Math 141V Midterm Test 1 **Name:** Solution Key

October 1st, 2020

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Signature: ____

You must show all of your work and reasoning to receive full credit.

[15] **1.** Evaluate the definite integral.

$$\int_0^1 \tan^{-1} x \,\mathrm{d}x$$

Solution: This can be most easily handled by integration by parts:

$$\begin{aligned} u &= \tan^{-1} x & \mathrm{d}v &= \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{1+x^2}\,\mathrm{d}x & v &= x \end{aligned} \tag{1}$$

$$x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \tag{2}$$

The remaining integral can now be handled by u-substitution (we'll use w here, though):

$$\begin{aligned} w &= 1+x^2 & x = 0 &\Rightarrow w = 1 \\ \mathrm{d}w &= 2x\,\mathrm{d}x & x = 1 &\Rightarrow w = 2 \end{aligned} \tag{3}$$

$$\begin{aligned} x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{w}\,\mathrm{d}w &= x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\ln w\Big|_1^2 \\ &= \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}(\ln 2 - 0) = \boxed{\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2}. \end{aligned} \tag{4}$$

[15] **2.** Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} \sec^6 t \; \mathrm{d}t$$

Solution: We use the trigonometric identity $\sec^2 t - \tan^2 t = 1$ to write

$$\int_0^{\frac{\pi}{4}} \left(1 + \tan^2 t\right)^2 \sec^2 t \; \mathrm{d}t. \tag{5}$$

This can now be handled by u-substitution:

$$\begin{aligned} u &= \tan t & t = 0 &\Rightarrow u = 0 \\ \mathrm{d}u &= \sec^2 t \; \mathrm{d}t & t = \tfrac{\pi}{4} &\Rightarrow u = 1 \end{aligned} \tag{6}$$

$$\begin{aligned} \int_0^1 \left(1 + u^2\right)^2 \mathrm{d}u = \int_0^1 1 + 2u^2 + u^4 \; \mathrm{d}u = u + \frac{2}{3}u^3 + \frac{1}{5}u^5 \Big|_0^1 \\ = \left(1 + \frac{2}{3} + \frac{1}{5}\right) - (0 + 0 + 0) = \boxed{\tfrac{28}{15}}. \end{aligned} \tag{7}$$

[15] **3.** Evaluate the indefinite integral.

$$\int \left(x^2 + 100\right)^{-2} \, \mathrm{d}x$$

Solution: This can be done using a trigonometric substitution:

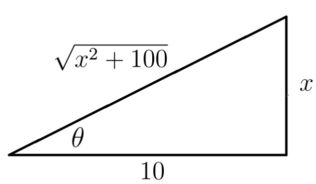
$$\begin{aligned} x &= 10 \tan \theta \\ \mathrm{d}x &= 10\sec^2\theta \, \mathrm{d}\theta \end{aligned} \tag{8}$$

$$\begin{aligned} \int \left((10 \tan \theta)^2 + 100\right)^{-2} 10\sec^2\theta \, \mathrm{d}\theta &= \int \frac{1}{\left(100\tan^2\theta + 100\right)^2} \, 10\sec^2\theta \, \mathrm{d}\theta \\ &= \int \frac{1}{\left(100 \left(\tan^2\theta + 1\right)\right)^2} 10\sec^2\theta \, \mathrm{d}\theta \\ &= \frac{1}{1000} \int \frac{\sec^2\theta}{\sec^4\theta} \, \mathrm{d}\theta = \frac{1}{1000} \int \cos^2\theta \, \mathrm{d}\theta. \end{aligned} \tag{9}$$

We now use the trigonometric identity $\cos^2\theta = \frac{1}{2}\left(1 + \cos(2\theta)\right)$:

$$\frac{1}{1000} \int \frac{1}{2} \left(1 + \cos(2\theta)\right) \, \mathrm{d}\theta = \frac{1}{2000} \left(\theta + \frac{1}{2} \sin(2\theta)\right) + C. \tag{10}$$

According to our substitution, $\tan \theta = \frac{x}{10}$. We illustrate this with a right triangle diagram:



The diagram indicates that $\sin\theta = \frac{x}{\sqrt{x^2+100}}$, and $\cos\theta = \frac{10}{\sqrt{x^2+100}}$. By the trigonometric identity $\sin(2\theta) = 2 \sin\theta \cos\theta$, our antiderivative can be written as:

$$\boxed{\tfrac{1}{2000} \left(\tan^{-1}\left(\tfrac{x}{10}\right) + \tfrac{10x}{x^2+100}\right) + C}. \tag{11}$$

[15] **4.** Evaluate the indefinite integral.

$$\int \frac{\mathrm{d}t}{t^3 + t^2 - 2t}\,\mathrm{d}t$$

This problem contains a typographical error. The problem was actually intended to be

$$\int \frac{1}{t^3 + t^2 - 2t}\,\mathrm{d}t. \tag{12}$$

Under this assumption, the solution is as follows: we first set up the partial fraction decomposition

$$\frac{1}{t^3 + t^2 - 2t} = \frac{1}{t\,(t+2)\,(t-1)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1}. \tag{13}$$

Clearing the denominators, distributing, and combining like terms in t yields

$$1 = (A + B + C)\,t^2 + (A - B + 2C)\,t + (-2A)\,. \tag{14}$$

This gives the system of equations

$$\begin{aligned} A + B + C &= 0 \\ A - B + 2C &= 0. \\ -2A &= 1 \end{aligned} \tag{15}$$

The third equation gives $A = -\frac{1}{2}$. Adding the first and second equations gives $2A + 3C = 0$, and so $-1 + 3C = 0$, or in other words, $C = \frac{1}{3}$. By the second equation, $B = A + 2C = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}$. We now have

$$\begin{aligned} &\int \frac{\left(-\frac{1}{2}\right)}{t} + \frac{\left(\frac{1}{6}\right)}{t+2} + \frac{\left(\frac{1}{3}\right)}{t-1} \\ &\qquad = -\frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t + \frac{1}{6}\int \frac{1}{t+2}\,\mathrm{d}t + \frac{1}{3}\int \frac{1}{t-1}\,\mathrm{d}t \\ &\qquad = \boxed{-\tfrac{1}{2}\ln|t| + \tfrac{1}{6}\ln|t+2| + \tfrac{1}{3}\ln|t-1| + C}. \end{aligned} \tag{16}$$

[20] **5.** Evaluate the indefinite integral.

$$\int e^{4x} \sin\left(e^{2x}+1\right)\ \mathrm{d}x$$

Solution: We begin with a u-substitution:

$$\begin{aligned} u &= e^{2x}+1 \\ \mathrm{d}u &= 2e^{2x}\,\mathrm{d}x \end{aligned}. \tag{17}$$

This allows us to re-write the integral as

$$\frac{1}{2}\int (u-1)\sin u\,\mathrm{d}u. \tag{18}$$

We now use integration by parts (we will use w and $\mathrm{d}v$):

$$\begin{aligned} w &= u-1 \quad & \mathrm{d}v &= \sin u\,\mathrm{d}u \\ \mathrm{d}w &= \mathrm{d}u & v &= -\cos u \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{1}{2}\left((1-u)\cos u + \int \cos u\,\mathrm{d}u\right) &= \frac{1}{2}\left((1-u)\cos u + \sin u\right) + C \\ &= \boxed{\tfrac{1}{2}\left(\sin\left(e^{2x}+1\right) - e^{2x}\cos\left(e^{2x}+1\right)\right) + C}. \end{aligned} \tag{20}$$

[20] **6.** Find all real values of p such that the integral

$$\int_1^\infty \frac{(\ln x)^p}{x}\,\mathrm{d}x$$

is convergent.

This problem ended up being significantly harder than I had intended. Solution: Consider the case that $p < 0$. This means that $\ln x$ is in the denominator, and since $\ln 1 = 0$, this would mean that the integral is improper for more than one reason. Now,

$$\int_1^\infty \frac{(\ln x)^p}{x}\,\mathrm{d}x = \lim_{s\to 1^+}\int_s^2 \frac{(\ln x)^p}{x}\,\mathrm{d}x + \lim_{t\to\infty}\int_2^t \frac{(\ln x)^p}{x}\,\mathrm{d}x. \tag{21}$$

For now, we will concern ourselves only with the second term. We proceed by u-substitution:

$$\begin{aligned} u &= \ln x & x = 2 &\Rightarrow u = \ln 2 \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x & x = t &\Rightarrow u = \ln t \end{aligned} \tag{22}$$

$$\lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^p\,\mathrm{d}u \tag{23}$$

Now, if $p = -1$, then this becomes

$$\lim_{t\to\infty} \ln u\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2) = \infty, \tag{24}$$

and so the integral diverges in this case. If $p < 0$ and $p \neq -1$, this becomes

$$\lim_{t\to\infty} \frac{u^{p+1}}{p+1}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{1}{p+1}\left((\ln t)^{p+1} - (\ln 2)^{p+1}\right). \tag{25}$$

Now, if $p + 1 > 0$, this diverges. On the other hand, if $p + 1 < 0$ (or in other words, $p < -1$), then this converges to $-\frac{(\ln 2)^{p+1}}{p+1}$. Thus, the second term of Equation 21 converges if and only if $p < -1$. We now consider the first term, still in the case that $p < 0$.

By the same u-substition as was previously used, the first term of Equation 21 becomes

$$\lim_{s \to 1^+} \int_{\ln s}^{\ln 2} u^p \, \mathrm{d}u. \tag{26}$$

We need not even consider the case that $p = -1$, since we have already seen that the second term of Equation 21 would diverge in that case. Therefore, assume that $p \neq -1$. In this case, the antiderivative is

$$\lim_{s \to 1^+} \left. \frac{u^{p+1}}{p+1} \right|_{\ln s}^{\ln 2} = \lim_{s \to 1^+} \frac{1}{p+1} \left((\ln 2)^{p+1} - (\ln s)^{p+1} \right). \tag{27}$$

Since $\lim\limits_{s \to 1^+} \ln s = 0$, this converges if and only if $p + 1 \geq 0$. To summarize: the first term of Equation 21 converges if and only if $p \geq -1$, while the second term converges if and only if $p < -1$. In other words, it is impossible for both terms to converge, and so the improper integral must diverge if $p < 0$.

Consider the case that $p \geq 0$. In this case, the integral is improper only because of the upper bound, so

$$\int_1^\infty \frac{(\ln x)^p}{x} \, \mathrm{d}x = \lim_{t \to \infty} \int_1^t \frac{(\ln x)^p}{x} \, \mathrm{d}x. \tag{28}$$

Using the same u-substitution as before, this becomes

$$\lim_{t \to \infty} \int_0^{\ln t} u^p \, \mathrm{d}u = \lim_{t \to \infty} \left. \frac{u^{p+1}}{p+1} \right|_0^{\ln t} = \lim_{t \to \infty} \frac{1}{p+1} \left((\ln t)^{p+1} - 0^{p+1} \right). \tag{29}$$

This diverges if $p + 1 > 0$ (which it is, because we are assuming that $p \geq 0$). Therefore, the integral diverges in this case.

$\boxed{\text{There are no real values of } p \text{ for which the integral converges.}}$

Math 141V Midterm Test 1 **Name:** Solution Key

October 6th, 2020

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[15] **1.** Evaluate the definite integral.

$$\int_1^e (\ln x)^2 \, \mathrm{d}x$$

Solution: We proceed by integration by parts:

$$\begin{aligned} u &= (\ln x)^2 & \mathrm{d}v &= \mathrm{d}x \\ \mathrm{d}u &= \tfrac{2\ln x}{x}\,\mathrm{d}x & v &= x \end{aligned} \tag{30}$$

$$x(\ln x)^2\Big|_1^e - 2\int_1^e \ln x \, \mathrm{d}x. \tag{31}$$

The remaining integral can be handled by a second integration by parts:

$$\begin{aligned} u &= \ln x & \mathrm{d}v &= \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x & v &= x \end{aligned} \tag{32}$$

$$\begin{aligned} x(\ln x)^2\Big|_1^e &- 2\left(x\ln x\Big|_1^e - \int_1^e 1 \, \mathrm{d}x\right) \\ &= x(\ln x)^2\Big|_1^e - 2x\ln x\Big|_1^e + 2x\Big|_1^e \\ &= (e-0) - 2\,(e-0) + 2\,(e-1) = \boxed{e-2}. \end{aligned} \tag{33}$$

[15] **2.** Evaluate the definite integral.

$$\int_0^{\frac{\pi}{2}} \cos^4 t \; \mathrm{d}t$$

Solution: We use the trigonometric identity $\cos^2 t = \frac{1}{2}\left(1 + \cos\left(2t\right)\right)$:

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\left(1 + \cos\left(2t\right)\right)\right)^2 \mathrm{d}t = \frac{1}{4}\int_0^{\frac{\pi}{2}} 1 + 2\cos\left(2t\right) + \cos^2\left(2t\right) \; \mathrm{d}t. \tag{34}$$

We use the same identity to re-write the last term:

$$\begin{aligned} &\frac{1}{4}\int_0^{\frac{\pi}{2}} 1 + 2\cos\left(2t\right) + \frac{1}{2}\left(1 + \cos\left(2t\right)\right) \; \mathrm{d}t \\ &\qquad = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{3}{2} + \frac{5}{2}\cos\left(2t\right) \; \mathrm{d}t = \frac{1}{4}\left(\frac{3}{2}t + \frac{5}{4}\sin\left(2t\right)\Big|_0^{\frac{\pi}{2}}\right) \\ &\qquad\qquad = \frac{1}{4}\left(\frac{3}{2}\left(\frac{\pi}{2}\right)\right) = \boxed{\tfrac{3\pi}{16}}. \end{aligned} \tag{35}$$

[15] **3.** Evaluate the indefinite integral.

$$\int \frac{1}{x^4\sqrt{x^2-4}}\,\mathrm{d}x$$

Solution: We begin with a trigonometric substitution:

$$\begin{aligned} x &= 2\sec\theta \\ \mathrm{d}x &= 2\sec\theta\tan\theta\,\mathrm{d}\theta \end{aligned} \tag{36}$$

$$\int \frac{2\sec\theta\tan\theta}{16\sec^4\theta\sqrt{4\sec^2\theta-4}}\,\mathrm{d}\theta = \frac{1}{16}\int \cos^3\theta\,\mathrm{d}\theta. \tag{37}$$

We now use the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$:

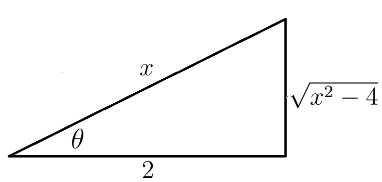
$$\frac{1}{16}\int \left(1-\sin^2\theta\right)\cos\theta\,\mathrm{d}\theta. \tag{38}$$

From here, a u-substitution is effective:

$$\begin{aligned} u &= \sin\theta \\ \mathrm{d}u &= \cos\theta\,\mathrm{d}\theta \end{aligned} \tag{39}$$

$$\frac{1}{16}\int 1-u^2\,\mathrm{d}u = \frac{1}{16}\left(u-\frac{1}{3}u^3\right)+C = \frac{1}{16}\left(\sin\theta-\frac{1}{3}\sin^3\theta\right)+C \tag{40}$$

In order to write this in terms of x, we now construct a right triangle diagram:



The diagram reveals that $\sin\theta = \frac{\sqrt{x^2-4}}{x}$, and so our final answer can be written as

$$\boxed{\frac{\sqrt{x^2-4}}{16x} - \frac{\left(x^2-4\right)^{\frac{3}{2}}}{48x^3} + C}. \tag{41}$$

[15] **4.** Evaluate the indefinite integral.

$$\int \frac{2t^3+2t-1}{t^4+t^2}\,\mathrm{d}t$$

Solution: As this is a rational function that has a numerator of lesser degree than the denominator, we can use partial fraction decomposition:

$$\frac{2t^3+2t-1}{t^2\left(t^2+1\right)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} \tag{42}$$

$$2t^3+2t-1 = A\left(t^3+t\right) + B\left(t^2+1\right) + \left(Ct+D\right)t^2 \tag{43}$$

$$2t^3+2t-1 = (A+C)\,t^3 + (B+D)\,t^2 + (A)\,t + (B) \tag{44}$$

This gives the system of simultaneous equations:

$$\begin{gathered} A+C=2 \\ B+D=0 \\ A=2 \\ B=-1 \end{gathered} \tag{45}$$

This shows that $D=1$ and $C=0$. Thus, our integral becomes

$$\int \frac{2}{t} - \frac{1}{t^2} + \frac{1}{t^2+1}\,\mathrm{d}t = \boxed{2\ln|t| + \tfrac{1}{t} + \tan^{-1}t + C}. \tag{46}$$

[20] **5.** Evaluate the indefinite integral.

$$\int \frac{e^{-\frac{1}{x}}}{x^3}\,\mathrm{d}x$$

Solution: We begin with a u-substitution:

$$\begin{aligned} u &= -\tfrac{1}{x} \\ \mathrm{d}u &= \tfrac{1}{x^2}\,\mathrm{d}x \end{aligned} \tag{47}$$

$$-\int ue^u\,\mathrm{d}u. \tag{48}$$

This can now be handled by integration by parts (we use w and $\mathrm{d}v$ here):

$$\begin{aligned} w &= u & \mathrm{d}v &= e^u\,\mathrm{d}u \\ \mathrm{d}w &= \mathrm{d}u & v &= e^u \end{aligned} \tag{49}$$

$$\begin{aligned} -\left(ue^u - \int e^u\,\mathrm{d}u\right) &= -\left(ue^u - e^u\right) + C \\ &= e^u\left(1-u\right) + C = \boxed{e^{-\frac{1}{x}}\left(1+\tfrac{1}{x}\right) + C}. \end{aligned} \tag{50}$$

[20] **6.** Find all real values of p such that the integral

$$\int_1^\infty xe^{-px}\,\mathrm{d}x$$

is convergent.

Solution: By definition,

$$\int_1^\infty xe^{-px}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t xe^{-px}\,\mathrm{d}x. \tag{51}$$

First, we consider the case that $p = 0$. Under this assumption,

$$\lim_{t\to\infty}\int_1^t xe^{-px}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t x\,\mathrm{d}x = \lim_{t\to\infty}\frac{1}{2}x^2\Big|_1^t = \frac{1}{2}\lim_{t\to\infty}\left(t^2-1\right) = \infty, \tag{52}$$

and so the integral diverges if $p = 0$. Next, suppose that $p \neq 0$. In this case, we use integration by parts:

$$\begin{array}{ll} u = x & \mathrm{d}v = e^{-px}\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = -\frac{1}{p}e^{-px} \end{array} \tag{53}$$

$$\begin{aligned}\lim_{t\to\infty}\left(-\frac{1}{p}xe^{-px}\Big|_1^t + \frac{1}{p}\int_1^t e^{-px}\,\mathrm{d}x\right) &= \lim_{t\to\infty} -\frac{1}{p}xe^{-px}\Big|_1^t - \frac{1}{p^2}e^{-px}\Big|_1^t \\ = \frac{1}{p}\lim_{t\to\infty}\left(-xe^{-px} - \frac{1}{p}e^{-px}\Big|_1^t\right) &= -\frac{1}{p}\lim_{t\to\infty}\left(\frac{px+1}{pe^{px}}\Big|_1^t\right) \\ &= -\frac{1}{p^2}\lim_{t\to\infty}\left(\frac{pt+1}{e^{pt}} - \frac{p+1}{e^p}\right).\end{aligned} \tag{54}$$

Now, if $p < 0$, then this diverges. If $p > 0$, then this becomes an indeterminate form of type $\frac{0}{0}$, in which case we can use L'Hopital's rule on the first term:

$$-\frac{1}{p^2}\left(\lim_{t\to\infty}\frac{p}{pe^{pt}} - \frac{p+1}{e^p}\right) = -\frac{1}{p^2}\left(0 - \frac{p+1}{e^p}\right) = \frac{p+1}{p^2e^p}. \tag{55}$$

Thus, the integral is convergent if and only if $\boxed{p > 0}$.

Math 141V Midterm Test 1 **Name:**

October 8th, 2020

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[15] **1.** Evaluate the definite integral.

$$\int_0^1 \tan^{-1}x \, \mathrm{d}x$$

Solution: This can be most easily handled by integration by parts:

$$\begin{aligned} u &= \tan^{-1}x & \mathrm{d}v &= \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{1+x^2}\,\mathrm{d}x & v &= x \end{aligned} \tag{56}$$

$$x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \tag{57}$$

The remaining integral can now be handled by u-substitution (we'll use w here, though):

$$\begin{aligned} w &= 1+x^2 & x = 0 &\Rightarrow w = 1 \\ \mathrm{d}w &= 2x\,\mathrm{d}x & x = 1 &\Rightarrow w = 2 \end{aligned} \tag{58}$$

$$\begin{aligned} x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{w}\,\mathrm{d}w &= x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln w\Big|_1^2 \\ &= \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}(\ln 2 - 0) = \boxed{\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2}. \end{aligned} \tag{59}$$

[15] **2.** Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} \sec t \, \tan^5 t \, \mathrm{d}t$$

Solution: First, we use the trigonometric identity $\sec^2 t - \tan^2 t = 1$:

$$\int_0^{\frac{\pi}{4}} \sec t \, \tan^4 t \tan t \, \mathrm{d}t = \int_0^{\frac{\pi}{4}} \left(\sec^2 t - 1\right)^2 \, \sec t \tan t \, \mathrm{d}t \tag{60}$$

Next, we use a u-substitution:

$$\begin{aligned} u &= \sec t & t = 0 &\Rightarrow u = 1 \\ \mathrm{d}u &= \sec t \tan t \, \mathrm{d}t & t = \tfrac{\pi}{4} &\Rightarrow u = \sqrt{2} \end{aligned} \tag{61}$$

$$\begin{aligned} \int_1^{\sqrt{2}} \left(u^2 - 1\right)^2 \mathrm{d}u &= \int_1^{\sqrt{2}} u^4 - 2u^2 + 1 \, \mathrm{d}u = \frac{1}{5}u^5 - \frac{2}{3}u^3 + u \bigg|_1^{\sqrt{2}} \\ &= \frac{1}{5}\left(4\sqrt{2} - 1\right) - \frac{2}{3}\left(2\sqrt{2} - 1\right) + \left(\sqrt{2} - 1\right) = \boxed{\tfrac{7\sqrt{2} - 8}{15}}. \end{aligned} \tag{62}$$

[15] **3.** Evaluate the indefinite integral.

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,\mathrm{d}x$$

Solution: We proceed by trigonometric substitution:

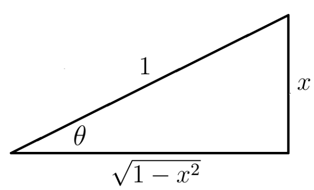
$$\begin{aligned} x &= \sin\theta \\ \mathrm{d}x &= \cos\theta\,\mathrm{d}\theta \end{aligned} \tag{63}$$

$$\int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta. \tag{64}$$

From here, we use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ to get

$$\int \frac{\cos\theta}{\sin^2\theta\sqrt{\cos^2\theta}}\,\mathrm{d}\theta = \int \frac{\cos\theta}{\sin^2\theta\ \cos\theta}\,\mathrm{d}\theta = \int \csc^2\theta\,\mathrm{d}\theta = -\cot\theta + C. \tag{65}$$

In order to write this in terms of x, we now construct a right triangle diagram. Based on our original substitution, $\sin\theta = \frac{x}{1}$, so there exists a right triangle with an acute angle of θ whose opposite side has length x and whose hypotenuse has length 1:



Based on this, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$, so $\cot\theta = \frac{\sqrt{1-x^2}}{x}$. Thus, our indefinite integral is

$$\boxed{-\frac{\sqrt{1-z^2}}{x} + C}. \tag{66}$$

[15] **4.** Evaluate the indefinite integral.

$$\int \frac{5t^2+t-2}{2t^3-2t^2}\,\mathrm{d}t$$

Solution: We proceed by partial fraction decomposition:

$$\frac{5t^2+t-2}{t^2\,(2t-2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{2t-2} \tag{67}$$

$$\begin{aligned} 5t^2+t-2 &= At\,(2t-2) + B\,(2t-2) + Ct^2 \\ 5t^2+t-2 &= 2At^2 - 2At + 2Bt - 2B + Ct^2 \\ 5t^2+t-2 &= (2A+C)\,t^2 + (2B-2A)\,t + (-2B) \end{aligned} . \tag{68}$$

This gives the system of equations

$$\begin{aligned} 2A + C &= 5 \\ 2B - 2A &= 1. \\ -2B &= -2 \end{aligned} \tag{69}$$

The last equation gives $B = 1$, so $A = \frac{1}{2}$, and $C = 4$. We can now re-write the integral as

$$\int \frac{\left(\frac{1}{2}\right)}{t} + \frac{1}{t^2} + \frac{4}{2t-2}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t + \int t^{-2}\,\mathrm{d}t + 2\int \frac{1}{t-1}\,\mathrm{d}t. \tag{70}$$

Taking antiderivatives, we get

$$\boxed{\tfrac{1}{2}\ln|t| - \tfrac{1}{t} + 2\ln|t-1| + C}. \tag{71}$$

[20] **5.** Evaluate the indefinite integral.

$$\int \frac{1}{\left(1+\sqrt{x}\right)^2}\,\mathrm{d}x$$

Solution: We proceed by u-substitution:

$$\begin{aligned} u &= 1+\sqrt{x} \\ \mathrm{d}u &= \tfrac{1}{2\sqrt{x}}\,\mathrm{d}x \end{aligned}. \tag{72}$$

The definition of $\mathrm{d}u$ can be re-written as $2\left(u-1\right)\,\mathrm{d}u = \mathrm{d}x$. Therefore, the integral becomes

$$\int \frac{2\left(u-1\right)}{u^2}\,\mathrm{d}u = 2\int \frac{1}{u} - \frac{1}{u^2}\,\mathrm{d}u = 2\ln|u| + \frac{2}{u} + C. \tag{73}$$

In terms of x, this is

$$\boxed{2\ln\left(1+\sqrt{x}\right) + \tfrac{2}{1+\sqrt{x}} + C}. \tag{74}$$

[20] **6.** Find all real values of p such that the integral

$$\int_1^\infty x^p \ln x \, \mathrm{d}x$$

is convergent.

Solution: By definition, this is

$$\lim_{t\to\infty} \int_1^t x^p \ln x \, \mathrm{d}x. \tag{75}$$

We proceed by integration by parts:

$$\begin{aligned} u &= \ln x \qquad \mathrm{d}v = x^p \, \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{x} \, \mathrm{d}x \end{aligned} \tag{76}$$

The function v will depend on the value of p. We consider two cases: either $p = -1$, or $p \neq -1$.

If $p = -1$, then $v = \ln |x|$, in which case this integral becomes

$$\lim_{t\to\infty} \left((\ln x)^2 \Big|_1^t - \int_1^t \frac{\ln x}{x} \, \mathrm{d}x \right). \tag{77}$$

We can now evaluate the remaining integral by a u-substitution:

$$\begin{aligned} u &= \ln x \qquad x = 1 \Rightarrow u = 0 \\ \mathrm{d}u &= \tfrac{1}{x} \, \mathrm{d}x \quad x = t \Rightarrow u = \ln t \end{aligned} \tag{78}$$

$$\begin{aligned} \lim_{t\to\infty} \left((\ln x)^2 \Big|_1^t - \int_0^{\ln t} u \, \mathrm{d}u \right) &= \lim_{t\to\infty} \left((\ln x)^2 \Big|_1^t - \frac{1}{2} u^2 \Big|_0^{\ln t} \right) \\ &= \lim_{t\to\infty} \left((\ln t)^2 - \frac{1}{2} (\ln t)^2 \right) = \lim_{t\to\infty} \frac{1}{2} (\ln t)^2 = \infty. \end{aligned} \tag{79}$$

Thus, if $p = -1$, then the improper integral diverges.

On the other hand, if $p \neq -1$, then $v = \frac{x^{p+1}}{p+1}$. In that case, the integral becomes

$$\begin{aligned}
\lim_{t\to\infty} & \left(\frac{x^{p+1}}{p+1} \ln x \bigg|_1^t - \int_1^t \frac{x^p}{p+1} \, \mathrm{d}x \right) \\
&= \lim_{t\to\infty} \left(\frac{x^{p+1}}{p+1} \ln x \bigg|_1^t - \frac{x^{p+1}}{(p+1)^2} \bigg|_1^t \right) \\
&= \lim_{t\to\infty} \left(\frac{t^{p+1} \ln t}{p+1} - \frac{t^{p+1}}{(p+1)^2} + \frac{1}{(p+1)^2} \right) \\
&= \frac{1}{(p+1)^2} + \lim_{t\to\infty} \frac{t^{p+1} \left((p+1) \ln t - 1 \right)}{(p+1)^2}. \qquad (80)
\end{aligned}$$

If $p+1 > 0$, then this approaches infinity, and so the improper integral diverges if $p > -1$. If $p+1 < 0$, then we can use L'Hopital's rule as follows:

$$\lim_{t\to\infty} \frac{(p+1)\ln t - 1}{(p+1)^2 t^{-p-1}} = \lim_{t\to\infty} \frac{\left(\frac{p+1}{t}\right)}{(p+1)^2 \left(-p-1\right) t^{-p-2}} = \lim_{t\to\infty} -\frac{t^{p+1}}{(p+1)^2} = 0. \qquad (81)$$

Thus, the integral converges if and only if $\boxed{p < -1}$.