Math 141V Quiz 2 **Name: Solution Key**

September 15th, 2020

You must show all of your work and reasoning to receive full credit.

1. [7] Evaluate the definite integral.

$$\int_0^{\frac{\pi}{2}} \tan^2 x + \tan^4 x \, \mathrm{d}x$$

This problem is actually impossible to do without knowing about improper integrals, which is a topic that we will discuss later. The *intention* was to put $\frac{\pi}{4}$ as the upper bound of the integral, not $\frac{\pi}{2}$. If that had been the case, this would have been the solution:

$$\int_0^{\frac{\pi}{4}} \tan^2 x + \tan^4 x \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^2 x \left(1 + \tan^2 x\right) \, \mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, \mathrm{d}x \tag{1}$$

$$\begin{aligned} u &= \tan x & x = 0 &\Rightarrow u = 0 \\ \mathrm{d}u &= \sec^2 x \, \mathrm{d}x & x = \tfrac{\pi}{4} &\Rightarrow u = 1 \end{aligned} \tag{2}$$

$$\int_0^1 u^2 \, \mathrm{d}u = \frac{1}{3} u^3 \bigg|_0^1 = \boxed{\tfrac{1}{3}}. \tag{3}$$

2. [13] Evaluate the indefinite integral. Your final answer should be in terms of x, with no inverse trigonometric functions involved.

$$\int \frac{\sqrt{x^2-100}}{x^4}\,\mathrm{d}x$$

Solution:

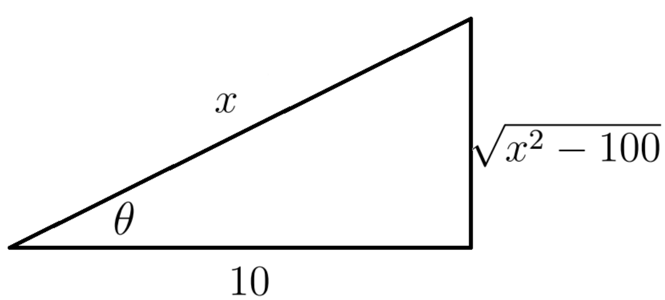
$$\begin{aligned} x &= 10\sec\theta \\ \mathrm{d}x &= 10\sec\theta\tan\theta\,\mathrm{d}\theta \end{aligned} \tag{4}$$

$$\begin{aligned} &\int \frac{\sqrt{100\sec^2\theta-100}}{10000\sec^4\theta} 10\sec\theta\tan\theta\,\mathrm{d}\theta = \frac{1}{100}\int\frac{\sqrt{\tan^2\theta}}{\sec^3\theta}\tan\theta\mathrm{d}\theta \\ &= \frac{1}{100}\int\frac{\tan^2\theta}{\sec^3\theta}\,\mathrm{d}\theta = \frac{1}{100}\int\frac{\sin^2\theta}{\cos^2\theta}\cos^3\theta\,\mathrm{d}\theta = \frac{1}{100}\int\sin^2\theta\,\cos\theta\,\mathrm{d}\theta \end{aligned} \tag{5}$$

$$\begin{aligned} u &= \sin\theta \\ \mathrm{d}u &= \cos\theta\,\mathrm{d}\theta \end{aligned} \tag{6}$$

$$\frac{1}{100}\int u^2\,\mathrm{d}u = \frac{1}{300}u^3 + C = \frac{1}{300}\sin^3\theta + C \tag{7}$$

$$\sec\theta = \frac{x}{10} \Rightarrow \cos\theta = \frac{10}{x} \tag{8}$$



The diagram shows that $\sin\theta = \frac{\sqrt{x^2-100}}{x}$, so our final answer is

$$\boxed{\frac{1}{300}\left(\frac{\sqrt{x^2-100}}{x}\right)^3 + C}. \tag{9}$$

Math 141V Quiz 2 **Name: Solution Key**

September 17th, 2020

Let's try this again. This time, with less of my incompetence.

You must show all of your work and reasoning to receive full credit.

1. [7] Evaluate the definite integral.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^5 x \sec^5 x \, \mathrm{d}x$$

Solution:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^5 x \sec^5 x \, \mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sec^2 x - 1\right)^2 \sec^4 x \ \sec x \ \tan x \, \mathrm{d}x \tag{10}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \sec x & x = \tfrac{\pi}{6} &\Rightarrow u = \sec\left(\tfrac{\pi}{6}\right) = \tfrac{2}{\sqrt{3}} \\ \mathrm{d}u &= \sec x \ \tan x \, \mathrm{d}x & x = \tfrac{\pi}{3} &\Rightarrow u = \sec\left(\tfrac{\pi}{3}\right) = 2 \end{aligned} \tag{11}$$

$$\begin{aligned} \int_{\frac{2}{\sqrt{3}}}^{2} \left(u^2 - 1\right)^2 u^4 \, \mathrm{d}u &= \int_{\frac{2}{\sqrt{3}}}^{2} u^8 - 2u^6 + u^4 \, \mathrm{d}u \\ &= \frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5 \Big|_{\frac{2}{\sqrt{3}}}^{2} \\ &= \boxed{\left(\tfrac{512}{9} - \tfrac{256}{7} + \tfrac{32}{5}\right) - \left(\tfrac{512}{729\sqrt{3}} - \tfrac{256}{189\sqrt{3}} + \tfrac{32}{45\sqrt{3}}\right)} \end{aligned} \tag{12}$$

2. [13] Evaluate the indefinite integral. Your final answer should be in terms of x, with no inverse trigonometric functions involved.

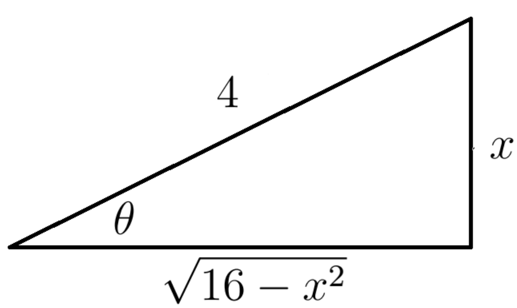
$$\int \frac{1}{x^2\sqrt{16-x^2}}\,\mathrm{d}x$$

Solution:

$$\begin{aligned} x &= 4\sin\theta \\ \mathrm{d}x &= 4\cos\theta\,\mathrm{d}\theta \end{aligned} \tag{13}$$

$$\begin{aligned} &\int \frac{1}{(4\sin\theta)^2\sqrt{16-(4\sin\theta)^2}} 4\cos\theta\,\mathrm{d}\theta \\ &= \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16\left(1-\sin^2\theta\right)}}\,\mathrm{d}\theta \\ &= \int \frac{4\cos\theta}{16\sin^2\theta\sqrt{16\cos^2\theta}}\,\mathrm{d}\theta = \int \frac{4\cos\theta}{16\sin^2\theta\;4\cos\theta}\,\mathrm{d}\theta \\ &= \frac{1}{16}\int \frac{1}{\sin^2\theta}\,\mathrm{d}\theta = \frac{1}{16}\int \csc^2\theta\,\mathrm{d}\theta = -\frac{1}{16}\cot\theta + C. \end{aligned} \tag{14}$$

Now, since $\sin\theta = \frac{x}{4}$,



The diagram shows that $\tan\theta = \frac{x}{\sqrt{16-x^2}}$, so $\cot\theta = \frac{\sqrt{16-x^2}}{x}$. Therefore, our indefinite integral is

$$\boxed{-\frac{\sqrt{16-x^2}}{16x} + C} \tag{15}$$