Math 141V Quiz 6 **Name:**

October 20th, 2020

You must show all of your work and reasoning to receive full credit.

1. [10] Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Solution: This is most easily handled by the integral test. In order to determine that the integral test applies, we consider the function $f(x) = xe^{-x^2}$. The derivative is

$$f'(x) = x\left(e^{-x^2}\right)(-2x) + e^{-x^2} = \left(1 - 2x^2\right)e^{-x^2}. \tag{1}$$

This is negative for $x > \frac{1}{\sqrt{2}}$, so the sequence of terms of the series is eventually decreasing. We consider the improper integral:

$$\int_1^{\infty} xe^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,\mathrm{d}x. \tag{2}$$

We use a u-substitution:

$$\begin{aligned} u &= x^2 & x = 1 &\Rightarrow u = 1 \\ \mathrm{d}u &= 2x\,\mathrm{d}x & x = t &\Rightarrow u = t^2 \end{aligned} \tag{3}$$

$$\lim_{t\to\infty}\int_1^{t^2} e^{-u}\,\mathrm{d}u = \lim_{t\to\infty} -e^{-u}\Big|_1^{t^2} = \lim_{t\to\infty} e^{-1} - e^{-t^2} = \frac{1}{e}. \tag{4}$$

Since the improper integral is convergent, the integral test indicates that the series $\sum_{n=1}^{\infty} ne^{-n^2}$ is $\boxed{\text{convergent}}$.

2. [10] Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{\sqrt{n}}$$

Solution: We note that

$$\frac{e^{\frac{1}{n}}}{\sqrt{n}} > \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}. \tag{5}$$

We note that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series. According to the p-series test, since $\frac{1}{2} \leq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent. Thus, the comparison test indicates that $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{\sqrt{n}}$ is also $\boxed{\text{divergent}}$.

Math 141V Quiz 6 **Name:**

October 22, 2020

This is a make-up quiz. If you have not received my permission to take this make-up quiz, then your submission will not be accepted.

You must show all of your work and reasoning to receive full credit.

1. [10] Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Solution: This can be handled by the integral test. In order to apply the integral test, we must verify that the sequence of terms is eventually decreasing. We do this by finding the derivative of $f(x) = \frac{\ln x}{x}$:

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x\,(1)}{x^2} = \frac{1 - \ln x}{x^2}. \tag{6}$$

This is negative for $x > e$, and so the sequence $\frac{\ln n}{n}$ is eventually decreasing. We now consider the improper integral

$$\int_1^{\infty} \frac{\ln x}{x}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x}\,\mathrm{d}x. \tag{7}$$

We do this by u-substitution:

$$\begin{array}{ll} u = \ln x & x = 1 \Rightarrow u = 0 \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & x = t \Rightarrow u = \ln t \end{array} \tag{8}$$

$$\lim_{t\to\infty} \int_0^{\ln t} u\,\mathrm{d}u = \lim_{t\to\infty} \frac{1}{2}u^2 \Big|_0^{\ln t} = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty. \tag{9}$$

As the improper integral is divergent, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also $\boxed{\text{divergent}}$.

2. [10] Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^3}$$

Solution: We note that

$$\frac{\tan^{-1} n}{n^3} < \frac{\left(\frac{\pi}{2}\right)}{n^3}. \tag{10}$$

We consider the series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{2}\right)}{n^3} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}. \tag{11}$$

This is convergent, since it is a constant multiple of a convergent p-series. The comparison test now indicates that $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^3}$ must also be $\boxed{\text{convergent}}$.