Math 142V Quiz 10 **Name:** Solution Key

December 1st, 2020

You must show all of your work and reasoning to receive full credit.

1. [10] Find the exact length of the curve.

$$y = (x-1)^{\frac{3}{2}}, \quad 1 \leq x \leq 9$$

Solution: As always, we need to use the arc length formula. To do this, we first find the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}(x-1)^{\frac{1}{2}} = \frac{3}{2}\sqrt{x-1}. \tag{1}$$

Now,

$$s = \int_1^9 \sqrt{1 + \left(\frac{3}{2}\sqrt{x-1}\right)^2}\,\mathrm{d}x = \int_1^9 \sqrt{1 + \frac{9}{4}(x-1)}\,\mathrm{d}x = \int_1^9 \sqrt{\frac{9}{4}x - \frac{5}{4}}\,\mathrm{d}x. \tag{2}$$

This can be handled by a u-substitution:

$$\begin{aligned} u &= \tfrac{9}{4}x - \tfrac{5}{4} \quad & x = 1 \Rightarrow u = 1 \\ \mathrm{d}u &= \tfrac{9}{4}\,\mathrm{d}x \quad & x = 9 \Rightarrow u = 19 \end{aligned} \tag{3}$$

$$s = \int_1^{19} \frac{4}{9}\sqrt{u}\,\mathrm{d}u = \frac{4}{9}\int_1^{19} u^{\frac{1}{2}}\,\mathrm{d}u = \frac{8}{27}u^{\frac{3}{2}}\Big|_1^{19} = \boxed{\tfrac{8}{27}\left(19^{\frac{3}{2}} - 1\right)}. \tag{4}$$

□

2. [10] Find the exact length of the curve.

$$y = \sqrt{1-x^2}, \quad 0 \le x \le 1$$

Solution: Again, we need the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}. \tag{5}$$

We now use the arc length formula:

$$\begin{aligned} s &= \int_0^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,\mathrm{d}x = \int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,\mathrm{d}x \\ &= \int_0^1 \sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}\,\mathrm{d}x = \int_0^1 \sqrt{\frac{1}{1-x^2}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\ &= \sin^{-1}x\Big|_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \boxed{\tfrac{\pi}{2}}. \end{aligned} \tag{6}$$

□

Math 142V Quiz 10

Name: Solution Key

December 3rd, 2020

This is a make-up quiz. If you have not received my permission to take this make-up quiz, then your submission will not be accepted.

You must show all of your work and reasoning to receive full credit.

1. [10] Find the exact length of the curve.

$$y = x^{\frac{3}{2}} - 1, \quad 1 \leq x \leq 4$$

Solution: As always, we need to use the arc length formula. To do this, we first find the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}. \tag{7}$$

Now,

$$s = \int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\,\mathrm{d}x = \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x. \tag{8}$$

This can be handled by a u-substitution:

$$\begin{array}{ll} u = 1 + \frac{9}{4}x & x = 1 \Rightarrow u = \frac{13}{4} \\ \mathrm{d}u = \frac{9}{4}\,\mathrm{d}x & x = 4 \Rightarrow u = 9 \end{array} \tag{9}$$

$$s = \int_{\frac{13}{4}}^9 \frac{4}{9}\sqrt{u}\,\mathrm{d}u = \frac{9}{4}\int_{\frac{13}{4}}^9 u^{\frac{1}{2}}\,\mathrm{d}u = \frac{3}{2}u^{\frac{3}{2}}\bigg|_{\frac{13}{4}}^9 = \boxed{\tfrac{3}{2}\left(27 - \left(\tfrac{13}{4}\right)^{\frac{3}{2}}\right)}. \tag{10}$$

□

2. [10] Find the exact length of the curve.

$$y = \tfrac{1}{2}x^2 - \ln\left(\sqrt[4]{x}\right), \quad 1 \le x \le e$$

Solution: Again, we need the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - \frac{1}{\sqrt[4]{x}}\frac{1}{4}x^{-\frac{3}{4}} = x - \frac{1}{4x}. \tag{11}$$

We now use the arc length formula:

$$\begin{aligned} s &= \int_1^e \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\,\mathrm{d}x = \int_1^e \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\,\mathrm{d}x \\ &= \int_1^e \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\,\mathrm{d}x = \int_1^e \sqrt{\frac{1}{16x^2}\left(16x^4 + 8x^2 + 1\right)}\,\mathrm{d}x \\ &= \int_1^e \sqrt{\frac{1}{16x^2}\left(4x^2 + 1\right)^2}\,\mathrm{d}x = \int_1^e \frac{4x^2+1}{4x}\,\mathrm{d}x = \int_1^e x + \frac{1}{4x}\,\mathrm{d}x \\ &= \frac{1}{2}x^2 + \frac{1}{4}\ln|x|\bigg|_1^e = \frac{1}{2}\left(e^2 - 1\right) + \frac{1}{4}\left(\ln e - \ln 1\right) = \boxed{\tfrac{1}{2}e^2 - \tfrac{1}{4}}. \end{aligned} \tag{12}$$

□

Math 142V Quiz 10

Name: Solution Key

December 4th, 2020

This is a make-up quiz. If you have not received my permission to take this make-up quiz, then your submission will not be accepted.

You must show all of your work and reasoning to receive full credit.

1. [10] Find the exact length of the curve.

$$y = 2(x+1)^{\frac{3}{2}} - 1, \quad 1 \leq x \leq 4$$

Solution: As always, we need to use the arc length formula. To do this, we first find the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x+1)^{\frac{1}{2}} = 3\sqrt{x+1}. \tag{13}$$

Now,

$$s = \int_1^4 \sqrt{1 + \left(3\sqrt{x+1}\right)^2}\,\mathrm{d}x = \int_1^4 \sqrt{9x+4}\,\mathrm{d}x \tag{14}$$

This can be handled by a u-substitution:

$$\begin{aligned} u &= 9x+4 \quad x = 1 \Rightarrow u = 13 \\ \mathrm{d}u &= \tfrac{9}{4}\,\mathrm{d}x \quad x = 4 \Rightarrow u = 40 \end{aligned} \tag{15}$$

$$s = \int_{13}^{40} \frac{4}{9}\sqrt{u}\,\mathrm{d}u = \frac{4}{9}\int_{13}^{40} u^{\frac{1}{2}}\,\mathrm{d}u = \frac{8}{27}u^{\frac{3}{2}}\bigg|_{13}^{40} = \boxed{\tfrac{8}{27}\left(40^{\frac{3}{2}} - 13^{\frac{3}{2}}\right)}. \tag{16}$$

□

2. [10] Find the exact length of the curve.

$$y = \tfrac{1}{6}x^6 + \tfrac{1}{x^4}, \quad 1 \leq x \leq 2$$

Solution: **This problem has a mistake. It was actually supposed to be**

$$y = \frac{1}{6}x^6 + \frac{1}{16x^4}. \tag{17}$$

Assuming this, we need the derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^5 - \frac{1}{4}x^{-5}. \tag{18}$$

We now use the arc length formula:

$$\begin{aligned} s = \int_1^2 \sqrt{1 + \left(x^5 - \frac{1}{4}x^{-5}\right)^2}\,\mathrm{d}x &= \int_1^2 \sqrt{1 + x^{10} - \frac{1}{2} + \frac{1}{16}x^{-10}}\,\mathrm{d}x \\ &= \int_1^2 \sqrt{x^{10} + \frac{1}{2} + \frac{1}{16}x^{-10}}\,\mathrm{d}x = \int_1^2 \sqrt{\frac{1}{16}\left(16x^{20} + 8x^{10} + 1\right)}\,\mathrm{d}x \\ &= \int_1^2 \sqrt{\frac{1}{16}(4x^{10}+1)^2}\,\mathrm{d}x = \int_1^2 \frac{4x^{10}+1}{4}\,\mathrm{d}x \\ &= \int_1^2 x^{10} + \frac{1}{4}\,\mathrm{d}x = \frac{1}{11}x^{11} + \frac{1}{4}x \\ &= \frac{1}{11}\left(2^{11} - 1\right) + \frac{1}{4}(2-1) = \boxed{\tfrac{2^{11}}{11} + \tfrac{7}{44}} \end{aligned} \tag{19}$$

□