Math 142V Midterm Test 2 **Name: Solution Key**

[10] **1.** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=0}^{\infty} \frac{1}{10^{n-2}}$$

Solution: This is a geometric series that can be handled in multiple different ways; here is at least one. We can first rewrite this as

$$\sum_{n=0}^{\infty} \frac{1}{10^{n-2}} = \sum_{n=0}^{\infty} \frac{10^2}{10^n} = \sum_{n=0}^{\infty} 100\left(\frac{1}{10}\right)^n. \tag{1}$$

Now, since $\left|\frac{1}{10}\right| < 1$, this is a $\boxed{\text{convergent}}$ geometric series. It sum is given by the characterization theorem for geometric series $\sum_{n=0}^{\infty} br^n = \frac{b}{1-r}$:

$$\sum_{n=0}^{\infty} \frac{1}{10^{n-2}} = \frac{100}{1-\frac{1}{10}} = \boxed{\frac{1000}{9}} \tag{2}$$

□

[15] **2.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} ne^{-n}$$

Solution: First, we construct the function $f(x) = xe^{-x}$. We note that

$$f'(x) = xe^{-x}(-1) + e^{-x} = e^{-x}(1-x), \tag{3}$$

which is negative for $x > 1$. This shows that f is an eventually decreasing function, and so the integral test applies. We now compute the improper integral of f:

$$\int_1^{\infty} f(x)\ \mathrm{d}x = \lim_{t\to\infty} \int_1^t xe^{-x}\ \mathrm{d}x. \tag{4}$$

This is an integration by parts problem:

$$\begin{aligned} u &= x & \mathrm{d}v &= e^{-x}\,\mathrm{d}x \\ \mathrm{d}u &= \mathrm{d}x & v &= -e^{-x} \end{aligned} \tag{5}$$

$$\begin{aligned} &\lim_{t\to\infty}\left(-xe^{-x}\Big|_1^t + \int_1^t e^{-x}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left(-xe^{-x}\Big|_1^t - e^{-x}\Big|_1^t\right) \\ &= \lim_{t\to\infty} -(x+1)\,e^{-x}\Big|_1^t = \lim_{t\to\infty} 2e^{-1} - (t+1)\,e^{-t} = \frac{2}{e} - \lim_{t\to\infty}\frac{t+1}{e^t} = \frac{2}{e}. \end{aligned} \tag{6}$$

As this is a convergent improper integral, the integral test indicates that $\sum_{n=1}^{\infty} ne^{-n}$ is also $\boxed{\text{convergent}}$. $\square$

[15] **3.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{4}\right)}{n^7}$$

Solution: The terms of this series are not all positive, but it is not an alternating series. As the result, this is quite difficult to deal with directly. Instead, we consider whether the series is absolutely convergent. We note that

$$\left|\frac{\cos\left(\frac{n\pi}{4}\right)}{n^7}\right| = \frac{\left|\cos\left(\frac{n\pi}{4}\right)\right|}{n^7} \le \frac{1}{n^7}. \tag{7}$$

As $\sum_{n=1}^{\infty}\frac{1}{n^7}$ is a convergent p-series, the comparison test indicates that $\sum_{n=1}^{\infty}\left|\frac{\cos\left(\frac{n\pi}{4}\right)}{n^7}\right|$ is also convergent. Ergo, $\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{4}\right)}{n^7}$ is absolutely convergent, and so it is $\boxed{\text{convergent}}$. $\square$

[15] **4.** Determine whether the series is absolutely convergent, conditionally

convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{(-2)^n}$$

Solution: The easiest way to understand the behavior of this series is by rewriting it as

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{(-2)^n} = \sum_{n=1}^{\infty} \frac{9}{-2}\left(-\frac{3}{2}\right)^{n-1}. \tag{8}$$

This is a geometric series. Since $\left|-\frac{3}{2}\right| \geq 1$, the characterization theorem for geometric series indicates that this series is $\boxed{\text{divergent}}$. $\square$

[15] **5.** Determine the radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} n!(3x-4)^n$$

Solution: Like most power series, this is most easily handled using the ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)!(3x-4)^{n+1}}{n!(3x-4)^n}\right| = \lim_{n\to\infty}(n+1)\,|3x-4|. \tag{9}$$

This limit will be ∞, unless $x = \frac{4}{3}$, in which case it is zero. Thus, the ratio test indicates that divergence is guaranteed except when $x = \frac{4}{3}$, and so the radius of convergence is $\boxed{0}$. $\square$

[15] **6.** Find a power series representation for the function and determine its radius of convergence.

$$f(x) = \frac{x}{5-x}$$

Solution: We first rewrite this function as

$$f(x) = x\frac{1}{5-x} = \frac{x}{5}\left(\frac{1}{1-\frac{1}{5}x}\right). \tag{10}$$

Using the characterization theorem for geometric series, this can be written as

$$f\left(x\right)=\frac{x}{5}\sum_{n=0}^{\infty}\left(\frac{1}{5}x\right)^{n}=\boxed{\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}}}, \tag{11}$$

as long as $\left|\frac{1}{5}x\right|<1$. This inequality can be rephrased as $|x|<5$, revealing that the radius of convergence is $\boxed{5}$. $\square$

[15] **7.** Find the Taylor series for f centered at the given value of a, and find the associated radius of convergence.

$$f\left(x\right)=e^{-3x}, \qquad a=1$$

Solution: As always, the Taylor series centered at a is given by the definition: $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^{n}$. To write this, we need information about the derivatives of f at $a=1$:

$$\begin{array}{rr} f^{(0)}\left(x\right)=e^{-3x} & f^{(0)}\left(1\right)=e^{-3} \\ f^{(1)}\left(x\right)=-3e^{-3x} & f^{(1)}\left(1\right)=-3e^{-3} \\ f^{(2)}\left(x\right)=9e^{-3x} & f^{(2)}\left(1\right)=9e^{-3} \\ f^{(3)}\left(x\right)=-27e^{-3x} & f^{(3)}\left(1\right)=-27e^{-3} \end{array}. \tag{12}$$

From this we deduce the pattern

$$f^{(n)}\left(1\right)=(-3)^{n}e^{-3}, \tag{13}$$

and so the Taylor series is

$$\boxed{\sum_{n=0}^{\infty}\frac{(-3)^{n}}{n!e^{3}}(x-1)^{n}}. \tag{14}$$

The radius of convergence can be found using the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-3)^{n+1}(x-1)^{n+1}}{(n+1)!e^{3}}\frac{n!e^{3}}{(-3)^{n}(x-1)^{n}}\right|=\lim_{n\to\infty}\frac{3|x-1|}{n+1}=0. \tag{15}$$

As this is less than one for every value of x, the ratio test indicates that the series convergence at every value of x, and so the radius of convergence is $\boxed{\infty}$. $\square$

Math 142V Midterm Test 2 **Name: Solution Key**

[10] **1.** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{10}\right)^n}$$

Solution: This is a geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{10}\right)^n} = \sum_{n=1}^{\infty} \left(\frac{10}{11}\right)^n. \tag{16}$$

By the characterization theorem for geometric series, since $\left|\frac{10}{11}\right| < 1$, we know that this series is $\boxed{\text{convergent}}$. As for its sum:

$$\sum_{n=1}^{\infty} \left(\frac{10}{11}\right)^n = \sum_{n=1}^{\infty} \frac{10}{11}\left(\frac{10}{11}\right)^{n-1} = \frac{\left(\frac{10}{11}\right)}{1-\frac{10}{11}} = \boxed{10}. \tag{17}$$

□

[15] **2.** Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution: We know that $\frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$, so the terms form a decreasing sequence. This means that the integral test can apply. We define the function $f(x) = \frac{1}{x \ln x}$ and take its improper integral:

$$\int_2^{\infty} \frac{1}{x \ln x}\,\mathrm{d}x = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\,\mathrm{d}x. \tag{18}$$

We use a u-substitution:

$$\begin{aligned} u &= \ln x & x = 2 &\Rightarrow u = \ln 2 \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x & x = t &\Rightarrow u = \ln t \end{aligned} \tag{19}$$

$$\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u}\,\mathrm{d}u = \lim_{t\to\infty}\ln|u|\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty}\ln|\ln t| - \ln(\ln 2) = \infty. \tag{20}$$

As the improper integral is divergent, the integral test indicates that the series is also $\boxed{\text{divergent}}$. □

[15] **3.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{1}{2^n+n^5}$$

Solution: There are two ways to do this problem: by comparing to a geometric series, or by comparing to a p-series. In the first approach:

$$\frac{1}{2^n+n^5} \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n. \tag{21}$$

As $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a convergent geometric series, the comparison test indicates that $\sum\limits_{n=1}^{\infty}\frac{1}{2^n+n^5}$ is also $\boxed{\text{convergent}}$. □

[15] **4.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^3+1}$$

Solution: To test for absolute convergence, we consider the series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n}{n^3+1}\right| = \sum_{n=1}^{\infty}\frac{n}{n^3+1}. \tag{22}$$

We note that

$$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}. \tag{23}$$

Further, $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series. By the comparison test, this indicates that

$\sum\limits_{n=1}^{\infty} \frac{n}{n^3+1}$ is also convergent, so $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$ is $\boxed{\text{absolutely convergent}}$. $\square$

[15] **5.** Determine the radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{n(5x+3)^n}{5^n}$$

Solution: As is common for power series, we use the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\left(\frac{(n+1)(5x+3)^{n+1}}{5^{n+1}}\right)}{\left(\frac{n(5x+3)^n}{5^n}\right)} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)(5x+3)^{n+1}}{5^{n+1}} \frac{5^n}{n(5x+3)^n} \right| \\ &= \lim_{n\to\infty} \frac{n+1}{5n} |5x+3| = \frac{1}{5}|5x+3|. \quad (24)\end{aligned}$$

The ratio test guarantees absolute convergence for $\frac{1}{5}|5x+3| < 1$. In other words, absolute convergence is guaranteed by the ratio test for all x satisfying $\left|x + \frac{3}{5}\right| < 1$. This reveals that the center of convergence is $-\frac{3}{5}$, and the radius of convergence is $\boxed{1}$. We now need to check the endpoints. When $x = \frac{2}{5}$, the series becomes:

$$\sum_{n=0}^{\infty} \frac{n\left(5\left(\frac{2}{5}\right)+3\right)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n5^n}{5^n} = \sum_{n=0}^{\infty} n, \quad (25)$$

which is divergent, by the test for divergence. When $x = -\frac{8}{5}$, the series becomes:

$$\sum_{n=0}^{\infty} \frac{n\left(5\left(-\frac{8}{5}\right)+3\right)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n n, \quad (26)$$

which is also divergent by the test for divergence. Thus, the interval of convergence is $\boxed{\left(-\frac{8}{5}, \frac{2}{5}\right)}$. $\square$

[15] **6.** Find a power series representation for the function and determine its radius of convergence.

$$f(x) = \frac{x}{x+1}$$

Solution: We note that

$$f(x) = x\left(\frac{1}{1-(-x)}\right). \tag{27}$$

We now use the characterization theorem for geometric series to re-write this as

$$x\sum_{n=0}^{\infty}(-x)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{n+1}}, \tag{28}$$

as long as $|-x| < 1$. This indicates that x must satisfy $|x| < 1$ in order for the equality to hold, so the radius of convergence is $\boxed{1}$. □

[15] **7.** Find the Taylor series for f centered at the given value of a, and find the associated radius of convergence.

$$f(x) = \frac{1}{x^2}, \qquad a = 1$$

Solution: As always, the Taylor series centered at a is given by the definition: $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$. To write this, we need information about the derivatives of f at $a = 1$:

$$\begin{array}{cc} f^{(0)}(x) = x^{-2} & f^{(0)}(1) = 1 \\ f^{(1)}(x) = -2x^{-3} & f^{(1)}(1) = -2 \\ f^{(2)}(x) = (-2)(-3)x^{-4} & f^{(2)}(1) = (2)(3) \\ f^{(3)}(x) = (-2)(-3)(-4)x^{-5} & f^{(3)}(1) = -(2)(3)(4) \end{array}. \tag{29}$$

From this we deduce the pattern

$$f^{(n)}(1) = (-1)^n(n+1)!, \tag{30}$$

and so the Taylor series is

$$\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)!}{n!}(x-1)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n}. \tag{31}$$

The radius of convergence can be found using the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n(n+1)(x-1)^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}|x-1| = |x-1|. \tag{32}$$

The ratio test indicates absolute convergence whenever $|x-1|<1$, and so the radius of convergence is $\boxed{1}$. $\square$

Math 142V Midterm Test 2 **Name: Solution Key**

[10] **1.** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{(3)^n}{(-2)^{n+2}}$$

Solution: This is a geometric series:

$$\sum_{n=1}^{\infty} \frac{3^n}{(-2)^{n+2}} = \sum_{n=1}^{\infty} \frac{1}{4}\left(-\frac{3}{2}\right)^n. \tag{33}$$

As $\left|-\frac{3}{2}\right| > 1$, the characterization theorem for geometric series indicates that this is $\boxed{\text{divergent}}$. $\square$

[15] **2.** Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution: We know that $\frac{1}{n\ln n} > \frac{1}{(n+1)\ln(n+1)}$, so the terms form a decreasing sequence. This means that the integral test can apply. We define the function $f(x) = \frac{1}{x \ln x}$ and take its improper integral:

$$\int_2^{\infty} \frac{1}{x \ln x}\,\mathrm{d}x = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\,\mathrm{d}x. \tag{34}$$

We use a u-substitution:

$$\begin{aligned} u &= \ln x \quad & x = 2 \Rightarrow u = \ln 2 \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x \quad & x = t \Rightarrow u = \ln t \end{aligned} \tag{35}$$

$$\lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,\mathrm{d}u = \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \ln|\ln t| - \ln(\ln 2) = \infty. \tag{36}$$

As the improper integral is divergent, the integral test indicates that the series is also $\boxed{\text{divergent}}$. $\square$

[15] **3.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{2^n + 1}$$

Solution: The terms of this series are not all positive, but it is not an alternating series. For this reason, it would be easiest to determine whether the series of absolute values is convergent:

$$\sum_{n=1}^{\infty} \left| \frac{\cos\left(\frac{n\pi}{3}\right)}{2^n + 1} \right| = \sum_{n=1}^{\infty} \frac{\left|\cos\left(\frac{n\pi}{3}\right)\right|}{2^n + 1}. \tag{37}$$

We note that, for any value of n, $\left|\cos\left(\frac{n\pi}{3}\right)\right| \leq 1$. As a result,

$$\frac{\left|\cos\left(\frac{n\pi}{3}\right)\right|}{2^n + 1} \leq \frac{1}{2^n + 1} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n. \tag{38}$$

As $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, the comparison test indicates that $\sum\limits_{n=1}^{\infty} \left| \frac{\cos\left(\frac{n\pi}{3}\right)}{2^n+1} \right|$ is also convergent. We deduce that $\sum\limits_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{2^n+1}$ is absolutely convergent, which implies that it is $\boxed{\text{convergent}}$. □

[15] **4.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^{n+1}}{5^n}$$

Solution: To test for absolute convergence, we take the series of absolute values:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{2^{n+1}}{5^n} \right| = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n. \tag{39}$$

This is a convergent geometric series, and so $\sum\limits_{n=1}^{\infty} (-1)^n \frac{2^{n+1}}{5^n}$ is $\boxed{\text{absolutely convergent}}$.
□

[15] **5.** Determine the radius and interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{(5x-1)^n}{e^n}$$

Solution: As usual when dealing with power series, we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(5x-1)^{n+1}}{e^{n+1}}\right)}{\left(\frac{(5x-1)^n}{e^n}\right)} \right| = \lim_{n\to\infty} \left| \frac{(5x-1)^{n+1}}{e^{n+1}} \frac{e^n}{(5x-1)^n} \right| = \frac{1}{e}|5x-1|. \tag{40}$$

The ratio test guarantees absolute convergence for $\frac{1}{e}|5x-1| < 1$, or in other words, $\left|x - \frac{1}{5}\right| < \frac{e}{5}$. This indicates that the ratio of convergence is $\boxed{\frac{e}{5}}$. We now check the endpoints of the interval of convergence. When $x = \frac{1}{5} - \frac{e}{5}$:

$$\sum_{n=0}^{\infty} \frac{\left(5\left(\frac{1-e}{5}\right) - 1\right)^n}{e^n} = \sum_{n=0}^{\infty} \frac{(-e)^n}{e^n} = \sum_{n=0}^{\infty} (-1)^n. \tag{41}$$

By the test for divergence, this is divergent. When $x = \frac{1}{5} + \frac{e}{5}$:

$$\sum_{n=0}^{\infty} \frac{\left(5\left(\frac{1+e}{5}\right) - 1\right)^n}{e^n} = \sum_{n=0}^{\infty} \frac{e^n}{e^n} = \sum_{n=0}^{\infty} 1, \tag{42}$$

which is also divergent, by the test for divergence. Thus, the interval of convergence is $\boxed{\left(\frac{1-e}{5}, \frac{1+e}{5}\right)}$. $\square$

[15] **6.** Find a power series representation for the function and determine its radius of convergence.

$$f(x) = \frac{1}{x+4}$$

Solution: We note that

$$f(x) = \frac{1}{4-(-x)} = \frac{1}{4}\left(\frac{1}{1-\left(-\frac{1}{4}x\right)}\right) \tag{43}$$

From here, we can use the characterization theorem for geometric series to say that

$$f(x) = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{4}x\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{4^{n+1}}}, \tag{44}$$

as long as $\left|-\frac{1}{4}x\right| < 1$, or in other words, $|x| < 4$. This reveals that the radius of convergence is $\boxed{4}$. $\square$

[15] **7.** Find the Taylor series for f centered at the given value of a, and find the associated radius of convergence.

$$f(x) = e^{-\pi x}, \qquad a = 0$$

Solution: As always, the Taylor series centered at a is given by the definition: $\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$. To write this, we need information about the derivatives of f at $a = 0$:

$$\begin{array}{rr} f^{(0)}(x) = e^{-\pi x} & f^{(0)}(0) = 1 \\ f^{(1)}(x) = -\pi e^{-\pi x} & f^{(1)}(0) = -\pi \\ f^{(2)}(x) = \pi^2 e^{-\pi x} & f^{(2)}(0) = \pi^2 \\ f^{(3)}(x) = -\pi^3 e^{-\pi x} & f^{(3)}(0) = -\pi^3 \end{array}. \tag{45}$$

From this we deduce the pattern

$$f^{(n)}(0) = (-1)^n \pi^n, \tag{46}$$

and so the Taylor series is

$$\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{n!}(x-1)^n}. \tag{47}$$

The radius of convergence can be found using the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\pi^{n+1}(x-1)^{n+1}}{(n+1)!}\frac{n!}{(-1)^n\pi^n(x-1)^n}\right| = \lim_{n\to\infty}\frac{|x-1|}{n+1} = 0. \tag{48}$$

Since this is less than 1 for all real values of x, the ratio test indicates that the radius of convergence is $\boxed{\infty}$ $\square$