Math 141V Quiz 4 **Name: Solution Key**

September 29th, 2020

You must show all of your work and reasoning to receive full credit.

1. [10] Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,\mathrm{d}x$$

Solution: By definition, this is

$$\lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{1+x^2}\,\mathrm{d}x. \tag{1}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \tan^{-1}x & x = 1 \Rightarrow u = \tfrac{\pi}{4} \\ \mathrm{d}u &= \tfrac{1}{1+x^2}\,\mathrm{d}x & x = t \Rightarrow u = \tan^{-1}t \end{aligned} \tag{2}$$

$$\begin{aligned} \lim_{t\to\infty}\int_{\frac{\pi}{4}}^{\tan^{-1}t} u\,\mathrm{d}u &= \lim_{t\to\infty}\frac{1}{2}u^2\Big|_{\frac{\pi}{4}}^{\tan^{-1}t} \\ &= \lim_{t\to\infty}\frac{1}{2}\left(\left(\tan^{-1}t\right)^2 - \frac{\pi^2}{16}\right) \\ &= \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \frac{\pi^2}{16}\right) = \boxed{\tfrac{3\pi^2}{32}}. \end{aligned} \tag{3}$$

2. [10] Use the comparison theorem to determine whether the integral is convergent or divergent.

$$\int_0^{\infty} e^{-x}\sin^{10}x\,\mathrm{d}x$$

Solution: We note that $0 \le \sin^{10}x \le 1$, so $0 \le e^{-x}\sin^{10}x \le e^{-x}$. Now,

$$\int_0^{\infty} e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^{t} e^{-x}\,\mathrm{d}x = \lim_{t\to\infty} -e^{-x}\Big|_0^t = \lim_{t\to\infty} 1 - e^{-t} = 1. \tag{4}$$

Since this integral is convergent, the comparison theorem indicates that $\int_0^{\infty} e^{-x}\sin^{10}x\,\mathrm{d}x$ is also $\boxed{\text{convergent}}$.

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This is a make-up quiz. If you have not received my permission to take this make-up quiz, then your submission will not be accepted.

You must show all of your work and reasoning to receive full credit.

1. [10] Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^\infty \frac{3x^2}{1+x^6}\,\mathrm{d}x$$

Solution: By definition, this is

$$\lim_{t\to\infty}\int_0^t \frac{3x^2}{1+\left(x^3\right)^2}\,\mathrm{d}x. \tag{5}$$

We now use u-substitution:

$$\begin{aligned} u &= x^3 & x = 0 &\Rightarrow u = 0 \\ \mathrm{d}u &= 3x^2\,\mathrm{d}x & x = t &\Rightarrow u = t^3 \end{aligned} \tag{6}$$

$$\lim_{t\to\infty}\int_0^{t^3} \frac{1}{1+u^2}\,\mathrm{d}u = \lim_{t\to\infty}\tan^{-1}u\Big|_0^{t^3} = \lim_{t\to\infty}\tan^{-1}\left(t^3\right) - \tan^{-1}0 = \boxed{\tfrac{\pi}{2}}. \tag{7}$$

2. [10] Use the comparison theorem to determine whether the integral is convergent or divergent.

$$\int_1^\infty \frac{1}{x^{100}e^x}\,\mathrm{d}x$$

Solution: We note that, for $x > 1$,

$$\frac{1}{x^{100}e^x} = \frac{e^{-x}}{x^{100}} < e^{-x}. \tag{8}$$

Now,

$$\int_1^\infty e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty} -e^{-x}\Big|_1^t = \lim_{t\to\infty} e - e^{-t} = e. \tag{9}$$

Since this integral is convergent, the comparison theorem indicates that $\int_1^\infty \frac{1}{x^{100}e^x}\,\mathrm{d}x$ is also $\boxed{\text{convergent}}$.