a real value. If $F(s) = \mathcal{L}(f(t))$ exists, then*

$$\mathcal{L}(u(t-a)\,f(t-a)) = e^{-as}F(s),$$

where $u(t)$ is the unit step function.

This theorem implies that $\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)\,f(t-a)$. For example, $\mathcal{L}^{-1}\left(e^{-2s}\frac{s}{s^2+9}\right) = u(t-2)\cos(3t)$.

What is $\mathcal{L}(f(t)g(t))$? The answer is **not** $\mathcal{L}(f(t))\,\mathcal{L}(g(t))$. However, the next theorem will give us something to help with this issue. First, we must define a new concept.

Definition 4.14 *Let f and g be piecewise continuous functions defined on the real line. The <u>convolution of f and g</u> is a function $h(t)$ defined for $t \geq 0$ as*

$$h(t) = \int_0^t f(r)g(t-r)\;\mathrm{d}r.$$

We denote the convolution of f and g by $f * g$. Note that $f * g = g * f$.

The following theorem reveals that convolution behaves like multiplication of Laplace transforms.

Theorem 4.15 *If f and g are piecewise continuous functions that are of exponential order, then*

$$\mathcal{L}\left(f(t) * g(t)\right) = \mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right). \tag{216}$$

This theorem implies that $\mathcal{L}^{-1}\left(\mathcal{L}\left(f(t)\right)\mathcal{L}\left(g(t)\right)\right) = f(t) * g(t)$. For example, $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\frac{s}{s^2+4}\right) = \cos(t) * \cos(2t)$.

The Dirac delta function

What function would have a Laplace transform of $F(s) = 1$? As it turns out, no function of exponential order could actually satisfy this condition, since, as we mentioned before, $\lim_{s\to\infty} F(s) = 0$. Nevertheless, we will now define the "Dirac delta function," which is not a *function*, but rather a *thing* satisfying some particular properties.

Definition 4.16 *Let $a \geq 0$. The Dirac delta function centered at a is an operation $\delta_a(t)$ such that for any function $f(t)$,*

$$\int_0^\infty f(t)\delta_a(t)\,\mathrm{d}t = f(a).$$

This property, which we take as the *definition* of the symbol $\delta_a(t)$, is sometimes called the "sifting property" of $\delta_a(t)$.

Based on this, we describe $\mathcal{L}\left(\delta_a(t)\right)$ as follows:

$$\mathcal{L}\left(\delta_a(t)\right) = \int_0^\infty e^{-st}\delta_a(t)\,\mathrm{d}t = e^{-sa}. \tag{217}$$

Therefore, if $a = 0$, then $\mathcal{L}\left(\delta_0(t)\right) = 1$. The symbol $\delta_0(t)$ is often written as $\delta(t)$, and $\delta_a(t)$ is understood via the relationship $\delta_a(t) = \delta(t-a)$.

In applications, $\delta_a(t)$ is typically used to describe an effect that lasts for approximately no time, like a sudden voltage spike in an electrical circuit, or the force of a bat hitting a baseball. This causes $\delta_a(t)$ to commonly appear as the inhomogeneous term of an ODE:

$$ax'' + bx' + cx = \delta_a(t). \tag{218}$$

In such cases, we *must* proceed by Laplace transform methods, since no other method can make sense of $\delta_a(t)$.

Here are some examples of initial value problems that involve delta functions, and their solutions:

$$\begin{array}{cc}
x'' + 4x = \delta(t) & x(t) = \frac{1}{2}\sin(2t) \\
x(0) = 0,\ x'(0) = 0 & \\
x'' + 2x' + x = t + \delta(t) & x(t) = t - 2 + (3t+2)\, e^{-t} \\
x(0) = 0,\ x'(0) = 1 & \\
x'' + 4x = \delta(t) + \delta(t-\pi) & x(t) = \frac{1}{2}\left(1 + u(t-\pi)\right)\sin(2t) \\
x(0) = 0,\ x'(0) = 0 & \\
x'' + 4x' + 4x = 1 + \delta(t-2) & x(t) = \frac{1}{4} + \frac{1}{4}e^{-2t}\left(4e^4 u(t-2)(t-2) - 2t - 1\right) \\
x(0) = 0,\ x'(0) = 0 & \\
x'' + 2x' + 2x = 2\delta(t-\pi) & x(t) = -2u(t-\pi)\, e^{-t+\pi}\sin t \\
x(0) = 0,\ x'(0) = 0 & \\
x'' + 4x' + 5x = \delta(t-\pi) + \delta(t-2\pi) & x(t) = \left(2 - e^{2\pi}u(t-\pi) + e^{4\pi}u(t-2\pi)\right) e^{-2t}\sin t \\
x(0) = 0,\ x'(0) = 2 &
\end{array} \tag{219}$$

Worked examples

To solve $x'' + x = \cos(3t)$ with $x(0) = 1$ and $x'(0) = 0$:

1. Take the Laplace transform of both sides (let $\mathcal{L}(x) = X(s)$):

$$\mathcal{L}\left(x'' + x\right) = \mathcal{L}\left(\cos(3t)\right). \tag{220}$$

2. Use the linearity of the Laplace transform to distribute the operator among the terms:

$$\mathcal{L}\left(x''\right) + \mathcal{L}(x) = \mathcal{L}\left(\cos(3t)\right). \tag{221}$$

3. Use the fact that $\mathcal{L}(x') = s\mathcal{L}(x) - x(0)$, as many times as necessary:

$$\begin{aligned} \left(s^2X - sx(0) - x'(0)\right) + X &= \mathcal{L}\left(\cos\left(3t\right)\right) \\ \left(s^2X - s - 0\right) + X &= \mathcal{L}\left(\cos\left(3t\right)\right) \end{aligned}. \tag{222}$$

4. Find the Laplace transform of the inhomogeneous term:

$$s^2X - s + X = \frac{s}{s^2+9} \tag{223}$$

5. Solve for $X(s)$:

$$\begin{aligned} \left(s^2+1\right)X - s &= \tfrac{s}{s^2+9} \\ \left(s^2+1\right)X &= \tfrac{s}{s^2+9} + s \\ X(s) &= \tfrac{1}{s^2+1}\left(\tfrac{s}{s^2+9} + s\right) \end{aligned} \tag{224}$$

6. Find the inverse Laplace transform of $X(s)$. In this case, we first distribute:

$$X(s) = \frac{s}{\left(s^2+1\right)\left(s^2+9\right)} + \frac{s}{s^2+1}. \tag{225}$$

We recognize $\frac{s}{s^2+1}$ as the Laplace transform of $\cos t$. As for the other term, we use partial fraction decomposition:

$$\frac{s}{\left(s^2+1\right)\left(s^2+9\right)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9} \tag{226}$$

This gives the solutions $A = \frac{1}{8}$, $B = 0$, $C = -\frac{1}{8}$ and $D = 0$. Thus,

$$X(s) = \frac{s}{8\left(s^2+1\right)} - \frac{s}{8\left(s^2+9\right)} + \frac{s}{s^2+1} = \frac{9}{8}\frac{s}{s^2+1} - \frac{1}{8}\frac{s}{s^2+9}. \tag{227}$$

We recognize $\frac{s}{s^2+9}$ as the Laplace transform of $\cos\left(3t\right)$. Thus,

$$\boxed{x(t) = \mathcal{L}^{-1}\left(X(s)\right) = \tfrac{9}{8}\cos t - \tfrac{1}{8}\cos\left(3t\right)}. \tag{228}$$

To solve $x^{(3)} + x'' - 6x' = 0$ with $x(0) = 0$, $x'(0) = 1$ and $x''(0) = 1$:

1. Take the Laplace transform of both sides (let $\mathcal{L}(x) = X(s)$):

$$\mathcal{L}\left(x^{(3)} + x'' - 6x'\right) = \mathcal{L}\left(0\right). \tag{229}$$

2. Use the linearity of the Laplace transform to distribute the operator among the terms:

$$\mathcal{L}\left(x^{(3)}\right) + \mathcal{L}\left(x''\right) - 6\mathcal{L}\left(x'\right) = \mathcal{L}\left(0\right). \tag{230}$$

3. Use the fact that $\mathcal{L}(x') = s\mathcal{L}(x) - x(0)$, as many times as necessary:

$$\begin{array}{c} \left(s^3X - s^2x(0) - sx'(0) - x''(0)\right) + \left(s^2X - sx(0) - x'(0)\right) - 6\left(sX - x(0)\right) = \mathcal{L}\left(0\right) \\ \left(s^3X - s - 1\right) + \left(s^2X - 1\right) - 6\left(sX\right) = \mathcal{L}\left(0\right) \end{array}. \tag{231}$$

4. Find the Laplace transform of the inhomogeneous term:

$$s^3X - s - 1 + s^2X - 1 - 6sX = 0 \tag{232}$$

5. Solve for $X(s)$:

$$\begin{array}{c} \left(s^3 + s^2 - 6s\right)X - s - 2 = 0 \\ \left(s^3 + s^2 - 6s\right)X = s + 2 \\ X(s) = \frac{s+2}{s^3+s^2-6s} \end{array} \tag{233}$$

6. Find the inverse Laplace transform of $X(s)$. In this case, we factor the denominator:

$$X(s) = \frac{s+2}{s\left(s+3\right)\left(s-2\right)} \tag{234}$$

Now we proceed by partial fraction decomposition:

$$\frac{s+2}{s\left(s+3\right)\left(s-2\right)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}. \tag{235}$$

This gives the solution $A = -\frac{1}{3}$, $B = -\frac{1}{15}$ and $C = \frac{2}{5}$. Thus,

$$X(s) = -\frac{1}{3}\frac{1}{s} - \frac{1}{15}\frac{1}{s+3} + \frac{2}{5}\frac{1}{s-2}. \tag{236}$$

We recognize $\frac{1}{s}$ as the Laplace transform of 1, $\frac{1}{s+3}$ as the Laplace transform of e^{-3t}, and $\frac{1}{s-2}$ as the Laplace transform of e^{2t}. Thus,

$$\boxed{x(t) = \mathcal{L}^{-1}\left(X(s)\right) = -\tfrac{1}{3} - \tfrac{1}{15}e^{-3t} + \tfrac{2}{5}e^{2t}}. \tag{237}$$

To solve $x'' + 2x' + x = \delta(t) - \delta\left(t-2\right)$ with $x(0) = 2$ and $x'(0) = 2$:

1. Take the Laplace transform of both sides (let $\mathcal{L}(x) = X(s)$):

$$\mathcal{L}\left(x'' + 2x' + x\right) = \mathcal{L}\left(\delta(t) - \delta(t-2)\right). \tag{238}$$

2. Use the linearity of the Laplace transform to distribute the operator among the terms:

$$\mathcal{L}\left(x''\right) + 2\mathcal{L}\left(x'\right) + \mathcal{L}\left(x\right) = \mathcal{L}\left(\delta\left(t\right)\right) - \mathcal{L}\left(\delta\left(t-2\right)\right). \tag{239}$$

3. Use the fact that $\mathcal{L}\left(x'\right) = s\mathcal{L}(x) - x(0)$, as many times as necessary:

$$\begin{aligned} \left(s^2X - sx(0) - x'(0)\right) + 2\left(sX - x(0)\right) + X &= \mathcal{L}\left(\delta(t)\right) - \mathcal{L}\left(\delta(t-2)\right) \\ \left(s^2X - 2s - 2\right) + 2\left(sX - 2\right) + X &= \mathcal{L}\left(\delta(t)\right) - \mathcal{L}\left(\delta(t-2)\right) \end{aligned}. \tag{240}$$

4. Find the Laplace transform of the inhomogeneous term:

$$s^2X - 2s - 2 + 2sX - 4 + X = 1 - e^{-2s}. \tag{241}$$

5. Solve for $X(s)$:

$$\begin{aligned} \left(s^2 + 2s + 1\right)X - 2s - 6 &= 1 - e^{-2s} \\ \left(s^2 + 2s + 1\right)X &= 7 + 2s - e^{-2s} \\ X(s) &= \tfrac{7+2s-e^{-2s}}{s^2+2s+1} \end{aligned}. \tag{242}$$

6. Find the inverse Laplace transform of $X(s)$. In this case, we first split the

fraction into three parts:

$$X(s) = \frac{7}{(s+1)^2} + \frac{2s}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}. \tag{243}$$

We now use partial fraction decomposition to write $\frac{2s}{(s+1)^2} = \frac{2}{s+1} - \frac{2}{(s+1)^2}$:

$$\begin{aligned} X(s) = \frac{7}{(s+1)^2} + \frac{2}{s+1} - \frac{2}{(s+1)^2} - e^{-2s}\frac{1}{(s+1)^2} \\ = \frac{5}{(s+1)^2} + \frac{2}{s+1} - e^{-2s}\frac{1}{(s+1)^2}. \end{aligned} \tag{244}$$

We recognize $\frac{1}{(s+1)^2}$ as $\frac{1}{s^2}$ shifted by -1. We recognize $\frac{1}{s+1}$ as the Laplace transform of e^{-t}. We recognize $e^{-2s}\frac{1}{(s+1)^2}$ as e^{-2s} times $\frac{1}{s}$ shifted by -1. Therefore,

$$\boxed{x(t) = \mathcal{L}^{-1}\left(X(s)\right) = 5e^{-t}t + 2e^{-t} - u(t-2)e^{-(t-2)}\left(t-2\right)}. \tag{245}$$

5 Systems of Ordinary Differential Equations

In this section, we'll deal with the situation of having *more than one* dependent variable with one independent variable.

5.1 Eigenvalue methods for homogeneous linear systems

In this section, we'll study a particular class of systems of ODEs: the "homogeneous linear systems."

Definition 5.1 *A* <u>*homogeneous linear system of ODEs*</u> *is a system of n ODEs in some independent variables x_1, x_2, ..., x_n and dependent variable t such that each ODE in the system can be expressed in the form $x_i' = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n$ for some functions $a_{i1}, a_{i2}, ..., a_{in}$ of t.*

These are systems of equations that look like:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{aligned} \tag{246}$$

We will only deal with situations in which each a_{ij} is a constant.

Here are some examples of homogeneous systems of ODEs, and their general solutions:

$$\begin{cases} x' = 2x + 2y \\ y' = 9x - y \end{cases} \quad \text{has the general solution:} \quad \begin{array}{c} x(t) = 2c_1e^{5t} + c_2e^{-4t} \\ y(t) = 3c_1e^{5t} - 3c_2e^{-4t} \end{array}$$

$$\begin{cases} x' = -5x + y \\ y' = 4x - 2y \end{cases} \quad \text{has the general solution:} \quad \begin{array}{c} x(t) = c_1e^{-6t} + c_2e^{-t} \\ y(t) = -c_1e^{-6t} + 4c_2e^{-t} \end{array}$$

$$\begin{cases} x_1' = x_1 - x_2 \\ x_2' = -x_1 - x_3 \\ x_3' = -x_2 + x_3 \end{cases} \quad \text{has the general solution:} \quad \begin{array}{c} x_1(t) = c_1e^{-t} + c_2e^{t} + c_3e^{2t} \\ x_2(t) = 2c_1e^{-t} - c_3e^{2t} \\ x_3(t) = c_1e^{-t} + c_2e^{t} + c_3e^{2t} \end{array}$$

$$\begin{cases} x' = -5x - 6y + 3z \\ y' = 3x + 4y - 3z \\ z' = -2z \end{cases} \quad \text{has the general solution:} \quad \begin{array}{c} x(t) = c_1e^{-2t} + c_3e^{t} \\ y(t) = c_2e^{-2t} - c_3e^{t} \\ z(t) = (c_1 + 2c_2)\, e^{-2t} \end{array} \tag{247}$$

Given a homogeneous system of ODEs, we can save some ink by writing it in matrix form:

$$\begin{array}{c} x_1' = a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{array} \to \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{248}$$

We will often abbreviate this even further by defining the following:

$$\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \overrightarrow{x}' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} \tag{249}$$

Continuing the examples:

$$
\begin{aligned}
\overrightarrow{x}' &= \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_1 \\ 0 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-3t} \\
\overrightarrow{x}' &= \begin{pmatrix} -50 & 20 \\ 100 & -60 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} e^{-10t} + \begin{pmatrix} 2c_2 \\ -5c_2 \end{pmatrix} e^{-100t} \\
\overrightarrow{x}' &= \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{-12t} \\
\overrightarrow{x}' &= \begin{pmatrix} 3 & 6 \\ -6 & 15 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_2 - 6c_1 \\ -6c_1 \end{pmatrix} e^{9t} + \begin{pmatrix} -6c_2 \\ -6c_2 \end{pmatrix} te^{9t} \\
\overrightarrow{x}' &= \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_2 - 4c_1 \\ 4c_1 \end{pmatrix} e^{5t} + \begin{pmatrix} -4c_2 \\ 4c_2 \end{pmatrix} te^{9t} \\
\overrightarrow{x}' &= \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} c_1 - c_2 \\ -3c_1 \end{pmatrix} \cos(3t) + \begin{pmatrix} c_1 + c_2 \\ -3c_2 \end{pmatrix} \sin(3t) \\
\overrightarrow{x} &= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \overrightarrow{x} & \text{solution: } \overrightarrow{x}(t) &= \begin{pmatrix} -2c_1 + c_2 \\ 2c_2 \\ c_3 \end{pmatrix} e^{t} + \begin{pmatrix} -2c_2 + c_3 \\ 2c_3 \\ 0 \end{pmatrix} te^{t} + \begin{pmatrix} -c_3 \\ 0 \\ 0 \end{pmatrix} t^2 e^{t}
\end{aligned}
\tag{250}
$$

In the same way that finding the characteristic equation was the key to finding solutions to linear, homogeneous ODEs with constant coefficients, so will we also need to find the characteristic equation of the coefficient matrix in order to solve linear homogeneous systems of ODEs with constant coefficient matrices. However, just as before, the steps following that will depend on what kinds of roots the characteristic equation has. We'll consider three cases: real and distinct roots, repeated real roots, and complex roots.

5.1.1 Real and distinct eigenvalues

If the roots of the characteristic equation are real and distinct, then the process of finding the general solution is quite simple: find the eigenvectors corresponding to each eigenvalue λ_i, select one eigenvector $\overrightarrow{v_i}$ for each, and then linearly independent solutions of the system will be in the form

$$\begin{aligned}\overrightarrow{x}_1 &= \overrightarrow{v_1} e^{\lambda_1 t} \\ \overrightarrow{x}_2 &= \overrightarrow{v_2} e^{\lambda_2 t} \\ &\vdots \\ \overrightarrow{x}_n &= \overrightarrow{v_n} e^{\lambda_n t}\end{aligned} . \tag{251}$$

We demonstrate the procedure below in the case of $\overrightarrow{x}' = A\overrightarrow{x}$, where

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{pmatrix} : \tag{252}$$

1. Find the characteristic equation of the matrix by setting $\det(A - \lambda I_n) = 0$ and solving for λ:

$$0 = \det(A - \lambda I_3) = \begin{vmatrix} 5-\lambda & 1 & 3 \\ 1 & 7-\lambda & 1 \\ 3 & 1 & 5-\lambda \end{vmatrix} = -\lambda^3 + 17\lambda^2 - 84\lambda + 108. \tag{253}$$

2. Find the roots of the characteristic equation (that is, the eigenvalues of the matrix). In this case, we solve the cubic equation by guessing (yes, literally guessing) the solution $\lambda = 2$. From here, we can use polynomial division to find that the equation becomes

$$\begin{aligned} 0 = -\lambda^3 + 17\lambda^2 - 84\lambda + 108 &= -(\lambda - 2)\left(\lambda^2 - 15\lambda + 54\right) \\ &= -(\lambda - 2)(\lambda - 6)(\lambda - 9). \end{aligned} \tag{254}$$

Therefore, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 6$, and $\lambda_3 = 9$, all with multiplicity 1.

3. For each eigenvalue λ_i, find the eigenvectors associated to that eigenvalue by solving the equation $(A - \lambda_i I_n)\overrightarrow{v} = \overrightarrow{0}$ for $\overrightarrow{v} \neq \overrightarrow{0}$. (Note that this is the same as solving $A\overrightarrow{v} = \lambda_i \overrightarrow{v}$ for $\overrightarrow{v} \neq \overrightarrow{0}$.) In problems with matrices larger than 2×2 matrices, this usually involves Gauss-Jordan elimination; for more information

about Gauss-Jordan elimination, see the document "Rudimentary Matrix Algebra." In this case, for $\lambda_1 = 2$:

$$
\begin{gathered}
(A - 2I_3)\overrightarrow{v}_1 = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\left(\begin{array}{ccc:c} 3 & 1 & 3 & 0 \\ 1 & 5 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc:c} 3 & 1 & 3 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 3R_2} \left(\begin{array}{ccc:c} 0 & -14 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\xrightarrow{R_1 \to -\frac{1}{14}R_1} \left(\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 5R_1} \left(\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
a_1 + c_1 = 0, \ b_1 = 0
\end{gathered} \tag{255}
$$

This tells us that the eigenvectors of A corresponding to $\lambda_1 = 2$ are all vectors of the form

$$
\overrightarrow{v}_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ -r \end{pmatrix}, \tag{256}
$$

where r is any nonzero real number. For $\lambda_2 = 6$:

$$(A - 6I_3)\overrightarrow{v}_2 = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} -1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 0 & 2 & 4 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$a_2 - c_2 = 0$$

$$b_2 + 2c_2 = 0 \tag{257}$$

Thus the eigenvectors corresponding to $\lambda_2 = 6$ are all vectors of the form

$$\overrightarrow{v}_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} s \\ -2s \\ s \end{pmatrix}, \tag{258}$$

where s is any nonzero real number. Finally, for $\lambda_3 = 9$,

$$(A - 9I_3)\overrightarrow{v}_3 = \begin{pmatrix} -4 & 1 & 3 \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc:c} -4 & 1 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & 1 & -4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 0 & -7 & 7 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$a_3 - c_3 = 0,\ b_3 - c_3 = 0 \tag{259}$$

Thus, the eigenvectors corresponding to $\lambda_3 = 9$ are all vectors of the form

$$\overrightarrow{v}_3 = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \tag{260}$$

where t is any nonzero real number.

4. For each eigenvalue, select a single fixed eigenvector. The choice is completely arbitrary. For this case, we'll select $r = 1$, $s = 1$ and $t = 1$, so that

$$\overrightarrow{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \overrightarrow{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \overrightarrow{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{261}$$

5. The chosen eigenvalue-eigenvector pairs form n linearly independent solutions:

$$\begin{aligned} \overrightarrow{x}_1(t) = \overrightarrow{v}_1 e^{\lambda_1 t} &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} \\ \overrightarrow{x}_2(t) = \overrightarrow{v}_2 e^{\lambda_2 t} &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t}, \\ \overrightarrow{x}_3(t) = \overrightarrow{v}_3 e^{\lambda_3 t} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t} \end{aligned} \tag{262}$$

allowing us to construct the general solution as their linear combinations:

$$\boxed{\overrightarrow{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{9t}} \tag{263}$$

5.1.2 Repeated eigenvalues

If the characteristic equation has repeated roots, then the situation can be a bit more nuanced. First of all, if all eigenvalues are complete (that is, none are defective), then the situation proceeds as before. For more information about complete and defective eigenvalues, see the document "Rudimentary Matrix Algebra."

We demonstrate the procedure for repeated complete eigenvalues below in the case of $\overrightarrow{x}' = A\overrightarrow{x}$, where

$$A = \begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix} : \tag{264}$$

1. Find the characteristic equation of the matrix by setting $\det(A - \lambda I_n) = 0$ and solving for λ:

$$0 = \det(A - \lambda I_3) = \begin{vmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-5) \tag{265}$$

2. Find the roots of the characteristic equation. In this case, the eigenvalues are $\lambda_1 = 3$, with multiplicity 2, and $\lambda_2 = 5$, with multiplicity 1.

3. For each eigenvalue λ_i, find the eigenvectors associated to that eigenvalue by solving the equation $(A - \lambda_i I_n)\overrightarrow{v} = \overrightarrow{0}$ for $\overrightarrow{v} \neq \overrightarrow{0}$. In this case, for $\lambda_1 = 3$:

$$\begin{gathered} (A - 3I_3)\overrightarrow{v} = \begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad . \\ a + \tfrac{2}{3}b = 0 \end{gathered} \tag{266}$$

Thus, the eigenvectors corresponding to $\lambda_1 = 3$ are all vectors of the form

$$\overrightarrow{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}b \\ b \\ c \end{pmatrix}, \tag{267}$$

where b and c are not both zero. For $\lambda_2 = 5$:

$$\begin{gathered} (A - 5I_3)\,\overrightarrow{u} = \begin{pmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ r - t = 0,\ s + t = 0 \end{gathered} \quad . \tag{268}$$

Thus, the eigenvectors corresponding to $\lambda_2 = 5$ are all vectors of the form

$$\overrightarrow{u} = \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} r \\ -r \\ r \end{pmatrix}, \tag{269}$$

where r is any nonzero real number.

4. For each eigenvalue λ_i, find the defect of λ_i. In this case, the eigenvectors corresponding to $\lambda_1 = 3$, which has multiplicity 2, have two free variables, and so the eigenspace of $\lambda_1 = 3$ has dimension 2. Therefore, the defect of $\lambda_1 = 3$ is $2 - 2 = 0$. On the other hand, the eigenvectors corresponding to $\lambda_2 = 5$, which has multiplicity 1, have one free variable, and so the eigenspace of $\lambda_2 = 5$ has dimension 1. Therefore, the defect of $\lambda_2 = 5$ is $1 - 1 = 0$.

5. If the defect of each eigenvalue is zero, then the general solution can be constructed only from the eigenvectors. In that case, for each eigenvalue, select as many linearly independent eigenvectors as the dimension of its eigenspace. For this

case, we'll select $b = 3$ and $c = 0$ for one eigenvector of $\lambda_1 = 3$, $b = 0$ and $c = 1$ for the other eigenvector of $\lambda_1 = 3$, and $r = 1$ for the eigenvector of $\lambda_2 = 5$:

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \tag{270}$$

6. If no eigenvalues are defective, then construct the linearly independent solutions in the form $\vec{v}e^{\lambda t}$:

$$\begin{aligned} \vec{x}_1(t) = \vec{v}_1 e^{\lambda_1 t} &= \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} e^{3t} \\ \vec{x}_2(t) = \vec{v}_2 e^{\lambda_1 t} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t} \;. \\ \vec{x}_3(t) = \vec{u} e^{\lambda_2 t} &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t} \end{aligned} \tag{271}$$

7. Construct the general solution as linear combinations of these linearly independent solutions:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}} \tag{272}$$

On the other hand, if there exist defective eigenvalues, then we need a concept known as a "generalized eigenvector."

Definition 5.2 *Let A be an $n \times n$ matrix. Given an eigenvalue λ of A, a <u>generalized eigenvector of A with rank k corresponding to λ</u> is a vector $\vec{v}_k$ such that*

$(A - \lambda I_n)^k \overrightarrow{v}_k = \overrightarrow{0}.$

In this case, we cannot construct the general solution using just eigenvectors alone, so we need to use these objects. In general, solutions gotten from generalized eigenvectors will look different from those gotten from eigenvectors; for a rank k generalized eigenvector, the solution it produces looks like

$$\overrightarrow{x}_k(t) = \left(\frac{1}{(k-1)!} \overrightarrow{v}_1 t^{k-1} + ... + \frac{1}{2!} \overrightarrow{v}_{k-2} t^2 + \frac{1}{1!} \overrightarrow{v}_{k-1} t + \frac{1}{0!} \overrightarrow{v}_k \right) e^{\lambda t}. \tag{273}$$

We demonstrate the strategy for defective eigenvalues below, in the specific case of $\overrightarrow{x}' = A\overrightarrow{x}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix} : \tag{274}$$

1. Find the characteristic equation of A:

$$0 = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & -2-\lambda & -3 \\ 2 & 3 & 4-\lambda \end{vmatrix} = -(\lambda - 1)^3. \tag{275}$$

2. Find the roots of the characteristic equation, the eigenvalues of A. In this case, the only eigenvalue is $\lambda = 1$, with multiplicity 3.

3. Find the eigenvectors corresponding to each eigenvalue λ_i by solving the equation $(A - \lambda_i I_n)\overrightarrow{v} = \overrightarrow{0}$ for $\overrightarrow{v} \neq \overrightarrow{0}$. In our case, for $\lambda = 1$:

$$\begin{gathered} (A - 1I_3)\overrightarrow{v} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad , \\ a + \tfrac{3}{2}b + \tfrac{3}{2}c = 0 \end{gathered} \tag{276}$$

and so the eigenvectors corresponding to $\lambda = 1$ are all vectors of the form

$$\overrightarrow{v} = \begin{pmatrix} -\frac{3}{2}(b+c) \\ b \\ c \end{pmatrix}, \tag{277}$$

where b and c are not both zero.

4. For each eigenvalue λ_i, find the defect of λ_i. In this case, $\lambda = 1$ has an eigenspace of dimension 2, but multiplicity 3, so the defect of $\lambda = 1$ is $3 - 2 = 1$.

5. For each eigenvalue λ_i that has a nonzero defect d, find the rank $d + 1$ generalized eigenvectors corresponding to λ_i. In our case, we need to find the rank 2 generalized eigenvectors corresponding to $\lambda = 1$. This requires solving the equation

$$\begin{gathered} (A - 1I_3)^2 \overrightarrow{v}_2 = \overrightarrow{0} \\ \begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix}^2 \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \overrightarrow{0} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{gathered}. \tag{278}$$

This particular equation is satisfied for any choice of a_2, b_2 and c_2.

6. Select a single generalized eigenvector corresponding to each defective eigenvalue. In this case, we select (arbitrarily)

$$\overrightarrow{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{279}$$

7. For each defective eigenvalue λ_i, create a chain of generalized eigenvectors down to an eigenvector $\overrightarrow{v}_1$ by defining $\overrightarrow{v}_{k-1} = (A - \lambda_i I_n)\overrightarrow{v}_k$ for each k. In this

case, we define

$$\overrightarrow{v}_1 = (A - 1I_3)\,\overrightarrow{v}_2 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & -3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}. \tag{280}$$

8. For each eigenvalue, select eigenvectors that are linearly independent from the $\overrightarrow{v}_1$ gotten by these chains. In our case, all eigenvectors corresponding to $\lambda = 1$ are of the form

$$\overrightarrow{v} = \begin{pmatrix} -\frac{3}{2}(b+c) \\ b \\ c \end{pmatrix}, \tag{281}$$

so we select $b = 2$ and $c = 0$ to find the eigenvector

$$\overrightarrow{u} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \tag{282}$$

which is linearly independent from the generalized eigenvector $\overrightarrow{v}_1$.

9. Write the linearly independent solutions corresponding to each eigenvector and generalized eigenvector. In this case, these are

$$\begin{aligned} \overrightarrow{x}_1(t) = \overrightarrow{u}e^{\lambda t} &= \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} e^t \\ \overrightarrow{x}_2(t) = \overrightarrow{v}_1 e^{\lambda t} &= \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} e^t \\ \overrightarrow{x}_3(t) = (\overrightarrow{v}_1 t + \overrightarrow{v}_2)\, e^{\lambda t} &= \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix} e^t \end{aligned} \tag{283}$$

10. Write the general solution as the linear combinations of these solutions:

$$\boxed{\overrightarrow{x}(t) = c_1 \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} e^t + c_3 \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix} e^t} \tag{284}$$

5.1.3 Complex eigenvalues

If the characteristic equation has complex roots, then the situation is quite similar, except that one must use Euler's formula to find real-valued solutions. We demonstrate the procedure below in the case of $\overrightarrow{x}' = A\overrightarrow{x}$, where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix} : \tag{285}$$

1. Find the characteristic equation of A:

$$0 = \det\left(A - \lambda I_3\right) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -4 & -3-\lambda & -1 \\ 4 & 4 & 2-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 - 4\lambda + 4. \tag{286}$$

2. Find the roots of the characteristic equation, the eigenvalues of A:

$$0 = -\lambda^3 + \lambda^2 - 4\lambda + 4 = -\left(\lambda - 1\right)\left(\lambda^2 + 4\right). \tag{287}$$

In this case we have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \pm 2i$, each with multiplicity 1.

3. For each eigenvalue λ_i, find the eigenvectors associated to λ_i. In this case,

for $\lambda_1 = 1$:

$$
\begin{aligned}
(A - 1I_3)\,\overrightarrow{v}_1 = \begin{pmatrix} 1 & 1 & -1 \\ -4 & -4 & -1 \\ 4 & 4 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \quad , \\
a_1 + b_1 = 0,\ c_1 = 0 &
\end{aligned}
\tag{288}
$$

and so the eigenvectors corresponding to $\lambda_1 = 1$ are all vectors of the form

$$
\overrightarrow{v}_1 = \begin{pmatrix} r \\ -r \\ 0 \end{pmatrix}, \tag{289}
$$

where r is any nonzero real number. For $\lambda_2 = 2i$:

$$\begin{aligned}
(A - 2iI_3)\,\overrightarrow{v}_2 &= \begin{pmatrix} 2-2i & 1 & -1 \\ -4 & -3-2i & -1 \\ 4 & 4 & 2-2i \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\xrightarrow{R_1 \to (2+2i)R_1} \left(\begin{array}{ccc|c} 8 & 2+2i & -2-2i & 0 \\ -4 & -3-2i & -1 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_3} \left(\begin{array}{ccc|c} 0 & -6+2i & -6+2i & 0 \\ -4 & -3-2i & -1 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right) \\
&\xrightarrow{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|c} 0 & -6+2i & -6+2i & 0 \\ 0 & 1-2i & 1-2i & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{1-2i}R_2} \left(\begin{array}{ccc|c} 0 & -6+2i & -6+2i & 0 \\ 0 & 1 & 1 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right) \\
&\xrightarrow{R_1 \to R_1 - (-6+2i)R_2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 4 & 4 & 2-2i & 0 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{4}R_3} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & \frac{1}{2}-\frac{1}{2}i & 0 \end{array}\right), \\
&\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -\frac{1}{2}-\frac{1}{2}i & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \\
&\begin{pmatrix} 1 & 0 & -\frac{1}{2}(1+i) \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&a_2 - \tfrac{1}{2}(1+i)\,c_2 = 0,\; b_2 + c_2 = 0
\end{aligned} \tag{290}$$

so the eigenvectors corresponding to $\lambda_2 = \pm 2i$ are of the form

$$\overrightarrow{v}_2 = \begin{pmatrix} \frac{1}{2}(1+i)\,s \\ -s \\ s \end{pmatrix}, \tag{291}$$

where s is any nonzero real number.

4. Select a single eigenvector for each eigenspace. The choice is completely

arbitrary. In this case, we choose $r = 1$ and $s = 2$, so that

$$\overrightarrow{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \overrightarrow{v}_2 = \begin{pmatrix} 1+i \\ -2 \\ 2 \end{pmatrix} \tag{292}$$

5. Multiply the complex eigenvector by $e^{\lambda_i t}$, using Euler's formula:

$$\begin{aligned}
\overrightarrow{v}_2 e^{\lambda_2 t} = \begin{pmatrix} 1+i \\ -2 \\ 2 \end{pmatrix} e^{(2i)t} &= \begin{pmatrix} 1+i \\ -2 \\ 2 \end{pmatrix} (\cos(2t) + i\sin(2t)) \\
&= \begin{pmatrix} \cos(2t) + i\sin(2t) + i\cos(2t) + i^2\sin(2t) \\ -2\cos(2t) - 2i\sin(2t) \\ 2\cos(2t) + 2i\sin(2t) \end{pmatrix} \\
&= \begin{pmatrix} \cos(2t) - \sin(2t) + i(\cos(2t) + \sin(2t)) \\ -2\cos(2t) + i(-2\sin(2t)) \\ 2\cos(2t) + i(2\sin(2t)) \end{pmatrix} \\
&= \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix} + i \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}.
\end{aligned} \tag{293}$$

6. Form the linearly independent solutions corresponding to the complex eigenvalue(s) by taking the real and imaginary parts of the product:

$$\begin{aligned}
\overrightarrow{x}_2(t) &= \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix} \\
\overrightarrow{x}_3(t) &= \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}
\end{aligned}. \tag{294}$$

7. Form the other linearly independent solutions corresponding to the other

eigenvalue(s):

$$\overrightarrow{x}_1(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^t$$

$$\overrightarrow{x}_2(t) = \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix}. \tag{295}$$

$$\overrightarrow{x}_3(t) = \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}$$

8. Form the general solution as the linear combinations of the linearly independent solutions:

$$\boxed{\overrightarrow{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos(2t) - \sin(2t) \\ -2\cos(2t) \\ 2\cos(2t) \end{pmatrix} + c_3 \begin{pmatrix} \cos(2t) + \sin(2t) \\ -2\sin(2t) \\ 2\sin(2t) \end{pmatrix}} \tag{296}$$

5.1.4 Additional worked examples

$$\overrightarrow{x}' = \underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}_{A} \overrightarrow{x} \tag{297}$$

1. First, we seek the characteristic polynomial:

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 \tag{298}$$

2. Next, we find the roots of the characteristic polynomial, the eigenvalues of the matrix:

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = 1 \pm 2i. \tag{299}$$

3. We seek the eigenvalues corresponding to $\lambda = 1 \pm 2i$:

$$\begin{gathered}
(A - (1+2i)\, I_2) \overrightarrow{v} = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\left(\begin{array}{cc:c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{2} R_2} \left(\begin{array}{cc:c} -2i & -2 & 0 \\ 1 & -i & 0 \end{array}\right) \\
\xrightarrow{R_1 \to R_1 + 2i R_2} \left(\begin{array}{cc:c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc:c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right) \quad . \\
\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
a - ib = 0
\end{gathered} \tag{300}$$

Therefore, the eigenvectors corresponding to $\lambda = 1 \pm 2i$ are of the form

$$\overrightarrow{v} = \begin{pmatrix} ir \\ r \end{pmatrix}, \tag{301}$$

where r is any nonzero real number.

4. We select the eigenvector corresponding to $r = 1$:

$$\overrightarrow{v} = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{302}$$

5. We multiply the complex eigenvector $\overrightarrow{v}$ and $e^{\lambda t}$:

$$\overrightarrow{v} e^{\lambda t} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1+2i)t} \tag{303}$$