Math 306: Introduction to Differential Equations

Mark Sullivan

August 3, 2021

Contents

0 Introduction

0.1 Motivation

Imagine pouring water from a vessel. Let's suppose the vessel has a single opening, exactly below every other point in the vessel. How quickly does the water escape?

This problem is primitive enough that ancient humans certainly thought about it. However, answering it is not necessarily easy. After some thinking, you may realize that the rate at which the water leaves the vessel is not constant; *the rate at which the volume of the water within the vessel decreases is dependent upon how much water is left in the vessel at any given time.*

Here's another (recently relevant) problem: a disease spreads among a population of organisms. What is the rate at which the disease finds new hosts? (In other words, how many new infections occur per day?)

Again, the question is primitive enough that ancient humans considered it (even without a reasonable theory of germs, they noticed that diseases were contagious). However, again, the answer is not easy. With some thought, we can realize that the rate of infection is not constant; *the rate at which the disease spreads is dependent upon how many organisms in the population are already infected.*

The common theme in these two parables is the following: the rate of change of a certain variable is dependent upon the value of that variable at any given time. This is exactly the sort of problem that the theory of differential equations was created to solve.

0.2 Multivariable functions

First, let's address a small matter of terminology. Calculus III is not a prerequisite for this course, so I won't assume that you already know the material discussed in that class. There is, however, a matter that we must discuss. In Calculus I and Calculus II, you considered only functions of the form $y = f(x)$. These are called "single-variable functions." In these functions, the value of the dependent variable (usually called "y") is completely determined by the value of the single independent variable (usually called "x"). For example, if you give me a function like

$$y = \frac{\sin^{-1}\left(e^{x}\right)}{x^{2} + \ln|x|}, \tag{1}$$

and you ask me to find a value for y, then I need only ask you which value of x you'd like me to consider. Once you tell me a value for x, I just put it into the function above, and figure out the numbers.

On the other hand, suppose you give me an expression like

$$z = x^{2} + y^{2}, \tag{2}$$

and then tell me to find a value for z. Here, giving me just a value for x would not be sufficient; there are plenty of z values that correspond to any particular value for x. In this case, you'd have to specify the values of both x and y. For this reason, we'd say that z is a "function of two [independent] variables." The typical notation for this is $z = f(x, y)$, where, in this example, $f(x, y) = x^{2} + y^{2}$.

By extension, given any positive integer n, it is certainly possible to define a function of n many variables. In this class and Calculus III, this n is usually either two or three, though.

Given a function f of n many variables, f may depend on each of the variables in different ways. For example, suppose you give me the three-variable function

$$f(x, y, z) = x^{2}y + \frac{z}{y} - \cos(xy + z), \tag{3}$$

and you ask me the rate at which the function changes as its independent variables (x, y and z) change. In that case, I'd need you to be more specific. In the case of a one-variable function, this question can be answered by the derivative. For a multi-variable function, though, it's not immediately clear which variable we're considering.

So, suppose you tell me that you want to know how quickly f changes as x changes. In that case, you're asking about the "partial derivative of f with respect to x." This value is denoted by the symbols $\frac{\partial f}{\partial x}$. It's computed in exactly the same way as you would compute a single-variable derivative if the other variables (y and

z) were constants:

$$\frac{\partial f}{\partial x} = 2xy + y \sin(xy + z) . \tag{4}$$

Similarly, there are partial derivatives of f with respect to y and z:

$$\frac{\partial f}{\partial y} = x^2 - \frac{z}{y^2} + x \sin(xy + z) \tag{5}$$

$$\frac{\partial f}{\partial z} = \frac{1}{y} + \sin(xy + z) . \tag{6}$$

There are also higher order partial derivatives, in the same way that single-variable functions have higher order derivatives.

These are concepts that we'll also need later on, so don't hesitate to refer to this section again later.

0.3 What is a solution?

If you look back at the problems given in the motivation, you'll quickly realize that they can't be completely answered by just one single number. For example, if we're talking about the rate at which water leaves a vessel, if I just tell you "The water leaves the vessel at 6.5 mL/s," then you might rightly think that I don't understand the nuance of the question. Perhaps the rate at which the water leaves the vessel is 6.5 mL/s at *some* point in time, but that doesn't tell you anything about any other point in time. It would be far more helpful if I gave you a function that can compute the rate of drainage at any particular time. For this sort of reason, *the solution to a differential equation will always be a function, not merely a real number.*

Generally speaking, a **differential equation is any equation that describes a function's derivatives.** This equation describes a certain property of a function. We say that a function is a solution of that differential equation if it does have that property.

Here's an example of a differential equation: $y' = x^2 y$. The idea here is that y is the dependent variable, which is determined completely by the independent variable x. This equation describes the following property of a function: its first derivative is equal to the product of itself and the square of its independent variable. A *solution* of such an equation would be a function. For example, $y = e^{\frac{1}{3}x^3}$ is a solution of this ODE, because it satisfies the aformentioned property; its first derivative is x^2 times the original function.

0.4 Classification, and why it's important

Classifying differential equations means coming up with a term for each type of differential equation, and (if possible) a strategy for finding the solution. The key here is that the classifying term should be applied unambiguously. That is, if two mathematicians look at the same differential equation (perhaps simplified or written in a different way), then they ought to come up with the same terms to describe it.

At this point, a very natural question may come to mind: why bother learning this terminology? Indeed, when dealing with regular algebraic equations that don't involve derivatives, there is very little, if any need for classification. The reason for this is simple: the essential strategy of solving an algebraic equation is almost always the same: get the variable by itself on one side of the equation.

However, for differential equations, the situation is not so simple. There is no single method or strategy for solving differential equations. This is reminiscent of something you've learned before. Consider the following integrals:

$$\int \frac{x}{x+1}\,\mathrm{d}x \qquad \int x^2 e^x\,\mathrm{d}x \qquad \int \frac{1}{x^2+5x+6}\,\mathrm{d}x \tag{7}$$

A different method is necessary for evaluating each of these. Therefore, we've come up with terms like "u-substitution," "integration by parts," and "integration by partial fractions," in order to quickly refer to these strategies. This necessity is exactly the reason that classification of differential equations is so important; each classification needs to be solved using a different method.

Our first step in classifying differential equations is to state the number of independent variables that we expect in the solution. To be more specific: if the function is a single-variable function, and the equation relates the function to its derivative (and/or the second derivative, and/or the third derivative, and/or so on), then we call it an ordinary differential equation.

Definition 0.1 *An ordinary differential equation (ODE) is a differential equation in which the solution has only one independent variable.*

By contrast, if an equation relates a multi-variable function to its partial derivatives (and/or its second partial derivatives, and/or its third partial derivatives, and/or so on), then we call it a partial differential equation.

Definition 0.2 *A partial differential equation (PDE) is a differential equation in which the solution could have more than one independent variable.*

The following are examples of ordinary differential equations, with an example of a solution associated to each one.

$$\begin{array}{cc} y' = 3x^2 + 5 & y = x^3 + 5x + 3 \\ y' = 2y & y(x) = e^{2x} \\ y'' = -y & y(x) = \cos x \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 9x = 10\cos(2t) & x(t) = 2\cos(2t) \end{array}. \tag{8}$$

The following are examples of some famous partial differential equations, with their names:

$$\begin{array}{cc} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 & \text{(Laplace's equation)} \\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \text{(Heat equation)} \\ i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi & \text{(Schrödinger equation)} \end{array} \tag{9}$$

In general, the theory of partial differential equation is **much** harder than the theory of ordinary differential equations. We will not consider partial differential equations in this class.

The next step in classification is "order." This refers to the highest derivative that is present in the equation.

Definition 0.3 *The order of a differential equation is the highest order of all of the derivatives upon which it depends.*

Here are some examples:

$$\begin{array}{cc} y' = \sqrt{x+y} & \text{first order} \\ y'' = \frac{x}{y} & \text{second order} \\ y'y = \sin\left(x^2+y\right) & \text{first order} \\ y''y + \left(y'\right)^2 = 0 & \text{second order} \\ y''''''''''''''''''' = x & \text{nineteenth order} \end{array} . \tag{10}$$

As the last example above demonstrates, it is an eyesore to write many primes for a high-order derivative. In general, for dealing with higher order derivatives than the second derivative, we will write $y^{(3)}$ instead of y''', $y^{(4)}$ instead of y'''', and so on.

Here are some more examples:

$$\begin{array}{cc} y^{(4)} + y'' + y = e^x & \text{fourth order} \\ y^{(3)} + x^2y'' + xy' + y = 0 & \text{third order} \\ y' + e^{\sin x}y = \frac{1}{x-\ln x} & \text{first order} \\ \frac{1}{y''} + \frac{1}{y'} + \frac{1}{y} = \tan^{-1}x & \text{second order} \\ y^{(100)} + xy^{(75)} + x^2y^{(50)} + x^3y^{(25)} + y' = y^{200} & \text{one hundredth order} \\ x^3y' - e^{\left(y^{(5)}\right)} = \frac{1}{x} & \text{fifth order} \\ \left(y'+y\right)^2 = 1 - x & \text{first order} \\ x^2y'' + 2xy' + y = 16y^2 & \text{second order} \\ y^{(3)} + \left(x^2 + xy^{(3)}\right)^4 = \sec(x) & \text{third order} \\ \cos\left(xy'' + \pi y\right) = y^{(4)} & \text{fourth order} \end{array} \tag{11}$$

1 First order equations

1.1 Integrals as solutions

The truth is that you've already dealt with some differential equations without realizing it. For example, if I seek a function which satisfies the differential equation $y' = \frac{1}{x^2+1}$, then, based on your experiences in Calculus I and II, you should recognize that any solution of this ODE will be an antiderivative of $\frac{1}{x^2+1}$. So, you compute an integral:

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \tan^{-1}x + C. \tag{12}$$

At this point, I ask "What is C?" As you know your calculus, you tell me that C is just any constant; any constant value for C would be a solution of the ODE that I presented.

In some sense, you've given me more than I've asked for in this example. I only wanted *some* function that satisfied my differential equation, but you found *all* of the functions that satisfy my differential equation. Technically, you've given me a description of the set of all solutions of my ODE. In the theory of differential equations, we have a term for this kind of answer.

Definition 1.1 *The general solution of a differential equation is a full description of the set of all functions that are solutions of the differential equation.*

On the other hand, giving me just one function that satisfies a differential equation is far less information. When we wish to emphasize that a single function (and not a set of functions) is a solution to a differential equation, we will call it a "particular solution," to distinguish it from the general solution. For example, the function $y = \tan^{-1}x - 1000$ is a particular solution to the ODE $y' = \frac{1}{x^2+1}$.

At this point, I'd like you to come to the realization that all of the integration techniques that you learned in Calculus II were just methods of finding the general solutions to first order ODEs of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x), \tag{13}$$

where f is some function. (In the example above, $f(x) = \frac{1}{x^2+1}$.) Therefore, now would be a good time to refresh yourself on the material on integration techniques from Calculus II, if you haven't thought of those in a while.

There's one more piece of basic terminology that will appear throughout the course.

Definition 1.2 *An* <u>*initial value problem*</u> *(IVP) is an nth-order differential equation coupled with n required values for the solution and its derivatives at some values of the independent variables.*

Since we're on the topic of first-order ODEs, we'll begin with those. An IVP for a first-order ODE would look like a first order ODE together with a condition that the function have a particular y-value at a particular x-value.

Example 1.3 *(Exercise 1.1.2) Solve the following initial value problem:*

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 + x \\ y(1) &= 3 \end{aligned} \tag{14}$$

The standard technique of finding the solution to an IVP is simple: find the general solution, and then find a particular solution that satisfies the condition. In this case, the general solution is an indefinite integral:

$$y = \int x^2 + x \, \mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C. \tag{15}$$

From here, we must recognize that not all of these solutions will satisfy the condition $y(1) = 3$. This allows us to set up an equation for C:

$$3 = y(1) = \frac{1}{3}(1)^3 + \frac{1}{2}(1)^2 + C = \frac{5}{6} + C. \tag{16}$$

From this, we deduce that $C = \frac{13}{6}$. Thus,

$$\boxed{y = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{13}{6}} \tag{17}$$

is the particular solution of the ODE that solves that IVP. □

1.2 Slope fields

In general, every first-order ODE can be written as

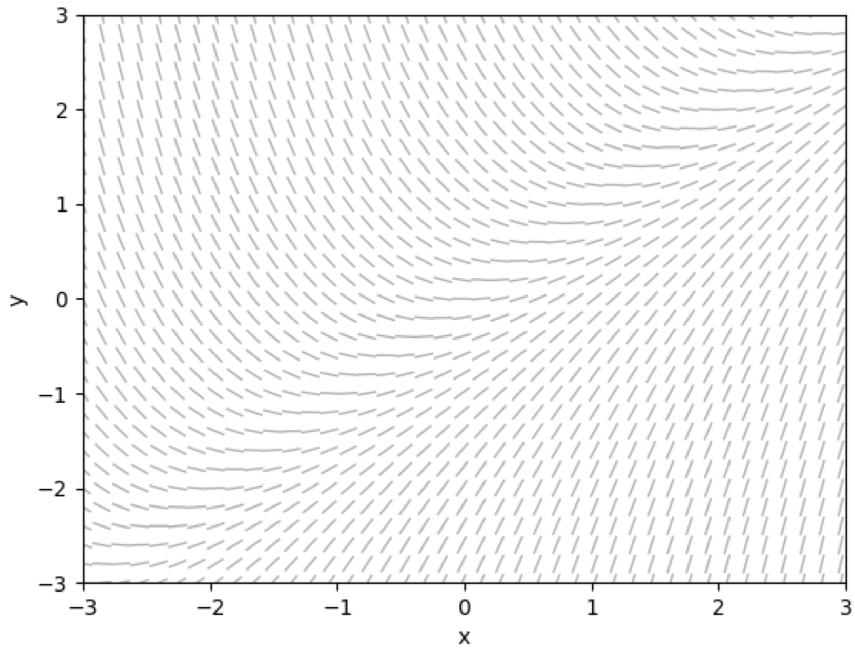
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \tag{18}$$

where f is some appropriate function of both x and y. In the previous section (and Calculus II), we only looked at the case that f doesn't actually depend on y. From now on, we'll be looking at the more general situation that y' could, in fact, depend on y.

1.2.1 Slope fields

As previously mentioned, there are many different functions that satisfy any given ODE. In fact, for any ODE, there are infinitely many particular solutions. (This can also be phrased as: the general solution describes an infinite set of functions.) How can one visualize an infinitely large set of functions? One way is by constructing a diagram called a slope field.

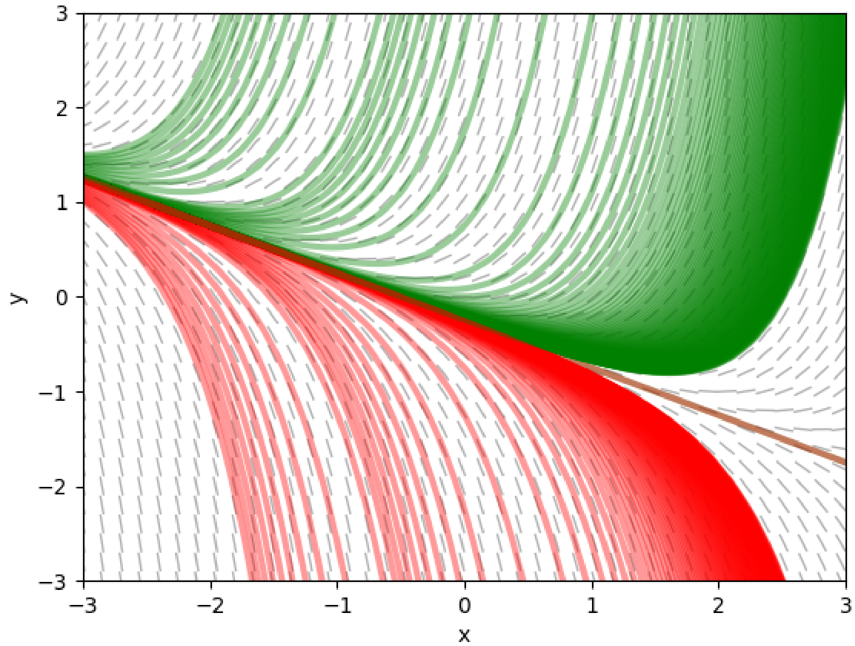
Consider the ODE $y' = f(x, y)$. This equation completely describes the slope of the tangent line (that is, the derivative) to a solution curve at any point based only on the x- and y-coordinates of that point. For example, given a solution to the ODE $y' = x - y$, we can figure out the slope of the tangent line to the solution curve at any point (x, y) simply by computing $x - y$. In other words, the slope of the tangent line is a function of the variables x and y. We can visualize the behavior of the ODE by simply computing $x - y$ at every point, and then drawing a small line segment with the slope $x - y$ at those points. This is called a slope field. The slope field of $y' = x - y$ is pictured below.



To summarize: a slope field is a diagram of tiny line segments whose slopes correspond to the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at any point (x, y). These tiny line segments are, literally, pieces of tangent lines to solution curves.

1.2.2 Existence and uniqueness

Another way to visualize the behavior of an ODE is simply to graph all of the solutions in the plane at the same time. The following diagram shows many solution curves to the ODE $y' = 2y + x$ (overlayed onto a slope field):



We notice two things here. First, it seems that for any and every point in the plane, there is a solution curve which passes through that point. Second, none of the solution curves cross each other (they certainly come close to each other, but they never overlap). To rephrase: for every point (a, b) in the plane, there exists a unique solution curve that passes through the point (a, b). To rephrase again, given any point (a, b), the following IVP has exactly one solution:

$$\begin{array}{c} y' = 2y + x \\ y(a) = b \end{array}. \tag{19}$$

Does this always happen?

The answer is no. For example, the following IVP has no solution:

$$\begin{array}{c} xy' = 1 \\ y(0) = 0 \end{array}. \tag{20}$$

On the other hand, the IVP

$$\begin{aligned} y' &= 2\sqrt{|y|} \\ y(0) &= 0 \end{aligned} \tag{21}$$

has at least two solutions: one is $y = 0$, and another is

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}. \tag{22}$$

So, the world is not perfect. Some IVPs have no solution, and some have more than one. In that case, when *can* we be sure that an IVP will have a unique solution? One answer is given by Picard's theorem.

Theorem 1.4 *(Picard's theorem) Let $y' = f(x, y)$ be an ODE, and let (a, b) be any point in the xy-plane. If f is continuous (as a function of two variables) in a neighborhood of (a, b) and the partial derivative $\frac{\partial f}{\partial y}$ is continuous (as a function of two variables) in a neighborhood of (a, b), then the IVP*

$$\begin{aligned} y' &= f(x, y) \\ y(a) &= b \end{aligned}$$

has a unique solution (on some interval).

What is a "neighborhood?" It's the interior of a rectangle contained in the xy-plane. In other words, if the region of continuity of f and $\frac{\partial f}{\partial y}$ contains a rectangle that contains the point (a, b), then the IVP will have a unique solution.

This theorem has theoretical significance: it indicates that for ODEs that satisfy the hypothesis of Picard's theorem, the solution curves will never overlap. However, it isn't particularly easy to find problems that relate to the theorem, and it is my opinion that those that do relate aren't particularly enlightening. Therefore, we will not dwell on this theorem too much, despite its importance.

1.3 Separable equations

Now, let's discuss how to solve ODEs of the form $y' = f(x, y)$. At first, it might seem like we could just do what we did in Section 1.1, but that won't work here. Let's consider what would happen if we tried to take the definite integral:

$$y = \int f(x, y)\ \mathrm{d}x. \tag{23}$$

Alright, so this tells us that in order to find y, we need the integral of $f(x, y)$. However, this requires putting y in terms of x (because the two are not both independent variables; y is dependent upon x). Of course, if we had that information, then we'd already be done.

There are different strategies for different types of ODEs, so we'll have to discuss them one at a time. (See, classification is important!)

1.3.1 Separable equations

We begin our study of the general solutions of non-trivial first-order ODEs by considering the classification called "separable equations."

Definition 1.5 *Let $y' = f(x, y)$ be an ODE. We say that $y' = f(x, y)$ is a separable ODE provided that there exist functions g and h such that $f(x, y) = g(x)\,h(y)$.*

In general, separable equations are among the easiest types of differential equations to solve. They are solved by the method of "separation of variables." To do this, we start by writing the ODE in the form prescribed by Definition 1.5:

$$\tfrac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,h(y) \tag{24}$$

Move everything involving y to the same side as y', and everything involving x to the other side:

$$\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x) \tag{25}$$

We now integrate both sides:

$$\int \frac{1}{h\left(y\right)}\;\mathrm{d}y = \int g\left(x\right)\;\mathrm{d}x. \tag{26}$$

From here, we will be able to find an algebraic equation that relates y to x.

Example 1.6 *(Exercise 1.3.2) Solve $y' = x^2 y$.*

This equation is separable, so we begin by putting it into Leibniz notation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y. \tag{27}$$

First, if $y = 0$, then the equation is satisfied. (This is a trivial, pathological case.) Assume now that $y \neq 0$. We move everything involving y to the left:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2. \tag{28}$$

Now we integrate:

$$\int \frac{1}{y}\;\mathrm{d}y = \int x^2\;\mathrm{d}x \tag{29}$$

This yields

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2, \tag{30}$$

where C_1 and C_2 are arbitrary constants of integration. This gives us

$$|y| = e^{\frac{1}{2}x^2 + C_2 - C_1}. \tag{31}$$

We note that this means

$$y = \pm e^{C_2 - C_1} e^{\frac{1}{2}x^2}. \tag{32}$$

We define $C_3 = \pm e^{C_2 - C_1}$, which could be any nonzero real number. (In fact, even if C_3 is zero, then the following will still be true.) This gives us

$$\boxed{y = C_3 e^{\frac{1}{2}x^2}}, \tag{33}$$

which is the general solution. □

Note: I was unreasonably careful in this last example. In the solution above, I held that C_1 and C_2 are not literally the same, and so I gave them different symbols. However, the truth is that while they may not be *numerically* equal, they are still *just constants*. For this reason, many mathematicians would just call both of them by the same symbol. (Usually, they choose "C.") I tend to avoid this conflation, but it's common enough that if you discuss differential equations with anyone, then you could expect them to not care about any differences between the constants. Moreover, I won't deduct points from your work if you don't bother to give the constants different names.

Example 1.7 *(Exercise 1.3.5) Solve $y' = xy + x + y + 1$.*

This isn't in the right form to separate variables, so we start by re-writing the ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(y+1\right) + \left(y+1\right) = \left(x+1\right)\left(y+1\right). \tag{34}$$

Now, we can proceed with separation: if $y \neq -1$, then

$$\frac{1}{y+1}\frac{\mathrm{d}y}{\mathrm{d}x} = x + 1, \tag{35}$$

so

$$\int \frac{1}{y+1}\,\mathrm{d}y = \int x + 1\,\mathrm{d}x. \tag{36}$$

This gives us

$$\ln\left|y+1\right| = \frac{1}{2}x^2 + x + C_1. \tag{37}$$

Ergo,

$$\left|y+1\right| = e^{\frac{1}{2}x^2 + x + C_1}, \tag{38}$$

and so

$$y + 1 = \pm e^{C_1} e^{\frac{1}{2}x^2 + x}. \tag{39}$$

Defining $C_2 = \pm e^{C_1}$, this becomes the general solution

$$\boxed{y = C_2 e^{\frac{1}{2}x^2 + x} - 1}. \tag{40}$$

□

Example 1.8 *(Exercise 1.3.8) Solve* $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2+1}{x^2+1}$.

This is a separable equation:

$$\frac{1}{y^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2+1}. \tag{41}$$

Thus,

$$\int \frac{1}{y^2+1}\,\mathrm{d}y = \int \frac{1}{x^2+1}\,\mathrm{d}x, \tag{42}$$

and so

$$\tan^{-1} y = \tan^{-1} x + C \tag{43}$$

Thus, our general solution is

$$\boxed{y = \tan\left(\tan^{-1} x + C\right)}. \tag{44}$$

□

1.3.2 Implicit solutions

In the course of solving ODEs, it may occasionally happen that we come to an algebraic equation that we cannot solve. In these cases, we simply leave the equation, calling it an "implicit [general] solution."

Example 1.9 *(Exercise 1.3.1) Solve* $y' = \frac{x}{y}$.

We need to realize that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{1}{y}\right). \tag{45}$$

Therefore,

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = x, \tag{46}$$

and so

$$\int y\,\mathrm{d}y = \int x\,\mathrm{d}x. \tag{47}$$

This gives us

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1. \tag{48}$$

Thus,

$$y^2 = x^2 + 2C_1. \tag{49}$$

Defining $C_2 = 2C_1$,

$$\boxed{y^2 = x^2 + C_2}. \tag{50}$$

is an implicit form of our general solution. □

Example 1.10 *(Exercise 1.3.105) Solve* $y' = \frac{\sin x}{\cos y}$.

We separate variables:

$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, \tag{51}$$

so

$$\int \cos y \, \mathrm{d}y = \int \sin x \, \mathrm{d}x. \tag{52}$$

This gives us

$$\boxed{\sin y = C - \cos x}. \tag{53}$$

At this point, there isn't much more that we can do to solve for y*, so we leave this and consider it an implicit solution.*

Note: it is not necessarily valid to write

$$y = \sin^{-1}\left(C - \cos x\right). \tag{54}$$

This is because the arcsine function only returns values between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$*. Unless it is known that* y *is in this interval, we cannot be sure of this equation.* □

1.3.3 Examples of separable equations

Definition 1.11 *A function* f *is* *<u>exponential</u>* *provided that there exists a nonzero real constant* k *such that* f *is a solution of* $y' = ky$*. If* $k > 0$*, then* f *is called an* *<u>exponential growth</u>*. *If* $k < 0$*, then* f *is called an* *<u>exponential decay</u>*.

Example 1.12 *Let* k *be a nonzero real constant. Find the general solution of the ODE* $y' = ky$.

This is a separable equation:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = k. \tag{55}$$

We integrate, getting

$$\int \frac{1}{y}\,\mathrm{d}y = \int k\,\mathrm{d}x, \tag{56}$$

which implies that

$$\ln|y| = kx + C_1. \tag{57}$$

Thus,

$$y = \pm e^{C_1} e^{kx}. \tag{58}$$

Defining $C_2 = e^{C_1}$ *gives us the following general solution:*

$$\boxed{y = C_2 e^{kx}}. \tag{59}$$

□

1.4 Linear equations and the integrating factor

Not all first-order ODEs are separable. For example,

$$y' + 6y = e^x \tag{60}$$

is not a separable ODE, since we cannot write y' as a product of a function that depends on x and a function that depends on y.

The next classification of first-order ODEs that can be easily solved are linear first-order ODEs. However, we'll be dealing with linear ODEs throughout the course, so we'll begin by defining what a linear ODE is in general.

Definition 1.13 *Given any non-negative integer i, the ith order basic differential operator is the operator $\frac{\mathrm{d}^i}{\mathrm{d}x^i}$.*

(For notation, we often denote f by $\frac{\mathrm{d}^0}{\mathrm{d}x^0} f$, calling it the "zeroth derivative of f.")

Definition 1.14 *Let $f_1, f_2, ..., f_n$ be mathematical objects, and let S be a set of mathematical objects. A linear combination of f_1, f_2, ..., and f_n with coefficients in S is any expression of the form $s_1 f_1 + s_2 f_2 + ... + s_n f_n$, where s_1, s_2, ..., and s_n are elements of S.*

For example,

$$3 \sin x + 2 \cos x \tag{61}$$

is "a linear combination of the functions $\sin x$ and $\cos x$ with constant coefficients." As another example,

$$x^2 \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} + x^{-1} \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} + 2^x \begin{pmatrix} 8 & -1 \\ 1 & 0 \end{pmatrix} \tag{62}$$

is "a linear combination of 2×2 matrices with functions of x as coefficients." As another example,

$$x^4 - 3x^3 + 6x^2 - 2x + 1 \tag{63}$$

is "a linear combination of $1, x, x^2, x^3$, and x^4 with constant coefficients." (In fact, every nth degree polynomial is, by definition, a linear combination of $1, x, x^2, ..., x^n$ with constant coefficients).

Definition 1.15 *A linear differential operator is a linear combination of basic differential operators with differentiable functions as coefficients.*

For example,

$$L = x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2^x \frac{\mathrm{d}}{\mathrm{d}x} - x \frac{\mathrm{d}^0}{\mathrm{d}x^0} \tag{64}$$

is a linear differential operator.

Definition 1.16 *A linear ODE is an ODE of the form $L(y) = q(x)$, where L is a linear differential operator and q is any function of x.*

For right now, we'll only be concerned with first-order linear ODEs in this section. These look like

$$p_2(x)\, y' + p_1(x)\, y = q(x), \tag{65}$$

where p_2, p_1 and q are functions of x. (Here p_2 must not be constantly zero.) An example could be

$$xy' + x^2 y = e^x. \tag{66}$$

On the other hand, the following are examples of non-linear first-order ODEs:

$$\begin{array}{c} yy' = x \\ (y')^2 = y \\ \frac{1}{y'} + \frac{1}{y} = x \end{array}. \tag{67}$$

The method of solving first-order linear ODEs uses the following technique. First, we put the equation into the following standard form:

$$y' + p(x)y = q(x). \tag{68}$$

Next, we define an "integrating factor" as follows:

$$\rho(x) = e^{\int p(x)\ \mathrm{d}x}. \tag{69}$$

Note that $\rho'(x) = \rho(x)\, p(x)$. We multiply this integrating factor by both sides of the equation:

$$\rho(x)\, y' + \rho(x)\, p(x)\, y = \rho(x)\, q(x)\,. \tag{70}$$

Now the left side of the equation is the result of a product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\, y\right) = \rho(x)\, q(x)\,. \tag{71}$$

From here, we can integrate both sides with respect to x and solve for y.

Example 1.17 *(Exercise 1.4.4) Solve $y' + xy = x$.*

This one is already in standard form, so we define the integrating factor:

$$\rho(x) = e^{\int x\ \mathrm{d}x} = e^{\frac{1}{2}x^2}. \tag{72}$$

We multiply this by both sides:

$$e^{\frac{1}{2}x^2} y' + e^{\frac{1}{2}x^2} xy = xe^{\frac{1}{2}x^2}. \tag{73}$$

Now we recognize that the left hand side is the derivative of $\rho(x)\, y$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{2}x^2} y\right) = xe^{\frac{1}{2}x^2}. \tag{74}$$

We now cancel the derivative by taking the integral of both sides:

$$e^{\frac{1}{2}x^2} y = \int xe^{\frac{1}{2}x^2}\ \mathrm{d}x \tag{75}$$

This gives us

$$e^{\frac{1}{2}x^2} y = e^{\frac{1}{2}x^2} + C, \tag{76}$$

and so

$$\boxed{y = 1 + Ce^{-\frac{1}{2}x^2}} \tag{77}$$

is our general solution. □

Example 1.18 *(Exercise 1.4.165) Solve the following IVP:*

$$\begin{array}{c} x^2y' + xy = 1 \\ y(1) = 3 \end{array}. \tag{78}$$

(Assume here that $x > 0$.)

Before we can define the integrating factor, we first need to put this into standard form by re-writing the equation so that the coefficient of y' is 1:

$$y' + \frac{1}{x}y = \frac{1}{x^2} \tag{79}$$

Now we define the integrating factor:

$$\rho(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln x} = x. \tag{80}$$

We multiply this by the entire standard form:

$$xy' + y = \frac{1}{x} \tag{81}$$

Again, the left side is a product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{1}{x}. \tag{82}$$

We integrate to cancel the derivative:

$$xy = \int \frac{1}{x}\,\mathrm{d}x = \ln x + C. \tag{83}$$

Thus, the general solution is

$$y = \frac{\ln x + C}{x}. \tag{84}$$

From here, the initial condition implies that

$$3 = y(1) = \frac{\ln 1 + C}{1} = C, \tag{85}$$

and so the particular solution that satisfies the IVP is

$$\boxed{y = \frac{\ln x + 3}{x}}. \tag{86}$$

□

Example 1.19 *(Exercise 1.4.163) Solve the following IVP:*

$$\begin{matrix} y' + (\cos x)\, y = 2\cos x \\ y\,(\pi) = 5 \end{matrix}. \tag{87}$$

As the ODE is in standard form, we can define the integrating factor:

$$\rho\,(x) = e^{\int \cos x \; \mathrm{d}x} = e^{\sin x}. \tag{88}$$

Now we multiply:

$$e^{\sin x} y' + e^{\sin x}\,(\cos x)\, y = 2e^{\sin x} \cos x. \tag{89}$$

Again, the left side is the result of the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\sin x} y\right) = 2e^{\sin x} \cos x. \tag{90}$$

We integrate:

$$e^{\sin x} y = \int 2e^{\sin x} \cos x \; \mathrm{d}x = 2e^{\sin x} + C. \tag{91}$$

Now we solve for y to get the following general solution:

$$y = 2 + Ce^{-\sin x}. \tag{92}$$

As for the initial condition:

$$5 = y\,(\pi) = 2 + Ce^{-\sin \pi} = 2 + C. \tag{93}$$

This reveals that $C = 3$, and so the particular solution that satisfies the IVP is

$$\boxed{y = 2e^{\sin x} + 3}. \tag{94}$$

□

Example 1.20 *(Exercise 1.4.168) Solve $y' = x - 2y$.* *We put this into standard form, so that y and y' are on the same side of the equation:*

$$y' + 2y = x. \tag{95}$$

We now define the integrating factor:

$$\rho(x) = e^{\int 2 \, \mathrm{d}x} = e^{2x}. \tag{96}$$

We multiply this by the standard form:

$$e^{2x}y' + 2e^{2x}y = xe^{2x}. \tag{97}$$

Again, the left side is a product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}y\right) = xe^{2x}. \tag{98}$$

We integrate:

$$e^{2x}y = \int xe^{2x} \, \mathrm{d}x = \frac{1}{4}e^{2x}(2x-1) + C. \tag{99}$$

Thus, the general solution is

$$\boxed{y = \tfrac{1}{4}(2x-1) + Ce^{-2x}}. \tag{100}$$

□

Example 1.21 *(Exercise 1.4.162) Solve $xy' + 2y = 5\sqrt{x}$.*

We first put this in standard form:

$$y' + \frac{2}{x}y = 5x^{-\frac{1}{2}}. \tag{101}$$

We define the integrating factor:

$$\rho(x) = e^{\int \frac{2}{x}\, \mathrm{d}x} = e^{2\ln|x|} = |x|^2 = x^2. \tag{102}$$

We multiply to get

$$x^2 y' + 2xy = 5x^{\frac{3}{2}}. \tag{103}$$

We recognize that the left side is a result of the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 y\right) = 5x^{\frac{3}{2}}. \tag{104}$$

Integrating,

$$x^2 y = \int 5x^{\frac{3}{2}}\, \mathrm{d}x = 2x^{\frac{5}{2}} + C, \tag{105}$$

and so our general solution is

$$\boxed{y = 2\sqrt{x} + \frac{C}{x^2}}. \tag{106}$$

□

1.5 Substitution

As in solving integrals, it is sometimes a good strategy to change to a different variable in order to solve an ODE. The best way to learn this is by example.

1.5.1 Substitution

Example 1.22 *(Exercise 1.5.5) Solve* $y' = (x + y - 1)^2$.

This ODE is neither separable nor linear. In order to solve it, we define the following substitution:

$$v(x) = x + y - 1. \tag{107}$$

In order to work with this, we need to put y' *in terms of this variable:*

$$v'(x) = 1 + y' \tag{108}$$

Thus, the ODE can be re-written as

$$v' - 1 = v^2. \tag{109}$$

We can use separation of variables here:

$$\int \frac{1}{v^2 + 1}\,\mathrm{d}v = \int \mathrm{d}x, \tag{110}$$

and so

$$\tan^{-1} v = x + C. \tag{111}$$

Therefore,

$$v = \tan\left(x + C\right). \tag{112}$$

We now reverse the substitution:

$$x + y - 1 = \tan\left(x + C\right). \tag{113}$$

This gives us the following general solution:

$$\boxed{y = 1 - x + \tan(x + C)}. \tag{114}$$

□

Example 1.23 *(Exercise 1.5.152) Solve $y' = \sqrt{x + y - 5}$.*

We make the substitution

$$\begin{aligned} v(x) &= x + y - 5 \\ v'(x) &= 1 + y' \end{aligned}. \tag{115}$$

This allows us to re-write the ODE as

$$v' - 1 = \sqrt{v}. \tag{116}$$

By separation of variables:

$$\int \frac{1}{\sqrt{v} + 1}\, \mathrm{d}v = \int\, \mathrm{d}x. \tag{117}$$

Bu u-substitution, this becomes

$$2\sqrt{v} - \ln\left(\sqrt{v} + 1\right) = x + C. \tag{118}$$

At this point, we reverse the substitution to get an implicit general solution:

$$\boxed{2\sqrt{x + y - 5} - \ln\left(\sqrt{x + y - 5} + 1\right) = x + C}. \tag{119}$$

□

Example 1.24 *(Exercise 1.5.104) Solve $2yy' = e^{y^2 - x^2} + 2x$.*

We define the substitution

$$\begin{aligned} v(x) &= y^2 - x^2 \\ v'(x) &= 2yy' - 2x \end{aligned}. \tag{120}$$

From here, the ODE becomes

$$v' = e^v. \tag{121}$$

This can be solved by separation of variables:

$$\int e^{-v}\,\mathrm{d}v = \int\,\mathrm{d}x. \tag{122}$$

Therefore,

$$-e^{-v} = x + C_1, \tag{123}$$

where C_1 is an arbitrary real constant. Defining $C_2 = -C_1$, this becomes

$$e^{-v} = C_2 - x. \tag{124}$$

Solving for v gives us

$$v = -\ln\left|C_2 - x\right|. \tag{125}$$

We now reverse the substitution to get an implict form of the general solution:

$$\boxed{y^2 - x^2 = -\ln\left|C_2 - x\right|}. \tag{126}$$

□

1.5.2 Bernoulli equations

Some substitution problems are more predictable.

Definition 1.25 *A Bernoulli differential equation is any ODE that can be written in the form $L(y) = q(x)\,y^n$, where L is a linear differential operator, and q is any differentiable function.*

Bernoulli equations can always be solved by making the substitution $v = y^{1-n}$. Doing this will allow you to re-write the equation as a first-order linear ODE.

Example 1.26 *(Exercise 1.5.151) Solve $x^2y' = y^2 + 3xy$.*

First, we re-write this as

$$y' - \frac{3}{x}y = \frac{1}{x^2}y^2. \tag{127}$$

This is a Bernoulli equation with $n = 2$*. We make the following substitution:*

$$\begin{array}{c} v(x) = y^{1-2} = y^{-1} \\ v'(x) = -y^{-2}y' \end{array}. \tag{128}$$

The latter equation implies that $v' = -v^2 y'$*, or in other words,* $y' = -v^{-2}v'$*. Now we can re-write the equation as*

$$-v^{-2}v' - \frac{3}{x}v^{-1} = \frac{1}{x^2}v^{-2}. \tag{129}$$

We multiply everything by $-v^2$ *to get*

$$v' + \frac{3}{x}v = \frac{1}{x^2}, \tag{130}$$

which is a first-order linear ODE in standard form. Using the integrating factor, we get the general solution

$$v = \frac{C}{x^3} + \frac{1}{2x}. \tag{131}$$

We now reverse the substitution:

$$y^{-1} = \frac{C}{x^3} + \frac{1}{2x}, \tag{132}$$

which gives us the general solution

$$\boxed{y = \frac{1}{\frac{C}{x^3} + \frac{1}{2x}} = \frac{2x^3}{C + x^2}}. \tag{133}$$

□

Example 1.27 *(Exercise 1.5.158) Solve* $y^4 y' = -3x^2 y^5 + x^2$*.*

We re-write this as

$$y' + 3x^2 y = x^2 y^{-4}. \tag{134}$$

This is a Bernoulli equation with $n = -4$. From here, we make the substitution

$$\begin{aligned} v(x) = y^{1-(-4)} = y^5. \\ v'(x) = 5y^4 y'. \end{aligned} \tag{135}$$

The latter equation gives us $y' = \frac{1}{5}y^{-4}v' = \frac{1}{5}v^{-\frac{4}{5}}v'$, and so the equation can be re-written as

$$\frac{1}{5}v^{-\frac{4}{5}}v' + 3x^2v^{\frac{1}{5}} = x^2v^{-\frac{4}{5}}. \tag{136}$$

We multiply everything by $5v^{\frac{4}{5}}$ to get

$$v' + 15x^2v = 5x^2, \tag{137}$$

which is a first-order linear ODE in standard form. We use the method of the integrating factor to get the general solution

$$v = \frac{1}{3} + Ce^{-5x^3}. \tag{138}$$

We reverse the substitution to get

$$y^5 = \frac{1}{3} + Ce^{-5x^3}, \tag{139}$$

and so the general solution is

$$\boxed{y = \sqrt[5]{\tfrac{1}{3} + Ce^{-5x^3}}}. \tag{140}$$

□

1.5.3 Homogeneous equations

WARNING: The term "homogeneous equation" has more than one meaning in the theory of ODEs. I don't make the rules. This section is about "homogeneous substitutions." We'll deal with an unrelated topic, "linear homogeneous equations," later.

A homogeneous substitution is a substitution of the form $v = \frac{y}{x}$. This is useful whenever an ODE can be written as $y' = f\left(\frac{y}{x}\right)$, for some function f.

Example 1.28 *(Exercise 1.5.151) Solve $x^2y' = y^2 + 3xy$.*

This ODE can be re-written as

$$y' = \left(\frac{y}{x}\right)^2 + 3\frac{y}{x}. \tag{141}$$

We make the substitution $v = \frac{y}{x}$. This gives $y = vx$, and so $y' = v'x + v$. This allows us to rewrite the ODE as

$$v'x + v = v^2 + 3v, \tag{142}$$

or

$$v'x = v^2 + 2v. \tag{143}$$

This can be solved by separation of variables, giving the general solution

$$v = -\frac{2x^2}{x^2 - C}, \tag{144}$$

where $C \neq 0$. We reverse the substitution to get

$$\frac{y}{x} = -\frac{2x^2}{x^2 - C}, \tag{145}$$

and so the general solution is

$$\boxed{y = -\frac{2x^3}{x^2 - C}}. \tag{146}$$

□

Example 1.29 *(Exercise 1.5.155) Solve the IVP*

$$\begin{array}{c} xy' = 2x + 3y \\ y(-1) = 3 \end{array}. \tag{147}$$

We re-write the equation as

$$y' = 2 + 3\frac{y}{x}. \tag{148}$$

We make the substitution $v = \frac{y}{x}$*, so that* $y = vx$*, and so* $y' = v'x + v$*. Thus,*

$$v'x + v = 2 + 3v. \tag{149}$$

We re-write this as

$$v' - \frac{2}{x}v = \frac{2}{x}, \tag{150}$$

which is a linear first-order ODE. We use the method of the integrating factor to get

$$v = Cx^2 - 1. \tag{151}$$

From here, we reverse the substitution to get the general solution

$$y = x\left(Cx^2 - 1\right). \tag{152}$$

To find the particular solution that solves the IVP, we use the initial condition:

$$3 = y(-1) = (-1)\left(C(-1)^2 - 1\right) = 1 - C. \tag{153}$$

Ergo, $C = -2$*, and so the solution of the IVP is*

$$\boxed{y = x\left(-2x^2 - 1\right)}. \tag{154}$$

□

Example 1.30 *Solve* $2xyy' = 4x^2 + 3y^2$*.*

We re-write the equation as

$$y' = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}. \tag{155}$$

We make the homogeneous substitution $v = \frac{y}{x}$*, so that* $y = vx$*, and so* $y' = v'x + v$*.*

This gives us

$$v'x + v = \frac{2}{v} + \frac{3}{2}v, \tag{156}$$

and so

$$v'x = \frac{2}{v} + \frac{v}{2} = \frac{4 + v^2}{2v}. \tag{157}$$

This can be done by separation of variables, giving us

$$v^2 = C|x| - 4. \tag{158}$$

Reversing the substitution now gives us

$$\boxed{y^2 = x^2 \left(C|x| - 4\right)}. \tag{159}$$

□

1.6 Autonomous equations

We'll now discuss a type of ODE in which the graphs of the solution curves are fairly predictable.

Definition 1.31 *A first-order ODE* $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$ *is called an* *autonomous ODE* *provided that* $\frac{\partial f}{\partial x} = 0$*. (In other words, the value of* $f(x,y)$ *does not depend on* x*.)*

These ODEs can always be solved by separation of variables (provided that $f(y)$ can be integrated). If we were to graph all of the solution curves of such an ODE, certain of them would stand out more than others.

Definition 1.32 *Let* $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$ *be an autonomous ODE. An* *equilibrium solution of the ODE* *is a solution of the form* $y(t) = k$*, where* k *is any constant value such that* $f(k) = 0$*.*

The equilibrium solutions of an autonomous ODE are constant solutions that partition the plane into segments. By Picard's theorem, no solution curves can cross, so no solution curve will be able to pass from one of these segments to another.

1.6.1 Sketching qualitatively-different solutions to autonomous DEs

We'll build up to drawing the solution curves to an autonomous ODE one step at a time. First, we'll start with the so-called "phase diagram."

Definition 1.33 *Let* $y' = f(y)$ *be an autonomous ODE. The* *phase diagram of the ODE* *is a number line that illustrates the value of* y' *on each side of the equilibrium solutions.*

For a phase diagram, we simply draw a vertical number line, mark the equilibrium solutions, and then draw arrows to indicate whether y' is positive or negative between those equilibrium solutions.

Example 1.34 *Sketch the phase diagram of the autonomous ODE* $y' = y^2 - 6y + 5$*.*

First, we need to determine the equilibrium solutions, so we must solve the equation $y^2 - 6y + 5 = 0$*. By solving this quadratic, we get the equilibrium solutions*

$y(t) = 1$ and $y(t) = 5$. Now, for y-values less than 1, $y' = y^2 - 6y + 5 > 0$, so we draw an upward arrow in the region where $y < 1$. For y-values between 1 and 5, $y' = y^2 - 6y + 5 < 0$, so we draw a downward arrow in that region. For y-values greater than 5, $y' = y^2 - 6y + 5 > 0$, so we draw an upward arrow in that region. This gives the following phase diagram.

□

Definition 1.35 *Let $y' = f(y)$ be an autonomous ODE, and let $y(t) = k$ be an equilibrium solution. We say that $y(t) = k$ is a stable equilibrium solution provided that both arrows on either side of the solution in the phase diagram point toward the solution. If this is not the case, we say that $y(t) = k$ is an unstable equilibrium solution.*

In our example above, $y(t) = 1$ is stable, while $y(t) = 5$ is unstable.

The phase diagram can serve as the y-axis for the graph of all of the solution curves.

Example 1.36 *Sketch all solution curves of the autonomous ODE $y' = y^2 - 6y + 5$.*

The phase diagram above forms the y-axis of the graph. The horizontal axis is t:

[Diagram]

□

Example 1.37 *Sketch all solution curves of the autonomous ODE $y' = y^2 - y$.*

We note that $y(t) = 0$ and $y(t) = 1$ are equilibrium solutions. The solution curves look like

[Diagram]

□

Example 1.38 *Sketch all solution curves of the autonomous ODE $y' = (y-1)^2$.*

The only equilibrium solution of this ODE is $y(t) = 1$. The solution curves look like

[Diagram]

□

Example 1.39 *Let $y(t)$ denote the population of a certain species at time t. Suppose that the population satisfies the ODE $y' = -y^3 + 6y^2 - 8y$. What are the minimum viable population and the maximum sustainable population of the species?*

First, we note that

$$y' = -y(y-1)(y-5). \tag{160}$$

This gives the equilibrium solutions $y(t) = 0$, $y(t) = 2$ and $y(t) = 4$. We examine the solution curves:

[Diagram]

The solution curves indicate that extinction is guaranteed if $0 \leq y(0) < 2$. Thus, the minimum viable population is $\boxed{2}$, in whatever units y is measured in. On the other hand, if $y(0) > 4$, then the population will gradually decrease to 4. Thus, the maximum sustainable population is $\boxed{4}$. □

2 Higher order linear ODEs

2.0 Background: Some complex analysis and linear algebra

Before moving on to higher order ODEs, we should review some math that is used in their study.

2.0.1 Euler's formula and complex solutions of real ODEs

As you probably learned in high school, in order to solve some equations involving real numbers, it is necessary to introduce objects called "imaginary numbers."

Definition 2.1 *The field of complex numbers is the Cartesian coordinate plane with the following definitions of addition and multiplication:*

$$\begin{aligned} (a,b)+(c,d) &= (a+c,b+d) \\ (a,b)\cdot(c,d) &= (ac-bd,ad+bc) \end{aligned}.$$

You might be wondering why we would define multiplication in such an odd way. The truth is that this is the *only* definition of multiplication that would allow for the usual properties of addition, subtraction, multiplication and division by all nonzero numbers.

Every complex number (a,b) can be written as

$$(a,b) = a\,(1,0) + b\,(0,1)\,. \tag{161}$$

The usual notation for complex numbers is to write $(1,0)$ as "1" and $(0,1)$ as "i," so that the number above can be written as

$$a+bi. \tag{162}$$

With these notations, it turns out that $i^2=-1$, and so we often refer to i as "$\sqrt{-1}$."

Definition 2.2 *Given a complex number $z=a+bi$, where a and b are real numbers, we call a the real part of z and b the imaginary part of z.*

The following theorem is one of the most important equations in all of mathematics.

Theorem 2.3 *(Euler's formula) Given any real number θ,*

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{163}$$

Proof Euler's formula follows from the following representations of the exponential, the cosine, and the sine as power series:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned} \quad . \tag{164}$$

(These representations come from the Maclaurin series, which is a topic of Calculus II.) Now, note that for any non-negative integer n,

$$i^n = \begin{cases} (-1)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} i & \text{if } n \text{ is odd} \end{cases} . \tag{165}$$

Based on this, for any real number θ,

$$\begin{aligned} e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!} &= \underbrace{\sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!}}_{n \text{ is even}} + \underbrace{\sum_{n=0}^{\infty} i^n \frac{\theta^n}{n!}}_{n \text{ is odd}} \\ &= \underbrace{\sum_{n=0}^{\infty} (-1)^{\frac{n}{2}} \frac{\theta^n}{n!}}_{n \text{ is even}} + \underbrace{\sum_{n=0}^{\infty} (-1)^{\frac{n-1}{2}} i \frac{\theta^n}{n!}}_{n \text{ is odd}} . \end{aligned} \tag{166}$$

Now, if n is even, then there exists an integer k such that $n = 2k$. At the same time, if n is odd, then there exists an integer l such that $n = 2l + 1$. Therefore, we can re-write this as

$$
\begin{aligned}
e^{i\theta} = \underbrace{\sum_{n=0}^{\infty} (-1)^{\frac{n}{2}} \frac{\theta^n}{n!}}_{n=2k} + \underbrace{\sum_{n=0}^{\infty} (-1)^{\frac{n-1}{2}} i \frac{\theta^n}{n!}}_{n=2l+1} \\
= \underbrace{\sum_{n=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!}}_{n=2k} + \underbrace{\sum_{n=0}^{\infty} (-1)^l i \frac{\theta^{2l+1}}{(2l+1)!}}_{n=2l+1} \\
= \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_{l=0}^{\infty} (-1)^l \frac{\theta^{2l+1}}{(2l+1)!} \\
= \cos\theta + i \sin\theta. \quad (167)
\end{aligned}
$$

□

(Note: I have given this proof in order to assuage your doubts, not you will need to be able to reproduce it on any assignment or test.)

Euler's formula allow us to convert between two different representations of a complex number: either as $re^{i\theta}$, where r is a real number (polar form), or as a linear combination of a real and imaginary part (Cartesian form).

Example 2.4 *Find the Cartesian forms of the following complex numbers:*

(i) $z = 2e^{i\frac{\pi}{4}}$

We apply Euler's formula:

$$
z = 2\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \boxed{\sqrt{2} + i\sqrt{2}}. \quad (168)
$$

(ii) $z = e^{i\pi}$

(iii) $z = 4e^{-i\frac{\pi}{3}}$

(iv) $z = \sqrt{i}$.

□

2.0.2 Linear independence

As mentioned in Section 1.4, a "linear combination" of finitely many mathematical objects $f_1, f_2, ..., f_n$ refers to an expression of the form $s_1 f_1 + s_2 f_2 + ... + s_n f_n$. One of our primary concerns will be linear combinations of functions with constants as coefficients.

Consider the following equation:

$$s_1 f_1(x) + s_2 f_2(x) + ... + s_n f_n(x) = 0. \tag{169}$$

Do there exist constants $s_1, s_2, ..., s_n$ that satisfy this equation for all values of x?

Well, yes. If $s_1 = s_2 = ... = s_n = 0$, then certainly this equation is satisfied. This is called the "trivial solution" of the equation above.

Alright, so, a harder question would be whether there exist constants that satisfy the above equation that are not all zero. For example, do there exist constants s_1, s_2, s_3, of which some are nonzero, such that

$$s_1 \sin x + s_2 \cos x + s_3 e^x = 0 \tag{170}$$

for all values of x? The answer to this question is dependent upon the functions being considered. Those sequences of functions which do have such constants are called "linearly dependent."

Definition 2.5 *Let f_1, f_2, ..., f_n be functions of x. We say that* <u>*f_1, f_2, ..., f_n are linearly dependent*</u> *provided that there exist constants $s_1, s_2, ..., s_n$ which are not all zero such that $s_1 f_1(x) + s_2 f_2(x) + ... + s_n f_n(x) = 0$ for all values of x. We say that the functions are* <u>*linearly independent*</u> *provided that they are not linearly dependent.*

To rephrase, $f_1, f_2, ..., f_n$ are linearly independent if and only if the only constants $s_1, s_2, ..., s_n$ such that $s_1 f_1(x) + s_2 f_2(x) + ... + s_n f_n(x) = 0$ for all x are the trivial solution.

2.0.3 Determinants

Definition 2.6 *An* <u>$m \times n$ *matrix*</u> *(pronounced "m by n matrix") is an assignment of some mathematical objects to each ordered pair (i, j), where $1 \le i \le m$ and $1 \le j \le n$. We say that an $m \times n$ matrix* <u>*has m rows*</u> *and* <u>*has n columns*</u>*.*

Matrices are generally visualized as rectangles made up of squares, each one labeled by (i, j), with larger values for i below smaller ones, and larger values for j to the right of smaller ones. In each square, the value corresponding to (i, j) is placed. (In other words, the value assigned to (i, j) is placed in the cell contained by the ith row and the jth column.)

For example, a 3×4 matrix of numbers making the following assignments:

$$\begin{array}{cc} (1,1) & 1 \\ (1,2) & 5 \\ (1,3) & -\frac{4}{3} \\ (1,4) & 4 \\ (2,1) & 0 \\ (2,2) & 4 \\ (2,3) & -2 \\ (2,4) & 3 \\ (3,1) & \sqrt{2} \\ (3,2) & 6 \\ (3,3) & e^2 \\ (3,4) & \pi \end{array} \tag{171}$$

would be visualized like this:

$$\begin{array}{|c|c|c|c|} \hline 1 & 5 & -\frac{4}{3} & 4 \\ \hline 0 & 4 & -2 & 3 \\ \hline \sqrt{2} & 6 & e^2 & \pi \\ \hline \end{array}. \tag{172}$$

However, we need not bother ourselves with drawing lines dividing the cells. Instead, we'll just write the assignments like this:

$$\begin{pmatrix} 1 & 5 & -\frac{4}{3} & 4 \\ 0 & 4 & -2 & 3 \\ \sqrt{2} & 6 & e^2 & \pi \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 5 & -\frac{4}{3} & 4 \\ 0 & 4 & -2 & 3 \\ \sqrt{2} & 6 & e^2 & \pi \end{bmatrix}. \tag{173}$$

We define two matrices as being equal if they are the same size and for each ordered pair (i, j), the two matrices associate (i, j) to equal mathematical objects. Thus, two matrices are equal only if *all* of their entries agree:

$$\begin{pmatrix} 1 & 0 & 9 \\ 0 & 6 & 7 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 9 \\ 0 & 6 & 6 \end{pmatrix}. \tag{174}$$

A common theme in mathematics is drawing information from expressions by associating simpler expressions to them.

For example, given a quadratic equation $ax^2 + bx + c = 0$, how can we tell whether the solutions will be real, non-real, or repeated? In this case, we look at the associated "discriminant," the expression $b^2 - 4ac$, and study it in order to learn about the original equation. Just from this single number, we can tell whether the equation has two real solutions (in the case that the discriminant is positive), and whether they are rational (if the discriminant is a nonzero perfect square) or irrational (if the discriminant is not a perfect square), one real solution (in the case that the discriminant is zero), or two non-real solutions (in the case that the discriminant is negative).

In this section, we'll see what we can learn about a square matrix from an associated expression known as its "determinant:"

Definition 2.7 *Let A be an $n \times n$ (square) matrix. The determinant of A is defined recursively as follows.*
(i) If $n = 2$, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{175}$$

then the determinant $\det(A) = ad - bc$.

(ii) If $n > 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \tag{176}$$

then the determinant

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{1i} \det(M_{1i}), \tag{177}$$

where M_{1i} is the submatrix defined via

$$M_{1i} = \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2(i-1)} & a_{2(i+1)} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3(i-1)} & a_{3(i+1)} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(i-1)} & a_{n(i+1)} & \dots & a_{nn} \end{pmatrix}. \tag{178}$$

This formula may seem somewhat complicated, and that's because it is. Thankfully, for this document, we will only consider determinants of 2×2 or 3×3 matrices, since computing determinants for larger matrices is intellectually easy, but deeply tiresome.

As the definition says, for a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{179}$$

the determinant is just

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{180}$$

On the other hand, for a 3×3 matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{181}$$

the determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{182}$$

To summarize: one takes each entry along the first row, with alternating $\pm$ signs, multiplies them by the determinant of the submatrix obtained by deleting the row and the column of the entry in question, and then adding it all up. With some practice, you'll see that the process is easy, but it can get very tedious.

Example 2.8 *Compute the following determinants:*

$$
\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5
$$

$$
\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0
$$

$$
\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = -7
$$

$$
\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2
$$

$$
\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1
$$

$$
\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -2
$$

$$
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0 \qquad . \qquad (183)
$$

$$
\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 24
$$

$$
\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 18
$$

$$
\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 30
$$

$$
\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0
$$

2.0.4 Wronskians

An important special case of determinants is the concept of the Wronskian.

Definition 2.9 *Let $f_1, f_2, ..., f_n$ be nth order differentiable functions. The* *Wronskian of $f_1, f_2, ..., f_n$* *is the determinant*

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}.$$

[*Lecture over; continued from here in the next lecture*]

As the following theorem indicates, computing the Wronskian of a sequence of functions can give some insight as to whether they are linearly independent.

Theorem 2.10 *Let $f_1, f_2, ..., f_n$ be nth order differentiable functions. If the Wronskian $W(f_1, f_2, ..., f_n) \neq 0$, then $f_1, f_2, ..., f_n$ are linearly independent.*

Example 2.11 *Determine whether the following sequences of functions are linearly independent:*

(i) $\sin x, \cos x$.

(ii) e^x, e^{2x}, e^{4x}.

(iii) $\sin^2 x, \cos^2 x, \cos(2x)$.

(iv) $-1, \tan^2 x, -\sec^2 x$.

(v) $1, t, t^2$.

(vi) $x^2, x|x|$.

□

2.1 Second order linear ODEs

Recall from Section 1.4 that a linear differential operator is a linear combination of derivative operators with functions as coefficients. For example,

$$L = e^x \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} - x^2 \frac{\mathrm{d}^0}{\mathrm{d}x^0} \tag{184}$$

is a linear differential operator. An ODE is called linear if it can be written in the form $Ly = q(x)$, where L is a linear differential operator and q is any function.

Definition 2.12 *Let $Ly = q(x)$ be a linear ODE. The* <u>*inhomogeneous term*</u> *of the ODE is the function $q(x)$.*

In other words, the inhomogeneous term of a linear ODE is the sum of all the functions that are not being multiplied by y or a derivative of y. For example:

$$\begin{array}{cc} y'' + x^2 y' + 9y = e^x & \text{inhomogeneous term: } e^x \\ y' - y = \sin x & \text{inhomogeneous term: } \sin x \\ y'' + y = 0 & \text{inhomogeneous term: } 0 \\ y'' - \frac{1}{x}y' + \frac{1}{x^2}y - \tan^{-1}x + x = 0 & \text{inhomogeneous term: } \tan^{-1}x - x \end{array} . \tag{185}$$

WARNING: As mentioned before, the following term has nothing to do with the so-called "homogeneous substitution" mentioned in Section 1.5.

Definition 2.13 *A linear ODE is called* <u>*homogeneous*</u> *provided that its inhomogeneous term is the constant function 0.*

For the time being, we will concentrate on homogeneous linear ODEs. We'll discuss non-homogeneous linear ODEs later.

The following theorems are essential to understanding the theory of linear homogeneous ODEs.

Theorem 2.14 *(Superposition) Let $Ly = 0$ be a linear homogeneous ODE. If $y_1, y_2, ..., y_n$ are solutions of $Ly = 0$, then any linear combination of $y_1, y_2, ..., y_n$ with constants as coefficients is also a solution of $Ly = 0$.*

For example, $y_1 = \sin x$ and $y_2 = \cos x$ are both solutions to the linear homogeneous ODE $y'' + y = 0$. For this ODE, the theorem simply states that, given any real constants c_1 and c_2, the function $y = c_1 \sin x + c_2 \cos x$ is also a solution. Further, the following theorem indicates that these linear combinations are the *only* solutions, as long as the solutions are linearly independent.

Theorem 2.15 *(Solution theorem for linear homogeneous ODEs) Let* $Ly = 0$ *be an* n*th order linear homogeneous ODE. If* $y_1, y_2, ..., y_n$ *are linearly independent solutions of* $Ly = 0$*, then every solution of* $Ly = 0$ *can be written in the form* $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ *for some constants* $c_1, c_2, ..., c_n$.

In other words, if we can find two linearly independent solutions of a second order linear homogeneous ODE, then the general solution is just all of the linear combinations of those two. For this reason, we often call the linearly independent solutions "basis solutions."

One more theorem about linear ODEs is that a unique solution of associated IVPs will always exist, given some conditions.

Theorem 2.16 *(Existence and uniqueness theorem for IVPs associated to linear ODEs) Suppose that* $p_0, p_1, ..., p_{n-1}$ *are continuous on some interval* I *on the real line. Suppose that* a *is a real number in* I*, and* $b_0, b_1, ..., b_{n-1}$ *are constants. Given any function* f*, the linear ODE*

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + p_{n-2}(x) y^{(n-2)} + ... + p_1(x) y' + p_0(x) y = f(x)$$

has exactly one solution defined on the interval I *that satisfies the initial conditions*

$$\begin{aligned} y(a) &= b_0 \\ y'(a) &= b_1 \\ y''(a) &= b_2 \\ &\vdots \\ y^{(n-1)}(a) &= b_{n-1} \end{aligned} \quad .$$

2.2 Constant coefficient second order linear ODEs

As it turns out, even taking the subset of ODEs that are "second order," "linear," and "homogeneous" is still too broad of a problem to easily approach. We'll need one more condition: constant coefficients. In other words, for the time being, we will demand that every linear ODE be writable in the form

$$p_2 y'' + p_1 y' + p_0 y = 0, \tag{186}$$

where p_0, p_1, and p_2 are constants.

2.2.1 Solving constant coefficient equations

The strategy for solving a second order linear homogeneous ODE with constant coefficients is as follows. Given the ODE in the form

$$ay'' + by' + cy = 0, \tag{187}$$

propose the solution $y = e^{rx}$ for some constant r. When we put this information in, the equation becomes

$$\begin{aligned} ar^2 e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ e^{rx}\left(ar^2 + br + c\right) &= 0 \end{aligned}. \tag{188}$$

At this point, we can recognize that $e^{rx} \neq 0$ for all x, so this is equivalent to the equation

$$ar^2 + br + c = 0, \tag{189}$$

a quadratic equation. This is called the "characteristic equation" of the ODE.

Definition 2.17 *Let $a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$ be a linear homogeneous ODE with constant coefficients. The characteristic equation of the ODE is the nth degree polynomial equation*

$$a_n r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0. \tag{190}$$

Each root of the characteristic equation will describe a basis solution. We'll use this to find the general solution.

Example 2.18 *(Exercise 2.2.6) Find the general solution of the linear homogeneous ODE* $2y'' + 2y' - 4y = 0$*.*

To do this, we'll first find the characteristic equation:

$$2r^2 + 2r - 4 = 0. \tag{191}$$

This can be factored as

$$2\left(r-1\right)\left(r+2\right) = 0. \tag{192}$$

From this we get the roots $r_1 = 1$*, and* $r_2 = -2$*. This describes two different solutions of the ODE:*

$$\begin{aligned} y_1 &= e^x \\ y_2 &= e^{-2x} \end{aligned}. \tag{193}$$

It turns out that these functions are linearly independent, as we can see by computing the Wronskian:

$$W\left(e^x, e^{-2x}\right) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = -2e^{-x} - e^{-x} = -3e^{-x} \neq 0. \tag{194}$$

(In fact, any family of functions of the form e^{kx} *with different values of* k *will be linearly independent, so we need not check this every time.) By the solution theorem for linear homogeneous ODEs, the general solution is*

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x}}. \tag{195}$$

□

As the previous example illustrates, the general solution of a second order linear homogeneous ODE will have two arbitrary real constants. Thus, any IVP involving a second order linear homogeneous ODE will require two different initial conditions.

Example 2.19 *Solve the following IVP:*

$$\begin{aligned} y'' + 5y' + 6 &= 0 \\ y(0) &= 1 \\ y'(0) &= -1 \end{aligned} . \tag{196}$$

Again, we first need to solve the characteristic equation:

$$r^2 + 5r + 6 = 0. \tag{197}$$

This factors as

$$(r+2)(r+3) = 0, \tag{198}$$

and so the roots are $r_1 = -2$ and $r_2 = -3$. This gives rise to two different solutions:

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^{-3x} \end{aligned} . \tag{199}$$

Again, these are linearly independent, so the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x}. \tag{200}$$

To solve the IVP, we first note that $y' = -2c_1 e^{-2x} - 3c_2 e^{-3x}$ and apply the initial conditions:

$$\begin{aligned} 1 &= y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \\ -1 &= -2c_1 e^0 - 3c_2 e^0 = -2c_1 - 3c_2 \end{aligned} . \tag{201}$$

Now finding c_1 and c_2 comes down to solving this system of equations. The solution is $c_1 = 2$, $c_2 = -1$, and so the particular solution that solves the IVP is

$$\boxed{y = 2e^{-2x} - e^{-3x}}. \tag{202}$$

□

In the previous examples, the characteristic equation always had distinct real roots. What happens when the roots are repeated? In this case, we end up with an equation that looks like

$$(r-k)^2 = 0, \tag{203}$$

Giving only the solution $y_1 = e^{kx}$. According to the solution theorem for linear homogeneous ODEs, we require one more linearly independent solution to find the general solution. This is acquired by simply multiplying by a factor of x: $y_2 = xe^{kx}$.

Example 2.20 *(Exercise 2.2.102) Find the general solution of $y'' - 6y' + 9y = 0$.*

The characteristic equation is

$$r^2 - 6r + 9 = 0. \tag{204}$$

This factors as

$$(r-3)^2 = 0, \tag{205}$$

giving the single root $r = 3$, with multiplicity 2. We acquire the solution $y_1 = e^{3x}$, and to find the other basis solution, we multiply by a factor of x: $y_2 = xe^{3x}$. The Wronskian shows that these are linearly independent:

$$W\left(e^{3x}, xe^{3x}\right) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)\,e^{3x} \end{vmatrix} = (3x+1)\,e^{6x} - 3xe^{6x} = e^{6x} \neq 0. \tag{206}$$

Thus, the solution theorem for linear homogeneous ODEs indicates that the general solution is

$$\boxed{y = c_1 e^{3x} + c_2 xe^{3x} = (c_1 + c_2 x)\,e^{3x}}. \tag{207}$$

□

Example 2.21 *(Exercise 2.2.8) Solve the IVP*

$$\begin{aligned} y'' - 8y' + 16y &= 0 \\ y(0) &= 2 \\ y'(0) &= 0 \end{aligned} \quad . \tag{208}$$

The characteristic equation is

$$r^2 - 8r + 16 = 0, \tag{209}$$

which factors as

$$(r-4)^2 = 0. \tag{210}$$

This gives us the single root $r = 4$*, with multiplicity* 2*. This gives rise to the basis solution* $y_1 = e^{4x}$*, but the other basis solution is found by multiplying by another factor of* x*:* $y_2 = xe^{4x}$*. Thus, the general solution is*

$$y = c_1 e^{4x} + c_2 x e^{4x} = (c_1 + c_2 x)\, e^{4x}. \tag{211}$$

To solve the IVP, we note that $y' = 4\,(c_1 + c_2 x)\, e^{4x} + c_2 e^{4x} = (4c_1 + c_2 + 4c_2 x)\, e^{4x}$ *and write the system of equations*

$$\begin{array}{c} 2 = (c_1 + 0)\, e^0 = c_1 \\ 0 = (4c_1 + c_2 + 0)\, e^0 = 4c_1 + c_2 \end{array}. \tag{212}$$

This gives us the solution $c_1 = 2$*.* $c_2 = -8$*, and so the particular solution that solves the IVP is*

$$\boxed{y = (2 - 8x)\, e^{4x}}. \tag{213}$$

□

2.2.3 Complex roots

What if the characteristic equation has no real roots? In this case, we take the complex roots and use Euler's formula to write the solution in Cartesian form. We then apply the following theorem:

Theorem 2.22 *If* $Ly = 0$ *is a linear homogeneous ODE with constant, real-valued coefficients, and* $u\,(x) + iv\,(x)$ *is a complex solution of the ODE, then* $u\,(x)$ *and* $v\,(x)$ *are solutions of the ODE.*

Example 2.23 *Find the general solution of the ODE* $y'' + 9y = 0$*.*

We first examine the characteristic equation:

$$r^2 + 9 = 0. \tag{214}$$

This has no real roots, but has the imaginary roots $r = \pm 3i$. Take one of these use Euler's formula to write e^{rx} in Cartesian form:

$$e^{3ix} = \cos(3x) + i\sin(3x). \tag{215}$$

Now, by the theorem above, since $\cos(3x) + i\sin(3x)$ are solutions, $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$ are solutions. Ergo, by the solution theorem for linear homogeneous ODEs, the general solution is

$$\boxed{y = c_1\cos(3x) + c_2\sin(3x)}. \tag{216}$$

□

Example 2.24 *Find the general solution of the ODE $y'' + y' + y = 0$.*

The characteristic equation is

$$r^2 + r + 1 = 0. \tag{217}$$

By the quadratic formula, the solutions of this equation are

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm i\sqrt{3}}{2}. \tag{218}$$

We take one of these and write e^{rx} using Euler's formula:

$$\begin{aligned} e^{-\frac{1+i\sqrt{3}}{2}x} &= e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} = e^{-\frac{1}{2}x}e^{i\left(-\frac{\sqrt{3}}{2}x\right)} \\ &= e^{-\frac{1}{2}x}\left(\cos\left(-\frac{\sqrt{3}}{2}x\right) + i\sin\left(-\frac{\sqrt{3}}{2}x\right)\right) \\ &= e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) - ie^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right). \end{aligned} \tag{219}$$

By the theorem, $y_1 = e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right)$ and $y_2 = -e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$ are solutions, *so the general solution is*

$$\boxed{y = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)}. \tag{220}$$

□

2.3 Higher order linear ODEs

2.3.2 Constant coefficient higher order ODEs

The procedure for solving a linear homogeneous ODE with constant coefficients of order larger than two is very much the same as the case for the second order equation: find the characteristic equation, factor it, and use the roots to construct as many basis solutions as the order of the ODE.

Example 2.25 *(Exercise 2.3.1) Find the general solution of* $y^{(3)} - y'' + y' - y = 0$.

Example 2.26 *(Exercise 2.3.2) Find the general solution of* $y^{(4)} - 5y^{(3)} + 6y'' = 0$.

Example 2.27 *(Exercise 2.3.3) Find the general solution of* $y^{(3)} + 2y'' + 2y' = 0$.

Example 2.28 *(Exercise 2.3.11) Find the general solution of* $y^{(5)} - y^{(4)} = 0$.

Example 2.29 *Solve the IVP*

$$\begin{gathered} 2y^{(3)} - 3y'' - 2y' = 0 \\ y(0) = 1 \\ y'(0) = 0 \\ y''(0) = 1 \end{gathered} \quad . \tag{221}$$

2.5 Nonhomogeneous equations

In the case that we are dealing with a linear homogeneous ODE with constant coefficients which is *not* homogeneous, the situation is a bit more complicated.

2.5.1 Solving nonhomogeneous equations

First, we need to introduce some terminology.

Definition 2.30 *Let $Ly = f(x)$ be a linear ODE. The* <u>*associated homogeneous ODE*</u> *is the linear homogeneous ODE $Ly = 0$.*

Definition 2.31 *Let $Ly = f(x)$ be a linear ODE. The* <u>*complementary solution of the ODE*</u> *is the general solution of the associated homogeneous ODE.*

Of course, if a linear ODE is homogeneous, then it is its own associated homogeneous ODE. Note the following linguistic annoyance: the complementary solution of a non-homogeneous ODE is a solution of its associated homogeneous ODE. This means that *the complementary solution of a non-homogeneous ODE is not actually a solution of the ODE.*

The Achilles' tendon of the method for solving linear homogeneous ODEs is the following theorem.

Theorem 2.32 *(Solution theorem for linear non-homogeneous ODEs) Suppose that $Ly = f(x)$ is a linear ODE. Let y_c be the complementary solution of the ODE, and let y_p be any particular solution of the ODE whatsoever. The general solution of the ODE is $y = y_c + y_p$.*

Therefore, as long as the associated homogeneous equation can be solved, and a single particular solution can be found, the ODE can be solved. We've already discussed how to solve the associated homogeneous equation. All that's left is how to figure out a particular solution.

2.5.2 Undetermined coefficients

The method we'll use for solving a linear non-homogeneous ODE with constant coefficients is called the "method of undetermined coefficients." (We'll also discuss Laplace transform methods, but not until the next chapter.) The method works for any non-homogeneous ODE $Ly = f(x)$, as long as the inhomogeneous term is a product and/or sum of only exponentials, polynomials, sines and/or cosines.

The method of undetermined coefficients is as follows: we will create a guess as to what the particular solution is, calling it y_T. This guess will have some unknown constants in it: the "undetermined coefficients." The guess is completely based on the inhomogeneous term. It goes like this:

If the inhomogeneous term is:	**then make the guess** $y_T =$
e^{kx},	Ae^{kx}.
a polynomial of degree m,	$A_m x^m + A_{m-1}x^{m-1} + ... + A_1 x + A_0$.
$\sin(kx)$ or $\cos(kx)$,	$A\cos(kx) + B\sin(kx)$.
a product of the forms above,	a product of the guesses above.
a sum of the forms above,	a sum of the forms above.

(222)

Here are some examples of linear non-homogeneous ODEs, and the initial guess that should be made to solve them:

$$
\begin{array}{cc}
y'' - 2y' - 3y = e^{4x} & y_T(x) = Ae^{4x} \\
y^{(4)} - 2y'' + y = \cos(3x) & y_T(x) = A\cos(3x) + B\sin(3x) \\
y'' - y' - 2y = x^2 + 4 & y_T(x) = Ax^2 + Bx + C \\
y'' + 5y' + 6y = (x+1)^3 & y_T(x) = Ax^3 + Bx^2 + Cx + D \\
y^{(3)} - 16y'' + 64y' = e^x \cos x & y_T(x) = (A\cos x + B\sin x)\, e^x \\
y'' + 3y' + 2y = e^x\left(x^2 + \cos(2x)\right) & y_T(x) = \left(Ax^2 + Bx + C + D\cos(2x) + E\sin(2x)\right) e^x \\
y'' + 2y' + 5y = x^2 e^x & y_T(x) = \left(Ax^2 + Bx + C\right) e^x \\
y^{(4)} + 32y'' + 256y = x^2 \sin x & y_T(x) = \left(Ax^2 + Bx + C\right)(D\cos x + E\sin x) \\
y^{(4)} - y = \cos(2x) - x & y_T(x) = A\cos(2x) + B\sin(2x) + Cx + D
\end{array}
. \tag{223}
$$

Example 2.33 *Find the general solution of $y'' + 3y' + 4y = 3x + 2$.*

$$\boxed{y = c_1 e^{-\frac{3}{2}x} \sin\left(\frac{\sqrt{7}}{2}x\right) + c_2 e^{-\frac{3}{2}x} \cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{3}{4}x - \frac{1}{16}}. \tag{224}$$

□

Example 2.34 *Find the general solution of $y'' + 2y' + 2y = \sin(3x)$.*

$$\boxed{y = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x - \frac{7}{85} \sin(3x) - \frac{6}{85} \cos(3x)}. \tag{225}$$

□

Example 2.35 *Find the general solution of $y'' - 3y' + 2y = 3e^{-x} - 10\cos(3x)$.*

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos(3x) + \frac{9}{13}\sin(3x)}. \tag{226}$$

□

There is a situation in which the guess will not work properly. This occurs whenever this is a nonzero function which could be described by both the complementary solution and the guess. This is called “duplication.” In order to deal with this situation, we simply multiply by factors of x until duplication no longer occurs.

Example 2.36 *(Exercise 2.5.3) Find the general solution of $y'' - 4y' + 4y = e^{2x}$.*

$$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2}x^2 e^{2x}}. \tag{227}$$

□

Example 2.37 *(Exercise 2.5.4) Solve the IVP*

$$\begin{aligned} y'' + 9y &= \cos(3x) + \sin(3x) \\ y(0) &= 2 \\ y'(0) &= 1 \end{aligned} \quad . \tag{228}$$

$$\boxed{y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{6}x\sin(3x) - \frac{1}{6}x\cos(3x)}. \tag{229}$$

□

Example 2.38 *Solve the IVP*

$$\begin{aligned} y^4 - y &= 5 \\ y(0) &= 0 \\ y'(0) &= 0 \\ y''(0) &= 0 \\ y^{(3)}(0) &= 0 \end{aligned} \quad . \tag{230}$$

$$\boxed{y = \frac{5}{4}e^{-x} + e^x + 2\cos x - 4}. \tag{231}$$

□

Example 2.39 *Find the general solution of* $y^{(3)} + y'' = 3e^x + 4x^2$.

$$\boxed{y = c_1 + c_2 x + c_3 e^{-x} + \frac{3}{2}e^x + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4}. \tag{232}$$

□

Example 2.40 *Solve the IVP*

$$\begin{aligned} y^{(4)} - 4y'' &= x^2 \\ y(0) &= 1 \\ y'(0) &= 1 \\ y''(0) &= -1 \\ y^{(3)}(0) &= -1 \end{aligned} \quad . \tag{233}$$

$$\boxed{y = -\frac{11}{64}e^{2x} - \frac{9}{192}e^{-2x} - \frac{1}{48}x^4 - \frac{1}{16}x^2 + \frac{5}{4}x + \frac{39}{32}}. \tag{234}$$

□

6 The Laplace transform

In this chapter, we'll discuss the Laplace transform, which grants another method for solving linear ODEs with constant coefficients. The important quality of this method is that it does not require that the inhomogeneous term of the ODE be differentiable, or even continuous.

6.1 The Laplace transform

6.1.1 The transform

Definition 6.1 *Let f be a function of the single variable t, defined for $t \geq 0$. The Laplace transform of f is the function F of the single variable s defined via*

$$F(s) = \int_0^\infty e^{-st} f(t)\ \mathrm{d}t, \tag{235}$$

where the improper integral converges.

We also sometimes denote the Laplace transform of a function f by $\mathcal{L}(f(t))$, but this notation comes with the disadvantage of not explicitly stating the independent variable s.

We begin by computing some examples.

Example 6.2 *Find the Laplace transform of the function $f(t) = 1$.*

$$\boxed{F(s) = \tfrac{1}{s} \text{ for } s > 0}. \tag{236}$$

□

Example 6.3 *Find the Laplace transform of the function $f(t) = t$.*

$$\boxed{F(s) = \tfrac{1}{s^2} \text{ for } s > 0}. \tag{237}$$

□

Example 6.4 *Given a real valued constant a, Find the Laplace transform of the function $f(t) = e^{at}$.*

$$\boxed{F(s) = \frac{1}{s-a} \textit{ for } s > a}. \tag{238}$$

□

Definition 6.5 *The* <u>*Heaviside function*</u>*, or* <u>*unit step function*</u> *is the piecewise function*

$$u(t) = \begin{cases} 0 & \textit{if } t < 0 \\ 1 & \textit{if } t \geq 0 \end{cases}. \tag{239}$$

Example 6.6 *Given a real valued constant a, find the Laplace transform of the function $f(t) = u(t-a)$.*

$$\boxed{F(s) = \frac{e^{-as}}{s} \textit{ for } s > 0}. \tag{240}$$

□

The following table summarizes a number of different Laplace transforms.

$$\begin{array}{cc} f(t) = 1 & F(s) = \frac{1}{s} \text{ for } s > 0 \\ f(t) = e^{at}, \text{ for a real value } a & F(s) = \frac{1}{s-a} \text{ for } s > a \\ f(t) = t^n, \text{ for an integer } n \geq 0 & F(s) = \frac{n!}{s^{n+1}} \text{ for } s > 0 \\ f(t) = u(t-a), \text{ for a real value } a & F(s) = \frac{e^{-as}}{s} \text{ for } s > 0\,. \\ f(t) = \cos(kt), \text{ for a real value } k & F(s) = \frac{s}{s^2+k^2} \text{ for } s > 0 \\ f(t) = \sin(kt), \text{ for a real value } k & F(s) = \frac{k}{s^2+k^2} \text{ for } s > 0 \\ f(t) = e^{at} g(t) & F(s) = G(s-a) \end{array} \tag{241}$$

For linear combinations of the functions above, we can use the following theorem.

Theorem 6.7 *Let $f(t)$ and $g(t)$ be functions, and let a and b be constants. In that case,*

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))\,. \tag{242}$$

The previous theorem is often called the "linearity" of the Laplace transform.

Example 6.8 *(Exercise 6.1.5) Determine the Laplace transform of the function* $f(t) = 3 + t^5 + \sin(\pi t)$.

$$\boxed{F(s) = \frac{3}{s} + \frac{120}{s^6} + \frac{\pi}{s^2+\pi^2} \textit{ for } s > 0}. \tag{243}$$

□

Example 6.9 *Find the Laplace transform of the function*

$$f(t) = \begin{cases} 1 & \textit{if } 1 \leq t \leq 2 \\ 0 & \textit{if } t < 1 \textit{ or } 2 < t \end{cases}. \tag{244}$$

This can be handled in multiple ways. One way would be to recognize that $f(t) = u(t-1) - u(t-2)$. *By linearity,*

$$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(u(t-1)) - \mathcal{L}(u(t-2)) = \boxed{\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}}, \tag{245}$$

for $s > 0$. □

Example 6.10 *Suppose that* $g(t)$ *is a function whose Laplace transform is* $G(s)$, *and* a *is a real value. Find the Laplace transform of* $f(t) = u(t-a)\, g(t-a)$.

$$\boxed{F(s) = e^{-as}G(s)}. \tag{246}$$

□

6.1.3 The inverse transform

It is often necessary to reverse the process of the Laplace transform, returning to the original function. We sometimes denote the inverse Laplace transform of a function $F(s)$ as $\mathcal{L}^{-1}(F(s))$.

Example 6.11 *Find the inverse Laplace transform of* $F(s) = \frac{1}{s+1}$.

$$\boxed{f(t) = e^{-t}}. \tag{247}$$

□

Example 6.12 *(Exercise 6.1.9) Find the inverse Laplace transform of the function* $F(s) = \frac{4}{s^2-9}$.

We note that

$$F(s) = \frac{4}{(s-3)(s+3)}. \tag{248}$$

We now use partial fraction decomposition to write this as

$$F(s) = \frac{2}{3}\frac{1}{s-3} - \frac{2}{3}\frac{1}{s+3}. \tag{249}$$

As the Laplace transform is linear, so too is the inverse Laplace transform. Therefore, the inverse Laplace transform is

$$\boxed{f(t) = \tfrac{2}{3}e^{3t} - \tfrac{2}{3}e^{-3t}}. \tag{250}$$

□

Example 6.13 *Find the inverse Laplace transform of* $F(s) = \frac{s^2+s+1}{s^3+s}$.

$$\boxed{f(t) = 1 + \sin t} \tag{251}$$

□

Example 6.14 *Find the inverse Laplace transform of* $F(s) = \frac{1}{s^2+4s+8}$.

$$\boxed{f(t) = \tfrac{1}{2}e^{-2t}\sin(2t)}. \tag{252}$$

□

Example 6.15 *(Exercise 6.1.13) Find the inverse Laplace transform of the function* $F(s) = \frac{s}{(s^2+s+2)(s+4)}$.

□

Example 6.16 *Find the inverse Laplace transform of* $F(s) = \frac{1}{(s-2)^4}$.

□

Example 6.17 *Find the inverse Laplace transform of* $F(s) = \frac{e^{-2s}}{s^2+1}$.

□

6.2 Transforms of derivatives and ODEs

Now that we know how to compute the Laplace transform, let's see how it can help in solving ODEs.

6.2.1 Transforms of derivatives

Let $f(t)$ be a function. Suppose that we desire to take the Laplace transform of $f'(t)$. We note that

$$\mathcal{L}\left(f'\left(t\right)\right) = \int_0^\infty e^{-st} f'\left(t\right)\ \mathrm{d}t = \lim_{b\to\infty} \int_0^b e^{-st} f'\left(t\right)\ \mathrm{d}t. \tag{253}$$

We compute this by integration by parts:

$$\begin{aligned} u &= e^{-st} & \mathrm{d}v &= f'\left(t\right)\ \mathrm{d}t \\ \mathrm{d}u &= -se^{-st}\,\mathrm{d}t & v &= f\left(t\right) \end{aligned} \tag{254}$$

$$\begin{aligned} &\lim_{b\to\infty}\left(e^{-st} f\left(t\right)\Big|_0^b + s\int_0^b e^{-st} f\left(t\right)\ \mathrm{d}t\right) \\ &\qquad = \lim_{b\to\infty}\left(e^{-sb} f\left(b\right) - e^{-s(0)} f\left(0\right)\right) + s\lim_{b\to\infty}\int_0^b e^{-st} f\left(t\right)\ \mathrm{d}t. \end{aligned} \tag{255}$$

If $\lim_{b\to\infty} e^{-sb} f\left(b\right) = 0$, then this becomes

$$-f\left(0\right) + s\int_0^\infty e^{-st} f\left(t\right)\ \mathrm{d}t = -f\left(0\right) + s\mathcal{L}\left(f\left(t\right)\right). \tag{256}$$

This allows us to compute the Laplace transform of a derivative:

$$\begin{aligned}\mathcal{L}\left(f'\left(t\right)\right) &= sF\left(s\right) - f\left(0\right)\\ \mathcal{L}\left(f''\left(t\right)\right) &= s^2F\left(s\right) - sf\left(0\right) - f'\left(0\right)\\ \mathcal{L}\left(f^{(3)}\left(t\right)\right) &= s^3F\left(s\right) - s^2f\left(0\right) - sf'\left(0\right) - f''\left(0\right)\\ &\vdots\\ \mathcal{L}\left(f^{(n)}\left(t\right)\right) &= s^nF\left(s\right) - \sum_{i=0}^{n-1} s^{n-i-i}f^{(i)}\left(0\right).\end{aligned} \tag{257}$$

6.2.2 Solving ODEs with the Laplace transform

Example 6.18 *Solve the following IVP:*

$$\begin{aligned}x'' + 4x &= 2\\ x(0) &= 3\\ x'(0) &= -1\end{aligned} \tag{258}$$

□

Example 6.19 *Solve the following IVP:*

$$\begin{aligned}y'' - 10y' + 9y &= 5t\\ y\left(0\right) &= -1 \quad .\\ y'\left(0\right) &= 2\end{aligned} \tag{259}$$

$$\boxed{y = \tfrac{50}{81} + \tfrac{5}{9}t + \tfrac{31}{81}e^{9t} - 2e^t}. \tag{260}$$

□

6.2.3 Using the Heaviside function

When given a linear ODE with constant coefficients that is non-homogeneous, if the inhomogeneous term is not continuous, then we have no choice but to use the Laplace transform methods.

Example 6.20 *Solve the following IVP:*

$$\begin{aligned} y' + 9y &= u\,(t-1) \\ y(0) &= 1 \end{aligned} \tag{261}$$

$$\boxed{y = u\,(t-1)\left(\tfrac{1}{9} + \tfrac{8}{9}e^{-9t}\right)}. \tag{262}$$

□

Example 6.21 *Solve the following IVP:*

$$\begin{aligned} x'' - x &= (t^2-1)\,u\,(t-1) \\ x(0) &= 1 \\ x'(0) &= 1 \end{aligned} \quad . \tag{263}$$

$$\boxed{x = e^t + u\,(t-1)\left(-2 - 2(t-1) - (t-1)^2 + 2e^{t-1}\right)}. \tag{264}$$

□

Example 6.22 *Given the function*

$$f(t) = \begin{cases} 0 & \textit{if } t < 1 \\ t-1 & \textit{if } 1 \le t < 2 \\ 1 & \textit{if } 2 \le t \end{cases}, \tag{265}$$

solve the following IVP:

$$\begin{aligned} x'' &= f(t) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned} \, . \tag{266}$$

□

6.4 Dirac delta and impulse response

6.4.1 Rectangular pulse

The "Dirac delta" is a concept first introduced by the physicist Paul Dirac. It's original **motivation** was the following. Imagine a function called d_s which is zero except over some closed interval $[-s, s]$, where it takes on the constant value $a > 0$:

$$r_s(t) = \begin{cases} 0 & \text{if } t < -s \text{ or } s < t \\ \frac{1}{2s} & \text{if } -s \leq t \leq s \end{cases}. \tag{267}$$

Notice that $\int_{-\infty}^{\infty} r_s \, \mathrm{d}t = \int_{-s}^{s} r_s \, \mathrm{d}t = 1$ for every value of s. The Dirac delta $\delta(t)$ was designed to be a kind of "limit" of these functions as s approaches 0. This brings up several issues, of which here a just a few:

1. What does a limit of functions even mean?
2. How do we know that such a limit exists?
3. If t is used to represent time, then how could $\delta(t)$ have a value of 1 for exactly zero time? Isn't that just the same as it always being zero?

There is no satisfying answer to these questions that doesn't involve very high level mathematics. For now, we will just say that the Dirac delta is not a function. It is a **thing** that satisfies some properties. Nonetheless, some humans still prefer to call it "the Dirac delta function," producing such absurd sentences as "The Dirac delta function is not a function." The particular human writing this text will abstain from this terminological monstrosity, preferring instead to simply call it "the Dirac delta." (Exception: the heading of the next subsection, which is taken from the textbook.)

6.4.2 The delta function

We will not go into great detail on the theoretical underpinnings of how the Dirac delta could exist, despite not being a function. The mathematical formalization of the Dirac delta does exist, but it's not the point of the class. For now, we will simply define the Dirac delta as a **thing** that has what is commonly called the "sifting property."

Definition 6.23 *The* <u>*Dirac delta*</u> *is a symbol* $\delta(t)$ *such that for any non-negative real value* a *and any continuous function* $f(t)$*,*

$$\int_0^\infty f(t)\,\delta(t-a)\ \mathrm{d}t = f(a)\,. \tag{268}$$

In particular, we can discuss the Laplace transform of a Dirac delta:

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a) = e^{-as}. \tag{269}$$

As a special case, this means that $\mathcal{L}(\delta(t)) = 1$.

The Dirac delta can certainly come up in applications of ODEs. For example, the Dirac delta can be used to model an effect that is significant, but lasts for almost no time, like a sudden voltage spike in an electrical circuit, or the force of a bat hitting a baseball. For this reason, we'll discuss IVPs that involve the Dirac delta. These types of IVPs must always be solved by Laplace transform methods.

Example 6.24 *Solve the following IVP:*

$$\begin{aligned} x'' + 4x &= \delta(t) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned} \tag{270}$$

$$\boxed{x(t) = \tfrac{1}{2}\sin(2t)}. \tag{271}$$

□

Example 6.25 *Solve the following IVP:*

$$\begin{aligned} x'' + 2x' + x &= t + \delta(t) \\ x(0) &= 0 \\ x'(0) &= 1 \end{aligned} \tag{272}$$

$$\boxed{x(t) = t - 2 + (3t+2)\,e^{-t}}. \tag{273}$$

□

Example 6.26 *Solve the following IVP:*

$$\begin{aligned} x'' + 4x' + 4x &= 1 + \delta(t-2) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned} \tag{274}$$

$$\boxed{x(t) = \tfrac{1}{4} + \tfrac{1}{4}e^{-2t}\left(4e^{4}(t-2)\,u(t-2) - 2t - 1\right)}. \tag{275}$$

□

Example 6.27 *Solve the following IVP:*

$$\begin{aligned} x'' + 4x &= \delta(t) + \delta(t-\pi) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned} \tag{276}$$

$$\boxed{x(t) = \tfrac{1}{2}\left(1 + u(t-\pi)\right)\sin(2t)}. \tag{277}$$

□

Example 6.28 *Solve the following IVP:*

$$\begin{aligned} x'' + 2x' + 2x &= 2\delta(t-\pi) \\ x(0) &= 0 \\ x'(0) &= 0 \end{aligned} \tag{278}$$

$$\boxed{x(t) = -2u(t-\pi)\,e^{\pi-t}\sin t}. \tag{279}$$

□

7 Power series methods

Every linear homogeneous ODE takes the form

$$p_n(x)\,y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \dots + p_2(x)\,y'' + p_1(x)\,y' + p_0(x)\,y = 0, \quad (280)$$

where the coefficients $p_0, p_1, \dots, p_n$ are all functions of x. Thus far, we have only discussed the case where these coefficients are constant functions. If they are not all constant, then the situation can be far more difficult. In this chapter, we'll discuss what is essentially a last resort for solving a linear homogeneous ODE when no other method can be effective, particularly when some of the coefficients are non-constants.

7.1 Power series

The entirety of this subsection is material from Calculus II. If you need to review this, I'd recommend the notes that I created when I taught the class in Fall of 2020:

`https://tinyurl.com/sullivanCalculus2Notes`

There is also an appendix at the end of these notes that lists a number of common power series.

7.2 Series solutions of linear second order ODEs

Suppose that we are dealing with a second-order linear homogeneous ODE:

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = 0, \tag{281}$$

where p, q and r are polynomials. The method of power series involves asserting a solution that is a power series:

$$y = \sum_{n=0}^{\infty} c_n (x-a)^n, \tag{282}$$

and then solving for the coefficients c_n.

Definition 7.1 *Let $p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$ be an ODE. Given a real number a, we say that $\underline{a \text{ is an ordinary point}}$ of the ODE if $p(a) \neq 0$. On the other hand, we say that $\underline{a \text{ is a singular point}}$ of the ODE if $p(a) = 0$.*

We start by considering only power series solutions centered at ordinary points.

Example 7.2 *Find the general solution of the ODE $y'' - y = 0$.*

Of course this example can be more easily handled by techniques that we discussed previously, but we will use the power series method to illustrate how it works. We begin by selecting an ordinary point of $y'' - y = 0$; $a = 0$ is a suitable choice (as any real number would be, since the coefficient of y'' is the constant function 1). We consider a power series of the form $y = \sum_{n=0}^{\infty} c_n (x-0)^n$. We note that

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\,c_n x^{n-2} \end{aligned}. \tag{283}$$

If this y is a solution, then this implies that

$$\sum_{n=2}^{\infty} n(n-1)\,c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0. \tag{284}$$

We desire to combine like terms in powers of x. In order to do this, we will write $k = n - 2$. In that case, the first term above can be written as

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1)\, c_{k+2} x^k. \tag{285}$$

Now, the index k has no intrinsic meaning to the series; it is merely a notational tool. In other words, there isn't a meaningful difference between the series above and

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n. \tag{286}$$

Thus, our equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0. \tag{287}$$

Now we can combine like terms:

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\, c_{n+2} - c_n\right) x^n = 0. \tag{288}$$

Of course, if this series is zero for all values of x, then it is constantly zero, meaning that its coefficients must all be zero. In other words, for every non-negative integer n,

$$(n+2)(n+1)\, c_{n+2} - c_n = 0. \tag{289}$$

This equation describes the coefficients of the power series; it is called the "recurrence relation" of the power series. To rephrase it,

$$c_{n+2} = \frac{c_n}{(n+2)(n+1)}. \tag{290}$$

Now, let us examine a few terms of this sequence:

$$\begin{array}{cc} c_0 = c_0 & c_1 = c_1 \\ c_2 = \frac{c_0}{(2)(1)} & c_3 = \frac{c_1}{(3)(2)} \\ c_4 = \frac{c_2}{(4)(3)} = \frac{c_0}{(4)(3)(2)(1)} & c_5 = \frac{c_3}{(5)(4)} = \frac{c_1}{(5)(4)(3)(2)(1)} \\ c_6 = \frac{c_4}{(6)(5)} = \frac{c_0}{(6)(5)(4)(3)(2)(1)} & c_7 = \frac{c_5}{(7)(6)} = \frac{c_1}{(7)(6)(5)(4)(3)(2)(1)} \\ \vdots & \vdots \end{array} . \tag{291}$$

After some time, we can notice the pattern

$$c_n = \begin{cases} \frac{c_0}{n!} & \textit{if } n \textit{ is even} \\ \frac{c_1}{n!} & \textit{if } n \textit{ is odd} \end{cases} . \tag{292}$$

Thus, the solution is

$$\begin{aligned} y = \sum_{n=0}^{\infty} c_n x^n &= \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n \textit{ is even}} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n \textit{ is odd}} = \underbrace{\sum_{n=0}^{\infty} \frac{c_0}{n!} x^n}_{n \textit{ is even}} + \underbrace{\sum_{n=0}^{\infty} \frac{c_1}{n!} x^n}_{n \textit{ is odd}} \\ &= \sum_{k=0}^{\infty} \frac{c_0}{(2k)!} x^{2k} + \sum_{l=0}^{\infty} \frac{c_1}{(2l+1)!} x^{2l+1} \\ &= \boxed{c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}} . \end{aligned} \tag{293}$$

(As you can see, c_0 and c_1 are arbitrary constants, exactly as we'd expect for the general solution of a second order ODE.) □

Example 7.3 *Find the general solution of the ODE $(x^2+1)\,y'' - 4xy' + 6y = 0$.*

$$\boxed{y = c_0\left(1 - 3x^2\right) + c_1\left(x - \frac{1}{3}x^3\right)} . \tag{294}$$

□

Example 7.4 *Find the general solution of the ODE* $(x^2-3)\,y''+2xy'=0$.

$$\boxed{y=c_0+c_1\sum_{n=0}^{\infty}\frac{x^{2n+1}}{3^n(2n+1)}}. \tag{295}$$

□

Example 7.5 *Find the general solution of the ODE* $y''+2xy'+4y=0$.

Example 7.6 *Find the general solution of the ODE* $(x^2-1)\,y''-6xy'+12y=0$.

$$\boxed{y=c_0\left(1+6x^2+x^4\right)+c_1\left(x+x^3\right)} \tag{296}$$

□

Example 7.7 *Find the general solution of the ODE* $(x^2+2)\,y''+4xy'+2y=0$.

$$\boxed{y=\frac{c_0+c_1x}{2+x^2}}. \tag{297}$$

□

Example 7.8 *Find the general solution of the ODE* $(x^2+1)\,y''+6xy'+4y=0$.

$$\boxed{y=c_0\sum_{n=0}^{\infty}(-1)^n\,(n+1)\,x^{2n}+c_1\sum_{n=0}^{\infty}(-1)^n\,(2n+3)\,x^{2n+1}} \tag{298}$$

□

7.3 Singular points and the method of Frobenius

What if we demand that a power series solution be centered at a, despite that a is a singular point?

7.3.1 The method of Frobenius

Definition 7.9 *Let $p(x)y'' + q(x)y' + r(x)y = 0$ be an ODE. Given a singular point a of the ODE, we say that $\underline{a \text{ is a regular singular point}}$ of the ODE if both of the following limits are finite:*

$$\lim_{x\to a}(x-a)\frac{q(x)}{p(x)} \text{ and } \lim_{x\to 0}(x-a)^2\frac{r(x)}{p(x)}.$$

The following theorem can be generalized to regular singular points other than 0, but we will only be concerned with $a = 0$ for our purposes.

Theorem 7.10 *(Method of Frobenius) Suppose that $p(x)y'' + q(x)y' + r(x)y = 0$ has a regular singular point at $a = 0$. In that case, there exists at least one solution of the form*

$$y = x^r \sum_{n=0}^{\infty} c_n x^n, \tag{299}$$

where $c_0 = 1$ and r is a real number.

A solution of this form is not exactly a power series, unless r turns out to be a non-negative integer. We call this type of function a "Frobenius-type solution."

The method of Frobenius works like this: first, propose a Frobenius-type solution, and put it into the original ODE. The first part of the recurrence relation will then contain a quadratic equation in r, called the "indicial equation." If the indicial equation has real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then there exist two linearly independent Frobenius-type solutions. If that is not the case, then things can get more complicated, but we will not be concerned with those situations in this class. (If you're interested in cases other than real roots not separated by an integer, feel free to look into Section 7.3 of the textbook.)

Example 7.11 *Find the general solution of the following ODE:*

$$x^2y'' - y = 0. \tag{300}$$

We note that $x = 0$ is a singular point of this ODE. However,

$$\begin{array}{c} \lim\limits_{x\to 0}(x-0)\frac{0}{x^2} = 0 \\ \lim\limits_{x\to 0}(x-0)^2\frac{-1}{x^2} = -1. \end{array} . \tag{301}$$

Both limits are finite, so $x = 0$ is a regular singular point of the ODE. We may therefore use the method of Frobenius, proposing a Frobenius series solution:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \tag{302}$$

where $c_0 = 1$. We note that

$$\begin{array}{c} y' = \sum\limits_{n=0}^{\infty}(n+r)\,c_n x^{n+r-1} \\ y'' = \sum\limits_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r-2} \end{array} . \tag{303}$$

Putting this information into the original ODE gives us

$$\begin{array}{c} x^2\sum\limits_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r-2} - \sum\limits_{n=0}^{\infty}c_n x^{n+r} = 0 \\ \sum\limits_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r} - \sum\limits_{n=0}^{\infty}c_n x^{n+r} = 0 \\ \sum\limits_{n=0}^{\infty}((n+r)(n+r-1)-1)\,c_n x^{n+r} = 0 \end{array} . \tag{304}$$

Thus, for all integers $n \geq 0$,

$$((n+r)(n+r-1)-1)\,c_n = 0. \tag{305}$$

In particular, since $c_0 = 1$, $n = 0$ gives us the indicial equation

$$r(r-1) - 1 = 0. \tag{306}$$

In other words, $r^2 - r - 1 = 0$. By the quadratic formula, the solutions of this equation are

$$r = \frac{1 \pm \sqrt{5}}{2}. \tag{307}$$

As for every integer value $n > 0$, the recurrence relation gives us

$$\left(r^2 + (2n-1)\,r + \left(n^2 - n - 1\right)\right) c_n = 0. \tag{308}$$

For $n = 1$, this becomes

$$\left(r^2 + r - 1\right) c_1 = 0. \tag{309}$$

Since neither $r^2 + r - 1 \neq 0$ when $r = \frac{1 \pm \sqrt{5}}{2}$, we must have that $c_1 = 0$. For $n = 2$, we have

$$\left(r^2 + 3r + 1\right) c_2 = 0. \tag{310}$$

Again, $r^2 + 3r + 1 \neq 0$, so $c_2 = 0$. This pattern continues, so that for every $n > 0$, $c_n = 0$.

Having found r and all of the coefficients, we can write down two different solutions of the ODE:

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} c_n x^{n+\frac{1+\sqrt{5}}{2}} = 1x^{\frac{1+\sqrt{5}}{2}} + 0x^{1+\frac{1+\sqrt{5}}{2}} + 0x^{2+\frac{1+\sqrt{5}}{2}} + \ldots = x^{\frac{1+\sqrt{5}}{2}} \\ y_2 &= \sum_{n=0}^{\infty} c_n x^{n+\frac{1-\sqrt{5}}{2}} = 1x^{\frac{1-\sqrt{5}}{2}} + 0x^{1+\frac{1-\sqrt{5}}{2}} + 0x^{2+\frac{1-\sqrt{5}}{2}} + \ldots = x^{\frac{1-\sqrt{5}}{2}} \end{aligned}. \tag{311}$$

One can show that these are linearly independent. Thus, the general solution of our ODE can be written as

$$\boxed{y = a_1 x^{\frac{1+\sqrt{5}}{2}} + a_2 x^{\frac{1-\sqrt{5}}{2}}}. \tag{312}$$

□

Example 7.12 *Find the general solution of the ODE $x^2 y'' + 4xy' + y = 0$.*

Example 7.13 *Find the general solution of the ODE $2x^2 y'' + 7xy' + 2y = 0$.*

Example 7.14 *Find the general solution of the ODE $2x^2 y'' - 3xy' + 2y = 0$.*

Example 7.15 *Find the general solution of the ODE $5x^2 y'' + y = 0$.*

Example 7.16 *Find the general solution of the ODE* $7x^2y'' + xy' + y = 0$.

3 Systems of ODEs

3.1 Introduction to systems of ODEs

At the beginning of this course, I said that ordinary differential equations deal with only one independent variable, while partial differential equations deal with multiple independent variables. This remains the case, but for systems of ODEs, there can be more than one *dependent* variable.

3.1.1 Systems

A system of ODEs with the single independent variable t and dependent variables $x_1, x_2, ..., x_n$ is exactly what the name suggests: a system of equations, which happen to be ODEs, for which we seek functions $x_1(t)$, $x_2(t)$, ..., and $x_n(t)$ that satisfy all of the ODEs simultaneously. The order of the system is the highest order of all of the ODEs involved.

Definition 3.1 *Let*

$$\begin{aligned} x_1' &= g_1(x_1, x_2, ..., x_n, t) \\ x_2' &= g_2(x_1, x_2, ..., x_n, t) \\ &\vdots \\ x_n' &= g_n(x_1, x_2, ..., x_n, t) \end{aligned} \tag{313}$$

be a first order system of ODEs. We say that the system above is a <u>*linear system*</u> *if all of its constituent ODEs are linear. We say that it is a* <u>*homogeneous system*</u> *if all of its constituent ODEs are homogeneous. We say that it has* <u>*constant coefficients*</u> *if all of its constitent ODEs have constant coefficients.*

Here are some first order, linear, homogeneous systems of ODEs with constant coefficients:

$$\begin{aligned} x' &= 3x + 2y - 4z \\ y' &= x + y + z \\ z' &= x \end{aligned} \qquad \begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= x_2 - x_3 \\ x_3' &= x_3 - x_4 \\ x_4' &= x_4 - x_1 \end{aligned} \qquad \begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= x_1 + x_2 \end{aligned} \tag{314}$$

Here are some first order, linear, non-homogeneous systems of ODEs with constant coefficients:

$$\begin{aligned} x' &= 3x + 2y - 4z + t \\ y' &= x + y + z + t^2 \\ z' &= x + t \end{aligned} \qquad \begin{aligned} x_1' &= x_1 - x_2 + \sin t \\ x_2' &= x_2 - x_3 + e^t \\ x_3' &= x_3 - x_4 - t^2 \\ x_4' &= x_4 - x_1 + 3^t \end{aligned} \qquad \begin{aligned} x_1' &= x_1 - x_2 + 6t \\ x_2' &= x_1 + x_2 \end{aligned} \tag{315}$$

Here are some first order ODEs which are neither linear nor homogeneous, and which do not have constant coefficients:

$$\begin{aligned} x'y + z &= 0 \\ x' &= 3tx \\ y' &= t \end{aligned} \qquad \begin{aligned} x_1' &= x_1 x_2 \\ x_2' &= x_2 - e^t x_3 \\ x_3' &= x_3 - x_4 \\ x_4' &= x_4 - x_1 + 4t \end{aligned} \qquad \begin{aligned} x_1' &= t^2 x_1 - x_2 - \ln t \\ x_2' &= x_1 + x_2 \end{aligned}. \tag{316}$$

Just like scalar ODEs, systems of ODEs have general solutions and particular solutions. For a first order system, there will be as many arbitrary constants in the general solution as dependent variables.

Example 3.2 *Find the general solution of the first order system*

$$\begin{aligned} x' &= 3x - y \\ y' &= x \end{aligned}. \tag{317}$$

Example 3.3 *(Exercise 3.1.2) Find the general solution of the first order system*

$$\begin{aligned} x_1' &= x_2 - x_1 + t \\ x_2' &= x_2 \end{aligned}. \tag{318}$$

3.1.2 Changing to first order

The previous two examples illustrate that some systems of ODEs can be rephrased as scalar ODEs. This is not true for all systems. However, the converse is true: every scalar ODE can be rephrased as a first order system of ODEs. Consider an nth order scalar ODE:

$$y^{(n)} = F\left(y, y', y'', ..., y^{(n-1)}, t\right). \tag{319}$$

We can rephrase this as system by defining $y_1 = y$, $y_2 = y'$, ..., and $y_n = y^{(n-1)}$. In that case, this becomes the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(y_1, y_2, ..., y_n, t) \end{aligned} \quad . \tag{320}$$

As you can see, the number of dependent variables is exactly the order of the original ODE.

Example 3.4 *Rewrite the scalar ODE $x'' + 3x' + 7x = t^2$ as a first order system.*

Example 3.5 *Rewrite the scalar ODE $y'' + 4y - y^3 = 0$ as a first order system.*

Example 3.6 *Rewrite the scalar ODE $y^{(4)} - y'' = 5e^{-3x}$ as a first order system.*

Example 3.7 *Rewrite the scalar ODE $t^2 x'' + tx' + (t^2 - 1)\, x = 0$ as a first order system.*

3.2 Matrices and linear systems

3.2.1 Matrices and vectors

To move forward with our discussion of linear systems of ODEs, we'll need to use a number of properties of matrices. First, given any scalar s (a scalar means just a number, real or complex), we define the "scalar multiplication" of a matrix as:

$$s\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} sa_{11} & sa_{12} & sa_{13} & \dots & sa_{1n} \\ sa_{21} & sa_{22} & sa_{23} & \dots & sa_{2n} \\ sa_{31} & sa_{32} & sa_{33} & \dots & sa_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ sa_{m1} & sa_{m2} & sa_{m3} & \dots & sa_{mn} \end{pmatrix}. \quad (321)$$

We also defined "addition of matrices" in the intuitive way. If two matrices have the same size (that is, the same number of rows and the same number of columns), then define their sum as

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}. \quad (322)$$

If two matrices do not have the same size, their sum is undefined.

3.2.2 Matrix multiplication

Matrix multiplication, on the other hand, is more complicated. To do this, we first define matrices that have only one row or only one column.

Definition 3.8 *A <u>[column] vector</u> is a matrix with exactly one column. A <u>row vector</u>*

is a matrix with exactly one row.

The "zero vector" is the vector whose entries are all 0:

$$\overrightarrow{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{323}$$

Definition 3.9 *Let $\overrightarrow{u}$ be a $1 \times n$ row vector, and let $\overrightarrow{v}$ be an $n \times 1$ column vector:*

$$\begin{aligned} \overrightarrow{u} &= \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \\ \overrightarrow{v} &= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \end{aligned} . \tag{324}$$

The <u>*dot product*</u> *of $\overrightarrow{u}$ and $\overrightarrow{v}$ is the scalar*

$$\overrightarrow{u} \cdot \overrightarrow{v} = \begin{pmatrix} a_1 & a_2 & ... & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + ... + a_n b_n. \tag{325}$$

Example 3.10 *Determine the following dot products:*

$$\begin{aligned} \begin{pmatrix} -5 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} &= -16 \\ \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} &= 6 \\ \begin{pmatrix} 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} &= 0 \end{aligned} \tag{326}$$

□

We now define multiplication of larger matrices. Suppose that A is an $m \times n$ matrix, and B is an $n \times p$ matrix:

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \\ B &= \begin{pmatrix} b_{11} & b_{12} & ... & b_{1p} \\ b_{21} & b_{22} & ... & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & ... & b_{np} \end{pmatrix} \end{aligned} \tag{327}$$

(That is, A has as many columns as B has rows.) In that case, we can consider each column of A as a vector. Let's call the ith row of A the vector $\overrightarrow{A_i}$. Similarly, let's call the jth column of B the vector $\overrightarrow{B_i}$. In that case, we define the product of A and B to be the following $m \times p$ matrix:

$$A \cdot B = \begin{pmatrix} \overrightarrow{A_1} \cdot \overrightarrow{B_1} & \overrightarrow{A_1} \cdot \overrightarrow{B_2} & ... & \overrightarrow{A_1} \cdot \overrightarrow{B_p} \\ \overrightarrow{A_2} \cdot \overrightarrow{B_1} & \overrightarrow{A_2} \cdot \overrightarrow{B_2} & ... & \overrightarrow{A_2} \cdot \overrightarrow{B_p} \\ \vdots & \vdots & \ddots & \vdots \\ \overrightarrow{A_m} \cdot \overrightarrow{B_1} & \overrightarrow{A_m} \cdot \overrightarrow{B_2} & ... & \overrightarrow{A_m} \cdot \overrightarrow{B_p} \end{pmatrix}. \tag{328}$$

Thus, the (i, j) entry of the product is the dot product of the ith row of A with the jth column of B. If A does not have the same number of columns as B has rows, then the product is undefined.

Example 3.11 *Determine the following matrix products.*

$$\begin{aligned} \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} &= \begin{pmatrix} -5 & 2 \\ -6 & 1 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} &= \begin{pmatrix} -2 & -4 \\ 0 & -3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} . \tag{329}$$

□

3.2.3 The determinant

Consider a linear homogeneous system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= 0 \end{aligned} . \tag{330}$$

This can be re-written as the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{331}$$

This is often significantly easier to work with, since it is only one equation, as opposed to m many equations. For this reason, we are often interested in solving equations of the form $A\overrightarrow{v} = \overrightarrow{0}$, where A is a known matrix and $\overrightarrow{v}$ is a vector that must be found. Of course, $\overrightarrow{v} = \overrightarrow{0}$ is one solution (called the "trivial solution"), but the question is whether others exist.

The following theorem from linear algebra is crucial to the theory:

Theorem 3.12 *Let A be an $n \times n$ matrix. The matrix equation $A\overrightarrow{v} = \overrightarrow{0}$ has a non-trivial solution if and only if* $\det(A) \neq 0$.

3.3 Linear systems of ODEs

Let's return to the topic of systems of ODEs. Every first order linear system of ODEs looks like

$$\begin{array}{c} x_1' = a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + ... + a_{1n}(t)\,x_n + b_1(t) \\ x_2' = a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + ... + a_{2n}(t)\,x_n + b_2(t) \\ \vdots \\ x_n' = a_{n1}(t)\,x_1 + a_{n2}(t)\,x_2 + ... + a_{nn}(t)\,x_n + b_n(t) \end{array}. \tag{332}$$

This can be written as the matrix equation

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & ... & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & ... & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & ... & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}. \tag{333}$$

We often write this as $\overrightarrow{x}' = A(t)\,\overrightarrow{x} + \overrightarrow{b}(t)$. The system is homogeneous whenever $\overrightarrow{b}(t) = \overrightarrow{0}$ for all t.

For this class, we will focus mainly on first order, linear, homogeneous systems of ODEs with constant coefficients. For this, we have the following two theorems.

Theorem 3.13 *(Superposition) Let $\overrightarrow{z}' = A(t)\,\overrightarrow{z}$ be a linear homogeneous system, where $A(t)$ is an $n \times n$ matrix. If $\overrightarrow{z_1}, \overrightarrow{z_2}, ..., \overrightarrow{z_k}$ are solutions, then every linear combination is also a solution.*

Theorem 3.14 *(Solution theorem for linear homogeneous systems of ODEs) Let $\overrightarrow{z}' = A(t)\,\overrightarrow{z}$ be a linear homogeneous system, where $A(t)$ is an $n \times n$ matrix. If $\overrightarrow{z_1}, \overrightarrow{z_2}, ..., \overrightarrow{z_n}$ are linearly independent solutions of the ODE, then every solution of the ODE can be written in the form $\overrightarrow{z} = c_1\overrightarrow{z_1} + c_2\overrightarrow{z_2} + ... + c_n\overrightarrow{z_n}$ for some constants $c_1, c_2, ..., c_n$.*

3.4 Eigenvalue method

3.4.1 Eigenvalues and eigenvectors of a matrix

Definition 3.15 *Let A be an $n \times n$ matrix. An eigenvalue of A is a complex number λ such that there exists a nonzero vector $\overrightarrow{v}$ satisfying $A\overrightarrow{v} = \lambda\overrightarrow{v}$. In that case, $\overrightarrow{v}$ is called an eigenvector of A associated to λ.*

Suppose that A is an $n \times n$ matrix that has an eigenvalue λ, with an associated eigenvector $\overrightarrow{v}$. In that case, $A\overrightarrow{v} = \lambda\overrightarrow{v}$. This can also be written as

$$(A - \lambda I_n)\overrightarrow{v} = \overrightarrow{0}. \tag{334}$$

By definition, in order for $\overrightarrow{v}$ to be an eigenvector of A corresponding to λ, we need that $\overrightarrow{v} \neq \overrightarrow{0}$. Now, according to Theorem 3.12, the equation above has a non-trivial solution if and only if

$$\det(A - \lambda I_n) = 0. \tag{335}$$

(The left side of the equation above is called the "characteristic polynomial" of the matrix, and the entire equation is called the "characteristic equation" of the matrix.) Using this fact, we can find the eigenvalues of any square matrix. From there, finding the eigenvectors associated to each eigenvalue is possible.

Example 3.16 *Find the eigenvalues and associated eigenvectors of the following matrix:*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{336}$$

Example 3.17 *Find the eigenvalues and associated eigenvectors of the following matrix:*

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{337}$$

Example 3.18 *Find the eigenvalues and associated eigenvectors of the following matrix:*

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}. \tag{338}$$

Example 3.19 *Find the eigenvalues and associated eigenvectors of the following matrix:*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{339}$$

3.4.2 The eigenvalue method with distinct real eigenvalues

Why are eigenvalues and eigenvectors important to systems of ODEs? Well, consider a first order, linear, homogeneous system with constant coefficients:

$$\overrightarrow{z}' = A\overrightarrow{z}. \tag{340}$$

Suppose that A is an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\overrightarrow{v}$. Consider the vector-valued function $\overrightarrow{z}(t) = \overrightarrow{v}e^{\lambda t}$. We note that

$$\overrightarrow{z}'(t) = \lambda\overrightarrow{v}e^{\lambda t} = A\overrightarrow{v}e^{\lambda t} = A\overrightarrow{z}(t). \tag{341}$$

This demonstrates that $\overrightarrow{z}(t)$ is a solution of the ODE. To summarize: *if we can figure out the eigenvalues and eigenvectors of a linear system, then we can construct solutions for it.*

Example 3.20 *Find the general solution of the following system of ODEs:*

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}. \tag{342}$$

Example 3.21 *Find the general solution of the following system of ODEs:*

$$\begin{aligned} x' &= x + 2y \\ y' &= 3x + 2y \end{aligned}. \tag{343}$$

Example 3.22 *Find the general solution of the following ODE:*

$$\overrightarrow{z}' = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix}\overrightarrow{z}. \tag{344}$$

3.4.3 Complex eigenvalues

When the eigenvalues of a system are complex, we need to use Euler's formula to separate the real and imaginary parts. Then, the real and imaginary parts will separately form basis solutions.

Example 3.23 *Find the general solution of the following ODE:*

$$\vec{z}\,' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \vec{z} \tag{345}$$

Example 3.24 *Find the general solution of the following system of ODEs:*

$$\begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= 6x_1 - x_2 \end{aligned}. \tag{346}$$

Example 3.25 *Solve the following IVP:*

$$\begin{aligned} x_1' &= 5x_1 + 7x_2 \\ x_2' &= -2x_1 - 4x_2 \\ x_1(0) &= -2,\ x_2(0) = -3 \end{aligned}. \tag{347}$$

Example 3.26 *Find the general solution of the following system of ODEs:*

$$\begin{aligned} x_1' &= x_1 - 5x_2 \\ x_2' &= x_1 - x_2 \end{aligned}. \tag{348}$$

3.7 Multiple eigenvalues

When the characteristic polynomial of a matrix has repeated roots, things can get somewhat complicated.

3.7.1 Geometric multiplicity

Definition 3.27 *Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A.*
(i) The <u>*algebraic multiplicity of λ*</u> *is the largest integer k such that $(x-\lambda)^k$ divides the characteristic polynomial of A.*
(ii) The <u>*geometric multiplicity of λ*</u> *is the largest possible number of linearly independent eigenvectors of λ.*
(iii) The <u>*defect of λ*</u> *is the algebraic multiplicity of λ minus the geometric multiplicity of λ.*
(iv) If the defect of an eigenvalue is zero, we say that the eigenvalue is <u>*complete*</u>*.*
(v) If the defect of an eigenvalue is greater than zero, we say that the eigenvalue is <u>*defective*</u>*.*

When an eigenvalue is complete, things go fairly smoothly.

Example 3.28 *Find the general solution of the vector ODE*

$$\vec{z}\,' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \vec{z}. \tag{349}$$

3.7.2 Defective eigenvalues

On the other hand, when an eigenvalue is defective, things get a bit weird. According to the solution theorem for linear homogeneous systems of ODEs, we need as many basis solutions as the dimension of the matrix, but there aren't enough linearly independent eigenvectors to construct them. In this case, we need to consider "generalized eigenvectors."

Definition 3.29 *Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Given a positive integer k, a* <u>*rank k generalized eigenvector corresponding to λ*</u> *is a vector $\vec{v}$ such that $(A-\lambda I_n)^k \vec{v} = \vec{0}$, but $(A-\lambda I_n)^{k-1} \vec{v} \neq \vec{0}$.*

(This means that a true eigenvector can be understood as a generalized eigenvector of rank 1.)

Example 3.30 *Find the general solution of the vector ODE*

$$\overrightarrow{z}' = \begin{pmatrix} 9 & 4 \\ -4 & 1 \end{pmatrix} \overrightarrow{z}. \tag{350}$$

We begin by finding the eigenvalues of the matrix, which we call A:

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 9-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2. \tag{351}$$

This makes it clear that $\lambda = 5$ is the only eigenvalue, with an algebraic multiplicity of 2. We now inspect the eigenvectors corresponding to 5:

$$\begin{gathered} (A - 5I_2)\overrightarrow{v_1} = \overrightarrow{0} \\ \begin{pmatrix} 4 & 4 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \\ 4a + 4b = 0 \\ -4a - 4b = 0 \end{gathered} \tag{352}$$

This makes it clear that $b = -a$, and so the eigenvectors corresponding to 5 all take the form

$$\overrightarrow{v_1} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{353}$$

where $a \neq 0$. Any two vectors matching this description will be linearly dependent, so the geometric multiplicity of $\lambda = 5$ is 1. Ergo, the defect of $\lambda = 5$ is $2 - 1 = 1$.

At this point, we consider generalized eigenvectors of rank 2. We consider

$$(A - 5I_2)^2 = \begin{pmatrix} 4 & 4 \\ -4 & -4 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{354}$$

In other words, we require a vector $\overrightarrow{v_2}$ that satisfies the equation

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \overrightarrow{v_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{355}$$

Literally any vector would satisfy this condition, but the definition of a rank 2 generalized eigenvector implies that $\overrightarrow{v_2}$ must also not be an eigenvector. Therefore, we need only take $\overrightarrow{v_2}$ to be a nonzero vector that is not an eigenvector. For no particular reason, we select

$$\overrightarrow{v_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{356}$$

Now, we need to select an eigenvector which is "compatible" with this choice, in the sense that $\overrightarrow{v_1} = (A - 5I_2)\,\overrightarrow{v_2}$:

$$\overrightarrow{v_1} = \begin{pmatrix} 4 & 4 \\ -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}. \tag{357}$$

We can now construct two basis solutions:

$$\begin{aligned} \overrightarrow{z_1} &= \overrightarrow{v_1} e^{\lambda t} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} e^{5t} \\ \overrightarrow{z_2} &= (\overrightarrow{v_1} t + \overrightarrow{v_2})\, e^{\lambda t} = \left(\begin{pmatrix} 4 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{5t} \end{aligned}, \tag{358}$$

and so the general solution is the set of linear combinations of these:

$$\boxed{\overrightarrow{z} = c_1 \begin{pmatrix} 4 \\ -4 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 4t + 1 \\ -4t \end{pmatrix} e^{5t}}. \tag{359}$$

□

Example 3.31 *Find the general solution of the system*

$$\begin{aligned} x_1' &= -5x_1 + x_2 \\ x_2' &= -x_1 - 3x_2 \end{aligned}. \tag{360}$$

Example 3.32 *Find the general solution of the system*

$$\begin{aligned} x_1' &= 7x_1 + x_2 \\ x_2' &= -4x_1 + 3x_2 \end{aligned}. \tag{361}$$

Example 3.33 *Find the general solution of the system*

$$\begin{aligned} x_1' &= -8x_1 - x_2 \\ x_2' &= 4x_1 - 4x_2 \end{aligned}. \tag{362}$$

Example 3.34 *Solve the IVP*

$$\begin{gathered} x_1' = -x_1 - 2x_2 \\ x_2' = 2x_1 - 5x_2 \\ x_1(0) = -1,\ x_2(0) = 3 \end{gathered} \tag{363}$$

Example 3.35 *Solve the IVP*

$$\begin{gathered} x_1' = 6x_1 + x_2 \\ x_2' = -x_1 + 8x_2 \\ x_1(0) = -2,\ x_2(0) = 5 \end{gathered} \tag{364}$$

Appendix: Exponents and logarithms

Exponential functions

Theorem 3.36 *Let a be a real number. The following statements are true.*
(i) For any real numbers x and y, $a^{x+y} = a^x a^y$.
(ii) For any real numbers x and y, $a^{xy} = (a^x)^y$.
(iii) If $a \neq 0$, then for any real number x, $a^{-x} = \frac{1}{a^x}$.
(iv) Given a positive integer n, $a^{\frac{1}{n}} = \sqrt[n]{a}$.
(v) If $a \neq 0$, then $a^0 = 1$.

Note: most sources agree that 0^0 should be defined as 1. This choice is made for notational convenience, not due to any important mathematical truth.

Theorem 3.37 *Let a and b be real numbers greater than 1. Given a real number x, $(ab)^x = a^x b^x$.*

Theorem 3.38 *(Binomial theorem) Let a and b be real numbers. Given a positive integer n,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \tag{365}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, the binomial coefficient.

Theorem 3.39 *Let a be a real number. If $a > 1$, then the following statements are true.*
(i) $\lim_{x\to\infty} a^x = \infty$.
(ii) $\lim_{x\to-\infty} a^x = 0$.

Logarithmic functions

Definition 3.40 *Let f and g be real-valued functions of a real variable. We say that f and g are inverse functions, or that g is the inverse function of f, or that f is the inverse function of g provided that for any real value x, $f(g(x)) = x$ and $g(f(x)) = x$.*

Definition 3.41 *Let a be a real number greater than 1. The base a logarithm is the inverse function $g(x) = \log_a x$ of the function $f(x) = a^x$.*

As a direct result of the definition of $\log_a$, for any real value x, $\log_a(a^x) = x$, and $a^{\log_a x} = x$ if $x > 0$. Additionally, since $a^x > 0$ for any $a > 0$ and any real number x, $\log_a(y)$ is not defined for any non-positive real number y.

Theorem 3.42 *Let a be a real number such that $a > 1$. The following statements are true.*
(i) For any positive real numbers x and y, $\log_a(xy) = \log_a(x) + \log_a(y)$.
(ii) For any real numbers x and y such that $x > 0$, $\log_a(x^y) = y\log_a(x)$.
(iii) For any positive real numbers x and y, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.

Theorem 3.43 *Let a be a real number such that $a > 1$. The following statements are true.*
(i) $\lim_{x\to\infty} \log_a x = \infty$.
(ii) $\lim_{x\to 0^+} \log_a x = -\infty$.

Calculus of exponential and logarithmic functions

Definition 3.44 *The natural exponential function is the real-valued function f of a real variable such that $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f(x)$ and $f(0) = 1$.*

Definition 3.45 *Euler's number is the real value e such that the function $f(x) = e^x$ is the natural exponential function.*

Definition 3.46 *The natural logarithm is the base e logarithm $\ln x = \log_e x$.*

Theorem 3.47 *The following statements are true.*
(i) If $a > 1$, then $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$.
(ii) $\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| = \frac{1}{x}$.

Appendix: Properties of trigonometric functions

List of trigonometric identities

This page will be provided on any test involving trigonometric functions.

Theorem 3.48 *(Pythagorean identity) Given a real number* θ*,*

$$\cos^2\theta + \sin^2\theta = 1.$$

Theorem 3.49 *(Angle sum formulas) Let* θ *and* ϕ *be real numbers. The following statements are true.*

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

Theorem 3.50 *(Double angle and half angle formulas) Let* θ *be a real number. The following statements are true.*

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta))$$

Definition 3.51 *Let* θ *be a real number.*
(i) If $\cos\theta \neq 0$*, then* $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
(ii) If $\cos\theta \neq 0$*, then* $\sec\theta = \frac{1}{\cos\theta}$.
(iii) If $\sin\theta \neq 0$*, then* $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
(iv) If $\sin\theta \neq 0$*, then* $\csc\theta = \frac{1}{\sin\theta}$.

Theorem 3.52 *Let* θ *be a real number. The following statements are true.*

$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$

Inverse trigonometric functions

Definition 3.53 *(i) The* <u>*arccosine function*</u> *is the function* $\cos^{-1}$ *whose domain is* $[0, \pi]$ *such that for any real value* x*, if* $-1 \leq x \leq 1$*, then* $\cos\left(\cos^{-1}(x)\right) = x$*, and for any real value* θ*, if* $0 \leq \theta \leq \pi$*, then* $\cos^{-1}(\cos(\theta)) = \theta$*.*

(ii) The <u>*arcsine function*</u> *is the function* $\sin^{-1}$ *whose domain is* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *such that for any real value* y*, if* $-1 \leq y \leq 1$*, then* $\sin\left(\sin^{-1}(y)\right) = y$*, and for any real value* θ*, if* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*, then* $\sin^{-1}(\sin(\theta)) = \theta$*.*

(iii) The <u>*arctangent function*</u> *is the function* $\tan^{-1}$*, whose domain is all real numbers, such that for any real value* z*,* $\tan\left(\tan^{-1}(z)\right) = z$*, and for any real value* θ*, if* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$*, then* $\tan^{-1}(\tan(\theta)) = \theta$*.*

(iv) The <u>*arcsecant function*</u> *is the function* $\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right)$*.*

(v) The <u>*arccosecant function*</u> *is the function* $\csc^{-1} z = \sin^{-1}\left(\frac{1}{z}\right)$*.*

(vi) The <u>*arccotangent function*</u> *is the function*

$$\cot^{-1} z = \begin{cases} \tan^{-1}\left(\frac{1}{z}\right) & \text{if } z > 0 \\ \pi + \tan^{-1}\left(\frac{1}{z}\right) & \text{if } z < 0 \end{cases}.$$

Derivatives of trigonometric functions and inverse trigonometric functions

Theorem 3.54 *The following statements are true.*

$$\begin{array}{cc} \frac{d}{d\theta}\sin\theta = \cos\theta & \frac{d}{d\theta}\cos\theta = -\sin\theta \\ \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta & \frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta \\ \frac{d}{d\theta}\tan\theta = \sec^2\theta & \frac{d}{d\theta}\cot\theta = -\csc^2\theta \end{array}$$

Theorem 3.55 *The following statements are true.*

$$\begin{array}{cc} \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1} & \frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1} \end{array}$$

Appendix: Some information on Laplace transforms

This page will be provided on any test involving Laplace transforms.

Definition:

$$\mathcal{L}\left(f\left(t\right)\right)=F\left(s\right)=\int_{0}^{\infty}e^{-st}f\left(t\right)\ \mathrm{d}t.$$

In the following table, a and k are real valued constants, $u\left(t\right)$ refers to the unit step function, and $g\left(t\right)$ is any function whose Laplace transform is $G\left(s\right)$.

	Function/expression	**Laplace transform**
1	$f(t)=1$	$F\left(s\right)=\frac{1}{s}$
2	$f(t)=t^{n},$ where n is a non-negative integer	$F\left(s\right)=\frac{n!}{s^{n+1}}$
3	$f(t)=e^{at}$	$F\left(s\right)=\frac{1}{s-a}$
4	$f(t)=\cos\left(kt\right)$	$F\left(s\right)=\frac{s}{s^{2}+k^{2}}$
5	$f\left(t\right)=\sin\left(kt\right)$	$F\left(s\right)=\frac{k}{s^{2}+k^{2}}$
6	$f\left(t\right)=u\left(t-a\right)$	$F\left(s\right)=\frac{e^{-as}}{s}$
7	$f\left(t\right)=u\left(t-a\right)g\left(t-a\right)$	$F\left(s\right)=e^{-as}G\left(s\right)$
8	$f\left(t\right)=e^{at}g\left(t\right)$	$F\left(s\right)=G\left(s-a\right)$
9	$f\left(t\right)=\delta\left(t-a\right)$	$F\left(s\right)=e^{-as}$

Linearity: given any functions $f\left(t\right)$ and $g\left(t\right)$, and any real valued constants a and b,

$$\mathcal{L}\left(af\left(t\right)+bg\left(t\right)\right)=a\mathcal{L}\left(f\left(t\right)\right)+b\mathcal{L}\left(g\left(t\right)\right).$$

Derivatives: given a differentiable function $y\left(t\right)$ whose Laplace transform is $Y\left(s\right)$, the following statements are true.

$$\begin{gathered}\mathcal{L}\left(y'\right)=sY-y\left(0\right)\\ \mathcal{L}\left(y''\right)=s^{2}Y-sy\left(0\right)-y'\left(0\right)\\ \mathcal{L}\left(y^{(3)}\right)=s^{3}Y-s^{2}y\left(0\right)-sy'\left(0\right)-y''\left(0\right)\\ \vdots\\ \mathcal{L}\left(y^{(n)}\right)=s^{n}Y-\sum_{i=1}^{n}s^{n-i}y^{(i-1)}\left(0\right)\end{gathered}$$