Contents

1 Exam 01

1.1 Summary

Exam 01 is based on Chapters 01, 02 and 06.

Chapter 01 focuses on first-order ODEs. The main two techniques for finding the general solutions of first-order ODEs are separation of variables and the integrating factor. Specifically, separation of variables is useful for solving "separable" ODEs, which are ODEs that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)\,, \tag{1}$$

where $f(x)$ is a function of x and $g(y)$ is a function of y. The integrating factor is helpful for solving first-order linear ODEs, which are ODEs that take the form

$$p_1(x)\,y' + p_0(x)\,y = q(x)\,. \tag{2}$$

The method of the integrating factor requires putting the first-order linear ODE in "standard form," or in other words, writing it in the form

$$y' + p(x)\,y = r(x)\,. \tag{3}$$

At this point, the integrating factor can be defined as

$$\mu(x) = e^{\int p(x)\ \mathrm{d}x}. \tag{4}$$

From here, one can multiply the integrating factor by both sides of the standard form equation and get

$$\mu(x)\,y' + \mu(x)\,p(x)\,y = \mu(x)\,r(x)\,. \tag{5}$$

By design, the left hand side of this equation is $\frac{\mathrm{d}}{\mathrm{d}x}(\mu(x)\,y)$, and so y can be found by integrating both sides.

These two main techniques also open up the possibility of solving a broader class of ODEs than just separable and linear first order ODEs. To do this, we discussed substitution methods. Substitution methods are used to re-write a first order ODE in such a form that it can be solved by either separation of variables or the integrating factor. There is an extremely wide variety of types of substitution problems.

Two families of substitution problems that were of particular interest to us were the "Bernoulli substitutions" and "homogeneous substitutions." The Bernoulli substitution is only useful in solving "Bernoulli equations," which are ODEs that can be written in the form

$$y' + p(x)\, y = q(x)\, y^n, \tag{6}$$

where $n \neq 1$. In these situations, we make the substitution $v = y^{1-n}$. When this substitution is implemented, the equation can be re-written as a first-order linear ODE whose dependent variable is v:

$$\begin{array}{c} v = y^{1-n} \Rightarrow y = v^{\frac{1}{1-n}} \\ y' = \frac{1}{1-n} v^{\frac{n}{1-n}} v' \end{array} \tag{7}$$

$$\begin{array}{c} \frac{1}{1-n} v^{\frac{n}{1-n}} v' + p(x)\, v^{\frac{1}{1-n}} = q(x)\, v^{\frac{n}{1-n}} \\ v' + (1-n)\, p(x)\, v = (1-n)\, q(x) \end{array}. \tag{8}$$

From here, the ODE can be solved as a linear first-order equation.

Homogeneous substitutions are useful for ODEs that can be written as

$$y' = f\left(\frac{y}{x}\right), \tag{9}$$

where f is a differentiable function. In these cases, we make the substitution $v = \frac{y}{x}$:

$$\begin{array}{c} v = \frac{y}{x} \Rightarrow y = vx \\ y' = v'x + v \end{array} \tag{10}$$

$$v'x + v = f(v). \tag{11}$$

From here, the equation is either separable or linear, and so it can be solved by either separation of variables or the integrating factor.

Finally, we also discussed methods for graphing all of the solution curves of a first order autonomous ODE $y' = f(y)$. This required finding the equilibrium solutions (which are those constant solutions for which $y' = 0$) and then choosing values for the initial condition that lie on either side of the equilibrium solutions. Determining the value of y' at these initial conditions allowed us to construct the phase diagram, which served as the y-axis of our graphs.

Chapters 02 and 06 were a study of linear ODEs with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f(t), \tag{12}$$

where $a_0, a_1, ..., a_{n-1}$, and a_n are constants. Linear ODEs of first order (with either constant or non-constant coefficients) could be solved using the integrating factor, but for this portion of the class, we were mainly interested in ODEs with orders higher than one.

Chapter 02 was about solving linear ODEs with constant coefficients by using the fact that the general solution to any linear ODE with constant coefficients can be written as

$$y = y_C + y_p, \tag{13}$$

where y_C is the complementary solution and y_p is any particular solution whatsoever.

The complementary solution is the general solution of the associated homogeneous equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0. \tag{14}$$

As the associated homogeneous equation is, by design, homogeneous, we spent the first few sections of Chapter 2 on solving linear homogeneous ODEs with constant coefficients. This involved constructing the characteristic equation of the homogeneous ODE by asserting the solution $y = e^{rt}$:

$$\begin{aligned} a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + ... + a_1 r e^{rt} + a_0 e^{rt} = 0 \\ e^{rt} \left(a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0\right) = 0 \\ a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0 \end{aligned} . \tag{15}$$

Solving this polynomial equation gave values for the roots r. From these values, we constructed n linearly independent basis solutions $y_1, y_2, ..., y_n$, although the exact method for doing so depended on whether the characteristic polynomial's roots were distinct and real, repeated, or non-real. The general solution of such a homogeneous ODE was then the set of all linear combinations of the basis solutions:

$$y = c_1 y_1 + c_2 y_2 + ... + c_n y_n. \tag{16}$$

At this point, we were able to construct the complementary solution y_C of any linear ODE with constant coefficients. The only remaining task was to construct a particular solution, y_p. The method we used for this was called the "method of undetermined coefficients." It involved making a "guess" for the structure of y_p based on the inhomogeneous term, $f(t)$. The exact form of the guess depended on the form of the function $f(t)$. We also studied the case of "duplication," in which the guess fails because it shares nonzero functions with the complementary solution y_C. In these cases, we modified the guess by multiplying by factors of the independent variable. Once the guess was finalized, it contained several unknown constants, which we needed to determine. This was accomplished by substituting the guess back into the original ODE. Typically, the task of finding the constants also hinged on the linear independence of several functions.

Chapter 06 was about solving linear ODEs with constant coefficients by the Laplace transform method. These methods came with the advantage that the inhomogeneous term $f(t)$ did not have to be continuous (or even a function) in order to work. The technique involved taking the Laplace transform of both sides of the equation:

$$\mathcal{L}\left(a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y\right) = \mathcal{L}\left(f(t)\right). \tag{17}$$

As the Laplace transform is a linear operator, this can be re-written as

$$a_n \mathcal{L}\left(y^{(n)}\right) + a_{n-1} \mathcal{L}\left(y^{(n-1)}\right) + ... + a_1 \mathcal{L}\left(y'\right) + a_0 \mathcal{L}\left(y\right) = F(s). \tag{18}$$

From here, we could find the Laplace transform of any derivative of y in terms of the Laplace transform of y. As a result, the left side of the equation would become an expression involving only constants and $\mathcal{L}(y) = Y(s)$. We then solved for Y, reducing the problem to finding the inverse Laplace transform of Y. This typically required referring to a table of Laplace transforms, which will be provided during the exam.

As previously mentioned, the Laplace transform method does not require the inhomogeneous term $f(t)$ to be a continuous function. Two expressions that were of particular interest to us as the inhomogeneous term are the unit step function:

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 \leq t \end{cases}, \tag{19}$$

and the Dirac delta, $\delta(t)$. We did not dwell on the theoretical underpinnings of how the Dirac delta could possibly exist or even what it actually is, but we simply defined the expression as having the property that for any continuous function $f(t)$ and any real-valued constant a,

$$\int_0^\infty f(t)\,\delta(t-a)\ \mathrm{d}t = f(a). \tag{20}$$

In particular, this meant that $\mathcal{L}(\delta(t-a)) = e^{-as}$.

1.2 Practice problems

1.

$$
\begin{gathered}
y' = 2x \\
y' = x^2 y \\
y' = xy^4 \\
\left(x^2+1\right) y' = 1 \\
y' \tan(x) = 2(y-1) \\
y' = \tfrac{2xy}{y+1} \\
(1+x)^2 y' = (1+y)^2 \\
y' + y = 0
\end{gathered} \tag{21}
$$

2.

$$
\begin{gathered}
y' - \tan(x) y = 2\sin(x) \\
xy' + 2\ln(x) y = x^{2-\ln(x)} \\
y' + \tfrac{1}{x} y = e^x
\end{gathered} \tag{22}
$$

3.

$$
\begin{gathered}
y' = \sqrt{x+y+1} \\
(\ln y)^2 + \left(\tfrac{y'}{y}\right) = 1 \\
y^2 y' + 2xy^3 = 6x \\
y' = y + y^3 \\
\left(2x\sin(y)\cos(y)\right) y' = 4x^2 + \sin^2 y \\
\left(x^2+1\right)\sec(y) y' + \sin(y) = 2\cos(y) \\
y' \tan(y) + 2xe^{\left(x^2\right)} = xe^{\left(x^2\right)} \csc(y) \\
2yy' \ln\left(y^2-4\right) = \left(y^2-4\right)\sqrt{e^{2x}+e^{-2x}+2}
\end{gathered} \tag{23}
$$

4.

$$
\begin{aligned}
xy^2y' &= x^3 + y^3 \\
xy' &= y + 2\sqrt{xy} \\
(x+y)y' &= x - y \\
(x-y)y' &= x + y \\
\left(x^2 - y^2\right) y' &= 2xy \\
xy' &= y + \sqrt{x^2 + y^2} \\
yy' + x &= \sqrt{x^2 + y^2}
\end{aligned} \tag{24}
$$

5.

$$
\begin{aligned}
3y^2y' + y^3 &= e^{-x} \\
xy' + 6y &= 3xy^{\frac{4}{3}} \\
y^2\left(xy' + y\right)\sqrt{1 + x^4} &= x \\
x^2y' + 2xy &= 5y^4
\end{aligned} \tag{25}
$$

6.

6.1. Plot the solution curves of the autonomous ODE $y' = 3y^2 - 15y$.

6.2. Plot the solution curves of the autonomous ODE $y' = 3 - y$.

7.

$$
\begin{gathered}
4y'' + 4y' + y = 0 \\
y'' + 11y' = 0 \\
y'' - y = 0 \\
y'' + y = 0 \\
y'' + y' + y = 0 \\
y'' + y' - y = 0 \\
y'' - y' + y = 0 \\
y'' - y' - y = 0 \\
y'' - y' - 2y = 0 \\
y'' + y' - 2y = 0 \\
y'' - y' + 2y = 0 \\
y'' + y' + 2y = 0 \\
y'' + 2y' + y = 0 \\
y'' + 5y' + 6y = 0 \\
y'' + y' - 6y = 0 \\
y'' - 5y' + 6y = 0 \\
y'' - y' - 6y = 0
\end{gathered} \tag{26}
$$

8.

$$
\begin{gathered}
y^{(4)} - 5y'' + 6y = 0 \\
y^{(3)} - y' = 0 \\
y^{(3)} + y' = 0 \\
y^{(3)} - 3y'' + 2y' = 0 \\
y^{(5)} - 15y^{(4)} + 85y^{(3)} - 225y'' + 274y' - 120y = 0 \\
y^{(100)} = 0 \\
y^{(3)} - 3y'' + 3y' - y = 0 \\
y^{(4)} - 3y^{(3)} + 3y'' - y' = 0 \\
y^{(6)} - 6y^{(5)} + 25y^{(4)} = 0 \\
y^{(5)} - 2y^{(4)} + y^{(3)} = 0 \\
y^{(4)} + 2y'' + y = 0 \\
y^{(5)} + 3y^{(4)} + 3y^{(3)} + y'' - 4y' + 2y = 0 \\
y^{(4)} - y = 0
\end{gathered}
$$

9.

$$
\begin{gathered}
y'' - 2y' - 3y = e^{4x} \\
y'' - 2y' - 3y = e^{-x} \\
y^{(4)} - 2y'' + y = \cos(3x) \\
y^{(3)} + y'' - y' - y = e^{x} + e^{-x} \\
y'' + 5y' + 6y = (x+1)^3 \\
y'' + 2y' + 5y = e^{x}\sin x \\
y'' + 3y' + 4y = 3x + 2 \\
y'' + 2y' + 2y = \sin(3x) \\
y'' - 3y' + 2y = e^{-x} - 10\cos(3x) \\
y'' - 4y' + 4y = e^{2x} \\
y'' + 9y = \cos(3x) + \sin(3x) \\
y^{(4)} - y = 5 \\
y^{(3)} + y'' = 3e^{x} + 4x^2 \\
y^{(4)} - 4y'' = x^2
\end{gathered} . \tag{27}
$$

10.

$$
\begin{gathered}
x'' + 4x = 0 \\
x(0) = 5,\ x'(0) = 0 \\
x'' - x' - 2x = 0 \\
x(0) = 0,\ x'(0) = 2 \\
x'' + x = \sin(2t) \\
x(0) = 0,\ x'(0) = 0 \\
x'' + x = \cos(3t) \\
x(0) = 1,\ x'(0) = 0 \\
x'' + 4x = 2 \\
x(0) = 3,\ x'(0) = -1 \\
y'' - 10y' + 9y = 5t \\
y(0) = -1,\ y'(0) = 2 \\
y' + 9y = u(t-1) \\
y(0) = 1 \\
x'' - x = (t^2 - 1)\,u(t-1) \\
x(0) = 1,\ x'(0) = 1
\end{gathered} \tag{28}
$$

11.

$$
\begin{gathered}
x'' + 4x = \delta(t)) \\
x(0) = 0,\ x'(0) = 0 \\
x'' + 2x' + x = t + \delta(t) \\
x(0) = 0,\ x'(0) = 1 \\
x'' + 4x = \delta(t) + \delta\left(t - \pi\right) \\
x(0) = 0,\ x'(0) = 0 \\
x'' + 4x' + 4x = 1 + \delta\left(t - 2\right) \\
x(0) = 0,\ x'(0) = 0 \\
x'' + 2x' + 2x = 2\delta\left(t - \pi\right) \\
x(0) = 0,\ x'(0) = 0 \\
x'' + 4x' + 5x = \delta\left(t - \pi\right) + \delta\left(t - 2\pi\right) \\
x(0) = 0,\ x'(0) = 2
\end{gathered}
\tag{29}
$$

1.3 Answers

1.

$$
\begin{gathered}
y(x) = x^2 + C \\
y(x) = Ce^{\frac{1}{3}x^3} \\
y(x) = \frac{8}{(C-3x^2)} \\
y(x) = \tan^{-1}x + C \\
y(x) = 1 + C\sin^2 x \\
\text{implicit: } y + \ln|y| = x^2 + C \\
y(x) = \frac{x+1}{C_2(x+1)+1} - 1 \\
y(x) = Ce^{-x}
\end{gathered} \tag{30}
$$

2.

$$
\begin{gathered}
y(x) = C\sec(x) - \cos(x) \\
y(x) = \left(\tfrac{1}{2}x^2 + C\right) x^{-\ln(x)} \\
y(x) = \frac{x-1}{x}e^x + \frac{C}{x}
\end{gathered} \tag{31}
$$

3.

$$
\begin{gathered}
y(x) = \left(\tfrac{1}{2}x + C\right)^2 - x - 1 \\
y(x) = e^{\sin(C \pm x)} \\
y(x) = \sqrt[3]{3 + Ce^{-3(x^2)}} \\
\text{implicit: } y^2 = \frac{Ce^{2x}}{1-Ce^{2x}} \\
\text{implicit: } \sin^2 y = 4x^2 + Cx \\
\text{implicit: } \tan(y) = Ce^{-\tan^{-1}(x)} + 2 \\
\text{implicit: } \sec(y) = e^{\left(e^{\left(x^2\right)}\right)}\left(C + \tfrac{1}{2}e^{\left(x^2\right)}\right) \\
\text{implicit: } y^2 = e^{\pm\sqrt{e^x + e^{-x} + C}} + 4
\end{gathered} \tag{32}
$$

4.

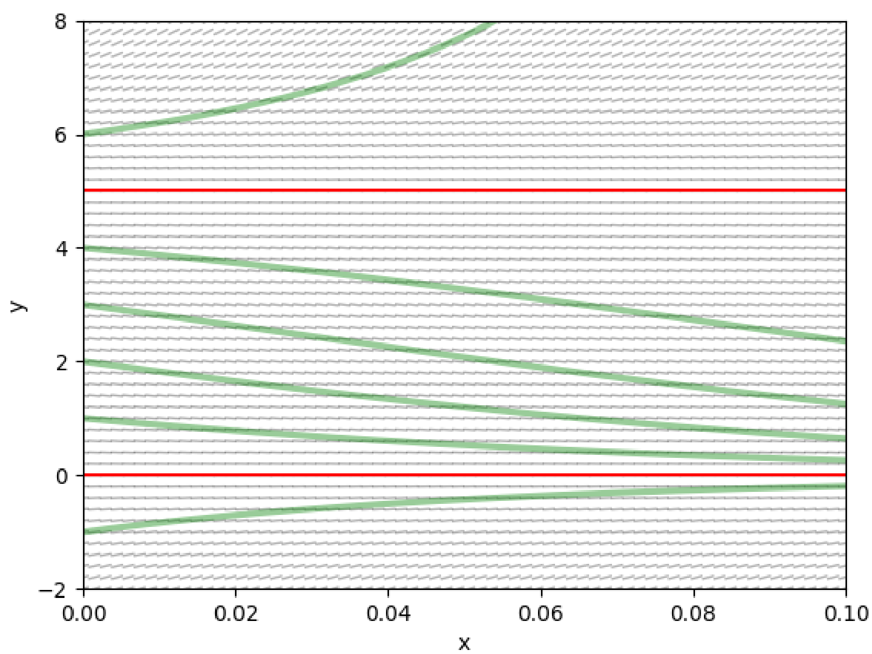
$$
\begin{gathered}
y(x) = x\sqrt[3]{C + 3\ln|x|} \\
y(x) = x(C + \ln|x|)^2 \\
\text{implicit: } y^2 + 2xy - x^2 = C \\
\text{implicit: } \tan^{-1}\left(\frac{y}{x}\right) = \ln\left(\sqrt{x^2+y^2}\right) + C \\
\text{implicit: } y = C\left(x^2 + y^2\right) \\
\text{implicit: } y + \sqrt{x^2 - y^2} = Cx^2 \\
\text{implicit: } y^2 = {C_3}^2 - 2C_3x
\end{gathered} \tag{33}
$$

5.

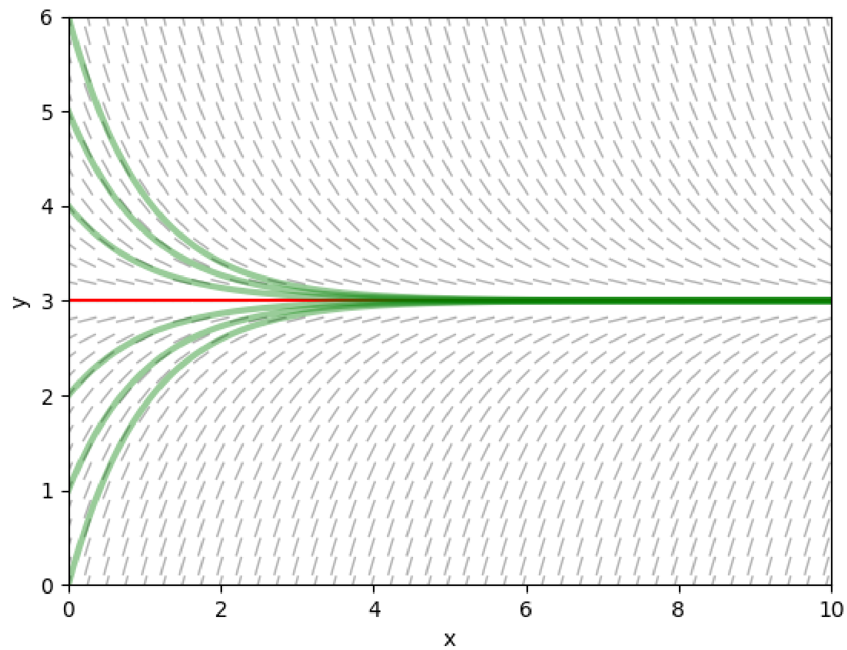
$$
\begin{gathered}
y(x) = e^{-3x}(x + C)^3 \\
y(x) = \frac{1}{x^3(x+C)^3} \\
\text{implicit: } 2x^3y^3 = 3\sqrt{1 + x^4} + C \\
y(x) = \sqrt[3]{\frac{7x}{15 + C_2x^7}}
\end{gathered} \tag{34}
$$

6.

6.1.



6.2.



7.

$$y(x) = (c_1 + c_2 x)\, e^{-\frac{1}{2}x}$$
$$y(x) = c_1 + c_2 e^{-11x}$$
$$y(x) = c_1 e^{x} + c_2 e^{-x}$$
$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$
$$y(x) = e^{-\frac{1}{2}x} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$
$$y(x) = c_1 e^{\frac{-1+\sqrt{5}}{2}x} + c_2 e^{\frac{-1-\sqrt{5}}{2}x}$$
$$y(x) = e^{\frac{1}{2}x} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$
$$y(x) = c_1 e^{\frac{1+\sqrt{5}}{2}x} + c_2 e^{\frac{1-\sqrt{5}}{2}x}$$
$$y(x) = c_1 e^{-x} + c_2 e^{2x}$$
$$y(x) = c_1 e^{x} + c_2 e^{-2x}$$
$$y(x) = e^{\frac{1}{2}x} \left(c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$
$$y(x) = e^{-\frac{1}{2}x} \left(c_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$
$$y(x) = (c_1 + c_2 x)\, e^{-x}$$
$$y(x) = c_1 e^{-3x} + c_2 e^{-2x}$$
$$y(x) = c_1 e^{-3x} + c_2 e^{2x}$$
$$y(x) = c_1 e^{3x} + c_2 e^{2x}$$
$$y(x) = c_1 e^{3x} + c_2 e^{-2x}$$

8.

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{3x} + c_4 e^{-3x}$$
$$y(x) = c_1 + c_2 e^{x} + c_3 e^{-x}$$
$$y(x) = c_1 + c_2 \cos x + c_3 \sin x$$
$$y(x) = c_1 + c_2 e^{x} + c_3 e^{2x}$$
$$y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x} + c_5 e^{5x}$$
$$y(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{98} x^{98} + c_{99} x^{99}$$
$$y(x) = \left(c_1 + c_2 x + c_3 x^2\right) e^{x}$$
$$y(x) = \left(c_1 + c_2 x + c_3 x^2\right) e^{x} + c_4$$
$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + e^{3x}\left(c_4 \cos(4x) + c_5 \sin(4x)\right)$$
$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{x} + c_5 x e^{x}$$
$$y(x) = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$$
$$y(x) = e^{x}\left(c_1 + c_2 x + c_3 \cos x + c_4 \sin x\right) + c_5 e^{-x}$$
$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

9.

$$y(x) = c_1 e^{-x} + c_2 e^{3x} + \tfrac{1}{5} e^{4x}$$
$$y(x) = c_1 e^{-x} + c_2 e^{3x} - \tfrac{1}{4} x e^{-x}$$
$$y(x) = (c_1 + c_2 x) e^{x} + (c_3 + c_4 x) e^{-x} + \tfrac{1}{82} \cos(3x)$$
$$y(x) = c_1 e^{x} + (c_2 + c_3 x) e^{-x} + \tfrac{1}{4} x e^{x} - \tfrac{1}{4} x^2 e^{-x}$$
$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \tfrac{1}{6} x^3 + \tfrac{1}{12} x^2 + \tfrac{7}{36} x - \tfrac{5}{216}$$
$$y(x) = e^{-x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right) + e^{x}\left(-\tfrac{4}{65} \cos x + \tfrac{7}{65} \sin x\right)$$
$$y(x) = c_1 e^{-\frac{3}{2}x} \sin\left(\tfrac{\sqrt{7}}{2} x\right) + c_2 e^{-\frac{3}{2}x} \cos\left(\tfrac{\sqrt{7}}{2} x\right) + \tfrac{3}{4} x - \tfrac{1}{16}$$
$$y(x) = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x - \tfrac{7}{85} \sin(3x) - \tfrac{6}{85} \cos(3x)$$
$$y(x) = c_1 e^{x} + c_2 e^{2x} + \tfrac{1}{2} e^{-x} + \tfrac{7}{13} \cos(3x) + \tfrac{9}{13} \sin(3x)$$
$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + \tfrac{1}{2} x^2 e^{2x}$$
$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \tfrac{1}{6} x \sin(3x) - \tfrac{1}{6} x \cos(3x)$$
$$y(x) = \tfrac{5}{4} e^{-x} + e^{x} + 2\cos x - 4$$
$$y(x) = c_1 + c_2 x + c_3 e^{-x} + \tfrac{3}{2} e^{x} + 4x^2 - \tfrac{4}{3} x^3 + \tfrac{1}{3} x^4$$
$$y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 + c_4 x - \tfrac{x^2}{16} - \tfrac{x^4}{48}$$

10.

$$
\begin{aligned}
x(t) &= 5\cos(2t) \\
x(t) &= \tfrac{2}{3}e^{2t} - \tfrac{2}{3}e^{-t} \\
x(t) &= \tfrac{2}{3}\sin(t) - \tfrac{1}{3}\sin(2t) \\
x(t) &= -\tfrac{1}{8}\cos(3t) + \tfrac{1}{8}\cos(t) \\
x(t) &= 3\cos(2t) + \tfrac{1}{2}\sin(2t) \\
y(t) &= \tfrac{50}{81} + \tfrac{5}{9}t + \tfrac{31}{81}e^{9t} - 2e^{t} \\
y(t) &= \left(\tfrac{1}{9} + \tfrac{8}{9}e^{-9t}\right)u(t-1) \\
x(t) &= e^{t} + \left(2e^{t-1} - t^2 - 1\right)u(t-1)
\end{aligned} \tag{35}
$$

11.

$$
\begin{aligned}
x(t) &= \tfrac{1}{2}\sin(2t) \\
x(t) &= t - 2 + (3t+2)\,e^{-t} \\
x(t) &= \tfrac{1}{2}\left(1 + u(t-\pi)\right)\sin(2t) \\
x(t) &= \tfrac{1}{4} + \tfrac{1}{4}e^{-2t}\left(4e^{4}u(t-2)(t-2) - 2t - 1\right) \\
x(t) &= -2u(t-\pi)\,e^{-t+\pi}\sin t \\
x(t) &= -\tfrac{1}{3}u(t-3\pi)\sin(3t) - \tfrac{1}{18}\sin^2(3t) \\
x(t) &= \left(2 - e^{2\pi}u(t-\pi) + e^{4\pi}u(t-2\pi)\right)e^{-2t}\sin t
\end{aligned} \tag{36}
$$

2 Exam 02

2.1 Summary

Exam 02 is based on Chapters 07 and 03.

Chapter 07 was about series methods for solving second order linear homogeneous ODEs. A linear homogeneous ODE with constant coefficients can be far more easily solved using the techniques of Chapters 02 or 06, but such ODEs with non-constant coefficients can be significantly more difficult. Series methods are complicated, so typically they are only used as a kind of last resort when other methods cannot be applied.

Given an ODE of the form

$$p\,(x)\,y'' + q\,(x)\,y' + r\,(x)\,y = 0, \tag{37}$$

a given value $x = a$ may be either an "ordinary point" or a "singular point" of the ODE, depending on whether $p\,(a) = 0$ or not. If $x = a$ is an ordinary point of the ODE, then there will exist at least one power series solution centered at $x = a$:

$$y = \sum_{n=0}^{\infty} c_n (x - a)^n. \tag{38}$$

On the other hand, if $x = a$ is a singular point, then such a solution is not guaranteed. If the value $x = a$ is a "regular singular point," (which means that the limits $\lim\limits_{x \to a} (x - a)\,\frac{q(x)}{p(x)}$ and $\lim\limits_{x \to a} (x - a)^2 \frac{r(x)}{p(x)}$ are both finite), then there will exist at least one solution that is a "Frobenius series:"

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \tag{39}$$

where $c_0 = 1$ and r is a real number.

In general, when attempting to find a solution of an ODE using series methods, one must propose the solution of the appropriate form, take several derivatives, and then put this equation back into the original ODE. When one combines like terms, this results in a power series which is equal to 0 for all values of x; in other words, a power series whose coefficients are all 0. This gives a system of infinitely many equations, known as the "recurrence relation" of the ODE. One then uses this information to find the coefficients c_n in terms of c_0 and c_1. This is the general solution.

Chapter 03 concerned systems of ODEs. We began our study of systems by first observing that every scalar ODE can be re-written as a first order linear system of ODEs. From here, we mainly concerned ourselves with solving first-order linear systems of ODEs with constant coefficients by the eigenvalue method. These were systems of the form

$$\overrightarrow{z}' = A\overrightarrow{z}, \tag{40}$$

where A was an $n \times n$ matrix. (We mainly only concerned ourselves with the case where $n = 2$.) This method involved several steps, always beginning with finding the characteristic polynomial of the matrix: $\det(A - \lambda I_n)$. The types of roots of the characteristic polynomial (the eigenvalues) then determined our next steps.

If the eigenvalues were distinct real numbers λ_1 and λ_2, the situation was mostly straightforward. In that case, we needed to find the eigenvectors of A corresponding to each eigenvalue. These were nonzero vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ that satisfied the equations

$$\begin{array}{c} (A - \lambda_1)\,\overrightarrow{v_1} = \overrightarrow{0} \\ (A - \lambda_2)\,\overrightarrow{v_2} = \overrightarrow{0} \end{array}. \tag{41}$$

We then selected two eigenvectors for each eigenvalue to create the basis solutions

$$\begin{array}{c} \overrightarrow{z_1}\,(t) = \overrightarrow{v_1}e^{\lambda_1 t} \\ \overrightarrow{z_2}\,(t) = \overrightarrow{v_2}e^{\lambda_2 t} \end{array}. \tag{42}$$

If the eigenvalues of the matrix were complex numbers $\lambda = a \pm ib$, then as usual, the (complex) eigenvectors $\overrightarrow{v_1}$ corresponding to one of the eigenvalues had to be found, and we needed to consider the product

$$\overrightarrow{v_1}e^{(a+ib)t}. \tag{43}$$

At this point, the task was to find the real and imaginary parts of this vector. This was accomplished by using Euler's formula. Once the real and imaginary parts were found, they formed separate real vector-valued basis solutions of the ODE.

Finally, if the eigenvalues of the matrix were repeated real numbers, then we needed to first find the "defect" of the eigenvalue. This was the difference between the algebraic multiplicity (which is just the multiplicity of the value as a root of the characteristic polynomial) and the geometric multiplicity (which is the number of free variables in the set of eigenvectors corresponding to the eigenvalue). If this defect was zero, then there would exist enough linearly independent eigenvectors to construct all of the basis solutions. On the other hand, if the defect was larger than zero, then we would need to consider "generalized eigenvectors" of ranks higher than one. A rank k generalized eigenvector $\overrightarrow{v_k}$ is a vector that matches both of the following descriptions:

$$(A - \lambda I_n)^k \overrightarrow{v_k} = \overrightarrow{0} \text{ and } (A - \lambda I_n)^{k-1} \overrightarrow{v_k} \neq \overrightarrow{0}. \tag{44}$$

For a 2×2 matrix, this required finding a nonzero vector $\overrightarrow{v_2}$ that was not an eigenvector such that $(A - \lambda I_2)^2 \overrightarrow{v_2} = \overrightarrow{0}$. At this point, we needed an eigenvector that was "compatible" with our choice of $\overrightarrow{v_2}$ in the sense that $\overrightarrow{v_1} = (A - \lambda I_2)\, \overrightarrow{v_2}$. From here, the two basis solutions could be constructed as

$$\begin{aligned} \overrightarrow{z_1} &= \overrightarrow{v_1} e^{\lambda t} \\ \overrightarrow{z_2} &= \left(\overrightarrow{v_1} t + \overrightarrow{v_2}\right) e^{\lambda t} \end{aligned}. \tag{45}$$

2.2 Practice problems

$$\begin{aligned}
(x^2+1)\,y'' + 6xy' + 4y &= 0\\
(x^2-3)\,y'' + 2xy' &= 0\\
(x^2-1)\,y'' - 6xy' + 12y &= 0\\
(x^2+1)\,y'' - 4xy' + 6y &= 0\\
(x^2+2)\,y'' + 4xy' + 2y &= 0\\
y'' + 2xy' + 4y &= 0
\end{aligned} \tag{46}$$

$$\begin{aligned}
2x^2y'' + 7xy' + 2y &= 0\\
2x^2y'' - 3xy' + 2y &= 0\\
5x^2y'' + y &= 0\\
7x^2y'' + xy' + y &= 0
\end{aligned} \tag{47}$$

$$\begin{aligned}
\overrightarrow{z}' &= \begin{pmatrix} 2 & 2 \\ 9 & -1 \end{pmatrix} \overrightarrow{z}\\
\overrightarrow{z}' &= \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \overrightarrow{z}\\
\overrightarrow{z}' &= \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} \overrightarrow{z}\\
\overrightarrow{z}' &= \begin{pmatrix} -50 & 20 \\ 100 & -60 \end{pmatrix} \overrightarrow{z}
\end{aligned} \tag{48}$$

$$\begin{aligned}
\overrightarrow{z}' &= \begin{pmatrix} -3 & -2 \\ 9 & 3 \end{pmatrix} \overrightarrow{z}\\
\overrightarrow{z}' &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \overrightarrow{z}\\
\overrightarrow{z}' &= \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \overrightarrow{z}
\end{aligned} \tag{49}$$

$$
\begin{aligned}
\overrightarrow{z}' &= \begin{pmatrix} -12 & 0 \\ 0 & -12 \end{pmatrix} \overrightarrow{z} \\
\overrightarrow{z}' &= \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \overrightarrow{z} \\
\overrightarrow{z}' &= \begin{pmatrix} -6 & 9 \\ -1 & -12 \end{pmatrix} \overrightarrow{z} \\
\overrightarrow{z}' &= \begin{pmatrix} 5 & 1 \\ -4 & 1 \end{pmatrix} \overrightarrow{z}
\end{aligned} \tag{50}
$$

2.3 Answers

$$\begin{gathered}
y = c_0 \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n} + c_1 \sum_{n=0}^{\infty} (-1)^n (2n+3) x^{2n+1} \\
y = c_0 + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{3^n (2n+1)} \\
y = c_0 (1 + 6x^2 + x^4) + c_1 (x + x^3) \\
y = c_0 (1 - 3x^2) + c_1 \left(x - \frac{1}{3} x^3\right) \\
y = \frac{c_0 + c_1 x}{2 + x^2} \\
y = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-2)^n}{(2n-1)(2n-3)...(3)(1)} x^{2n}\right) + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}
\end{gathered} \tag{51}$$

$$\begin{gathered}
y = c_0 x^{-2} + c_1 x^{-\frac{1}{2}} \\
y = c_0 x^2 + c_1 x^{\frac{1}{2}} \\
y = c_0 x^{\frac{1}{2} + \frac{\sqrt{5}}{10}} + c_1 x^{\frac{1}{2} - \frac{\sqrt{5}}{10}} \\
y = c_0 x^{\frac{3}{7} + \frac{\sqrt{2}}{7}} + c_1 x^{\frac{3}{7} - \frac{\sqrt{2}}{7}}
\end{gathered} \tag{52}$$

$$\begin{gathered}
\overrightarrow{z}(t) = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-4t} \\
\overrightarrow{z}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t} \\
\overrightarrow{z}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} \\
\overrightarrow{z}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-10t} + c_2 \begin{pmatrix} 2 \\ -5 \end{pmatrix} e^{-100t}
\end{gathered} \tag{53}$$

$$\begin{gathered}
\overrightarrow{z}(t) = c_1 \begin{pmatrix} \cos(3t) + \sin(3t) \\ -3\cos(3t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(3t) - \cos(3t) \\ -3\sin(3t) \end{pmatrix} \\
\overrightarrow{z}(t) = c_1 e^t \begin{pmatrix} -\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} \\
\overrightarrow{z}(t) = c_1 \begin{pmatrix} 3\cos\left(3\sqrt{3}t\right) \\ -\cos\left(3\sqrt{3}t\right) - \sqrt{3}\sin\left(3\sqrt{3}t\right) \end{pmatrix} + c_2 \begin{pmatrix} 3\sin\left(3\sqrt{3}t\right) \\ \sqrt{3}\cos\left(3\sqrt{3}t\right) - \sin\left(3\sqrt{3}t\right) \end{pmatrix}
\end{gathered} \tag{54}$$

$$
\begin{aligned}
\overrightarrow{z}(t) &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-12t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-12t} \\
\overrightarrow{z}(t) &= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} t \\ -2t+1 \end{pmatrix} e^{5t} \\
\overrightarrow{z}(t) &= c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-9t} + c_2 \begin{pmatrix} 3t+1 \\ -t \end{pmatrix} e^{-9t} \\
\overrightarrow{z}(t) &= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} \frac{1}{2}+t \\ -2t \end{pmatrix} e^{3}t
\end{aligned}
\tag{55}
$$