Math 142: College Calculus II

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Contents

Calculus II is organized into four parts, corresponding to four goals:

1. To provide practical strategies for solving integrals.
2. To introduce the theory of infinite series.
3. To extend calculus to curves that are not described by functions.

(4.) To apply integration strategies to problems in geometry.

In Calculus I, we explained the theory of the solutions of the tangent line problem and the area problem. These were related by the fundamental theorem of calculus, which states:

Theorem 0.1 *(Fundamental theorem of calculus) If f is a continuous function on the interval $[a, b]$, then the following statements are true.*

(i)

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x). \tag{1}$$

(ii) If F is an antiderivative of f, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2}$$

This tells us that integrals can be solved by using antiderivatives. However, finding antiderivatives is **much** harder than finding derivatives. Therefore, the first part of Calculus II will focus on strategies for finding antiderivatives. First, we review what we know from Calculus I.

Let k be a real-valued constant.

$$\begin{aligned}
\int 1 \, \mathrm{d}x &= x + C \\
\int x^k \, \mathrm{d}x &= \tfrac{x^{k+1}}{k+1} + C \qquad\qquad \text{if } k \neq -1 \\
\int \tfrac{1}{x} \, \mathrm{d}x &= \ln|x| + C \\
\int e^{kx} \, \mathrm{d}x &= \tfrac{e^{kx}}{k} + C \qquad\qquad \text{if } k \neq 0 \\
\int \cos x \, \mathrm{d}x &= \sin x + C \\
\int \sin x \, \mathrm{d}x &= -\cos x + C \\
\int \sec^2 x \, \mathrm{d}x &= \tan x + C \\
\int \sec x \tan x \, \mathrm{d}x &= \sec x + C \\
\int \csc x \cot x \, \mathrm{d}x &= -\csc x + C \\
\int \csc^2 x \, \mathrm{d}x &= -\cot x + C \\
\int \tfrac{1}{x^2+1} \, \mathrm{d}x &= \tan^{-1} x + C \\
\int \tfrac{1}{\sqrt{1-x^2}} \, \mathrm{d}x &= \sin^{-1} x + C \\
\int \tfrac{1}{x\sqrt{x^2-1}} \, \mathrm{d}x &= \sec^{-1} x + C
\end{aligned} \tag{3}$$

We also know that

$$\int k f(x) \, \mathrm{d}x = k \int f(x) \, \mathrm{d}x \tag{4}$$

and

$$\int f(x) + g(x) \, \mathrm{d}x = \int f(x) \, \mathrm{d}x + \int g(x) \, \mathrm{d}x. \tag{5}$$

As for definite integrals, we also [hopefully] learned:

$$\int_a^b f(x) \, \mathrm{d}x = -\int_b^a f(x) \, \mathrm{d}x \tag{6}$$

and if $a \leq t \leq b$, then

$$\int_a^b f(x) \, \mathrm{d}x = \int_a^t f(x) \, \mathrm{d}x + \int_t^b f(x) \, \mathrm{d}x. \tag{7}$$

5.5 The substitution rule

The last thing we [should have] learned in Calculus I was u-substitution.

Example 5.2 *Evaluate the integral.*

$$\int \frac{x}{x^2+1}\,\mathrm{d}x \tag{8}$$

Set

$$\begin{aligned} u &= x^2+1 \\ \mathrm{d}u &= 2x\,\mathrm{d}x \end{aligned}. \tag{9}$$

Therefore, the integral becomes

$$\int \frac{1}{2}\frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C = \boxed{\tfrac{1}{2}\ln|x^2+1| + C}. \tag{10}$$

□

Example 5.3 *Evaluate the integral.*

$$\int_0^{\frac{\pi}{2}} e^{\sin x}\cos x\,\mathrm{d}x \tag{11}$$

Set

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x\,\mathrm{d}x \end{aligned}. \tag{12}$$

The integral becomes

$$\int_0^1 e^u\,\mathrm{d}u = e^u\Big|_0^1 = e^1 - e^0 = \boxed{e-1}. \tag{13}$$

□

Example 5.4 *Evaluate the integral.*

$$\int_{\frac{1}{2}}^1 \frac{e^{\left(\frac{1}{x}\right)}}{x^2}\,\mathrm{d}x \tag{14}$$

Set

$$\begin{aligned} u &= \tfrac{1}{x} \\ \mathrm{d}u &= -\tfrac{1}{x^2}\,\mathrm{d}x \end{aligned}. \tag{15}$$

The integral becomes

$$\int_2^1 -e^u\,\mathrm{d}u = -\int_2^1 e^u\,\mathrm{d}u = \int_1^2 e^u\,\mathrm{d}u = e^u\Big|_1^2 = e^2 - e = \boxed{e\,(e-1)}. \tag{16}$$

□

Example 5.5

$$\int \frac{\ln x}{x}\,\mathrm{d}x \tag{17}$$

Set

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x \end{aligned}. \tag{18}$$

The integral becomes

$$\int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \boxed{\tfrac{1}{2}(\ln x)^2 + C}. \tag{19}$$

□

Example 5.6

$$\int \cot x\,\mathrm{d}x \tag{20}$$

Notice that

$$\int \cot x\,\mathrm{d}x = \int \frac{\cos x}{\sin x}\,\mathrm{d}x. \tag{21}$$

Set

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x\,\mathrm{d}x \end{aligned} \tag{22}$$

The integral becomes

$$\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \boxed{\ln|\sin x| + C}. \tag{23}$$

Does u-substitution always work? **Absolutely not**.

Example 5.7 *Consider the integrals*

$$\int x \sin x \, \mathrm{d}x \tag{24}$$

$$\int x \ln x \, \mathrm{d}x \tag{25}$$

$$\int x^2 e^x \, \mathrm{d}x \tag{26}$$

$$\int \ln x \, \mathrm{d}x \tag{27}$$

$$\int e^x \cos x \, \mathrm{d}x \tag{28}$$

We need a new method to solve these integrals. $\square$

7 Techniques of integration

7.1 Integration by parts

The product rule for derivatives is:

$$\frac{\mathrm{d}}{\mathrm{d}x}uv = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{29}$$

What if we took the antiderivative of both sides?

$$uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x. \tag{30}$$

However, we often write $\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$ as $\mathrm{d}v$, and $\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$ as $\mathrm{d}u$. Therefore, we can re-arrange and write this as

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u}. \tag{31}$$

This is called "integration by parts." We will illustrate this method of integration with an example.

Example 7.1 *Evaluate the integral*

$$\int x\sin x\,\mathrm{d}x. \tag{32}$$

Integration by parts is most effective when we can write the integrand as a product of two familiar functions, as above. Select one of these functions and name it u*. Select the other function and* $\mathrm{d}x$ *and name that* $\mathrm{d}v$*.*

$$u = x \quad \mathrm{d}v = \sin x\,\mathrm{d}x. \tag{33}$$

Now, take the derivative of u *and the antiderivative of* $\mathrm{d}v$*.*

$$\mathrm{d}u = \mathrm{d}x \quad v = -\cos x. \tag{34}$$

Now plug these "parts" into the integration by parts formula.

$$\begin{gathered}\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \\ \int x \sin x \, \mathrm{d}x = -x \cos x - \int -\cos x \, \mathrm{d}x.\end{gathered} \tag{35}$$

Notice that we can solve the integral on the right:

$$\int x \sin x \, \mathrm{d}x = -x \cos x + \int \cos x \, \mathrm{d}x = \boxed{-x \cos x + \sin x + C}. \tag{36}$$

□

Example 7.2 *Evaluate the integral*

$$\int x \ln x \, \mathrm{d}x. \tag{37}$$

We select

$$\begin{aligned} u &= \ln x & \mathrm{d}v &= x \, \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{x} \, \mathrm{d}x & v &= \tfrac{1}{2}x^2 \end{aligned} \tag{38}$$

Now put these into the integration by parts formula:

$$\begin{gathered}\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \\ \int x \ln x \, \mathrm{d}x = \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x^2 \tfrac{1}{x} \, \mathrm{d}x\end{gathered} \tag{39}$$

We can solve the integral on the right:

$$\int x \ln x \, \mathrm{d}x = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, \mathrm{d}x = \boxed{\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C}. \tag{40}$$

□

Sometimes, you need to use integration by parts more than once.

Example 7.3 *Evaluate the integral*

$$\int x^2 e^x \, \mathrm{d}x. \tag{41}$$

We select

$$\begin{array}{cc} u = x^2 & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = 2x\,\mathrm{d}x & v = e^x \end{array} \tag{42}$$

Now put these into the integration by parts formula:

$$\begin{aligned} \textstyle\int u\,\mathrm{d}v &= uv - \textstyle\int v\,\mathrm{d}u \\ \textstyle\int x^2e^x\,\mathrm{d}x &= x^2e^x - \textstyle\int 2xe^x\,\mathrm{d}x. \end{aligned} \tag{43}$$

Thus,

$$\int x^2e^x\,\mathrm{d}x = x^2e^x - 2\int xe^x\,\mathrm{d}x. \tag{44}$$

To solve the integral on the right, we use integration by parts yet again:

$$\int xe^x\,\mathrm{d}x \tag{45}$$

We select

$$\begin{array}{cc} u = x & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = e^x \end{array}. \tag{46}$$

Now,

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x + C. \tag{47}$$

We plug this result in to find

$$\begin{aligned} \int x^2e^x\,\mathrm{d}x &= x^2e^x - 2\int xe^x\,\mathrm{d}x \\ &= x^2e^x - 2\left(xe^x - e^x + C\right) = \boxed{x^2e^x - 2xe^x + 2e^x - 2C}. \end{aligned} \tag{48}$$

In finding antiderivatives, we are uninterested in the exact value of the constant C. Therefore, whether it is C or $-2C$ makes no difference to us; either way, it is a constant. We call it by the name C_1:

$$\boxed{x^2e^x - 2xe^x + 2e^x + C_1} \tag{49}$$

□

Sometimes, integration by parts can be weird.

Example 7.4 *Evaluate the integral*

$$\int \ln x \, \mathrm{d}x. \tag{50}$$

We select

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & v = x \end{array}. \tag{51}$$

Now put these into the integration by parts formula:

$$\begin{gathered} \int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u \\ \int \ln x \,\mathrm{d}x = x \ln x - \int x \tfrac{1}{x}\,\mathrm{d}x \end{gathered} \tag{52}$$

We can solve the integral on the right:

$$\int \ln x \,\mathrm{d}x = x \ln x - \int 1 \,\mathrm{d}x = \boxed{x \ln x - x + C}. \tag{53}$$

Example 7.5 *Evaluate the integral*

$$\int e^x \cos x \,\mathrm{d}x. \tag{54}$$

We select

$$\begin{array}{cc} u = \cos x & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = -\sin x\,\mathrm{d}x & v = e^x \end{array} \tag{55}$$

Putting this into the integration by parts formula,

$$\begin{gathered} \int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u \\ \int e^x \cos x \,\mathrm{d}x = e^x \cos x + \int e^x \sin x \,\mathrm{d}x \end{gathered} \tag{56}$$

We still cannot solve the integral on the right, so we try integration by parts again:

$$\begin{array}{cc} u = \sin x & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = \cos x\,\mathrm{d}x & v = e^x \end{array}. \tag{57}$$

Now we get

$$\int e^x \cos x \, \mathrm{d}x = e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, \mathrm{d}x \right) \tag{58}$$

Ergo,

$$2 \int e^x \cos x \, \mathrm{d}x = e^x \cos x + e^x \sin x + C, \tag{59}$$

and so

$$\int e^x \cos x \, \mathrm{d}x = \boxed{\tfrac{1}{2} e^x (\cos x + \sin x) + C_1}. \tag{60}$$

□

For a definite integral, the integration by parts formula becomes

$$\int_a^b u \, \mathrm{d}v = uv \Big|_a^b - \int_a^b v \, \mathrm{d}u. \tag{61}$$

Example 7.6 *(Problem 7.1.26) Evaluate the integral.*

$$\int_1^2 x^2 \ln x \, \mathrm{d}x \tag{62}$$

We select

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = x^2 \, \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x} \, \mathrm{d}x & v = \frac{1}{3} x^3. \end{array} \tag{63}$$

Now we put these into the integration by parts formula:

$$\begin{array}{c} \int_a^b u \, \mathrm{d}v = uv\big|_a^b - \int_a^b v \, \mathrm{d}u \\ \int_1^2 x^2 \ln x \, \mathrm{d}x = \frac{1}{3} x^3 \ln x\big|_1^2 - \int_1^2 \frac{1}{3} x^3 \frac{1}{x} \, \mathrm{d}x. \end{array} \tag{64}$$

We can solve the integral on the right:

$$\int_1^2 x^2 \ln x \, \mathrm{d}x = \frac{1}{3}x^3 \ln x\Big|_1^2 - \frac{1}{3}\int_1^2 x^2 \, \mathrm{d}x$$
$$= \frac{1}{3}x^3 \ln x\Big|_1^2 - \frac{1}{3}\left(\frac{1}{3}x^3\Big|_1^2\right) = \left(\frac{8}{3}\ln 2 - \frac{1}{3}\ln 1\right) - \frac{1}{3}\left(\frac{1}{3}2^3 - \frac{1}{3}1^3\right)$$
$$= \frac{8}{3}\ln 2 - \frac{7}{9} \quad (65)$$

□

Does integration by parts always work? **NO**.

Example 7.7

$$\int \sin^2 x \cos^2 x \, \mathrm{d}x. \quad (66)$$

This cannot be done by parts. □

7.2 Trigonometric integrals

We need some strategies for integrals involving trigonometric functions.

Example 7.8 *Evaluate the integral.*

$$\int \sec x \, \mathrm{d}x \tag{67}$$

In order to do this, we first multiply by $\frac{\sec x + \tan x}{\tan x + \sec x}$*:*

$$\int \sec x \left(\frac{\sec x + \tan x}{\tan x + \sec x} \right) \, \mathrm{d}x = \int \frac{\sec^2 x + \sec x \ \tan x}{\tan x + \sec x} \, \mathrm{d}x. \tag{68}$$

Now we use u*-substitution:*

$$\begin{aligned} u &= \tan x + \sec x \\ \mathrm{d}u &= \sec^2 x + \sec x \ \tan x \end{aligned}. \tag{69}$$

The integral now becomes

$$\int \frac{1}{u} \, \mathrm{d}u = \ln |u| + C = \boxed{\ln \left| \sec x + \tan x \right| + C}. \tag{70}$$

□

How should we integrate

$$\int \sin^m x \cos^n x \, \mathrm{d}x \tag{71}$$

where m and n are non-negative integers? It depends on which of m and n are even, and which are odd.

Strategy Consider the integral above. We have three cases to consider.

Case 1: n is odd. In that case, $n = 2k + 1$ for some integer k. Therefore, we have

$$\int \sin^m x \cos^{2k+1} x \, \mathrm{d}x = \int \sin^m x \left(\cos^2 x \right)^k \, \cos x \, \mathrm{d}x. \tag{72}$$

At this point, we can use the trigonometric identity $\sin^2 x + \cos^2 x = 1$. This identity implies that $\cos^2 x = 1 - \sin^2 x$. Therefore,

$$\int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, \mathrm{d}x. \tag{73}$$

At this point, we can use u-substitution.

Case 2: n is even, and m is odd. In that case, $m = 2l + 1$ for some integer l. Therefore, we have

$$\int \sin^{2l+1} x \cos^n x \, \mathrm{d}x = \int \left(\sin^2 x\right)^l \cos^n x \ \sin x \, \mathrm{d}x. \tag{74}$$

Again, we can use $\cos^2 x + \sin^2 x = 1$, so we get

$$\int \left(1 - \cos^2 x\right)^l \cos^n x \ \sin x \, \mathrm{d}x. \tag{75}$$

At this point, we can use u-substitution.

Case 3: m and n are both even. In this case, $m = 2l$ and $n = 2k$ for some integers k and l. Therefore, we have

$$\int \left(\sin^2 x\right)^l \left(\cos^2 x\right)^k \mathrm{d}x. \tag{76}$$

We need to use the trigonometric identities

$$\begin{aligned} \sin^2 x &= \tfrac{1}{2}\left(1 - \cos\left(2x\right)\right) \\ \cos^2 x &= \tfrac{1}{2}\left(1 + \cos\left(2x\right)\right). \\ \sin x \cos x &= \tfrac{1}{2}\sin\left(2x\right) \end{aligned} \tag{77}$$

Distribute, and reduce further if necessary.

Example 7.9 *Evaluate the integral*

$$\int \sin^4 x \cos^5 x \, \mathrm{d}x. \tag{78}$$

This is Case 1. Therefore, we separate one factor of cosine:

$$\int \sin^4 x \cos^4 x \ \cos x \, \mathrm{d}x \tag{79}$$

Now we use the identity $\sin^2 x + \cos^2 x = 1$*:*

$$\int \sin^4 x \left(1 - \sin^2 x\right)^2 \ \cos x \, \mathrm{d}x. \tag{80}$$

We now use u*-substitution:*

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x, \end{aligned} \tag{81}$$

so this becomes

$$\int u^4 \left(1 - u^2\right)^2 \mathrm{d}u. \tag{82}$$

We distribute:

$$\int u^4 \left(1 - 2u^2 + u^4\right) \ \mathrm{d}u = \int u^4 - 2u^6 + u^8 \, \mathrm{d}u, \tag{83}$$

and take the antiderivative:

$$\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C. \tag{84}$$

In terms of x*, this becomes:*

$$\boxed{\tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C}. \tag{85}$$

□

Example 7.10 *(Problem 7.2.2) Evaluate the integral.*

$$\int \sin^3\theta \cos^4\theta \, \mathrm{d}\theta. \tag{86}$$

This is Case 2. Now we can separate one factor of sine:

$$\int \sin^2\theta \cos^4\theta \ \sin\theta \,\mathrm{d}\theta. \tag{87}$$

We apply the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ *to get*

$$\int \left(1 - \cos^2\theta\right) \cos^4\theta \ \sin\theta \,\mathrm{d}\theta. \tag{88}$$

Now we use u*-substitution:*

$$\begin{aligned} u &= \cos\theta \\ \mathrm{d}u &= -\sin\theta \,\mathrm{d}\theta\text{'} \end{aligned} \tag{89}$$

so this becomes

$$-\int \left(1 - u^2\right) u^4 \,\mathrm{d}u = \int u^6 - u^4 \,\mathrm{d}u = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C. \tag{90}$$

In terms of x*, this is*

$$\boxed{\tfrac{1}{7}\cos^7\theta - \tfrac{1}{5}\cos^5\theta + C}. \tag{91}$$

□

Example 7.11 *(Problem 7.2.10) Evaluate the integral.*

$$\int_0^\pi \sin^2 t \cos^4 t \,\mathrm{d}t \tag{92}$$

This is Case 3. We notice that $\cos t$ *has a larger power, so we will begin by separating those extra factors from the rest of the integrand:*

$$\int_0^\pi (\sin t \cos t)^2 \cos^2 t \,\mathrm{d}t. \tag{93}$$

We use the fact that $\sin t \cos t = \frac{1}{2}\sin(2t)$ *and* $\cos^2 t = \frac{1}{2}\left(1 + \cos(2t)\right)$.

$$\int_0^\pi \left(\frac{1}{2}\sin(2t)\right)^2 \frac{1}{2}\left(1 + \cos(2t)\right) \,\mathrm{d}t. \tag{94}$$

We distribute:

$$\frac{1}{8}\int_0^{\pi} \sin^2(2t) + \sin^2(2t)\cos(2t)\ \mathrm{d}t. \tag{95}$$

We can now separate the integral into two terms:

$$\frac{1}{8}\int_0^{\pi} \sin^2(2t)\ \mathrm{d}t + \frac{1}{8}\int_0^{\pi} \sin^2(2t)\cos(2t)\ \mathrm{d}t. \tag{96}$$

For the left integral, we will use the trigonometric identity $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$*:*

$$\frac{1}{8}\int_0^{\pi} \frac{1}{2}(1-\cos(4t))\ \mathrm{d}t + \frac{1}{8}\int_0^{\pi} \sin^2(2t)\cos(2t)\ \mathrm{d}t \tag{97}$$

The right integral can be done by u*-substitution:*

$$\begin{gathered} u = \sin(2t) \\ \mathrm{d}u = 2\cos(2t)\,\mathrm{d}t^{,} \end{gathered} \tag{98}$$

giving us

$$\frac{1}{16}\int_0^{\pi} 1-\cos(4t)\ \mathrm{d}t + \frac{1}{16}\int_0^{0} u^2\ \mathrm{d}u \tag{99}$$

Any definite integral with equal bounds is zero, so this becomes

$$\frac{1}{16}\int_0^{\pi} 1-\cos(4t)\ \mathrm{d}t \tag{100}$$

Taking the antiderivative, we get

$$\frac{1}{16}\left(t - \frac{1}{4}\sin(4t)\right)\Bigg|_0^{\pi} = \boxed{\tfrac{\pi}{16}}. \tag{101}$$

□

How should we integrate

$$\int \tan^m x \sec^n x\ \mathrm{d}x, \tag{102}$$

where m and n are non-negative integers? This also depends on which of m and n

are even, and which are odd.

Strategy Consider the integral above. We have two important cases to consider.

Case 1: n is even. In that case, $n = 2k$ for some integer k. Therefore, we have

$$\int \tan^m x \sec^{2k} x \, \mathrm{d}x = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \, \mathrm{d}x \tag{103}$$

Now we can use the trigonometric identity $\tan^2 x + 1 = \sec^2 x$. This now becomes

$$\int \tan^m x \left(\tan^2 x + 1\right)^{k-1} \sec^2 x \, \mathrm{d}x. \tag{104}$$

At this point, we can use u-substitution.

Case 2: m is odd, and $n \geq 1$. In that case, $m = 2l + 1$ for some integer l. Therefore, we have

$$\int \left(\tan^2 x\right)^l \sec^{n-1} x \, \sec x \tan x \, \mathrm{d}x. \tag{105}$$

Again, we use the trigonometric identity $\tan^2 x + 1 = \sec^2 x$. This becomes

$$\int \left(\sec^2 x - 1\right)^l \sec^{n-1} x \, \sec x \tan x \, \mathrm{d}x. \tag{106}$$

Now we can use u-substitution.

Example 7.12 *(Problem 7.2.22) Evaluate the integral*

$$\int \tan^2\theta \sec^4\theta \, \mathrm{d}\theta \tag{107}$$

This is Case 1. We can write this as

$$\int \tan^2\theta \sec^2\theta \sec^2\theta \, \mathrm{d}\theta. \tag{108}$$

Using the identity $\tan^2\theta + 1 = \sec^2\theta$*, this becomes*

$$\int \tan^2\theta \left(\tan^2\theta + 1\right) \sec^2\theta \, \mathrm{d}\theta. \tag{109}$$

We now proceed by u*-substitution:*

$$u = \tan\theta \mathrm{d}u = \sec^2\theta \, \mathrm{d}\theta, \tag{110}$$

which gives

$$\int u^2 \left(u^2 + 1\right) \, \mathrm{d}u = \int u^4 + u^2 \, \mathrm{d}u = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C, \tag{111}$$

which, in terms of x*, is*

$$\boxed{\tfrac{1}{5}\tan^5\theta + \tfrac{1}{3}\tan^3\theta + C}. \tag{112}$$

□

Example 7.13 *(Problem 7.2.28) Evaluate the integral*

$$\int \tan^5 x \sec^3 x \, \mathrm{d}x \tag{113}$$

This is Case 2. We can write this as

$$\int \left(\tan^2 x\right)^2 \sec^2 x \sec x \tan x \, \mathrm{d}x \tag{114}$$

We use the identity $\tan^2 x + 1 = \sec^2 x$ *to get*

$$\int \left(\sec^2 x - 1\right)^2 \sec^2 x \, \sec x \tan x \, \mathrm{d}x. \tag{115}$$

Now we use u*-substitution:*

$$\begin{aligned} u &= \sec x \\ \mathrm{d}u &= \sec x \tan x \, \mathrm{d}x \end{aligned}, \tag{116}$$

to get

$$\int \left(u^2-1\right)^2 u^2 \, \mathrm{d}u = \int \left(u^4 - 2u^2 + 1\right) u^2 \, \mathrm{d}u = \int u^6 - 2u^4 + u^2 \, \mathrm{d}u. \tag{117}$$

Taking the antiderivative,

$$\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C. \tag{118}$$

In terms of x, this becomes

$$\boxed{\tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C}. \tag{119}$$

□

Integrals that fall into neither case will require creativity to solve.

Example 7.14 *Find*

$$\int \sec^3 x \, \mathrm{d}x. \tag{120}$$

This falls into neither case. In order to do this, we need to use integration by parts:

$$\begin{array}{cc} u = \sec x & \mathrm{d}v = \sec^2 x \, \mathrm{d}x \\ \mathrm{d}u = \sec x \tan x \, \mathrm{d}x & v = \tan x \end{array}. \tag{121}$$

The integral becomes

$$\int \sec^3 x \, \mathrm{d}x = \sec x \tan x - \int \tan^2 x \sec x \, \mathrm{d}x. \tag{122}$$

We apply the trigonometric identity $\tan^2 x + 1 = \sec^2 x$*:*

$$\begin{aligned} \int \sec^3 x \, \mathrm{d}x &= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x - \sec x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x. \end{aligned} \tag{123}$$

Now we add $\int \sec^3 x \,\mathrm{d}x$ *to both sides:*

$$2\int \sec^3 x \,\mathrm{d}x = \sec x \tan x + \int \sec x \,\mathrm{d}x. \tag{124}$$

As done in a previous example, $\int \sec x \,\mathrm{d}x = \ln|\sec x + \tan x| + C_1$*, so this becomes*

$$\int \sec^3 x \,\mathrm{d}x = \boxed{\tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C_2}. \tag{125}$$

□

7.3 Trigonometric substitution

When an integral involves one of the following expressions:

$$\begin{aligned} &x^2 - a^2 \\ &x^2 + a^2 \\ &a^2 - x^2 \end{aligned} \tag{126}$$

(where a is any nonzero real valued constant), we can use the following strategy to find the integral: substitute $x = f(\theta)$, where f is some trigonometric function.

$$\begin{array}{ll} x^2 - a^2 & x = a\sec\theta \text{ for } 0 \leq \theta < \frac{\pi}{2} \\ x^2 + a^2 & x = a\tan\theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \\ a^2 - x^2 & x = a\sin\theta \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array} \tag{127}$$

The strategy is best illustrated by some examples.

Example 7.15 *(Problem 7.3.8) Evaluate the integral*

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2 - 16}}. \tag{128}$$

This involves the form $x^2 - a^2$, so we substitute:

$$\begin{aligned} x &= 4\sec\theta \\ \mathrm{d}x &= 4\sec\theta\tan\theta\,\mathrm{d}\theta, \end{aligned} \tag{129}$$

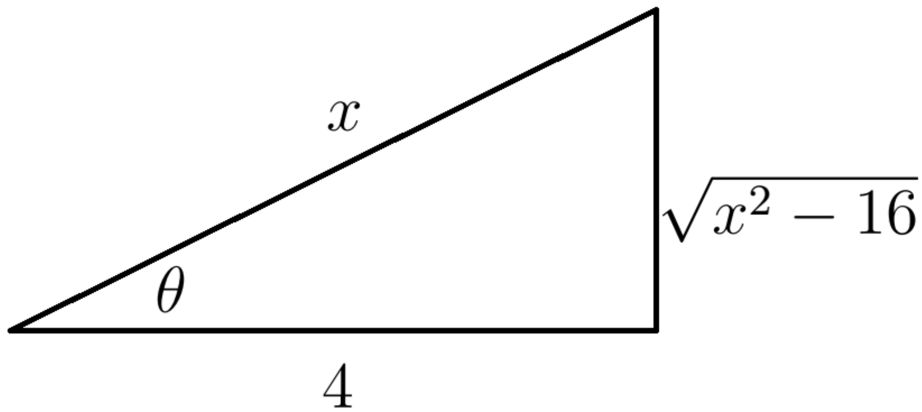
where $0 \leq \theta < \frac{\pi}{2}$. The integral now becomes

$$\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}\,\mathrm{d}\theta = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\left(\sec^2\theta - 1\right)}}. \tag{130}$$

We use the identity $\sec^2\theta - 1 = \tan^2\theta$:

$$\begin{aligned} \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,\mathrm{d}\theta &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\;4\tan\theta}\,\mathrm{d}\theta \\ &= \int \frac{1}{16\;\sec\theta}\,\mathrm{d}\theta = \frac{1}{16}\int \cos\theta\,\mathrm{d}\theta = \frac{1}{16}\sin\theta + C. \end{aligned} \tag{131}$$

Now we must put this answer back in terms of x. By definition, we know that $x = 4\sec\theta$. Therefore, $\cos\theta = \frac{4}{x}$. We construct a right triangle to illustrate this fact.



This triangle gives us that $\sin\theta = \frac{\sqrt{x^2-16}}{x}$. Therefore, our answer is

$$\boxed{\frac{1}{16}\frac{\sqrt{x^2-16}}{x} + C}. \tag{132}$$

□

Example 7.16 *Evaluate the integral.*

$$\int \sqrt{4 - x^2}\,\mathrm{d}x \tag{133}$$

This involves the form $a^2 - x^2$, so we make the substitution:

$$\begin{aligned} x &= 2\sin\theta \\ \mathrm{d}x &= 2\cos\theta\,\mathrm{d}\theta\text{'} \end{aligned} \tag{134}$$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. The integral now becomes

$$\begin{aligned} \int \sqrt{4 - (2\sin\theta)^2}\,2\cos\theta\,\mathrm{d}\theta &= \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,\mathrm{d}\theta \\ &= \int \sqrt{4\left(1 - \sin^2\theta\right)}\,2\cos\theta\,\mathrm{d}\theta. \end{aligned} \tag{135}$$

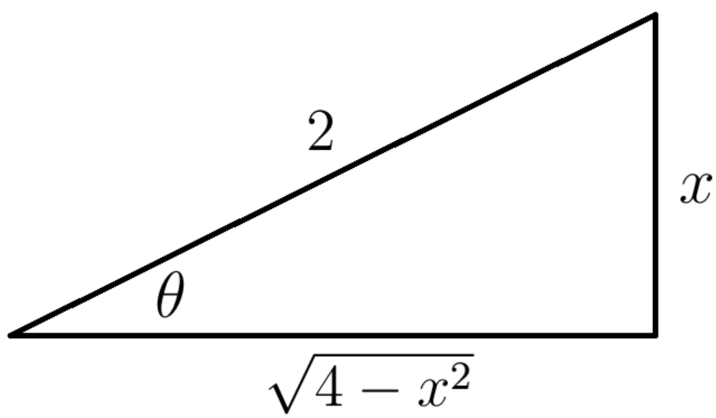
We use the identity $1 - \sin^2\theta = \cos^2\theta$ *to write this as*

$$\int \sqrt{4\cos^2\theta}\, 2\cos\theta \, \mathrm{d}\theta = \int 2\cos\theta \, 2\cos\theta \, \mathrm{d}\theta = 4 \int \cos^2\theta \, \mathrm{d}\theta. \tag{136}$$

In order to integrate this, we use the identity $\cos^2\theta = \frac{1}{2}\left(1 + \cos\left(2\theta\right)\right)$*:*

$$\begin{aligned} 4\int \frac{1}{2}\left(1 + \cos\left(2\theta\right)\right) \, \mathrm{d}\theta &= 2\int 1 + \cos\left(2\theta\right) \, \mathrm{d}\theta \\ &= 2\left(\theta + \frac{1}{2}\sin\left(2\theta\right)\right) + C = 2\theta + \sin\left(2\theta\right) + C. \end{aligned} \tag{137}$$

Again, we must write this in terms of x*. By definition, we know that* $x = 2\sin\theta$*. Therefore,* $\sin\theta = \frac{x}{2}$*. We construct a right triangle to illustrate this fact.*



At first, this doesn't seem to help us with $\sin\left(2\theta\right)$*. However, we can use the trigonometric identity* $\sin\left(2\theta\right) = 2\sin\theta\cos\theta$*, so that now our antiderivative is*

$$2\theta + 2\sin\theta\cos\theta + C. \tag{138}$$

Now, from our triangle, we know that $\sin\theta = \frac{x}{2}$ *and* $\cos\theta = \frac{\sqrt{4-x^2}}{2}$*. At the same time, if* $\sin\theta = \frac{x}{2}$ *and* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*, then* $\theta = \sin^{-1}\left(\frac{x}{2}\right)$*. Therefore, our final answer is*

$$\boxed{2\sin^{-1}\left(\tfrac{x}{2}\right) + x\tfrac{\sqrt{4-x^2}}{2} + C}. \tag{139}$$

□

Example 7.17 *Evaluate the integral.*

$$\int \frac{1}{\sqrt{x^2+9}}\,\mathrm{d}x \tag{140}$$

We recognize the form x^2+a^2 and make the substitution:

$$\begin{aligned} x &= 3\tan\theta \\ \mathrm{d}x &= 3\sec^2\theta\,\mathrm{d}\theta' \end{aligned} \tag{141}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. The integral now becomes

$$\int \frac{3\sec^2\theta}{\sqrt{(3\tan\theta)^2+9}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{\sqrt{9\left(\tan^2\theta+1\right)}}\,\mathrm{d}\theta. \tag{142}$$

We use the identity $\tan^2\theta+1=\sec^2\theta$ *to write this as*

$$\int \frac{3\sec^2\theta}{\sqrt{9\sec^2\theta}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{3\,\sec\theta}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta. \tag{143}$$

We know from a previous example how to integrate $\sec\theta$*:*

$$\int \sec\theta\left(\frac{\sec\theta+\tan\theta}{\tan\theta+\sec\theta}\right)\,\mathrm{d}\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\tan\theta+\sec\theta}\,\mathrm{d}\theta. \tag{144}$$

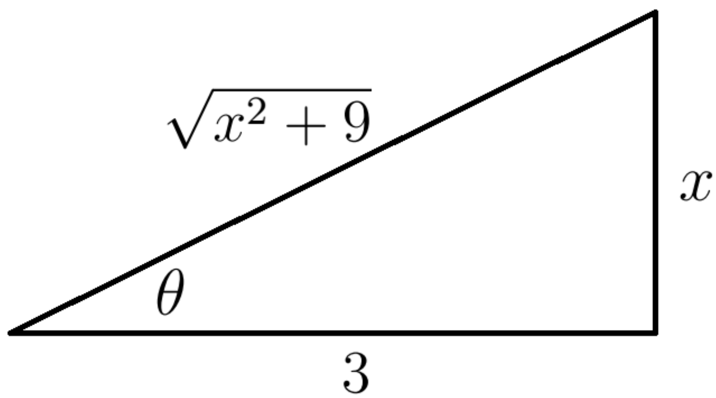
We can now use u-substitution:

$$\begin{aligned} u &= \tan\theta+\sec\theta \\ \mathrm{d}u &= \sec^2\theta+\sec\theta\tan\theta' \end{aligned} \tag{145}$$

to get

$$\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln\big|\tan\theta+\sec\theta\big| + C. \tag{146}$$

Again, we need to write this in terms of x. By definition, $x = 3\tan\theta$, so we can draw a right triangle to illustrate this fact:



The diagram indicates that $\sec\theta = \frac{\sqrt{x^2+9}}{3}$, *so our final answer becomes*

$$\boxed{\ln\left|\frac{x}{3} + \frac{\sqrt{x^2+9}}{3}\right| + C} \tag{147}$$

□

Example 7.18 *(Problem 7.3.14) Evaluate the integral*

$$\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x. \tag{148}$$

We recognize the form $x^2 + a^2$ *and make the substitution:*

$$\begin{aligned} x &= \tan\theta \\ \mathrm{d}x &= \sec^2\theta\,\mathrm{d}\theta, \end{aligned} \tag{149}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *The integral now becomes*

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(\tan^2\theta+1\right)^2}\,\mathrm{d}\theta. \tag{150}$$

(We have changed the bounds by recognizing that $x = 0$ *exactly when* $\theta = 0$ *and* $x = 1$ *exactly when* $\theta = \frac{\pi}{4}$.*) We use the identity* $\tan^2\theta + 1 = \sec^2\theta$ *to write this as*

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(\sec^2\theta\right)^2}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\,\mathrm{d}\theta. \tag{151}$$

In order to integrate this, we use the identity $\cos^2\theta = \frac{1}{2}\left(1+\cos\left(2\theta\right)\right)$*:*

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1+\cos\left(2\theta\right)\right)\,\mathrm{d}\theta &= \frac{1}{2}\left(\theta+\frac{1}{2}\sin\left(2\theta\right)\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\sin\left(2\frac{\pi}{4}\right)\right)-\frac{1}{2}\left(0+\frac{1}{2}\sin\left(0\right)\right) \\ &= \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\pi}{8}+\frac{1}{4}=\boxed{\tfrac{\pi+2}{8}}. \qquad (152)\end{aligned}$$

(For this definite integral, no triangle diagram is needed, because we have changed the bounds.) $\square$

7.4 Integration of rational functions by partial fractions

In this section, we'll discuss integration of rational functions.

Definition 7.19 *A polynomial is a function $f(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0$, where $c_n, c_{n-1}, ..., c_1$ and c_0 are real numbers.*

Definition 7.20 *Given a nonzero polynomial*

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_1 x + c_0$$

with $c_n \neq 0$, the degree of f is the value n.

Definition 7.21 *A rational function is a ratio $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.*

Any rational function can be written as a sum of a polynomial and a rational function whose numerator has a smaller degree than its denominator. For example:

$$\frac{x^2 + x + 1}{x + 1} = \frac{x(x + 1) + 1}{x + 1} = x + \frac{1}{x + 1}. \tag{153}$$

Given a rational function $f(x)$ whose numerator has a smaller degree than its denominator, there is a method of integrating f by factoring the denominator and then writing the function as a sum of fractions. This method is called "integration by partial fraction decomposition." It has three cases:

Case 1: the denominator $Q(x)$ is a product of distinct linear factors.

Case 2: the denominator $Q(x)$ has repeated linear factors.

Case 3: the denominator $Q(x)$ contains an irreducible quadratic factor.

We'll go through each of these cases individually.

Case 1: the denominator $Q(x)$ is a product of distinct linear factors.

In this case, we can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) ... (a_k x + b_k) \tag{154}$$

for some appropriate constants $a_1, a_2, ..., a_k$ and $b_1, b_2, ..., b_k$. In this case, we can find constants $A_1, A_2, ..., A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + ... + \frac{A_k}{a_kx + b_k}. \tag{155}$$

This will make the function easier to integrate.

Example 7.22 *Evaluate the integral.*

$$\int \frac{x+1}{x^2+x-2}\,\mathrm{d}x \tag{156}$$

First, we can factor the denominator as $x^2 + x - 2 = (x-1)(x+2)$*. This is a product of distinct linear factors, so we are working in Case 1.*

We can find constants A *and* B *such that*

$$\frac{x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}. \tag{157}$$

We proceed by multiplying both sides by the denominator on the left to get:

$$x + 1 = A(x+2) + B(x-1). \tag{158}$$

Now we distribute, and combine like terms in powers of x *to get:*

$$1x + 1 = Ax + 2A + Bx - B = (A+B)x + (2A - B). \tag{159}$$

We can now equate the corresponding coefficients of the polynomials on the left and right sides of this equation to get a system of equations:

$$\begin{aligned} 1 &= A + B \\ 1 &= 2A - B \end{aligned}. \tag{160}$$

There are many ways to solve this system of equations. My favorite is to add the two equations together to cancel the B *terms:* $2 = 3A$*. This gives* $A = \frac{2}{3}$*, and so*

$B = \frac{1}{3}$. *Our equation now becomes*

$$\frac{x+1}{(x-1)(x+2)} = \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}. \tag{161}$$

This we can now integrate:

$$\begin{aligned}\int \frac{x+1}{x^2+x-2}\,\mathrm{d}x &= \int \frac{\frac{2}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}\,\mathrm{d}x \\ &= \frac{2}{3}\int \frac{1}{x-1}\,\mathrm{d}x + \frac{1}{3}\int \frac{1}{x+2}\,\mathrm{d}x \\ &= \boxed{\tfrac{2}{3}\ln\left|x-1\right| + \tfrac{1}{3}\ln\left|x+2\right| + C}. \end{aligned} \tag{162}$$

Case 2: the denominator $Q(x)$ has repeated linear factors. In this case, the factor $(a_i + xb_i)^r$ occurs in the factorization of $Q(x)$. This results in additional terms in the partial fraction decomposition:

$$\frac{A_1}{(a_i + xb_i)^1} + \frac{A_2}{(a_i + xb_i)^2} + ... + \frac{A_r}{(a_i + xb_i)^r}. \tag{163}$$

We illustrate with an example.

Example 7.23 *Evaluate the integral.*

$$\int \frac{4x}{(x-1)\left(x^2-1\right)}. \tag{164}$$

We can write the denominator as $(x+1)(x-1)^2$*. This has a repeated linear factor of* $x-1$*, so this is Case 2.*

We seek constants A*,* B *and* C *such that*

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \tag{165}$$

In order to find these constants, we multiply each side of the equation by the denominator on the left:

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1). \tag{166}$$

We distribute:

$$4x = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C, \tag{167}$$

and we combine like terms in powers of x:

$$0x^2 + 4x + 0 = (A + B)\, x^2 + (C - 2A)\, x + (A - B + C)\,. \tag{168}$$

Now we can equate the corresponding coefficients of these polynomials to obtain a system of three equations:

$$\begin{aligned} 0 &= A + B \\ 4 &= C - 2A \\ 0 &= A - B + C \end{aligned}\;. \tag{169}$$

The first equation tells us that $-B = A$, and so by the third equation, $0 = 2A + C$, hence $C = -2A$. By the second equation, we deduce that $-4A = 4$, and so $A = -1$. This implies that $B = 1$ and $C = 2$. Thus, we have

$$\frac{4x}{(x+1)\,(x-1)^2} = \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}. \tag{170}$$

This is a function that we can integrate:

$$\begin{aligned} \int \frac{4x}{(x+1)\,(x-1)^2}\,\mathrm{d}x &= \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,\mathrm{d}x \\ &= -\int \frac{1}{x+1}\,\mathrm{d}x + \int \frac{1}{x-1}\,\mathrm{d}x + 2\int \frac{1}{(x-1)^2}\,\mathrm{d}x \\ &= -\ln|x+1| + \ln|x-1| + 2\int u^{-2}\,\mathrm{d}u, \end{aligned} \tag{171}$$

where in the last term we have used the u-substitution

$$\begin{aligned} u &= x - 1 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned}\,. \tag{172}$$

Our antiderivative now becomes

$$-\ln|x+1|+\ln|x-1|-2u^{-1}+C=\boxed{\ln\left|\tfrac{x-1}{x+1}\right|-\tfrac{2}{x-1}+C}. \tag{173}$$

□

Case 3 The denominator $Q(x)$ contains an irreducible quadratic factor. In this case, ax^2+bx+c will appear in the factorization of $Q(x)$. This factor will contribute a term of the form

$$\frac{Ax+B}{ax^2+bx+c}. \tag{174}$$

We illustrate with an example.

Example 7.24 *Evaluate the integral.*

$$\int \frac{2x^2-x+4}{x^3+4x}\,\mathrm{d}x. \tag{175}$$

We factor the denominator into $x\left(x^2+4\right)$. This cannot be factored further (over the real numbers), so this is Case 3.

We seek constants A, B and C such that

$$\frac{2x^2-x+4}{x\left(x^2+4\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}. \tag{176}$$

To find these constants, we'll use the same procedure we used before: multiply by the denominator on the left:

$$2x^2-x+4=A\left(x^2+4\right)+(Bx+C)\,x, \tag{177}$$

distribute:

$$2x^2-x+4=Ax^2+4A+Bx^2+Cx, \tag{178}$$

then combine like terms in powers of x:

$$2x^2-x+4=(A+B)\,x^2+Cx+4A. \tag{179}$$

Equating coefficients of these polynomials, we receive a system of three equations:

$$\begin{aligned} 2 &= A + B \\ -1 &= C \\ 4 &= 4A \end{aligned} \quad . \tag{180}$$

This tells us that $A = 1$*,* $C = -1$*, and* $B = 1$*. Thus, we have*

$$\frac{2x^2 - x + 4}{x\left(x^2 + 4\right)} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}. \tag{181}$$

This is a function that we can integrate:

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x}\,\mathrm{d}x = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x + \int \frac{x-1}{x^2+4}\,\mathrm{d}x \\ = \int \frac{1}{x}\,\mathrm{d}x + \int \frac{x}{x^2+4}\,\mathrm{d}x - \int \frac{1}{x^2+4}\,\mathrm{d}x \end{aligned} \tag{182}$$

The first term is just $\ln|x|$*. The second term can be done by a* u*-susbstitution with* $u = x^2 + 4$*. The third term can be done by trigonometric substitution, or it can be rewritten as follows:*

$$\int \frac{1}{x}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u - \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\,\mathrm{d}x \tag{183}$$

Now we get

$$\ln|x| + \frac{1}{2}\ln\left|x^2 + 4\right| - \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\,\mathrm{d}x \tag{184}$$

By doing another u*-substitution with* $u = \frac{x}{2}$*, this becomes*

$$\ln|x| + \frac{1}{2}\ln\left|x^2 + 4\right| - \frac{1}{2}\int \frac{1}{u^2 + 1}\,\mathrm{d}x \tag{185}$$

But this is familiar to us:

$$\boxed{\ln|x| + \tfrac{1}{2}\ln\left|x^2 + 4\right| - \tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right) + C}. \tag{186}$$

□

Example 7.25 *(Problem 7.4.12) Evaluate the integral.*

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x \tag{187}$$

We factor the denominator as $(x-2)(x-3)$, which reveals that we are dealing with Case 1: distinct linear factors. Thus,

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}. \tag{188}$$

We multiply both sides by the denominator on the left to get

$$x-4 = A(x-3) + B(x-2). \tag{189}$$

Combining like terms in powers of x:

$$x-4 = (A+B)x + (-3A-2B). \tag{190}$$

Equating corresponding coefficients:

$$\begin{aligned} A+B &= 1 \\ -3A-2B &= -4 \end{aligned}. \tag{191}$$

We add twice the first equation to the second equation to get $-A = -2$. Thus,

$A = 2$, and so $B = -1$, by the first equation. Thus,

$$\begin{aligned}
\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x &= \int_0^1 \frac{2}{x-2} - \frac{1}{x-3}\,\mathrm{d}x \\
&= 2\ln|x-2| - \ln|x-3|\Big|_0^1 \\
&= (2\ln|1-2| - \ln|1-3|) - (2\ln|0-2| - \ln|0-3|) \\
&= 2\ln 1 - \ln 2 - 2\ln 2 + \ln 3 = -3\ln 2 + \ln 3 \\
&= \ln 2^{-3} + \ln 3 = \ln\frac{1}{8} + \ln 3 = \boxed{\ln\tfrac{3}{8}}. \qquad (192)
\end{aligned}$$

□

Example 7.26 *(Problem 7.4.20) Evaluate the integral.*

$$\int \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x. \qquad (193)$$

This is Case 2, since $x-1$ is a repeated linear factor of the denominator. Therefore, we can write this integrand as

$$\frac{-5x^2+3x}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \qquad (194)$$

We multiply by the denominator on the left to get:

$$-5x^2+3x = A(x-1)^2 + B\,(3x-1)\,(x-1) + C\,(3x-1)\,. \qquad (195)$$

We distribute:

$$-5x^2+3x = A\left(x^2-2x+1\right) + B\left(3x^2-4x+1\right) + C\,(3x-1)\,. \qquad (196)$$

Combining like terms in powers of x:

$$-5x^2+3x = (A+3B)\,x^2 + (-2A-4B+3C)\,x + (A+B-C)\,. \qquad (197)$$

Equating corresponding coefficients gives the system of equations

$$\begin{aligned} A + 3B &= -5 \\ -2A - 4B + 3C &= 3. \\ A + B - C &= 0 \end{aligned} \tag{198}$$

Multiply the last equation by 3*:*

$$\begin{aligned} A + 3B &= -5 \\ -2A - 4B + 3C &= 3 \\ 3A + 3B - 3C &= 0 \end{aligned} \tag{199}$$

Now add the second and third equations to get $A - B = 3$*. Therefore,* $A = 3 + B$*, and so the first equation becomes* $3 + B + 3B = -5$*, hence* $B = -2$*. This tells us that* $A = 3 + B = 3 + (-2) = 1$*. Finally, the equation* $A + B - C = 0$ *now becomes* $C = A + B = 1 + (-2) = -1$*. We deduce*

$$\begin{aligned} \int \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x &= \int \frac{1}{3x-1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\,\mathrm{d}x \\ &= \int \frac{1}{3x-1}\,\mathrm{d}x - 2\int \frac{1}{x-1}\,\mathrm{d}x - \int (x-1)^{-2}\,\mathrm{d}x. \end{aligned} \tag{200}$$

We can do each of these three integrals separately. The first integral can be done by u*-substitution with* $u = 3x - 1$*. The second can be done by* u*-substitution with* $u = x - 1$*. The third can be done by* u*-substitution with* $u = x - 1$*. In the end, we get*

$$\int \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x = \boxed{\tfrac{1}{3}\ln|3x-1| - 2\ln|x-1| + \tfrac{1}{x-1} + C}. \tag{201}$$

□

Example 7.27 *(Problem 7.4.28) Evaluate the integral.*

$$\int \frac{x^3 + 6x - 2}{x^4 + 6x^2}\,\mathrm{d}x \tag{202}$$

We factor the denominator as $x^2\left(x^2+6\right)$. This is a hybrid of Cases 2 and 3; we have both a repeated linear factor of x and an irreducible quadratic factor of x^2+6. Thus, we use both guidelines.

We seek constants A, B, C and D such that

$$\frac{x^3+6x-2}{x^2\left(x^2+6\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+6}. \tag{203}$$

As before, we multiply by the denominator on the left:

$$x^3+6x-2=Ax\left(x^2+6\right)+B\left(x^2+6\right)+(Cx+D)\,x^2. \tag{204}$$

Now we distribute and combine like terms in powers of x:

$$x^3+0x^2+6x-2=(A+C)x^3+(B+D)x^2+(6A)x+(6B). \tag{205}$$

We equate corresponding coefficients of these two polynomials to get a system of four equations:

$$\begin{aligned} A+C&=1\\ B+D&=0\\ 6A&=6\\ 6B&=-2 \end{aligned}. \tag{206}$$

We immediately deduce from the third equation that $A=1$ and from the fourth equation that $B=-\frac{1}{3}$. The first equation now implies that $C=0$. The second equation implies that $D=\frac{1}{3}$. Thus,

$$\frac{x^3+6x-2}{x^2\left(x^2+6\right)}=\frac{1}{x}+\frac{-\frac{1}{3}}{x^2}+\frac{0x+\frac{1}{3}}{x^2+6}. \tag{207}$$

This is a function that we can integrate:

$$\begin{aligned}
\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x &= \int \frac{1}{x} - \frac{1}{3x^2} + \frac{1}{3\left(x^2+6\right)}\,\mathrm{d}x \\
&= \int \frac{1}{x}\,\mathrm{d}x - \frac{1}{3}\int x^{-2}\,\mathrm{d}x + \frac{1}{3}\int \frac{1}{x^2+6}\,\mathrm{d}x \\
&\boxed{= \ln|x| + \tfrac{1}{3x} + \tfrac{\sqrt{6}}{18}\tan^{-1}\left(\tfrac{x}{\sqrt{6}}\right) + C}. \qquad (208)
\end{aligned}$$

□

Example 7.28 *Evaluate the integral.*

$$\int \frac{x^5+7x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x \qquad (209)$$

The degree of the numerator is not less than the degree of the denominator, so we cannot use partial fractions immediately. First, we need to write this as a polynomial plus a rational function with a numerator of smaller degree than the denominator. We do this by polynomial division.

$$\begin{array}{r l}
 & \qquad\qquad\qquad\qquad\qquad\quad x \\
x^4+6x^2 \,\big) & \overline{x^5+0x^4+7x^3+0x^2+6x-2} \\
 - & \underline{\left(x^5+0x^4+6x^3\right)} \\
 & \qquad\qquad\quad x^3+0x^2+6x-2
\end{array}$$

This shows that

$$x^5+7x^3+6x-2 = x\left(x^4+6x^2\right) + \left(x^3+6x-2\right). \qquad (210)$$

Therefore, our integral becomes

$$\begin{aligned}\int \frac{x\left(x^4+6x^2\right)+\left(x^3+6x-2\right)}{x^4+6x^2}\,\mathrm{d}x &= \int x+\frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x \\ &= \int x\,\mathrm{d}x+\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x = \frac{1}{2}x^2+\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x. \end{aligned} \tag{211}$$

The previous example now indicates that our answer is

$$\boxed{\tfrac{1}{2}x^2+\ln|x|+\tfrac{1}{3x}+\tfrac{\sqrt{6}}{18}\tan^{-1}\left(\tfrac{x}{\sqrt{6}}\right)+C} \tag{212}$$

7.5 Strategy for integration (?)

Sometimes, more than one method for finding antiderivatives must be used to solve a problem. Especially, sometimes a clever u-substitution will reveal a separate method for evaluating an integral.

Example 7.29 *Evaluate the integral.*

$$\int \frac{e^x}{e^{2x} + 3e^x + 2} \, \mathrm{d}x \tag{213}$$

We begin with a u-substitution:

$$\begin{aligned} u &= e^x \\ \mathrm{d}u &= e^x \, \mathrm{d}x \end{aligned} \tag{214}$$

The integral now becomes

$$\int \frac{1}{u^2 + 3u + 2} \, \mathrm{d}u = \int \frac{1}{(u+1)(u+2)} \, \mathrm{d}u. \tag{215}$$

We can now use partial fraction decomposition:

$$\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}. \tag{216}$$

Multiplying both sides by the denominator on the left gives:

$$1 = A(u+2) + B(u+1) = (A+B)u + (2A+B). \tag{217}$$

Equating coefficients of u on the left and right gives the system of equations

$$A + B = 0 2A + B = 1 \tag{218}$$

From the first equation, we gather that $B = -A$, so the second equation reads

$2A - A = 1$, hence $A = 1$, and so $B = -1$:

$$\int \frac{1}{u^2+3u+2}\,\mathrm{d}u = \int \frac{1}{u+1}\,\mathrm{d}u - \int \frac{1}{u+2}\,\mathrm{d}u$$
$$= \ln|u+1| - \ln|u+2| + C = \ln\left|\frac{u+1}{u+2}\right| + C = \boxed{\ln\left|\tfrac{e^x+1}{e^x+2}\right| + C}. \quad (219)$$

□

Example 7.30 *Evaluate the integral.*

$$\int \sin\sqrt{2t}\,\mathrm{d}t \quad (220)$$

We begin with a u-substitution:

$$\begin{aligned} u &= \sqrt{2t} \\ \mathrm{d}u &= \tfrac{1}{\sqrt{2t}}\,\mathrm{d}t \end{aligned}. \quad (221)$$

From this, we deduce that $u\,\mathrm{d}u = \sqrt{2t}\,\mathrm{d}u = \mathrm{d}t$. Therefore, the integral becomes

$$\int u\sin u\,\mathrm{d}u. \quad (222)$$

From here we can use integration by parts.

$$\begin{array}{ll} u_1 = u & \mathrm{d}v_1 = \sin u\,\mathrm{d}u \\ \mathrm{d}u_1 = \mathrm{d}u & v_1 = -\cos u \end{array}. \quad (223)$$

Now,

$$\begin{aligned} \textstyle\int u_1\,\mathrm{d}v_1 &= u_1v_1 - \textstyle\int v_1\,\mathrm{d}u_1 \\ \textstyle\int u\sin u\,\mathrm{d}u &= -u\cos u + \textstyle\int \cos u\,\mathrm{d}u = -u\cos u + \sin u + C. \end{aligned} \quad (224)$$

This becomes

$$\int \sin\sqrt{2t}\,\mathrm{d}t = \sin u - u\cos u = \boxed{\sin\sqrt{2t} - \sqrt{2t}\cos\sqrt{2t} + C}. \quad (225)$$

□

Example 7.31 *(Problem 7.5.10) Evaluate the integral.*

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\,\mathrm{d}x \tag{226}$$

We begin with a u-substitution:

$$\begin{aligned} u &= \tfrac{1}{x} \\ \mathrm{d}u &= -\tfrac{1}{x^2}\,\mathrm{d}x \end{aligned}. \tag{227}$$

Now $\frac{1}{x^3}\,\mathrm{d}x = -\frac{1}{x}\,\mathrm{d}u = -u\,\mathrm{d}u$, so the integral becomes

$$-\int u\cos u\,\mathrm{d}u. \tag{228}$$

We can now use integration by parts:

$$\begin{array}{cc} u_1 = u & \mathrm{d}v_1 = \cos u\,\mathrm{d}u \\ \mathrm{d}u_1 = \mathrm{d}u & v_1 = \sin u \end{array}. \tag{229}$$

The integral now becomes

$$-\int u\cos u\,\mathrm{d}u = -u\sin u + \int \sin u\,\mathrm{d}u = -u\sin u - \cos u + C. \tag{230}$$

This gives us

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\,\mathrm{d}x = \boxed{-\frac{\sin\left(\frac{1}{x}\right)}{x} - \cos\left(\frac{1}{x}\right) + C}. \tag{231}$$

□

Example 7.32 *(Problem 7.5.42) Evaluate the integral.*

$$\int \frac{\tan^{-1}x}{x^2}\,\mathrm{d}x \tag{232}$$

We begin with integration by parts:

$$\begin{array}{cc} u = \tan^{-1} x & \mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x \\ \mathrm{d}u = \frac{1}{x^2+1}\,\mathrm{d}x & v = -\frac{1}{x} \end{array}. \tag{233}$$

The integral becomes

$$\int \frac{\tan^{-1} x}{x^2}\,\mathrm{d}x = -\frac{\tan^{-1} x}{x} + \int \frac{1}{x\left(x^2+1\right)}\,\mathrm{d}x. \tag{234}$$

We can now proceed by partial fraction decomposition:

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}. \tag{235}$$

Multiplying both sides by the denominator on the left, we get:

$$1 = A\left(x^2+1\right) + \left(Bx+C\right)x = \left(A+B\right)x^2 + Cx + A. \tag{236}$$

Equating coefficients in powers of x, we get the following system of equations:

$$\begin{array}{r} A+B=0 \\ C=0 \\ A=1 \end{array}. \tag{237}$$

From this, it's clear that $A = 1$, $B = -1$ and $C = 0$, so

$$\begin{aligned} \int \frac{1}{x\left(x^2+1\right)}\,\mathrm{d}x = \int \frac{1}{x} - \frac{x}{x^2+1}\,\mathrm{d}x &= \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + C \\ &= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned} \tag{238}$$

Thus,

$$\int \frac{\tan^{-1} x}{x^2}\,\mathrm{d}x = \boxed{-\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C}. \tag{239}$$

□

Moral of the story: antiderivatives are harder than derivatives.

7.8 Improper integrals

Definite integrals find the signed area between a curve and the x-axis. (By "signed area," we mean that the definite integral could be negative, if the curve dips below the x-axis.) Usually, this is the signed area of a bounded region. However, sometimes, it is possible to find a finite signed area of an unbounded region, using limits. Integrals over unbounded regions are called improper integrals.

There are two types of improper integrals: integrals over infinite intervals, and integrals of functions with vertical asymptotes in a finite interval. We'll discuss integrals over infinite intervals first.

Definition 7.33 *Let f be a function, and let a be a real value.*
(i) The improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ is defined as the limit

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x. \tag{240}$$

(ii) The improper integral $\int_{-\infty}^a f(x)\,\mathrm{d}x$ is defined as the limit

$$\int_{-\infty}^a f(x)\,\mathrm{d}x = \lim_{s\to-\infty} \int_s^a f(x)\,\mathrm{d}x. \tag{241}$$

(iii) We say that an improper integral is <u>*convergent*</u> *if the limit has a real value, and* <u>*divergent*</u> *if it does not have a real value.*
(iv) If $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x. \tag{242}$$

Example 7.34 *(Problem 7.8.8) Evaluate the improper integral.*

$$\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x \tag{243}$$

We write this as

$$\lim_{t\to\infty} \int_1^t (2x+1)^{-3}\,\mathrm{d}x, \tag{244}$$

and use u-substitution:

$$\begin{aligned} u &= 2x + 1 \\ \mathrm{d}u &= 2\,\mathrm{d}x \end{aligned} \tag{245}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to\infty} \int_3^{2t+1} \frac{1}{2} u^{-3}\,\mathrm{d}u &= \lim_{t\to\infty} -\frac{1}{4} u^{-2} \bigg|_3^{2t+1} \\ &= \lim_{t\to\infty} \frac{1}{4} \left(\frac{1}{3^2} - \frac{1}{(2t+1)^2} \right) = \frac{1}{4} \left(\frac{1}{9} - 0 \right) = \boxed{\tfrac{1}{36}}. \end{aligned} \tag{246}$$

$\square$

Example 7.35 *(Problem 7.8.6) Evaluate the improper integral.*

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,\mathrm{d}x. \tag{247}$$

We write this as

$$\lim_{t\to\infty} \int_0^t (1+x)^{-\frac{1}{4}}\,\mathrm{d}x. \tag{248}$$

We use u-substitution:

$$\begin{aligned} u &= x + 1 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned} \tag{249}$$

so this integral becomes

$$\lim_{t\to\infty} \int_1^{t+1} u^{-\frac{1}{4}}\,\mathrm{d}u = \lim_{t\to\infty} \frac{4}{3} u^{\frac{3}{4}} \bigg|_1^{t+1} = \lim_{t\to\infty} \frac{4}{3} \left((t+1)^{\frac{3}{4}} - 1^{\frac{3}{4}} \right) = \infty. \tag{250}$$

Thus, the integral is divergent. $\square$

Example 7.36 *(Problem 7.8.16) Evaluate the improper integral.*

$$\int_0^\infty \sin\theta\, e^{\cos\theta}\,\mathrm{d}\theta \tag{251}$$

We write this as

$$\lim_{t\to\infty} \int_0^t \sin\theta\, e^{\cos\theta}\,\mathrm{d}\theta. \tag{252}$$

We now use u-substitution:

$$\begin{aligned} u &= \cos\theta \\ \mathrm{d}u &= -\sin\theta \, \mathrm{d}\theta \end{aligned}. \tag{253}$$

The integral now becomes

$$\lim_{t\to\infty} \int_1^{\cos t} -e^u \, \mathrm{d}u = \lim_{t\to\infty} -e^u \Big|_1^{\cos t} = \lim_{t\to\infty} e^1 - e^{\cos t}. \tag{254}$$

However, as $t \to \infty$, $\cos t$ oscillates between -1 and 1. Therefore, $e^{\cos t}$ oscillates between e^{-1} and e^1. The limit has no real value, so the integral is divergent. $\square$

Example 7.37 *(Problem 7.8.12) Evaluate the improper integral.*

$$\int_{-\infty}^{\infty} y^3 - 3y^2 \, \mathrm{d}y. \tag{255}$$

We first write this as

$$\int_{-\infty}^{\infty} y^3 - 3y^2 \, \mathrm{d}y = \int_{-\infty}^{0} y^3 - 3y^2 \, \mathrm{d}y + \int_{0}^{\infty} y^3 - 3y^2 \, \mathrm{d}y. \tag{256}$$

(The number 0 was chosen arbitrarily; if this integral exists, then any number can be used.) We consider first

$$\begin{aligned} \int_{-\infty}^{0} y^3 - 3y^2 \, \mathrm{d}y &= \lim_{s\to-\infty} \int_s^0 y^3 - 3y^2 \, \mathrm{d}y \\ &= \lim_{s\to-\infty} \frac{1}{4}y^4 - y^3 \Big|_s^0 = \lim_{s\to-\infty} -\frac{1}{4}s^4 + s^3 = -\infty, \end{aligned} \tag{257}$$

and so the integral is divergent. $\square$

Example 7.38 *(Problem 7.8.22) Evaluate the improper integral*

$$\int_1^{\infty} \frac{\ln x}{x^2} \, \mathrm{d}x \tag{258}$$

We first write this as

$$\lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} \, \mathrm{d}x. \tag{259}$$

We use integration by parts:

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & v = -\frac{1}{x} \end{array}. \tag{260}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x &= \lim_{t\to\infty}\left(-\frac{\ln x}{x}\bigg|_1^t + \int_1^t \frac{1}{x^2}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\bigg|_1^t \\ &= \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right) = \lim_{t\to\infty} -\frac{\ln t}{t} - \frac{1}{t} + 1 \end{aligned} \tag{261}$$

Now, by L'Hopital's rule,

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} -\frac{\frac{1}{t}}{1} - \frac{1}{t} + 1 = \boxed{1}. \tag{262}$$

□

Example 7.39 *For what real values of p, if any, does the improper integral $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ converge?*

In order to answer this, we write the improper integral by its definition:

$$\lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,\mathrm{d}x. \tag{263}$$

If $p = 1$, then this becomes

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty} \ln x\bigg|_1^t = \lim_{t\to\infty} \ln t - \ln 1 = -\infty. \tag{264}$$

Therefore, if $p = 1$, then the integral is divergent.

If $p \neq 1$, then we can use the power rule for antiderivatives:

$$\lim_{t\to\infty}\int_1^t x^{-p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^t = \frac{1}{1-p}\lim_{t\to\infty}\left(t^{1-p} - 1\right) \tag{265}$$

From here, there are two possibilities. If $1 - p > 0$, then $\lim_{t\to\infty} t^{1-p} = \infty$, and so *if $p < 1$, then the integral is divergent. On the other hand, if $1 - p < 0$, then this becomes*

$$\frac{1}{1-p} \lim_{t\to\infty} \left(\frac{1}{t^{p-1}} - 1 \right) = \frac{1}{1-p} (0-1) = \frac{1}{p-1}. \tag{266}$$

This shows that the integral is convergent if and only if $p > 1$, in which case the integral converges to the real value $\frac{1}{p-1}$. □

Our final comments about integrals of functions over infinite intervals concerns situations in which is it inconvenient or impossible to actually find the antiderivative of the function. In these cases, we can use the following "comparison tests."

Theorem 7.40 *(The comparison tests for improper integrals) Let a be a real-valued constant, and let f be a continuous function such that for all real numbers $x \geq a$, $f(x) \geq 0$.*
(i) Suppose that for all $x \geq a$, $f(x) \leq g(x)$. If $\int_a^\infty g(x)\ \mathrm{d}x$ is convergent, then $\int_a^\infty f(x)\ \mathrm{d}x$ is also convergent.
(ii) Suppose that for all $x \geq a$, $0 \leq g(x) \leq f(x)$. If $\int_a^\infty g(x)\ \mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\ \mathrm{d}x$ is also divergent.

Example 7.41 *(Problem 7.8.52) Use the comparison theorem to determine whether the integral is convergent or divergent.*

$$\int_0^\infty \frac{\tan^{-1}x}{2+e^x}\ \mathrm{d}x \tag{267}$$

Finding the antiderivative of $\frac{\tan^{-1}x}{2+e^x}$ would be impractical. Instead, we can use the following fact: since $\tan^{-1}x < \frac{\pi}{2}$ for every real value of x,

$$\frac{\tan^{-1}x}{2+e^x} < \frac{\frac{\pi}{2}}{2+e^x} < \frac{\frac{\pi}{2}}{e^x} = \frac{\pi}{2}e^{-x}. \tag{268}$$

Now,

$$\int_0^\infty \frac{\pi}{2}e^{-x}\ \mathrm{d}x = \frac{\pi}{2}\lim_{t\to\infty}\int_0^t e^{-x}\ \mathrm{d}x = \frac{\pi}{2}\lim_{t\to\infty} -e^{-x}\Big|_0^t = \frac{\pi}{2}\left(1 - \lim_{t\to\infty} e^{-t}\right) = \frac{\pi}{2}. \tag{269}$$

Thus, $\int_0^\infty \frac{\pi}{2}e^{-x}\,\mathrm{d}x$ is convergent. Therefore, the comparison test for improper integrals indicates that $\int_0^\infty \frac{\tan^{-1}x}{2+e^x}\,\mathrm{d}x$ must also be convergent. (The comparison test does not tell us which real value this improper integral converges to, though.) □

Example 7.42 *(Problem 7.8.50) Use the comparison theorem to determine whether the integral is convergent or divergent.*

$$\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,\mathrm{d}x \tag{270}$$

Again, finding the antiderivative of $\frac{1+\sin^2 x}{\sqrt{x}}$ would be impractical. Instead, we can use the following fact: for every real value x, $\sin^2 x \geq 0$. Therefore,

$$\frac{1+\sin^2 x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \tag{271}$$

However, by Example 7.39, $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x$ diverges. The comparison theorem now indicates that $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}}\,\mathrm{d}x$ must also diverge. □

Now for the second type of improper integral: integrals of functions with vertical asymptotes over a finite interval.

Definition 7.43 *Let a and b be real values such that $a \leq b$, and let f be a function. (i) If f is defined and continuous on the interval $[a, b)$, and has a vertical asymptote at $x = b$, then we define the improper integral*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\,\mathrm{d}x. \tag{272}$$

(ii) If f is defined and continuous on the interval $(a, b]$, and has a vertical asymptote at $x = a$, then we defined the improper integral

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s \to a^+} \int_s^b f(x)\,\mathrm{d}x. \tag{273}$$

(iii) Given a real value c such that $a < c < b$, if f has a vertical asymptote at $x = c$

and both $\int_a^c f(x)\,\mathrm{d}x$ *and* $\int_c^b f(x)\,\mathrm{d}x$ *exist, then we define the improper integral*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{274}$$

Example 7.44 *(Problem 7.8.28) Evaluate the improper integral.*

$$\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,\mathrm{d}x \tag{275}$$

We write this as

$$\lim_{t\to 5^-} \int_0^t (5-x)^{-\frac{1}{3}}\,\mathrm{d}x. \tag{276}$$

We use u*-substitution:*

$$\begin{aligned} u &= 5-x \\ \mathrm{d}u &= -\mathrm{d}x \end{aligned}. \tag{277}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to 5^-} \int_5^{5-t} -u^{-\frac{1}{3}}\,\mathrm{d}u &= \lim_{t\to 5^-} -\frac{3}{2}u^{\frac{2}{3}}\Big|_5^{5-t} \\ &= \lim_{t\to 5^-} \frac{3}{2}\left(5^{\frac{2}{3}} - (5-t)^{\frac{2}{3}}\right) = \frac{3}{2}\left(5^{\frac{2}{3}} - 0^{\frac{2}{3}}\right) = \boxed{\tfrac{3}{2}5^{\frac{2}{3}}}. \end{aligned} \tag{278}$$

□

Example 7.45 *(Problem 7.8.34) Evaluate the improper integral.*

$$\int_0^5 \frac{x}{x-2}\,\mathrm{d}x \tag{279}$$

We write this as

$$\lim_{t\to 2^-} \int_0^t \frac{x}{x-2}\,\mathrm{d}x + \lim_{s\to 2^+} \int_s^5 \frac{x}{x-2}\,\mathrm{d}x \tag{280}$$

We can do this antiderivative either by a u*-substitution or by recognizing:*

$$\frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}. \tag{281}$$

The integral now becomes

$$\lim_{t \to 2^-} \int_0^t 1 + \frac{2}{x-2} \, \mathrm{d}x + \lim_{s \to 2^+} \int_s^5 1 + \frac{2}{x-2} \, \mathrm{d}x. \tag{282}$$

Let's focus first on the second term.

$$\begin{aligned} \lim_{s \to 2^+} \int_s^5 1 + \frac{2}{x-2} \, \mathrm{d}x &= \lim_{s \to 2^+} x + 2 \ln |x-2| \Big|_s^5 \\ &= \lim_{s \to 2^+} (5 + 2\ln 3) - (s + 2 \ln |s-2|) = \infty \end{aligned} \tag{283}$$

The integral is divergent. $\square$

11 Infinite series

11.1 Sequences

Definition 11.1 *A* <u>*sequence of real numbers*</u> *is an infinite list of real numbers with a defined order.*

Example 11.2 *Find a formula for the general term* a_n *of the sequence whose terms are* $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 16$, $a_5 = 32$, *.... (In other words, find a formula for* a_n *in terms of* n*.)*

Here the formula is $a_n = 2^n$. $\square$

Example 11.3 *(Problem 11.1.14) Find a formula for the general term* a_n *of the sequence whose terms are* $a_1 = 4$, $a_2 = -1$, $a_3 = \frac{1}{4}$, $a_4 = -\frac{1}{16}$, $a_5 = \frac{1}{64}$, *....*

Here the formula is $a_n = (-1)^{n+1}4^{2-n}$. $\square$

We will be largely concerned with limits of sequences.

Definition 11.4 *Let* (a_n) *be a sequence of real numbers. Given a real number* L*, we say that* <u>L *is the limit of* (a_n) *as* n *approaches infinity*</u>*, or that* <u>(a_n) *converges to* L</u> *provided that the following is true: by taking* n *to be sufficiently large, we can make the distance* $|a_n - L|$ *arbitrarily small.*

Notation: $\lim_{n\to\infty} a_n = L$.

If a sequence has a real-valued limit, we say that the sequence is "convergent." If it has no real-valued limit, we say that it is "divergent."

We take limits of sequences in the same way we took limits of functions in Calculus I.

Example 11.5 *Determine whether the sequence* $a_n = \frac{3n^4+5}{4n^4-7n^2+9}$ *converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{3n^4+5}{4n^4-7n^2+9} = \lim_{n\to\infty} \frac{3+\frac{5}{n^4}}{4-\frac{7n^2}{n^4}+\frac{9}{n^4}} = \frac{3+0}{4-0+0} = \boxed{\frac{3}{4}}. \tag{284}$$

□

Example 11.6 *(Problem 11.1.28) Determine whether the sequence* $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$ *converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{3}{1+\frac{2}{\sqrt{n}}} = \frac{3}{1+0} = \boxed{3}. \tag{285}$$

□

Example 11.7 *Determine whether the sequence* $a_n = \frac{n^2-1}{n}$ *converges or diverges. If it converges, find the limit.*

We know that

$$\lim_{n\to\infty} \frac{n^2-1}{n} = \infty, \tag{286}$$

so the sequence is divergent. □

Example 11.8 *Determine whether the sequence* $a_n = 1 + (-1)^n$ *converges or diverges. If it converges, find the limit.*

The terms in the sequence are $0, 2, 0, 2, 0, 2, \ldots$*, so the sequence has no limit; it is divergent.* □

Theorem 11.9 *If* $\lim\limits_{n\to\infty} a_n = L$ *and* f *is continuous at* L*, then* $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

Example 11.10 *(Problem 11.1.32) Determine whether the sequence* $a_n = \cos\left(\frac{n\pi}{n+1}\right)$ *converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{n\pi}{n+1} = \lim_{n\to\infty} \frac{\pi}{1+\frac{1}{n}} = \pi. \tag{287}$$

Now $\cos(\pi) = -1$*, so by the theorem,* $\lim\limits_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right) = \boxed{-1}$. □

Definition 11.11 *A* <u>*geometric sequence*</u> *is a sequence* (a_n) *in which* $a_n = br^n$ *for some real values* b *and* r*. We call* r *the* <u>*common ratio of* a_n</u>.

Theorem 11.12 *The geometric sequence* (br^n) *is:*
(i) Convergent to 0 *for* $-1 < r < 1$.
(ii) Convergent to b *for* $r = 1$.
(iii) Divergent for $r \leq -1$ *or* $r > 1$.

Definition 11.13 *A sequence a_n of real numbers is called:*
(i) decreasing if $a_{n+1} \leq a_n$ for all n.
(ii) increasing if $a_{n+1} \geq a_n$ for all n.
(iii) monotone if it is either decreasing or increasing.
(iv) eventually decreasing if $a_{n+1} \leq a_n$ for all n greater than a certain value.
(v) eventually increasing if $a_{n+1} \geq a_n$ for all n greater than a certain value.

Example 11.14 *Determine whether the sequence $a_n = \frac{1}{2n+3}$ is increasing, decreasing, or not monotone.*

We know that $2(n+1)+3 > 2n+3$ for any positive integer n. Therefore, $a_n = \frac{1}{2n+3} > \frac{1}{2(n+1)+3} = a_{n+1}$, so the sequence is $\boxed{\textit{decreasing}}$. □

Example 11.15 *Determine whether the sequence $a_n = \frac{n}{n^2+1}$ is increasing, decreasing, or not monotone.*

Consider the function

$$f(x) = \frac{x}{x^2+1}. \tag{288}$$

Taking the derivative,

$$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}. \tag{289}$$

This is negative when $x > 1$, so f is decreasing on $(1, \infty)$. This tells us that $a_{n+1} = f(n+1) < f(n) = a_n$ for any positive integer n. Ergo, a_n is $\boxed{\textit{decreasing}}$.
□

Definition 11.16 *The harmonic sequence is the sequence $a_n = \frac{1}{n}$.*

The harmonic sequence is a decreasing sequence which converges to 0.

11.2 Series

It makes sense to discuss $a+b$, because addition is defined for two real numbers. You can extend this to discussing $a_1+a_2+...+a_n$, for any positive integer n. What would be the meaning of a sum of infinitely many numbers?

Definition 11.17 *Let a_n be a sequence of real numbers. Given a positive integer k, the kth partial sum of a_n is the value $s_k = \sum_{n=1}^{k} a_n$.*

An "infinite series" is the limit of the partial sums of a sequence.

Definition 11.18 *Let a_n be a sequence of real numbers, and let s_n be the nth partial sum of a_n. If the sequence (s_n) converges to a real number s, then we say that the series $\sum_{n=1}^{\infty} a_n$ is convergent to s, and that s is the sum of the series. If no such real number exists, we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.*

For the first part of this chapter, we will be mostly concerned with series whose terms are all positive. At first, this may sound like nonsense. Our intuition tells us that a sum of infinitely many numbers, all of which are greater than zero, must be infinitely large, right? This is a false intuition for more than one reason. First, we should note that we are dealing with an infinite series whenever we have a non-terminating decimal. Second, and more importantly, refusing to accept the concept of the infinite series leads to some rather bizarre conclusions. (See: Zeno's paradoxes.)

We begin by discussing one of the most important examples.

Definition 11.19 *A geometric series is a series $\sum_{n=1}^{\infty} a_n$, where a_n is a geometric sequence.*

Since every geometric sequence looks like $a_n = br^n$, every geometric series will take the form $\sum_{n=1}^{\infty} br^n$, for some appropriate values of the constants b and r.

Do geometric series converge? It depends.

Consider the geometric series $\sum_{n=1}^{\infty} br^{n-1}$. (Note: $\sum_{n=1}^{\infty} br^{n-1} = \sum_{n=0}^{\infty} br^{n}$.)

If the common ratio $r = 1$, then the kth partial sum of $a_n = br^{n-1}$ is

$$s_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} br^{n-1} = \sum_{n=1}^{k} b = kb. \tag{290}$$

Now, $\lim_{k\to\infty} kb = \infty$, so the series $\sum_{n=1}^{\infty} br^{n-1}$ is divergent in this case.

If the common ratio $r \neq 1$, then the kth partial sum is

$$s_k = \sum_{n=1}^{k} br^{n-1} = b + br + br^2 + ... + br^{k-1}. \tag{291}$$

We notice:

$$rs_k = br + br^2 + br^3 + ... + br^k = s_k - b + br^k. \tag{292}$$

Therefore,

$$\begin{aligned} rs_k - s_k &= -b + br^k \\ (r-1)\, s_k &= b\left(-1 + r^k\right). \\ s_k &= \tfrac{b\left(r^k - 1\right)}{r-1} = \tfrac{b\left(1 - r^k\right)}{1-r}. \end{aligned} \tag{293}$$

Now, if $-1 < r < 1$, then

$$\sum_{n=1}^{\infty} br^{n-1} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{b\left(1-r^k\right)}{1-r} = \frac{b\,(1-0)}{1-r} = \frac{b}{1-r}. \tag{294}$$

On the other hand, if $r \leq -1$ or $r > 1$, then $\lim_{k\to\infty} r^k$ diverges, so therefore $\sum_{n=1}^{\infty} br^{n-1}$ also diverges. To summarize:

Theorem 11.20 *(Characterization theorem for geometric series) Let b and r be real numbers.*
(i) The geometric series $\sum_{n=1}^{\infty} br^{n-1}$ diverges if $|r| \geq 1$.
(ii) The geometric series $\sum_{n=1}^{\infty} br^{n-1} = \frac{b}{1-r}$ if $-1 < r < 1$.

What can we say about non-geometric series?

Some series are called "telescoping series," and have terms that cancel pairwise in the partial sums.

Example 11.21 *Determine whether the series* $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ *is convergent or divergent. If it is convergent, find its sum.*

We notice that the k*th partial sum of this series is*

$$s_k = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+...+\left(\frac{1}{k}-\frac{1}{k+1}\right). \qquad (295)$$

Most of these terms cancel to give us:

$$s_k = 1-\frac{1}{k+1}. \qquad (296)$$

Now,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty} s_k = \lim_{k\to\infty} 1-\frac{1}{k+1} = \boxed{1}. \qquad (297)$$

□

Theorem 11.22 *(Test for divergence) If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Example 11.23 *(Problem 11.2.30) Determine whether the series* $\sum_{n=1}^{\infty}\frac{n^2}{n^2-2n+5}$ *is convergent or divergent. If it is convergent, find its sum.*

We notice that

$$\lim_{n\to\infty}\frac{n^2}{n^2-2n+5} = \lim_{n\to\infty}\frac{1}{1-\frac{2}{n}+\frac{5}{n^2}} = \frac{1}{1-0+0} = 1. \qquad (298)$$

Since the limit of the terms is nonzero, the test for divergence dictates that the series cannot converge; it is $\boxed{\textit{divergent}}$. □

Example 11.24 *Determine whether the series* $\sum_{n=1}^{\infty}\ln n$ *is convergent or divergent. If it is convergent, find its sum.*

We know that $\lim_{n\to\infty} \ln n = \infty$*, so by the test for divergence, the series* $\sum_{n=1}^{\infty} \ln n$ *is* $\boxed{\textit{divergent.}}$ $\square$

Example 11.25 *Determine whether the series* $\sum_{n=1}^{\infty} \sin^2\left(\frac{n\pi}{2}\right)$ *is convergent or divergent. If it is convergent, find its sum.*

First, we note that if n *is even, then* $\sin^2\left(\frac{n\pi}{2}\right) = 0$*. On the other hand, if* n *is odd, then* $\sin^2\left(\frac{n\pi}{2}\right) = 1$*. Thus, as* n *approaches infinity,* $\sin^2\left(\frac{n\pi}{2}\right)$ *continues to oscillate between* 0 *and* 1*, and so* $\lim_{n\to\infty} \sin^2\left(\frac{n\pi}{2}\right)$ *does not even exist. By the test for divergence, the infinite series* $\sum_{n=1}^{\infty} \sin^2\left(\frac{n\pi}{2}\right)$ *is* $\boxed{\textit{divergent}}$. $\square$

The test for divergence proposes the following test: if $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ cannot be convergent. The converse is untrue:

Theorem 11.26 *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Why? We'll see in the next section.

11.3 The integral test

Finding the exact sum of a series is a difficult problem in general. For the next few sections, we'll concentrate on testing whether a series is convergent or divergent without finding the sum.

Theorem 11.27 *(The integral test) Let (a_n) be a positive, eventually decreasing sequence of real numbers. Given a continuous function f defined on $[1, \infty)$ such that for each positive integer n, $f(n) = a_n$, the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,\mathrm{d}x$ is convergent.*

Example 11.28 *Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$ is convergent or divergent.*

In order to use the integral test, we must verify that the function $f(x) = \frac{\ln x}{x}$ is eventually decreasing. We do so by examining the derivative:

$$f'(x) = \frac{1 - \ln x}{x^2} \tag{299}$$

This is negative when $1 - \ln x < 0$, or in other words, when $\ln x > 1$, hence $x > e$. Thus, f is eventually decreasing, and so the integral test can be used.

$$\int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,\mathrm{d}x \tag{300}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x \end{aligned}. \tag{301}$$

Now,

$$\int_1^\infty \frac{\ln x}{x}\,\mathrm{d}x = \lim_{t\to\infty} \int_0^{\ln t} u\,\mathrm{d}u = \lim_{t\to\infty} \frac{1}{2}u^2\Big|_0^{\ln t} = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty. \tag{302}$$

This shows that the series is $\boxed{\text{divergent}}$. □

Example 11.29 *(Problem 11.3.8) Determine whether the series $\sum\limits_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent or divergent.*

First we need to verify that $f(x) = x^2 e^{-x^3}$ is eventually decreasing. We examine the derivative:

$$f'(x) = -3x^4 e^{-x^3} + 2xe^{-x^3} = xe^{-x^3}\left(2 - 3x^3\right). \tag{303}$$

For $x > \sqrt[3]{\frac{2}{3}}$, this is negative, so the sequence of terms is eventually decreasing. We can now use the integral test:

$$\int_1^\infty x^2 e^{-x^3}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t x^2 e^{-x^3}. \tag{304}$$

We proceed by u-substitution:

$$\begin{aligned} u &= x^3 \\ \mathrm{d}u &= 3x^2\,\mathrm{d}x \end{aligned}. \tag{305}$$

Now,

$$\begin{aligned} \int_1^\infty x^2 e^{-x^3}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^{t^3} \frac{1}{3}e^{-u}\,\mathrm{d}u \\ = \lim_{t\to\infty} -\frac{1}{3}e^{-u}\bigg|_1^{t^3} = \lim_{t\to\infty}\frac{1}{3}e^{-1} - \frac{1}{3}e^{-t^3} = \frac{1}{3e}. \end{aligned} \tag{306}$$

This shows that the series is $\boxed{\textit{convergent}}$*. (It does not tell us the sum.)* □

Example 11.30 *(Problem 11.3.6) Determine whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{(3n-1)^4}$ is convergent or divergent.*

First we verify that $f(x) = \frac{1}{(3x-1)^4}$ is eventually decreasing:

$$f'(x) = -4(3x-1)^{-5}(3) = \frac{-12}{(3x-1)^5}. \tag{307}$$

This is negative for $x > \frac{1}{3}$, so the sequence of terms is eventually decreasing. We *can now use the integral test:*

$$\int_1^\infty \frac{1}{(3x-1)^4}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t (3x-1)^{-4}\,\mathrm{d}x. \tag{308}$$

We proceed by u-substitution:

$$\begin{aligned} u &= 3x-1 \\ \mathrm{d}u &= 3\,\mathrm{d}x \end{aligned}, \tag{309}$$

so

$$\begin{aligned} \int_1^\infty \frac{1}{(3x-1)^4}\,\mathrm{d}x &= \lim_{t\to\infty}\int_2^{3t-1}\frac{1}{3}u^{-4}\,\mathrm{d}u \\ &= \lim_{t\to\infty} -\frac{1}{9}u^{-3}\Big|_2^{3t-1} = \lim_{t\to\infty}\frac{1}{9}\left(\frac{1}{8}-\frac{1}{(3t-1)^3}\right) = \frac{1}{9}\frac{1}{8} = \frac{1}{72}. \end{aligned} \tag{310}$$

This shows that the series is $\boxed{\textit{convergent}}$. $\square$

Example 11.31 *The series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ for an integer p is called the "p-series." For what values of p does the p-series converge?*

If $p<0$, then $\lim\limits_{n\to\infty}\frac{1}{n^p}=\infty$, in which case the test for divergence reveals that the p-series is divergent.

If $p=0$, then $\lim\limits_{n\to\infty}\frac{1}{n^p}=\lim\limits_{n\to\infty}1=1$, so again the test for divergence dictates that the p-series is divergent.

If $p=1$, then this is the harmonic series. We will now show that the harmonic series is divergent, using the integral test. We know that $n<n+1$, so $\frac{1}{n+1}<\frac{1}{n}$ for any positive integer n. Thus, the terms of the harmonic series form a decreasing sequence. Consider the function $f(x)=\frac{1}{x}$. Now use the integral test:

$$\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty}\ln x\Big|_1^t = \lim_{t\to\infty}\ln t - \ln 1 = \lim_{t\to\infty}\ln t = \infty. \tag{311}$$

As this is divergent, we see that the harmonic series is also divergent. Thus, the p-series is divergent for $p=1$.

If $p>0$ and $p\neq 1$, then we notice that $n^p\le(n+1)^p$, so $\frac{1}{(n+1)^p}\le\frac{1}{n^p}$ for any positive integer n. This shows that the terms of the p-series form a decreasing *sequence. We consider the function* $f(x) = \frac{1}{x^p}$*. Now, we use the integral test:*

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty x^{-p}\,\mathrm{d}x = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^t \\ = \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p}-1\right) = \lim_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right)\end{aligned} \tag{312}$$

This improper integral is convergent if and only if $p-1>0$*. This shows that the* p*-series is convergent exactly when* $\boxed{p>1}$*.* □

From the previous example, we can state the following theorem.

Theorem 11.32 *(*p*-series test) Let* p *be a real number. The* p*-series* $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ *is convergent if and only if* $p>1$*.*

Example 11.33 *(Problem 11.3.4) Determine whether the series* $\sum\limits_{n=1}^{\infty} n^{-0.3}$ *is convergent or divergent.*

This is a p*-series:*

$$\sum_{n=1}^{\infty} n^{-0.3} = \sum_{n=1}^{\infty}\frac{1}{n^{0.3}}. \tag{313}$$

Since $0.3 \le 1$*, the series is* $\boxed{\text{divergent}}$*.* □

11.4 The comparison tests

If you know that a given series with positive terms is convergent or divergent, you can sometimes use that information to deduce whether another series with positive terms is convergent or divergent.

Theorem 11.34 *(The comparison tests for infinite series) Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.*
(i) Suppose that for all positive integers n, $a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) Suppose that for all positive integers n, $c_n \leq a_n$. If $\sum_{n=1}^{\infty} c_n$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

This is exactly analogous to the comparison test for improper integrals.

Example 11.35 *Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges or diverges.*

We notice that

$$\frac{1}{2^n+1} < \frac{1}{2^n}, \tag{314}$$

so since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series, the series is $\boxed{\text{convergent}}$.
□

Example 11.36 *(Problem 11.4.4) Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ converges or diverges.*

We notice that

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}, \tag{315}$$

so since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a divergent p-series, the series is $\boxed{\text{divergent}}$. □

Example 11.37 *(Problem 11.4.6) Determine whether the series $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ is convergent or divergent.*

We notice that

$$\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}. \tag{316}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the series is* $\boxed{\text{convergent}}$. $\square$

Example 11.38 *(Problem 11.4.8) Determine whether the series* $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ *is convergent or divergent.*

We notice that

$$\frac{6^n}{5^n-1} > \frac{6^n}{5^n}. \tag{317}$$

As $\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ *is a divergent geometric series, the series is also* $\boxed{\text{divergent}}$. $\square$

Example 11.39 *(Problem 11.4.14) Determine whether the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ *is convergent or divergent.*

We notice that

$$\frac{1}{\sqrt[3]{3n^4+1}} < \frac{1}{\sqrt[3]{3n^4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{\frac{4}{3}}}. \tag{318}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ *is a convergent* p*-series, the series is* $\boxed{\text{convergent}}$. $\square$

Theorem 11.40 *(Limit comparison test) Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. If* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *is positive and finite, then either both converge series or both diverge.*

Example 11.41 *(Problem 11.4.22) Determine whether the series* $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ *converges or diverges.*

We use the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n^2}$*:*

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{n+2}{(n+1)^3}\right)} = \lim_{n\to\infty} \frac{(n+1)^3}{n^2(n+2)} = \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{n^3+2n^2} = 1. \tag{319}$$

Since 1 *is positive and finite, and since* $\sum\limits_{n=3}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the limit comparison test indicates that* $\sum\limits_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ *is also* $\boxed{\textit{convergent}}$.

Example 11.42 *Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{1}{2n+1}$ *converges or diverges.*

We use the limit comparison test with the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$*:*

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2n+1}\right)} = \lim_{n\to\infty} \frac{2n+1}{n} = 2. \tag{320}$$

Since 2 *is positive and finite, and since* $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ *is divergent, the limit comparison test indicates that* $\sum\limits_{n=1}^{\infty} \frac{1}{2n+1}$ *is also* $\boxed{\textit{divergent}}$.

11.5 Alternating series

Definition 11.43 *An alternating series is a series whose successive terms are alternately positive and negative.*

Example 11.44 *The following are alternating series:*

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ... \tag{321}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^2 = 1 - 4 + 9 - 16 + 25 - ... \tag{322}$$

$$\sum_{n=1}^{\infty} \cos(n\pi) = (-1) + 1 + (-1) + 1 + (-1) + ... \tag{323}$$

□

Theorem 11.45 *(Alternating series test) Let* $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ *be an alternating series, where for each positive integer* n, $b_n > 0$. *If the sequence* (b_n) *is eventually decreasing and* $\lim_{n\to\infty} b_n = 0$, *then the series* $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ *is convergent.*

Example 11.46 *(Problem 11.5.2) Determine whether the series* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{2n+1}$ *is convergent or divergent.*

We note that

$$\frac{2}{2n+1} > \frac{2}{2(n+1)+1}. \tag{324}$$

Additionally,

$$\lim_{n\to\infty} \frac{2}{2n+1} = 0. \tag{325}$$

Therefore, by the alternating series test, the series is $\boxed{\textit{convergent}}$. □

Example 11.47 *(Problem 11.5.4) Determine whether the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(2n+1)}$ *is convergent or divergent.*

We note that

$$\frac{1}{\ln(2n+1)} > \frac{1}{\ln(2(n+1)+1)}. \tag{326}$$

Additionally,

$$\lim_{n\to\infty} \frac{1}{\ln(2n+1)} = 0. \tag{327}$$

Therefore, by the alternating series test, the series is $\boxed{\textit{convergent}}$. □

Example 11.48 *(Problem 11.5.8) Determine whether the series* $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$ *is convergent or divergent.*

We note that

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1. \tag{328}$$

Therefore, $\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+n+1}$ *does not exist. By the test for divergence, the series is* $\boxed{\textit{divergent}}$. □

Example 11.49 *Determine whether the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *is convergent or divergent.*

We note that

$$\frac{1}{n} > \frac{1}{n+1}. \tag{329}$$

Further,

$$\lim_{n\to\infty} \frac{1}{n} = 0. \tag{330}$$

Therefore, by the alternating series test, the series is $\boxed{\textit{convergent}}$. □

Example 11.50 *Determine whether the series* $\sum_{n=1}^{\infty} (-1)^n e^{\frac{1}{n}}$ *is convergent or divergent.*

We note that

$$\lim_{n\to\infty} e^{\frac{1}{n}} = 1, \tag{331}$$

so the limit $\lim_{n\to\infty} (-1)^n e^{\frac{1}{n}}$ *does not exist. Therefore, by the test for divergence, the series is* $\boxed{\textit{divergent}}$. □

Note: both of the assumptions (eventual decreasing and a limit of zero) are necessary for the alternating series test to be applicable. It is possible for only one, or neither, of these conditions to be satisfied.

Example 11.51 *Determine whether the series* $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$ *is convergent or divergent.*

The sequence $b_n = \frac{n+1}{n}$ *is (eventually) decreasing, but*

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n+1}{n} = 1 \neq 0, \tag{332}$$

so the alternating series test does not apply. However, $\lim_{n\to\infty}(-1)^{n+1}\frac{n+1}{n}$ *does not exist, since the absolute value is not approaching zero, but the sign continues to alternate. Thus, the test for divergence indicates that the series is* $\boxed{\text{divergent}}$.

Example 11.52 *Determine whether the series* $\sum_{n=1}^{\infty}(-1)^{n}\frac{1+\frac{1}{2}(-1)^n}{n}$ *is convergent or divergent.*

Define the sequence $b_n = \frac{1+\frac{1}{2}(-1)^n}{n}$. *We know that* $\frac{\left(\frac{1}{2}\right)}{n} \leq \frac{1+\frac{1}{2}(-1)^n}{n} \leq \frac{\left(\frac{3}{2}\right)}{n}$. *Further,* $\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)}{n} = 0$ *and* $\lim_{n\to\infty}\frac{\left(\frac{3}{2}\right)}{n} = 0$. *By the squeeze theorem (from Calculus I), this means that* $\lim_{n\to\infty}\frac{1+\frac{1}{2}(-1)^n}{n} = 0$. *However,*

$$b_n = \frac{1+\frac{1}{2}(-1)^n}{n} = \begin{cases} \frac{1}{2n} & \text{if } n \text{ is odd} \\ \frac{3}{2n} & \text{if } n \text{ is even} \end{cases}. \tag{333}$$

Based on this, it can be shown that if n *is odd, then* $b_n < b_{n+1}$, *and if* n *is even, then* $b_n > b_{n+1}$. *This indicates that the sequence is not eventually decreasing. Thus, the alternating series test does not apply.*

In order to determine the behavior of this series, we need to consider the following two sequences: $c_n = \frac{1}{n}$ *and* $d_n = \frac{(-1)^n}{2n}$. *We note that*

$$b_n = \frac{1+\frac{1}{2}(-1)^n}{n} = \frac{1}{n} + \frac{\frac{1}{2}(-1)^n}{n} = c_n + d_n. \tag{334}$$

By the alternating series test, $\sum_{n=1}^{\infty}(-1)^n c_n$ *is convergent. Also,*

$$\sum_{n=1}^{\infty}(-1)^n d_n = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2n} = \sum_{n=1}^{\infty}\frac{\left((-1)^2\right)^n}{2n} = \sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}, \tag{335}$$

which is a constant times the harmonic series, which is divergent. Finally, if the series $\sum_{n=1}^{\infty}(-1)^n c_n$ *is convergent and* $\sum_{n=1}^{\infty}(-1)^n d_n$ *is divergent, then*

$$\sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{\infty}(-1)^n (c_n + d_n) \tag{336}$$

must also be $\boxed{\textit{divergent}}$. □

It's true that all convergent series can be approximated by taking the nth partial sum, for a large enough n, but an approximation is only useful if the amount of its accuracy is known. For alternating series that pass the alternating series test, the following theorem can be used to find the size of the error.

Theorem 11.53 *(Alternating series estimation theorem) Let* $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *be an alternating series, where for each positive integer* n, $b_n > 0$. *Assume that the sequence* (b_n) *is decreasing and that* $\lim_{n\to\infty} b_n = 0$. *Let* s *be the real number to which the series converges, and for each positive integer* k, *define*

$$R_k = s - \sum_{n=1}^{k}(-1)^{n+1}b_n. \tag{337}$$

(In other words, R_k *is the difference between the sum of the series and the* k*th partial sum.) In that case,* $|R_k| \le b_{k+1}$.

To summarize, the error R_k in the approximation $s \approx \sum_{n=1}^{k}(-1)^{n+1}b_n$ has an absolute value of, at most, b_{k+1}, which is the absolute value of the first term that is being cut off.

Example 11.54 *Approximate the sum of the series* $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n4^n}$ *correct to three decimal places.*

The series certainly satisfies the conditions of the alternating series test, and so it is convergent. To get the first three decimal places correct, we need the error, R_k, *to be less than* $0.001 = \frac{1}{1000}$. *We define* $b_n = \frac{1}{n4^n}$, *and examine a few of its terms:*

$$\begin{matrix} b_1 = \frac{1}{4} \\ b_2 = \frac{1}{32} \\ b_3 = \frac{1}{192} \\ b_4 = \frac{1}{1024} \end{matrix}. \tag{338}$$

The alternating series estimation theorem indicates that $|R_3| \leq b_4 = \frac{1}{1024} < \frac{1}{1000}$. *Thus, only three terms are needed to get an error less than* 0.001*:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n} \approx s_3 = \sum_{n=1}^{3} \frac{(-1)^{n-1}}{n4^n} = \boxed{\frac{1}{4} - \frac{1}{32} + \frac{1}{192} = \frac{43}{192}}. \tag{339}$$

□

Example 11.55 *(Problem 11.5.28) Approximate the sum of the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ *correct to four decimal places.*

The series certainly satisfies the conditions of the alternating series test, so it is convergent. As for the sum, to get the first four decimal places correct, we need the error, R_k*, to be less than* $0.0001 = \frac{1}{10000}$. *We define* $b_n = \frac{1}{n^6}$, *and examine a few of its terms:*

$$\begin{matrix} b_1 = 1 \\ b_2 = \frac{1}{2^6} = \frac{1}{64} \\ b_3 = \frac{1}{3^6} = \frac{1}{729} \\ b_4 = \frac{1}{4^6} = \frac{1}{4096} \\ b_5 = \frac{1}{5^6} = \frac{1}{15625} \end{matrix}. \tag{340}$$

The alternating series estimation theorem indicates that $|R_4| \leq b_5 = \frac{1}{15625} < \frac{1}{10000}$. *Thus, only four terms are needed to get an error that does not affect the first four decimal places:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \approx s_4 = \sum_{n=1}^{4} \frac{(-1)^{n+1}}{n^6} = \boxed{1 - \frac{1}{64} + \frac{1}{729} - \frac{1}{4096} = \frac{2{,}942{,}695}{2{,}985{,}984}} \tag{341}$$

□

11.6 Absolute convergence and the ratio and root tests

There is a useful concept that can allow us to determine the behavior of a series whose terms are not necessarily positive by studying the behavior of a related series whose terms are positive.

Definition 11.56 *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series. We say that* $\sum_{n=1}^{\infty} a_n$ *is* *absolutely convergent* *provided that the series* $\sum_{n=1}^{\infty} |a_n|$ *is convergent.*

Note: $\sum_{n=1}^{\infty} |a_n|$ is definitely **NOT** the same as $\left|\sum_{n=1}^{\infty} a_n\right|$.

Theorem 11.57 *If a series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

The converse of the theorem above is false; there are some series which converge, but do not converge absolutely.

Example 11.58 *Consider the alternating harmonic series,* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. *This is not absolutely convergent, because*

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}, \tag{342}$$

which is the harmonic series, a divergent series. However, by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *is, in fact, convergent.* □

For series like these, we have the following term.

Definition 11.59 *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series. We say that* $\sum_{n=1}^{\infty} a_n$ *is* *conditionally convergent* *if it is convergent, but not absolutely convergent.*

Nevertheless, it is often viable and easy to show that a series is convergent as a result of it being absolutely convergent, as the following examples show.

Example 11.60 *Determine whether* $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$ *converges absolutely, converges conditionally, or is divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right|. \tag{343}$$

We notice that $\left| \sin\left(\frac{n\pi}{3}\right) \right| \leq 1$*, so*

$$\left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right| \leq \frac{1}{n^2}. \tag{344}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, so by the comparison test,* $\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right|$ *converges. Ergo,* $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$ *is* $\boxed{\textit{absolutely convergent}}$*, which implies that it is convergent.* □

Example 11.61 *(Problem 11.6.36) Determine whether the series* $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}}$ *is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right| \tag{345}$$

Since $\left| \sin\left(\frac{n\pi}{6}\right) \right| \leq 1$*,*

$$\left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right| \leq \frac{1}{1+n\sqrt{n}} < \frac{1}{n^{\frac{3}{2}}}. \tag{346}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ *is a convergent* p*-series, so by the comparison test,* $\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right|$ *converges. Ergo,* $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}}$ *is* $\boxed{\textit{absolutely convergent}}$*.* □

Example 11.62 *(Problem 11.6.4) Determine whether the series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ *is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}. \tag{347}$$

We note that

$$\frac{1}{n^3+1} < \frac{1}{n^3}. \tag{348}$$

Now $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ *is a convergent* p*-series, so by the comparison test,* $\sum\limits_{n=1}^{\infty} \frac{1}{n^3+1}$ *is convergent. Therefore, the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ *is* $\boxed{\textit{absolutely convergent}}$. $\square$

Example 11.63 *(Problem 11.6.2) Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}. \tag{349}$$

This is a divergent p*-series, so the series is not absolutely convergent. However,*

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}, \tag{350}$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0, \tag{351}$$

so by the alternating series test, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *is convergent; the series is* $\boxed{\textit{conditionally convergent}}$. $\square$

Theorem 11.64 *(Ratio test) Let* $\sum\limits_{n=1}^{\infty} a_n$ *be an infinite series.*

(i) If $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$*, then the series converges absolutely.*

(ii) If $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$*, then the series diverges.*

Notice that we have no conclusion if $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Example 11.65 *(Problem 11.6.12) Determine whether the series* $\sum\limits_{n=1}^{\infty} ne^{-n}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{e} = \frac{1}{e} < 1. \tag{352}$$

By the ratio test, the series is $\boxed{\textit{absolutely convergent}}$. □

Example 11.66 *(Problem 11.6.14) Determine whether the series* $\sum\limits_{n=1}^{\infty}\frac{n!}{100^n}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{100^{n+1}}\frac{100^n}{n!} = \lim_{n\to\infty}\frac{n+1}{100} = \infty \tag{353}$$

By the ratio test, the series is $\boxed{\textit{divergent}}$. □

Example 11.67 *(Problem 11.6.8) Determine whether the series* $\sum\limits_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^2}\frac{n^2}{2^n} = \lim_{n\to\infty}\frac{2n^2}{n^2+2n+1} = 2 > 1. \tag{354}$$

By the ratio test, the series is $\boxed{\textit{divergent}}$. □

Theorem 11.68 *(Root test) Let* $\sum\limits_{n=1}^{\infty} a_n$ *be an infinite series.*
(i) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$*, then the series converges absolutely.*
(ii) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$*, then the series diverges.*

Again, notice that no conclusion can be drawn if $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$.

Example 11.69 *(Problem 10.6.32) Determine whether the series* $\sum\limits_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the root test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{1-n}{2+3n}\right| = \frac{1}{3} < 1. \tag{355}$$

Therefore, by the root test, the series is $\boxed{\textit{absolutely convergent}}$. $\square$

11.7 Strategy for testing series (?)

Example 11.70 *(Problem 11.7.2) Test the series* $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ *for convergence or divergence.*

We note that

$$\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}. \tag{356}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the series* $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ $\boxed{\textit{converges}}$ *by the comparison test.* $\square$

Example 11.71 *(Problem 11.7.4) Test the series* $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1. \tag{357}$$

Therefore, $\lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1}$ *does not exist. By the test for divergence, the series* $\boxed{\textit{diverges}}$. $\square$

Example 11.72 *(Problem 11.7.6) Test the series* $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{n^2}{(1+n)^3} = 0 < 1, \tag{358}$$

so by the root test, the series $\boxed{\textit{converges}}$. $\square$

Example 11.73 *(Problen 11.7.8) Test the series* $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{(n+1)^4}{4^{n+1}} \frac{4^n}{n^4} = \lim_{n\to\infty} \frac{(n+1)^4}{n^4} \frac{1}{4} = \frac{1}{4} < 1. \tag{359}$$

Thus, by the ratio test, the series $\boxed{\textit{converges}}$. $\square$

Example 11.74 *(Problem 11.7.12) Test the series* $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ *for convergence or divergence.*

We note that

$$\frac{1}{k\sqrt{k^2+1}} < \frac{1}{k\sqrt{k^2}} = \frac{1}{k^2}. \tag{360}$$

Also, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *is a convergent* p*-series. Thus, by the comparison test,* $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ *is* $\boxed{\textit{convergent}}$. □

Example 11.75 *(Problem 11.7.14) Test the series* $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ *for convergence or divergence.*

We consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{1+2^n} \right| = \sum_{n=1}^{\infty} \frac{|\sin(2n)|}{1+2^n}. \tag{361}$$

We note that

$$\frac{|\sin(2n)|}{1+2^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n. \tag{362}$$

However, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ *is a convergent geometric series. Thus, by the comparison test,* $\sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{1+2^n} \right|$ *is convergent. Ergo,* $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ *is absolutely convergent, which indicates that it is* $\boxed{\textit{convergent}}$. □

Example 11.76 *(Problem 11.7.16) Test the series* $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ *for convergence or divergence.*

We note that

$$\frac{\sqrt{n^4+1}}{n^3+n} > \frac{\sqrt{n^4}}{n^3+n} = \frac{n^2}{n^3+n} = \frac{n}{n^2+1}. \tag{363}$$

Now, we use the limit comparison test between $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *and* $\sum_{n=1}^{\infty} \frac{1}{n}$*:*

$$\lim_{n\to\infty} \frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1. \tag{364}$$

Since this is positive and finite, the limit comparison test indicates that $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *and*

$\sum_{n=1}^{\infty} \frac{1}{n}$ *must have the same behavior. As the harmonic series is divergent,* $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *is also divergent. The comparison test now implies that* $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ *is* $\boxed{\textit{divergent}}$. $\square$

Example 11.77 *(Problem 11.7.18) Test the series* $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ *for convergence or divergence.*

We note that

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n+1}-1} \tag{365}$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0. \tag{366}$$

Thus, by the alternating series test, the series $\boxed{\textit{converges}}$. $\square$

Example 11.78 *(Problem 11.7.20) Test the series* $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}$ *for convergence or divergence.*

We note that

$$\frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)} < \frac{\sqrt[3]{k}}{k\left(\sqrt{k}+1\right)} < \frac{\sqrt[3]{k}}{k\sqrt{k}} = \frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}} = \frac{1}{k^{\frac{7}{6}}}. \tag{367}$$

However, $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{7}{6}}}$ *is a convergent* p*-series. Thus, by the comparison test,* $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}$ *is* $\boxed{\textit{convergent}}$. $\square$

Example 11.79 *(Problem 11.7.22) Test the series* $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ *for convergence or divergence.*

We note that $1 \le 2 + \sin k \le 3$, *so*

$$\frac{1}{3} \le \frac{1}{2+\sin k}. \tag{368}$$

However, $\sum_{k=1}^{\infty} \frac{1}{3}$ *is divergent, by the test for divergence. Thus, the comparison test indicates that* $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$ *is also* $\boxed{\textit{divergent}}$. $\square$

Example 11.80 *(Problem 11.7.24) Test the series* $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}. \tag{369}$$

This is an indeterminate form of type $\frac{0}{0}$*, so we can use L'Hopital's rule to get*

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\left(-\frac{1}{n^2}\right)\cos\left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos 0 = 1. \tag{370}$$

Therefore, by the test for divergence, the series is $\boxed{\textit{divergent}}$. □

Example 11.81 *(Problem 11.7.26) Test the series* $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ *for convergence or divergence.*

We use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2+1}{5^{n+1}} \frac{5^n}{n^2+1}\right| = \lim_{n\to\infty} \frac{n^2+2n+2}{5n^2+5} = \frac{1}{5} < 1. \tag{371}$$

By the ratio test, the series is absolutely convergent, and therefore $\boxed{\textit{convergent}}$. □

Example 11.82 *(Problem 11.7.30) Test the series* $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$ *for convergence or divergence.*

We note that this is an alternating series. First of all, to show that $b_j = \frac{\sqrt{j}}{j+5}$ *is an eventually decreasing sequence, we consider the function* $f(x) = \frac{\sqrt{x}}{x+5}$*. We find its derivative by the quotient rule:*

$$f'(x) = \frac{(x+5)\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}}{(x+5)^2}. \tag{372}$$

This is negative when $x > 5$*, so* f *is decreasing on the interval* $(5, \infty)$*, which* *indicates that b_j is eventually decreasing. At the same time,*

$$\lim_{j\to\infty} b_j = \lim_{j\to\infty} \frac{\sqrt{j}}{j+5} = \lim_{j\to\infty} \frac{\left(\frac{1}{\sqrt{j}}\right)}{1+\frac{1}{j}} = 0. \tag{373}$$

Thus, by the alternating series test, the series is $\boxed{\textit{convergent}}$. $\square$

Example 11.83 *(Problem 11.7.34) Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ for convergence or divergence.*

We note that $1+\cos^2 n \leq 2$. Therefore,

$$\frac{1}{n+n\cos^2 n} = \frac{1}{n\left(1+\cos^2 n\right)} \geq \frac{1}{2n}. \tag{374}$$

At the same time, $\sum\limits_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum\limits_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series. By the comparison test, this indicates that $\sum\limits_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ is also $\boxed{\textit{divergent}}$. $\square$

11.8 Power series

So far, we've only dealt with infinite series where all terms were constants. Now, we'll discuss functions defined by infinite series.

Definition 11.84 *Let a be a real value. A* <u>*power series centered at a*</u> *is a function f defined via $f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ for some sequence of real numbers (c_n).*

The domain of a power series is the set of all x-values at which $f(x)$ is convergent. The following theorem indicates that the domain of a power series is always a single point, an interval, or all real numbers.

Theorem 11.85 *Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series. Exactly one of the following statements is true.*
(i) $f(x)$ converges only when $x = a$.
(ii) $f(x)$ converges for all real values x.
(iii) There exists a positive real number R (called the "radius of convergence") such that $f(x)$ converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(If $f(x)$ converges for only one point, we say that "the radius of convergence is 0". If $f(x)$ converges for all real x, we say that "the radius of convergence is ∞.")

Definition 11.86 *Let $f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n$ be a power series. The* <u>*interval of convergence of f*</u> *is the interval of real numbers x such that $f(x)$ converges (in other words, the domain of f).*

To find the interval of convergence, use the ratio test (or root test) to determine the x-values for which $f(x)$ converges absolutely. This will give the radius of convergence, and some open interval of values of x. Next, test the endpoints of this interval.

Example 11.87 *Find the radius of convergence and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} x^n$.*

We note that the series converges absolutely when the ratio test gives a limit less than 1*:*

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+3}\frac{n+2}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+3}|x| = |x| < 1. \tag{375}$$

This gives us a radius of convergence $\boxed{R=1}$*. We deduce that the power series converges for* $-1 < x < 1$*. What about for* $x = \pm 1$*? We check these separately. First, for* $x = -1$*:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}(-1)^n = \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{n+2} = \sum_{n=0}^{\infty}\frac{1}{n+2}. \tag{376}$$

This series is

$$\sum_{n=0}^{\infty}\frac{1}{n+2} = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+... = \left(\sum_{n=1}^{\infty}\frac{1}{n}\right) - 1, \tag{377}$$

which diverges, since $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ *diverges. Next, for* $x = 1$*:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}1^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}. \tag{378}$$

This converges by the alternating series test. Therefore, the power series converges only for $-1 < x \leq 1$*, and so the interval of convergence is* $\boxed{(-1,1]}$*.* $\square$

Example 11.88 *Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=0}^{\infty}\frac{(x-2)^n}{2^n}$*.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{2^{n+1}}\frac{2^n}{(x-2)^n}\right| = \frac{|x-2|}{2} < 1. \tag{379}$$

This tells us that $|x-2| < 2$*, so the radius of convergence is* $\boxed{R=2}$*. We deduce* *that the power series converges for $0 < x < 4$. We test $x = 0$:*

$$\sum_{n=0}^{\infty} \frac{(0-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n. \tag{380}$$

This is divergent, by the test for divergence. We test $x = 4$:

$$\sum_{n=0}^{\infty} \frac{(4-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1, \tag{381}$$

which is also divergent, by the test for divergence. Therefore, the power series converges only for $0 < x < 4$, and so the interval of convergence is $\boxed{(0,4)}$. $\square$

Example 11.89 *(Problem 11.8.4) Find the radius of convergence and interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$.*

We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{\sqrt[3]{n+1}} \frac{\sqrt[3]{n}}{(-1)^n x^n} \right| = \lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} |x| = |x| < 1 \tag{382}$$

The radius of convergence is $\boxed{R = 1}$. We deduce that the power series converges for $-1 < x < 1$. We test $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}. \tag{383}$$

This is a divergent p-series. We test $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}. \tag{384}$$

This is convergent, by the alternating series test. Therefore, the power series converges only for $-1 < x \leq 1$, and so the interval of convergence is $\boxed{(-1,1]}$. $\square$

Example 11.90 *(Problem 11.8.6) Find the radius of convergence and interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}\frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1}|x| = |x| < 1. \tag{385}$$

The radius of convergence is $\boxed{R=1}$*. We deduce that the power series converges for* $-1 < x < 1$*. We test* $x=-1$*:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}. \tag{386}$$

This is a convergent p*-series. We test* $x=1$*:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n 1^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}. \tag{387}$$

This converges by the alternating series test. Therefore, the power series converges only for $-1 \le x \le 1$*, and so the interval of convergence is* $\boxed{[-1,1]}$*.* $\square$

Example 11.91 *(Problem 11.8.8) Find the radius and interval of convergence of the power series* $\sum\limits_{n=1}^{\infty} n^n x^n$*.*

This one is easier to handle by the root test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|n^n x^n|} = \lim_{n\to\infty}(n^n|x|^n)^{\frac{1}{n}} = \lim_{n\to\infty} n|x|. \tag{388}$$

This limit will always be ∞ *unless* $x=0$*. In that case, and no other, convergence is guaranteed. Thus, the radius of convergence is* $\boxed{0}$ *and the interval of convergence is* $\boxed{\{0\}}$*.* $\square$

Example 11.92 *(Problem 11.8.10) Find the radius and interval of convergence of the power series* $\sum\limits_{n=1}^{\infty} 2^n n^2 x^n$*.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)^2 x^{n+1}}{2^n n^2 x^n}\right| = \lim_{n\to\infty} 2\frac{(n+1)^2}{n^2}|x| = 2|x| < 1. \tag{389}$$

This tells us that $|x| < \frac{1}{2}$, so the radius of convergence is $\boxed{R = \frac{1}{2}}$. We deduce that the power series converges absolutely for $-\frac{1}{2} < x < \frac{1}{2}$. We test $x = -\frac{1}{2}$:

$$\sum_{n=1}^{\infty} 2^n n^2 \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n n^2. \tag{390}$$

This is divergent, by the test for divergence. We test $x = \frac{1}{2}$:

$$\sum_{n=1}^{\infty} 2^n n^2 \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n^2. \tag{391}$$

This is divergent, also by the test for divergence. Therefore, the power series converges only for $-\frac{1}{2} < x < \frac{1}{2}$, and so its interval of convergence is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$. □

Example 11.93 *(Problem 11.8.12) Find the radius and interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$.*

We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)5^{n+1}} \frac{n5^n}{(-1)^{n-1} x^n} \right| = \lim_{n\to\infty} \frac{n}{5(n+1)}|x| = \frac{1}{5}|x| < 1. \tag{392}$$

This gives $|x| < 5$, and so the radius of convergence is $\boxed{R = 5}$. The ratio test guarantees absolute convergence in the interval $-5 < x < 5$. As for the endpoints, we test $x = -5$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} (-5)^n = \sum_{n=1}^{\infty} \frac{-1}{n}. \tag{393}$$

This is a constant times the harmonic series, and so must be divergent. As for $x = 5$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} 5^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}. \tag{394}$$

This is the alternating harmonic series, which is convergent, by the alternating series test. Thus, the power series converges only when $-5 < x \leq 5$, and so the interval of convergence is $\boxed{(-5, 5]}$. □

Example 11.94 *(Problem 11.8.14) Find the radius and interval of convergence of* *the power series* $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$.

We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \frac{n!}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{n+1} = 0 <!. \tag{395}$$

We see that, regardless of the actual value of x, the ratio test guarantees absolute convergence. Thus, the radius of convergence is $\boxed{R = \infty}$ *and the interval of convergence is* $\boxed{\text{all real numbers, } (-\infty, \infty)}$. □

Example 11.95 *(Problem 11.8.16) Find the radius and interval of convergence of the power series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$.

We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(2n+1)\,2^{n+1}} \frac{(2n-1)\,2^n}{(-1)^n(x-1)^n} \right| = \lim_{n\to\infty} \frac{2n-1}{2\,(2n+1)}|x-1| = \frac{1}{2}|x-1| < 1. \tag{396}$$

This gives $|x-1| < 2$, and so the radius of convergence is $\boxed{R = 2}$*, and the ratio test guarantees absolute convergence for $-1 < x < 3$. For the endpoints, we test $x = -1$:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\,2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{1}{2n-1}. \tag{397}$$

This can be shown to be divergent, by limit comparison with the harmonic series. For $x = 3$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\,2^n}2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}. \tag{398}$$

By the alternating series test, this is convergent. Ergo, we have convergence only for $-1 < x \le 3$, and so the interval of convergence is $\boxed{(-1, 3]}$. □

Example 11.96 *(Problem 11.8.18) Find the radius and interval of convergence of the power series* $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$.

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{\sqrt{n+1}(x+6)^{n+1}}{8^{n+1}}\frac{8^n}{\sqrt{n}(x+6)^n}\right| = \lim_{n\to\infty}\frac{1}{8}\frac{\sqrt{n+1}}{\sqrt{n}}|x+6| = \frac{1}{8}|x+6| < 1. \tag{399}$$

This implies that $|x+6|<8$*, and so the radius of convergence is* $\boxed{R=8}$*. Ergo, the ratio test guarantees absolute convergence for* $-14<x<2$*. For the endpoints, we test* $x=-14$*:*

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(-8)^n = \sum_{n=1}^{\infty}(-1)^n\sqrt{n}. \tag{400}$$

This is divergent, by the test for divergence. As for $x=2$*:*

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}8^n = \sum_{n=1}^{\infty}\sqrt{n}. \tag{401}$$

This is also divergent, by the test for divergence. We deduce that convergence occurs only for $-14<x<2$*, and so the interval of convergence is* $\boxed{(-14,2)}$*.* $\square$

Example 11.97 *(Problem 11.8.20) Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}}$*.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{\sqrt{n}}{5\sqrt{n+1}}|2x-1| = \frac{1}{5}|2x-1| < 1. \tag{402}$$

This gives $|2x-1|<5$*, so* $\left|x-\frac{1}{2}\right|<\frac{5}{2}$*, so the radius of convergence is* $\boxed{R=\frac{5}{2}}$*. We deduce that the power series converges for* $-\frac{5}{2}<x-\frac{1}{2}<\frac{5}{2}$*, or in other words,* $-2<x<3$*.*

We test $x=-2$*:*

$$\sum_{n=1}^{\infty}\frac{(2(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}. \tag{403}$$

This converges by the alternating series test. We test $x = 3$:

$$\sum_{n=1}^{\infty} \frac{2\,(3) - 1^{n}}{5^{n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{5^{n}}{5^{n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}. \tag{404}$$

This is a divergent p-series. Thus, the power series converges only for $-2 \leq x < 3$, and so the interval of convergence is $\boxed{[-2, 3)}$. $\square$

11.9 Representations of functions as power series

Certain functions can be expressed as power series.

Recall that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $-1 < r < 1$. Therefore,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \tag{405}$$

whose interval of convergence is $(-1, 1)$.

Example 11.98 *(Problem 11.9.4) Find a power series representation of the function $f(x) = \frac{5}{1-4x^2}$ and determine the interval of convergence.*

We note that

$$\frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty} \left(4x^2\right)^n = \boxed{\sum_{n=0}^{\infty} 5\left(4^n\right) x^{2n}}. \tag{406}$$

As for the interval of convergence: the geometric series $\sum_{n=0}^{\infty} (4x^2)^n$ converges if and only if $|4x^2| < 1$. This means that $x^2 < \frac{1}{4}$, so $-\frac{1}{2} < x < \frac{1}{2}$. The interval of convergence is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$. $\square$

Example 11.99 *(Problem 11.9.6) Find a power series representation of the function $f(x) = \frac{4}{2x+3}$ and determine the interval of convergence.*

First, we may re-write this function as

$$f(x) = \frac{\left(\frac{4}{3}\right)}{\frac{2}{3}x + 1} = \frac{\left(\frac{4}{3}\right)}{1 - \left(-\frac{2}{3}x\right)} = \frac{4}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{4}{3}\left(-\frac{2}{3}x\right)^n}. \tag{407}$$

This is convergent whenever $\left|-\frac{2}{3}x\right| < 1$, or, in other words, whenever $|x| < \frac{3}{2}$, so $-\frac{3}{2} < x < \frac{3}{2}$. The interval of convergence is, therefore, $\boxed{\left(-\frac{3}{2}, \frac{3}{2}\right)}$. $\square$

Example 11.100 *(Problem 11.9.8) Find a power series representation of the function $f(x) = \frac{x}{2x^2+1}$ and determine the interval of convergence.*

We note that

$$\frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty}\left(-2x^2\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}}. \quad (408)$$

As for the interval of convergence: the geometric series $\sum\limits_{n=0}^{\infty}\left(-2x^2\right)^n$ *converges if and only if* $|-2x^2| < 1$, *or in other words if* $x^2 < \frac{1}{2}$. *Ergo,* $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, *so the interval of convergence is* $\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$. □

Theorem 11.101 *Let* $f(x) = \sum\limits_{n=0}^{\infty} c_n(x-a)^n$ *be a power series with a radius of convergence* $R > 0$.
(i) This function is differentiable on $(a-R, a+R)$ *and*

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}, \quad (409)$$

with radius of convergence R.
(ii) This function is integrable on $(a-R, a+R)$ *and*

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}, \quad (410)$$

with radius of convergence R.

Example 11.102 *Find a power series representation of* $f(x) = \frac{1}{x^2-2x+1}$ *and its radius of convergence.*

We note that

$$f(x) = \frac{1}{x^2-2x+1} = \frac{1}{(1-x)^2}. \quad (411)$$

This function happens to be the derivative of

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (412)$$

which has a radius of convergence of 1*. By the theorem,*

$$f(x) = g'(x) = \boxed{\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)\, x^n}, \tag{413}$$

with radius of convergence $\boxed{1}$. □

Example 11.103 *Find a power series representation for* $f(x) = \tan^{-1}x$ *and find its radius of convergence.*

We know that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = (-1)^n x^{2n}. \tag{414}$$

This is convergent if and only if $|-x^2| < 1$*, so* $|x| < 1$ *and thus the radius of convergence is* $\boxed{R = 1}$*. Further,*

$$\tan^{-1}x = \int f'(x)\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{415}$$

To find the value of C*, we substitute* $x = 0$*:*

$$C = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = \tan^{-1}0 = 0. \tag{416}$$

Thus,

$$\boxed{\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}. \tag{417}$$

□

Example 11.104 *Find a power series representation for* $f(x) = \ln(1+x)$ *and its radius of convergence.*

We know that

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n. \tag{418}$$

This is convergent if and only if $|-x| < 1$*, so* $|x| < 1$ *and thus the radius of convergence is* $\boxed{R = 1}$*. Further,*

$$\ln(1+x) = \int f'(x)\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-1)^n x^n\,\mathrm{d}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \tag{419}$$

To find C*, we substitute* $x = 0$*:*

$$C = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1} = \ln(1+0) = \ln 1 = 0. \tag{420}$$

Therefore,

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}. \tag{421}$$

□

In some cases, you can approximate a function's values or definite integrals using the alternating series estimation theorem.

Example 11.105 *Find an approximation for* $\ln\left(\frac{3}{2}\right)$ *that is correct to two decimal places.*

The previous example indicates that

$$\ln\left(\frac{3}{2}\right) = \ln\left(1 + \frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)\,2^{n+1}}. \tag{422}$$

Therefore, if we desire to find an approximation of $\ln\left(\frac{3}{2}\right)$*, then it is sufficient to get an approximation of the sum of this series. As it is, this is an alternating series which passes the alternating series test, so we can apply the alternating series estimation theorem.*

We require that the error in our approximation not affect the first four decimal places. In other words, the error must be less than $0.01 = \frac{1}{100}$*. To accomplish this,* *we examine the first few terms of the sequence* $b_n = \frac{1}{(n+1)2^{n+1}}$*:*

$$\begin{aligned}
b_0 &= \tfrac{1}{(1)2^1} = \tfrac{1}{2} \\
b_1 &= \tfrac{1}{(2)2^2} = \tfrac{1}{8} \\
b_2 &= \tfrac{1}{(3)2^3} = \tfrac{1}{24} \\
b_3 &= \tfrac{1}{(4)2^4} = \tfrac{1}{64} \\
b_4 &= \tfrac{1}{(5)2^5} = \tfrac{1}{160}
\end{aligned} \tag{423}$$

The alternating series estimation theorem indicates that the error, R_3*, in taking the third partial sum satisfies* $|R_3| \leq b_4 = \frac{1}{160} < \frac{1}{100}$*. Therefore, the following is an approximation of* $\ln\left(\frac{3}{2}\right)$ *with an error in acceptable bounds:*

$$\ln\left(\frac{3}{2}\right) \approx \boxed{\tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64}} \tag{424}$$

□

11.10 Taylor and Maclaurin series

There is a flaw in the Lagrange notation for derivatives. Suppose we desire to take the nineteenth derivative of a function f. We would need to write

$$f'''''''''''''''''''(x)\,. \tag{425}$$

This is an eyesore. In order to avoid this absurdity, we use a different notation for derivatives that could be of orders larger than 3: parentheses around a superscript. For example, the nineteenth derivative of f would be $f^{(19)}(x)$.

Definition 11.106 *Let f be a function that is differentiable of all orders, and let a be a real number. The Taylor series of f centered at $x = a$ is the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{426}$$

The Maclaurin series of f is the Taylor series of f centered at $x = 0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{427}$$

(Note here that $f^{(0)} = f$ and $0! = 1$.)

Theorem 11.107 *If f has a power series representation about $x = a$, then f is equal to its Taylor series about $x = a$.*

Note that the previous theorem depends on f being able to be represented by a power series about $x = a$ in the first place. This is not always the case.

Example 11.108 *Find the Maclaurin series for $f(x) = e^x$, and its radius of convergence.*

We note that for any positive integer n, $f^{(n)}(x) = e^x$. Therefore, $f^{(n)}(0) = 1$. Ergo, the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}. \tag{428}$$

The radius of convergence can be found through the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0. \tag{429}$$

As this is less than 1 *for every real value of* x*, the radius of convergence is* $\boxed{\infty}$. $\square$

Example 11.109 *(Problem 11.10.22) Find the Taylor series for* $f(x) = \frac{1}{x}$ *centered at* $a = -3$*, and its associated radius of convergence.*

We note that

$$\begin{array}{rcl} f(x) = x^{-1} & & f(-3) = -\frac{1}{3} \\ f'(x) = -x^{-2} & & f'(-3) = -\frac{1}{9} \\ f''(x) = 2x^{-3} & & f''(-3) = -\frac{2}{27} \\ f^{(3)}(x) = -6x^{-4} & & f^{(3)}(-3) = -\frac{6}{81} \\ \vdots & & \vdots \\ & & f^{(n)}(-3) = -\frac{n!}{3^{n+1}} \end{array} \tag{430}$$

Thus, the Taylor series about $a = -3$ *is*

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} -\frac{1}{n!}\frac{n!}{3^{n+1}}(x+3)^n = \boxed{\sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}}. \tag{431}$$

As for the radius of convergence, we use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{3^{n+2}}\frac{3^{n+1}}{(x+3)^n}\right| = \lim_{n\to\infty}\frac{|x+3|}{3} = \frac{|x+3|}{3}. \tag{432}$$

This is less than 1 *exactly when* $|x+3| < 3$*, and so the radius of convergence is* $\boxed{3}$.
$\square$

Example 11.110 *(Problem 11.10.14) Find the Maclaurin series for* $f(x) = e^{-2x}$ *and its radius of convergence.*

We note that

$$\begin{array}{cc} f^{(0)}(x) = e^{-2x} & f^{(0)}(0) = 1 \\ f^{(1)}(x) = -2e^{-2x} & f^{(1)}(0) = -2 \\ f^{(2)}(x) = 2^2 e^{-2x} & f^{(2)}(0) = 2^2 \\ f^{(3)}(x) = -2^3 e^{-2x} & f^{(3)}(0) = -2^3 \\ f^{(4)}(x) = 2^4 e^{-2x} & f^{(4)}(0) = 2^4 \\ \vdots & \vdots \\ f^{(n)}(x) = (-1)^n 2^n e^{-2x} & f^{(n)}(0) = (-1)^n 2^n \end{array}. \tag{433}$$

Thus, the Maclaurin series is

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n}. \tag{434}$$

Its radius of convergence can be found through the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{n+1} x^{n+1}}{(n+1)!} \frac{n!}{(-1)^n 2^n x^n} \right| = \lim_{n\to\infty} \frac{2}{n+1} |x| = 0. \tag{435}$$

As this is less than 1 *for all real values of* x*, the radius of convergence is* $\boxed{\infty}$. □

Example 11.111 *Find the Maclaurin series for* $f(x) = \tan^{-1} x$ *and its radius of convergence.*

We note that

$$\begin{array}{cc} f^{(0)}(x) = \tan^{-1} x & f^{(0)}(0) = 0 \\ f^{(1)}(x) = \frac{1}{x^2+1} & f^{(1)}(0) = 1 \\ f^{(2)}(x) = \frac{-2x}{(x^2+1)^2} & f^{(2)}(0) = 0 \\ f^{(3)}(x) = \frac{6x^2-2}{(x^2+1)^3} & f^{(3)}(0) = -2 \\ f^{(4)}(x) = \frac{24x\left(1-x^2\right)}{(x^2+1)^4} & f^{(4)}(0) = 0 \\ f^{(5)}(x) = \frac{120x^4-240x^2+24}{(x^2+1)^5} & f^{(5)}(0) = 24 \end{array} \tag{436}$$

The pattern is harder to see here, but it turns out that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}(n-1)! & \text{if } n \text{ is odd} \end{cases}. \tag{437}$$

Therefore, the Maclaurin series is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}_{\textit{for even } n} + \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}_{\textit{for odd } n} \\ &= \underbrace{\sum_{n=0}^{\infty} \frac{0}{n!} x^n}_{\textit{for even } n} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n-1}{2}}(n-1)!}{n!} x^n}_{\textit{for odd } n} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n} x^n}_{\textit{for odd } n}. \end{aligned} \tag{438}$$

Now, n is odd if and only if there exists an integer k such that $n = 2k+1$, so this can be re-written as

$$\tan^{-1} x = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}}, \tag{439}$$

which coincides with the power series representation of $\tan^{-1} x$ that we found in Example 11.103. The radius of convergence can be found through the ratio test:

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2(k+1)+1}}{2(k+1)+1} \frac{2k+1}{(-1)^k x^{2k+1}} \right| = \lim_{k\to\infty} \frac{2k+1}{2k+3} x^2 = x^2. \tag{440}$$

To guarantee convergence by the ratio test, we must have $x^2 < 1$, or in other words, $|x| < 1$. Therefore, the radius of convergence is $\boxed{1}$. $\square$

Example 11.112 *(Problem 11.10.48) Find the Maclaurin series for the function $f(x) = \tan^{-1}(x^2)$ and its radius of convergence.*

By the previous example, the Maclaurin series would be

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(x^2\right)^{2k+1} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{4k+2}}. \tag{441}$$

The radius of convergence can be found either by recognizing that we must have $|x^2| < 1$, or by the ratio test. Either way, the radius of convergence is $\boxed{1}$. $\square$

Example 11.113 *Let k be a real number. Find the Maclaurin series for the function $f(x) = (1+x)^k$, and find its associated radius of convergence.*

We note that

$$\begin{aligned} f(x) &= (1+x)^k & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k\,(k-1) \\ f^{(3)}(x) &= k(k-1)(k-2)(1+x)^{k-3} & f^{(3)}(0) &= k(k-1)(k-2) \\ &\vdots & &\vdots \\ f^{(n)}(x) &= k(k-1)(k-2)...(k-n+1)(1+x)^{k-n} & f^{(n)} &= k(k-1)(k-2)...(k-n+1) \end{aligned} . \tag{442}$$

Therefore, the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0) = \boxed{\sum_{n=0}^{\infty} \frac{k(k-1)(k-2)...(k-n+1)}{n!} x^n} = \sum_{n=0}^{\infty} \binom{k}{n} x^n. \tag{443}$$

(This particular series is called the binomial series.) To find the radius of convergence, we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{k\,(k-1)\,...\,(k-n+1)\,(k-n)\,x^{n+1}}{(n+1)!} \frac{n!}{k\,(k-1)\,...\,(k-n+1)\,x^n} \right| \\ = \lim_{n\to\infty} \left| \frac{k-n}{n+1} x \right| = |x|. \tag{444}$$

The ratio test indicates absolute convergence if $|x| < 1$ and divergence if $|x| > 1$, so the radius of convergence is $\boxed{1}$. $\square$

Example 11.114 *Find the Maclaurin series for $f(x) = \sin x$, and find its associated radius of convergence.*

We note that

$$\begin{array}{rr} f(x)=\sin x & f(0)=0 \\ f'(x)=\cos x & f'(0)=1 \\ f''(x)=-\sin x & f''(0)=0 \\ f^{(3)}(x)=-\cos x & f^{(3)}(0)=-1 \\ f^{(4)}(x)=\sin x & f^{(4)}(0)=0 \end{array}. \tag{445}$$

Again, the pattern is a bit difficult to write, but it turns out to be

$$f^{(n)}(0)=\begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}. \tag{446}$$

Therefore, the Maclaurin series is

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n &= \underbrace{\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}_{\text{for even } n}+\underbrace{\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}_{\text{for odd } n} \\ &= \underbrace{\sum_{n=0}^{\infty}\frac{0}{n!}x^n}_{\text{for even } n}+\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n!}x^n}_{\text{for odd } n}=\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n!}x^n}_{\text{for odd } n}. \end{aligned} \tag{447}$$

An integer n is odd if and only if $n = 2k+1$ for some appropriate integer k. Therefore, this can be re-written as

$$\boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}}. \tag{448}$$

As for the radius of convergence, we use the ratio test:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| &= \lim_{n\to\infty}\left|\frac{-x^{2n+3}}{x^{2n+1}}\frac{(2n+1)!}{(2n+3)!}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+3)(2n+2)}=0. \end{aligned} \tag{449}$$

Since this is less than 1 for all real values of x, the radius of convergence is $\boxed{\infty}$. $\square$

11.11 Applications of Taylor polynomials

If a function can be represented as a power series about $x = a$, then according to the theorem of the previous section, it will be equal to its own Taylor series. As always, it may be difficult to determine the sum of a Taylor series, so we may need to take a partial sum as an approximation.

Definition 11.115 *Let f be a function that can be represented by a power series centered at $x = a$ over some interval of convergence. Given a non-negative integer k, the kth degree Taylor polynomial of f at a is $T_k = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$.*

In other words, the kth degree Taylor polynomial of f at a is the $n = k$th partial sum of the Taylor series of f at a. This is a generalization of the linearization of a function. In fact, the 1st degree Taylor polynomial of a function at a is always the linearization of the function near a.

We will not be concerned with measuring the accuracy of these approximations in this class. If you are interested, the book has details on this topic in Sections 11.10 and 11.11.

Example 11.116 *Find the fourth-degree Taylor polynomial of $f(x) = e^x$ at 1.*

In the previous section, in Example 11.108, we determined that the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{450}$$

Thus, the fourth-degree Taylor polynomial is just the $n = 4$ partial sum:

$$\boxed{\sum_{n=0}^{4} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4}. \tag{451}$$

□

Example 11.117 *Find the second-degree Taylor polynomial of $f(x) = \sqrt[3]{x}$ at 8.*

We need two derivatives of f:

$$\begin{array}{rr} f^{(0)}(x) = x^{\frac{1}{3}} & f^{(0)}(8) = 2 \\ f^{(1)}(x) = \frac{1}{3}x^{-\frac{2}{3}} & f^{(1)}(8) = \frac{1}{12} \\ f^{(2)}(x) = -\frac{2}{9}x^{-\frac{5}{3}} & f^{(2)}(8) = -\frac{1}{144} \end{array}. \tag{452}$$

Thus, the second-degree Taylor polynomial of f at 8 is

$$\boxed{\sum_{n=0}^{2} \frac{f^{(n)}(8)}{n!}(x-8)^n = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2}. \tag{453}$$

□

Example 11.118 *(Problem 11.11.6) Find the third-degree Taylor polynomial of the function $f(x) = e^{-x}\sin x$ at 0.*

We need three derivatives of f:

$$\begin{array}{cc} f^{(0)}(x) = e^{-x}\sin x & f^{(0)}(0) = 0 \\ f^{(1)}(x) = e^{-x}(\cos x - \sin x) & f^{(1)}(0) = 1 \\ f^{(2)}(x) = -2e^{-x}\cos x & f^{(2)}(0) = -2 \\ f^{(3)}(x) = 2e^{-x}(\cos x + \sin x) & f^{(3)}(0) = 2 \end{array}. \tag{454}$$

Thus, the third-degree Taylor polynomial of f at 0 is

$$\boxed{\sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = x - x^2 + \frac{1}{3}x^3}. \tag{455}$$

□

8 Further applications of integration

8.1 Arc length

The length of a line segment is simply the distance between the two endpoints. However, this is a very specific situation. What if, instead, we have a curve that goes between two points? How would we find its length? We'll come back to this question a few times, but for now, we'll just concentrate on finding the lengths of curves that happen to be described by functions.

Definition 8.1 *Let f be a continuous function defined on an interval $[a, b]$. The arc length of f from a to b is the length of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$.*

The arc length of a continuous function f from $x = a$ to $x = b$ is given by the following formula:

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x. \tag{456}$$

The arc length function beginning at $x = a$ is defined as

$$s(x) = \int_a^x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t. \tag{457}$$

Example 8.2 *(Problem 8.1.2) Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$ from 0 to 1.*

We note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(2 - x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{2 - x^2}}. \tag{458}$$

Therefore,

$$\begin{aligned} s &= \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{2 - x^2}}\right)^2}\, \mathrm{d}x = \int_0^1 \sqrt{1 + \frac{x^2}{2 - x^2}}\, \mathrm{d}x \\ &= \int_0^1 \sqrt{\frac{2}{2 - x^2}}\, \mathrm{d}x = \int_0^1 \sqrt{\frac{1}{1 - \frac{x^2}{2}}}\, \mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1 - \left(\frac{x}{\sqrt{2}}\right)^2}}\, \mathrm{d}x. \end{aligned} \tag{459}$$

We now use a u-substitution:

$$\begin{array}{ll} u = \frac{x}{\sqrt{2}} & x = 0 \Rightarrow u = 0 \\ \mathrm{d}u = \frac{1}{\sqrt{2}}\,\mathrm{d}x & x = 1 \Rightarrow u = \frac{1}{\sqrt{2}} \end{array} \tag{460}$$

$$\begin{aligned} s = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{2}\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u &= \sqrt{2}\sin^{-1}u\Big|_0^{\frac{1}{\sqrt{2}}} \\ &= \sqrt{2}\left(\sin^1\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}(0)\right) = \boxed{\tfrac{\pi\sqrt{2}}{4}}. \end{aligned} \tag{461}$$

□

Example 8.3 *(Problem 8.1.12) Find the exact length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ over $1 \leq x \leq 2$.*

We note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}. \tag{462}$$

Therefore,

$$\begin{aligned} s &= \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2}\,\mathrm{d}x = \int_1^2 \sqrt{1 + \left(\frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}\right)}\,\mathrm{d}x \\ &= \int_1^2 \sqrt{\frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6}}\,\mathrm{d}x = \int_1^2 \sqrt{\frac{1}{4}x^{-6}\left(x^{12} + 2x^6 + 1\right)}\,\mathrm{d}x \\ &= \int_1^2 \sqrt{\frac{\left(x^6+1\right)^2}{4x^6}}\,\mathrm{d}x = \int_1^2 \frac{x^6+1}{2x^3}\,\mathrm{d}x = \frac{1}{2}\int_1^2 x^3 + x^{-3}\,\mathrm{d}x \\ &= \frac{1}{2}\left(\frac{1}{4}x^4 - \frac{1}{2}x^{-2}\Big|_1^2\right) = \frac{1}{2}\left(4 - \frac{1}{8}\right) - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2}\right) = \boxed{\tfrac{33}{16}}. \end{aligned} \tag{463}$$

□

Example 8.4 *(Problem 8.1.14) Find the exact length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{3}$.*

First, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan x$. *Now,*

$$\begin{aligned} s = \int_0^{\frac{\pi}{3}} \sqrt{1+(-\tan x)^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{3}} \sec x\,\mathrm{d}x = \ln\left|\sec x + \tan x\right|\Big|_0^{\frac{\pi}{3}} \\ &= \ln\left|\sec\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right| - \ln\left|\sec 0 + \tan 0\right| \\ &= \ln\left|2+\sqrt{3}\right| - \ln\left|1+0\right| = \boxed{\ln\left|2+\sqrt{3}\right|}. \end{aligned} \tag{464}$$

□

Example 8.5 *(Problem 8.1.18) Find the exact length of the curve*

$$y = \sqrt{x-x^2} + \sin^{-1}\left(\sqrt{x}\right). \tag{465}$$

First, we note that the problem is well-formed, despite not giving us an interval, because the function's entire domain is the finite interval $[0,1]$. *We note that*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(x-x^2\right)^{-\frac{1}{2}}(1-2x) + \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{1}{2}x^{-\frac{1}{2}} = \sqrt{\frac{1-x}{x}}. \tag{466}$$

Therefore,

$$s = \int_0^1 \sqrt{1+\left(\sqrt{\frac{1-x}{x}}\right)^2}\,\mathrm{d}x = \int_0^1 \sqrt{1+\frac{1-x}{x}}\,\mathrm{d}x = \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x. \tag{467}$$

At this point, we need to realize that this is an improper integral; $\frac{1}{\sqrt{x}}$ *is not continuous (or even defined) at* $x = 0$. *Thus,*

$$s = \lim_{a\to 0^+}\int_a^1 x^{-\frac{1}{2}}\,\mathrm{d}x = \lim_{a\to 0^+} 2x^{\frac{1}{2}}\Big|_a^1 = \lim_{a\to 0^+} 2 - 2\sqrt{a} = \boxed{2}. \tag{468}$$

□

Example 8.6 *(Problem 8.1.20) Find the exact length of the curve* $y = 1 - e^{-x}$ *over*

$0 \le x \le 2$.

We note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^{-x}. \tag{469}$$

Therefore,

$$s = \int_0^2 \sqrt{1 + \left(e^{-x}\right)^2}\ \mathrm{d}x. \tag{470}$$

We proceed by u*-substitution:*

$$\begin{aligned} u &= e^{-x} & x = 0 \Rightarrow u = 1 \\ \mathrm{d}u &= -e^{-x}\ \mathrm{d}x & x = 2 \Rightarrow u = e^{-2} \end{aligned} \tag{471}$$

$$s = -\int_1^{e^{-2}} \frac{\sqrt{1+u^2}}{u}\ \mathrm{d}u. \tag{472}$$

This is most easily done by trigonometric substitution:

$$\begin{aligned} u &= \tan\theta & u = 1 \Rightarrow \theta = \tfrac{\pi}{4} \\ \mathrm{d}u &= \sec^2\theta\ \mathrm{d}\theta & u = e^{-2} \Rightarrow \theta = \tan^{-1}\left(e^{-2}\right) \end{aligned} \tag{473}$$

$$\begin{aligned} s &= -\int_{\frac{\pi}{4}}^{\tan^{-1}\left(e^{-2}\right)} \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\ \mathrm{d}\theta = \int_{\tan^{-1}(e^{-2})}^{\frac{\pi}{2}} \frac{\sec^3\theta}{\tan\theta}\ \mathrm{d}\theta \\ &= \int_{\tan^{-1}(e^{-2})}^{\frac{\pi}{4}} \frac{\tan^2\theta + 1}{\tan\theta}\sec\theta\ \mathrm{d}\theta = \int_{\tan^{-1}(e^{-2})}^{\frac{\pi}{4}} \sec\theta\tan\theta + \csc\theta\ \mathrm{d}\theta. \end{aligned} \tag{474}$$

We know that $\sec\theta\tan\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta$*. As for* $\csc\theta$*, we can use a strategy analogous to that used in Example 7.8 to find that* $\int \csc\theta\ \mathrm{d}\theta = -\ln\left|\csc\theta + \cot\theta\right| + C$*. Therefore,*

$$\begin{aligned} s &= \sec\theta - \ln\left|\csc\theta + \cot\theta\right|\Big|_{\tan^{-1}(e^{-2})}^{\frac{\pi}{4}} \\ &= \boxed{\left(\sqrt{2} - \tfrac{\sqrt{e^4+1}}{e^2}\right) - \left(\ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{e^4+1} + e^2\right)\right)}. \end{aligned} \tag{475}$$

□

8.2 Option: Area of a surface of revolution

To find the surface area of a surface of revolution created by revolving $y = f(x)$ about the x-axis, we take a typical point on the curve:

[Draw diagram]

When revolved about the x-axis, this point forms a circle with circumference $2\pi f(x)$. To find the surface area, we integrate this circumference with respect to arc length:

$$A = \int_a^b 2\pi f(x)\, \mathrm{d}s. \tag{476}$$

Since $s(x) = \int_a^x \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$, we know, by the fundamental theorem of calculus, that $\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (f'(x))^2}$. Thus, $\mathrm{d}s = \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$, and so the surface area is given by:

$$A = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x. \tag{477}$$

Example 8.7 *Find the exact area of the surface obtained by rotating the curve $y = \sqrt{5 - x}$, for $3 \leq x \leq 5$, about the x-axis.*

First, we find that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{2\sqrt{5-x}}$. Thus, the surface area is

$$\begin{aligned} A = \int_3^5 2\pi \sqrt{5 - x} \sqrt{1 + \left(\frac{-1}{2\sqrt{5 - x}} \right)^2}\, \mathrm{d}x \\ = 2\pi \int_3^5 \sqrt{5 - x} \sqrt{1 + \frac{1}{4(5 - x)}}\, \mathrm{d}x \\ = 2\pi \int_3^5 \sqrt{5 - x + \frac{1}{4}}\, \mathrm{d}x = 2\pi \int_3^5 \sqrt{\frac{21}{4} - x}\, \mathrm{d}x. \end{aligned} \tag{478}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \tfrac{21}{4} - x \\ \mathrm{d}u &= -\, \mathrm{d}x \end{aligned} \tag{479}$$

The integral becomes

$$-2\pi \int_{\frac{9}{4}}^{\frac{1}{4}} u^{\frac{1}{2}} \, \mathrm{d}u = \frac{4}{3}\pi u^{\frac{3}{2}} \Big|_{\frac{1}{4}}^{\frac{9}{4}} = \boxed{\frac{13\pi}{4}}. \tag{480}$$

□

Example 8.8 *Find the area of the surface generated by rotating the curve $y = e^x$ for $0 \leq x \leq 1$ about the x-axis.*

First, we notice that $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$. Therefore,

$$A = \int_0^1 2\pi e^x \sqrt{1 + e^{2x}} \, \mathrm{d}x. \tag{481}$$

We proceed by u-substitution:

$$\begin{aligned} u &= e^x \\ \mathrm{d}u &= e^x \, \mathrm{d}x \end{aligned}. \tag{482}$$

The integral becomes

$$2\pi \int_1^e \sqrt{1 + u^2} \, \mathrm{d}u. \tag{483}$$

We now proceed by trigonometric substitution:

$$\begin{aligned} u &= \tan\theta \\ \mathrm{d}u &= \sec^2\theta \, \mathrm{d}\theta, \end{aligned} \tag{484}$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Define a real value α in this interval such that $\tan\alpha = e$. Now, the integral becomes

$$2\pi \int_{\frac{\pi}{4}}^{\alpha} \sqrt{1 + \tan^2\theta} \sec^2\theta \, \mathrm{d}\theta = 2\pi \int_{\frac{\pi}{4}}^{\alpha} \sec^3\theta \, \mathrm{d}\theta. \tag{485}$$

From a previous example, this is

$$\begin{aligned} &\pi \left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\right) \Big|_{\frac{\pi}{4}}^{\alpha} \\ &\quad = \pi \left(\sec\alpha \tan\alpha + \ln|\sec\alpha + \tan\alpha| - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right). \end{aligned} \tag{486}$$

Since $\tan\alpha = e$, $e^2+1 = \sec^2\alpha$ *and so* $\sqrt{e^2+1} = \sec\alpha$*:*

$$\boxed{\pi\left(e\sqrt{e^2+1}+\ln\left|e+\sqrt{e^2+1}\right|-\sqrt{2}-\ln\left(\sqrt{2}+1\right)\right)} \tag{487}$$

□

Example 8.9 *Find the surface area of a sphere with a radius of* R*.*

We can understand a sphere centered at the origin with a radius of R *as a revolution of the upper half of the circle* $x^2+y^2=R^2$ *about the* x*-axis. This upper half is given by* $y=\sqrt{R^2-x^2}$*. We compute*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(R^2-x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{R^2-x^2}}. \tag{488}$$

Now, the surface area is

$$\begin{aligned} A = \int_{-R}^{R} 2\pi\sqrt{R^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\,\mathrm{d}x \\ = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,\mathrm{d}x \\ = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2+x^2}\,\mathrm{d}x = 2\pi\int_{-R}^{R}R\,\mathrm{d}x = 2\pi R\int_{-R}^{R}\mathrm{d}x \\ = 2\pi R\left(R-(-R)\right) = 2\pi R\,(2R) = \boxed{4\pi R^2}. \end{aligned} \tag{489}$$

□

10 Parametric equations and polar coordinates

10.1 Curves defined by parametric equations

In the past, you've always defined curves as points (x, y) in the xy-plane which satisfy some kind of equation. (Typically, this was the graph of a function of the form $y = f(x)$.) This is called a "Cartesian description" of a curve. There is an alternative method to describe a curve: one can understand a curve as the path of a moving particle taken over a given time period. In this scheme, we describe the x- and y- coordinates of the particle's position separately as functions of a different variable, called the "parameter," which is usually denoted t, and often corresponds to time. These functions are called "parametric equations."

Example 10.1 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 4\sin t \\ y(t) &= 5\cos t \end{aligned} \tag{490}$$

for $0 \leq t \leq 2\pi$.

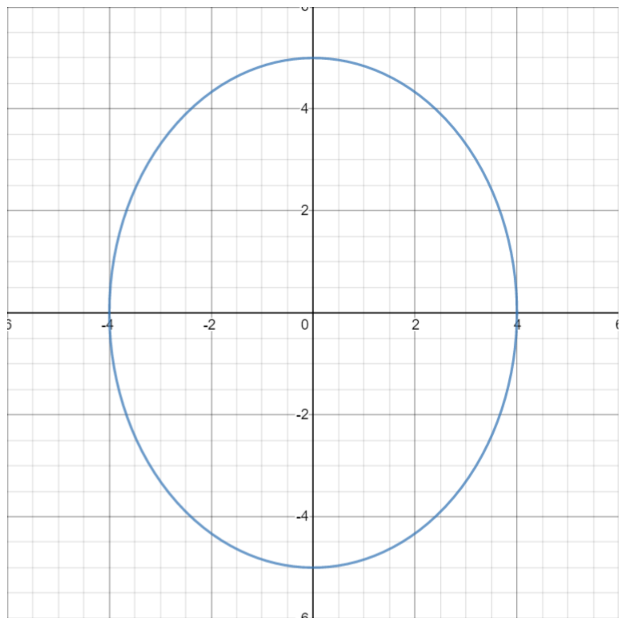
We can write $\frac{x}{4} = \sin t$ *and* $\frac{y}{5} = \cos t$*, so*

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \sin^2 t + \cos^2 t = 1. \tag{491}$$

Thus, the equation can be written in the Cartesian form as

$$\frac{x^2}{16} + \frac{y^2}{25} = 1. \tag{492}$$

(This process of coming up with an equation in terms of x and y but not t is often called "eliminating the parameter.") This is an ellipse with x-intercepts $x = \pm 4$ and y-intercepts $y = \pm 5$:



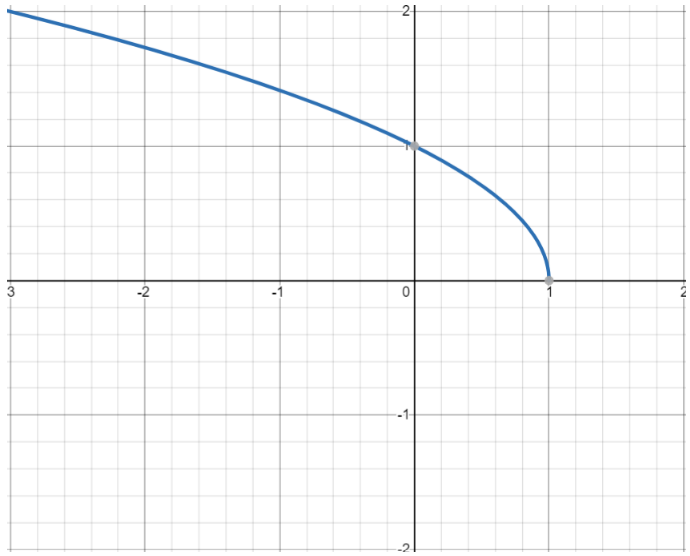
The parameter t begins at 0, corresponding to the point $(0, 5)$. As t increases, the x-coordinate increases and the y-coordinate decreases, creating a clockwise motion. $\square$

Example 10.2 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 1 - t \\ y(t) &= \sqrt{t} \end{aligned} \tag{493}$$

for $t \geq 0$.

We notice that $t = 1 - x$, so $y = \sqrt{1 - x}$. This is a half of a parabola:



The parameter t begins at 0, corresponding to the point $(1, 0)$. As t increases, the x-coordinate decreases while the y-coordinate increases, creating the upper half of the parabola. □

Example 10.3 *Given a curve which is the graph of a function $y = f(x)$ defined on the entire real line, we can describe the curve parametrically as*

$$\begin{aligned} x(t) &= t \\ y(t) &= f(t) \end{aligned} \tag{494}$$

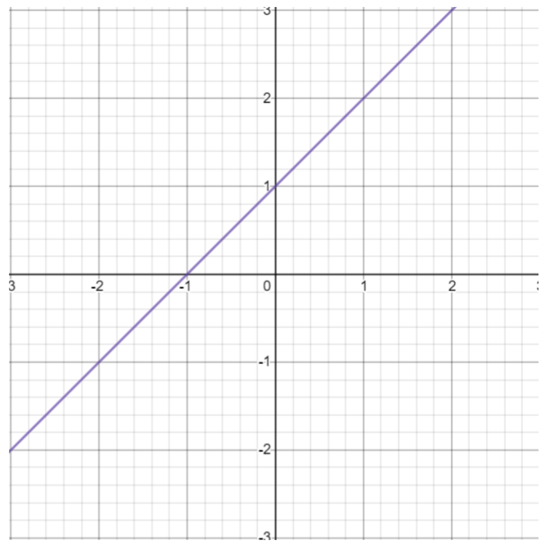
for all real values of t. □

Example 10.4 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 5t + 3 \\ y(t) &= 5t + 4 \end{aligned} \tag{495}$$

for all real values of t.

We notice that $x - 3 = 5t = y - 4$, so $y = x + 1$. This is a line:



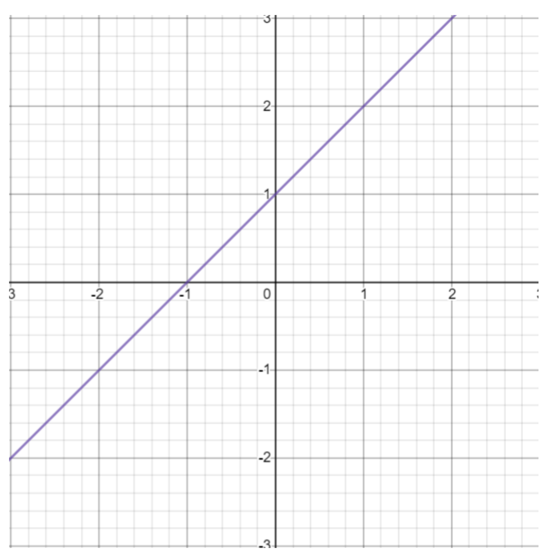
□

Example 10.5 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 3 - \tfrac{1}{3}t \\ y(t) &= 4 - \tfrac{1}{3}t \end{aligned} \tag{496}$$

for all real values of t*.*

We notice that $3 - x = \frac{1}{3}t = 4 - y$*, so* $y = x + 1$*. This is a line, in fact the same one as in the previous example:*



This example illustrates that the parametric description of a given curve is not unique; there could be multiple parametric descriptions that all define the same curve. □

Example 10.6 *Describe the curve given by the following parametric equations:*

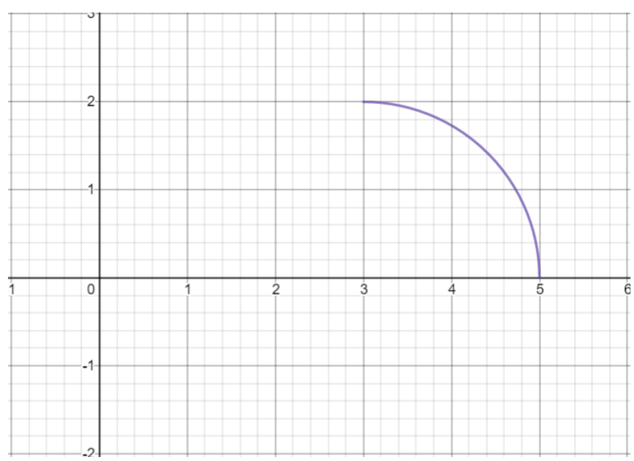
$$\begin{aligned} x(t) &= 3 + 2\cos t \\ y(t) &= 2\sin t \end{aligned} \tag{497}$$

for $0 \leq t \leq \frac{\pi}{2}$*.*

We note that

$$(x-3)^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4. \tag{498}$$

This is a portion of the circle $(x-3)^2 + y^2 = 4$*:*



This is not the full circle $(x-3)^2 + y^2 = 4$*, but rather only the portion of the circle over which* $x = 3 + 2\cos t$ *is between* 3 *and* 4*, and* $y = 2\sin t$ *is between* 0 *and* 1*.*

The parameter t *begins at* 0*, describing the point* $(5, 0)$*. As* t *increases, the* x*-coordinate decreases while the* y*-coordinate increases, creating a counterclockwise motion.*

This example illustrates that the Cartesian description of a curve can often fail to specify information that is present in a parametric description. □

Example 10.7 *(10.1.8) Describe the curve given by the parametric equations*

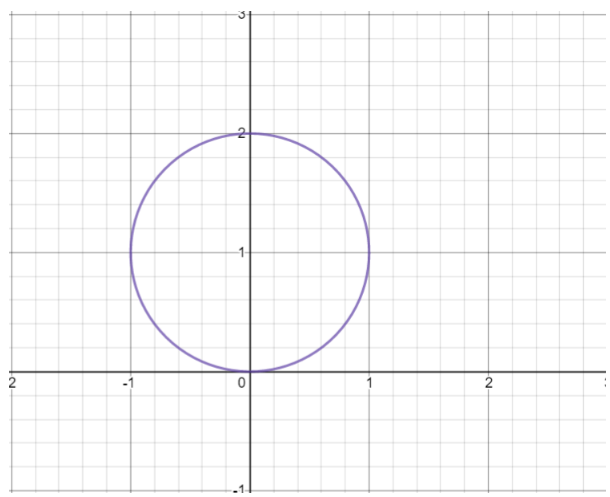
$$\begin{aligned} x(t) &= \sin t \\ y(t) &= 1 - \cos t \end{aligned} \tag{499}$$

for $0 \leq t \leq 2\pi$.

We note that

$$x^2 + (1-y)^2 = \sin^2 t + \cos^2 t = 1. \tag{500}$$

This describes a circle of radius 1 *centered at* $(0, 1)$*:*



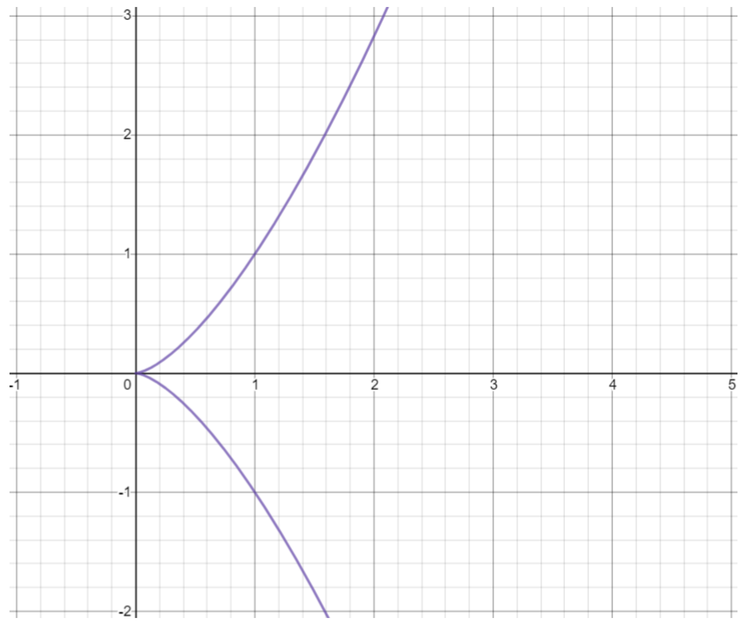
At $t = 0$*,* $x = 0$ *and* $y = 0$*, so this description begins at the origin. As* t *increases,* x *will initially increase and* y *will also initially increase, giving a counter-clockwise motion.* $\square$

Example 10.8 *(10.1.10) Describe the curve given by the parametric equations*

$$\begin{aligned} x(t) &= t^2 \\ y(t) &= t^3 \end{aligned} \tag{501}$$

for all real values of t.

The easiest way to handle this is to note that $t = \sqrt[3]{y}$*, and so* $x = \left(\sqrt[3]{y}\right)^2 = y^{\frac{2}{3}}$*:*



As t and y have the same sign, the motion of this description is upward. $\square$

Example 10.9 *(10.1.14) Describe the curve given by the parametric equations*

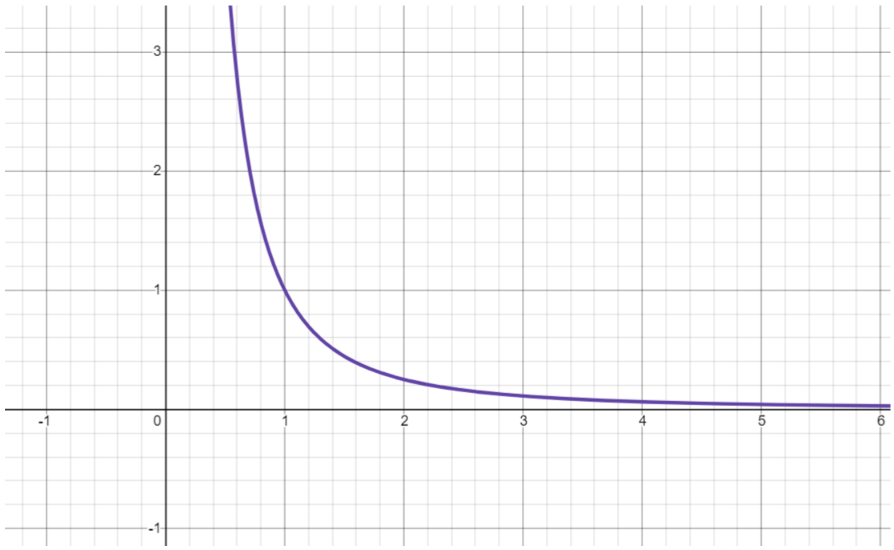
$$\begin{aligned} x(t) &= e^t \\ y(t) &= e^{-2t} \end{aligned} \tag{502}$$

for all real values of t.

We note that

$$x^2 = e^{2t} = \frac{1}{e^{-2t}} = \frac{1}{y}. \tag{503}$$

In other words, $y = \frac{1}{x^2}$. Since $e^t > 0$ for all real values of t, this includes only the region where $x > 0$:



As t increases, so does x, and so the motion of this particle is from left to right. □

Example 10.10 *(10.1.16) Describe the curve given by the parametric equations*

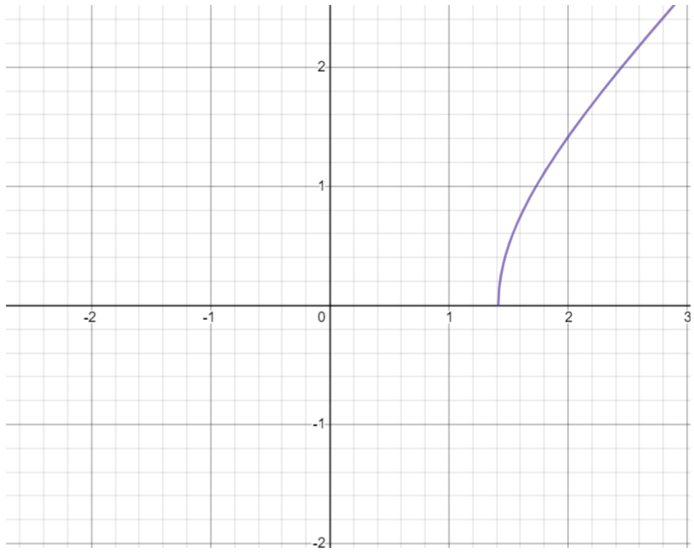
$$\begin{aligned} x(t) &= \sqrt{t+1} \\ y(t) &= \sqrt{t-1} \end{aligned} \tag{504}$$

for all real values of t.

We note that

$$x^2 - 1 = t = y^2 + 1. \tag{505}$$

This gives us part of the curve $x^2 - y^2 = 2$, which is a hyperbola:



We only get this piece of the curve, since both $\sqrt{t+1}$ *and* $\sqrt{t-1}$ *are non-negative for all real values of* t*. As* t *increases, so do* x *and* y*, giving an upward and rightward motion.* $\square$

Example 10.11 *(10.1.18) Describe the curve given by the parametric equations*

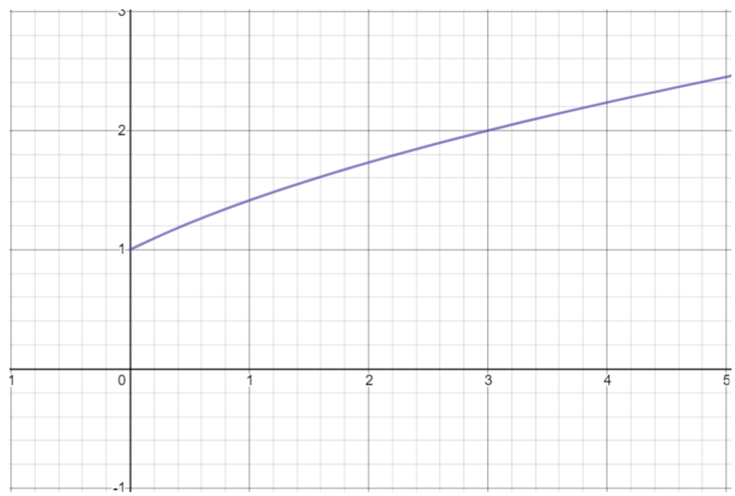
$$\begin{aligned} x(\theta) &= \tan^2\theta \\ y(\theta) &= \sec\theta \end{aligned} \tag{506}$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

We note that

$$x + 1 = \tan^2\theta + 1 = \sec^2\theta = y^2. \tag{507}$$

This gives a portion of the parabola $x = y^2 - 1$*:*



We get this portion because $x = \tan^2\theta \geq 0$*, and for* θ *in the interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*,* $y = \sec\theta > 0$*. As* θ *increases in* $\left(-\frac{\pi}{2}, 0\right)$*,* x *decreases, giving a leftward motion. As* θ *increases in* $\left(0, \frac{\pi}{2}\right)$*,* x *increases, giving a rightward motion. In other words, the particle traverses this curve leftward, bounces off the* y*-axis, and continues rightward along the same path.* $\square$

10.2 Calculus with parametric curves

In this section, we have two goals: to describe the tangent lines and find the arc lengths of curves that are described by parametric equations.

The slope of the tangent line is always given by $\frac{\mathrm{d}y}{\mathrm{d}x}$. In order to find this quantity for a parametric curve, use the following formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}. \tag{508}$$

The concavity is always given by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. To find this quantity for a parametric curve, use the following formula:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}t}}. \tag{509}$$

Example 10.12 *(Problem 10.2.2) Find the slope of the tangent line to the curve*

$$\begin{aligned} x(t) &= te^t \\ y(t) &= t + \sin t \end{aligned} \tag{510}$$

for $t \geq 0$.

We know that $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \cos t$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = te^t + e^t = (1+t)e^t$, so the slope of the tangent line at any non-negative value of t is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\frac{1+\cos t}{(1+t)e^t}}. \tag{511}$$

□

Example 10.13 *(Problem 10.2.4) Find an equation of the tangent line to the curve*

$$\begin{aligned} x(t) &= \sqrt{t} \\ y(t) &= t^2 - 2t \end{aligned} \tag{512}$$

at the point corresponding to $t = 4$.

First, we need the slope of the tangent line:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-2}{\frac{1}{2}t^{-\frac{1}{2}}} = 4\left(t-1\right)\sqrt{t}. \tag{513}$$

At $t = 4$,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=4} = 4\left(4-1\right)\sqrt{4} = 24. \tag{514}$$

Now we need a point on the line. We notice that

$$\begin{aligned} x(4) &= \sqrt{4} = 2 \\ y(4) &= (4)^2 - 2(4) = 8 \end{aligned}. \tag{515}$$

This gives the point $(2, 8)$. Thus, the equation of the tangent line is $y-8 = 24(x-2)$, or $\boxed{y = 24x - 40}$. $\square$

Example 10.14 *Find the points on the curve where the tangent is horizontal or vertical:*

$$\begin{aligned} x\left(t\right) &= e^{\sin t} \\ y\left(t\right) &= e^{\cos t} \end{aligned} \tag{516}$$

for $0 \leq t < 2\pi$.

To do this problem, we need to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{e^{\sin t}\cos t}{e^{\cos t}\left(-\sin t\right)} = -\tan t. \tag{517}$$

The tangent line to the curve will be horizontal when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$: this occurs exactly when $-\tan t = 0$: specifically, $t = 0$ and $t = \pi$. In order to find the points to which these values of the parameter correspond, we need only put them into $x\left(t\right)$ and $y\left(t\right)$:

$$\begin{array}{ll} x\left(0\right) = 1 & x\left(\pi\right) = 1 \\ y\left(0\right) = e & y\left(\pi\right) = \frac{1}{e} \end{array}. \tag{518}$$

Therefore, the tangent line is horizontal at two points: $\boxed{(1, e) \text{ and } \left(1, \frac{1}{e}\right)}$. The curve has a vertical tangent line when $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan t$ is undefined: $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

Again, to find the points to which these values of the parameter correspond, we only put them into $x(t)$ and $y(t)$:

$$\begin{array}{ll} x\left(\frac{\pi}{2}\right)=e & x\left(\frac{3\pi}{2}\right)=\frac{1}{e} \\ y\left(\frac{\pi}{2}\right)=1 & y\left(\frac{3\pi}{2}\right)=1 \end{array}. \tag{519}$$

Thus, the tangent line is vertical at two points: $\boxed{(e,1) \textit{ and } \left(\frac{1}{e},1\right)}$. $\square$

Example 10.15 *(Problem 10.2.18) Find the points on the curve where the tangent is horizontal or vertical:*

$$\begin{array}{c} x(t)=t^3-3t \\ y(t)=t^3-3t^2 \end{array} \tag{520}$$

To do this problem, we need to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}=\frac{3t^2-6t}{3t^2-3}. \tag{521}$$

Now, the tangent line to the curve will be horizontal when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$: this occurs exactly when $3t^2-6t=0$, or in other words, $3t(t-2)=0$. This has two solutions: $t=0$ and $t=2$. To convert these values for the parameter into actual points, we only need to put them into the equations for x and y:

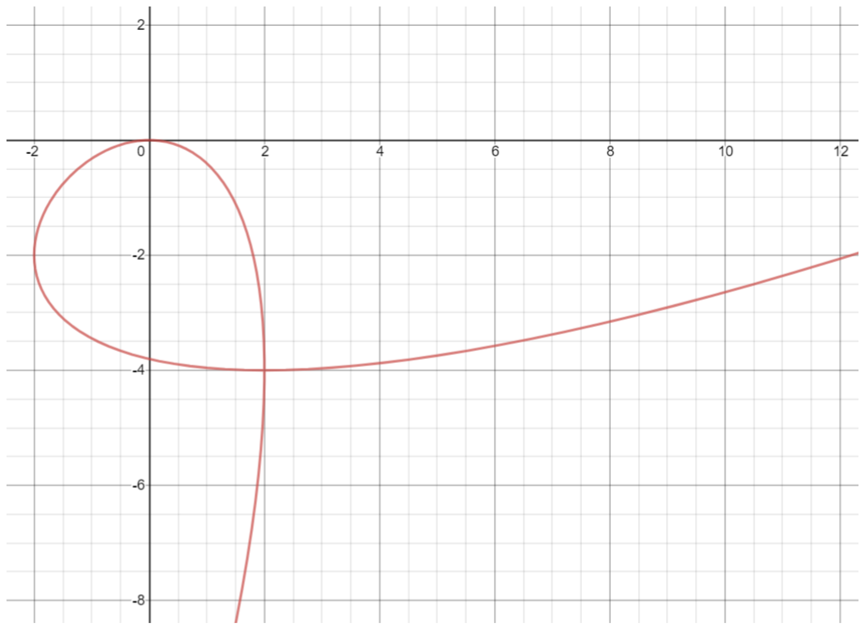
$$\begin{array}{ll} x(0)=0 & x(2)=2 \\ y(0)=0 & y(2)=-4 \end{array}. \tag{522}$$

Thus, the curve has a horizontal tangent line at the points $\boxed{(0,0) \textit{ and } (2,-4)}$. *The curve has a vertical tangent line when $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. Since $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3t^2-6t}{3t^2-3}$, this occurs if and only if $3t^2-3=0$. This has two solutions: $t=-1$ and $t=1$. Again, to figure out the points to which these two values correspond, we need only put them into x and y:*

$$\begin{array}{ll} x(-1)=2 & x(1)=-2 \\ y(-1)=-4 & y(1)=-2 \end{array}. \tag{523}$$

Thus, the curve has a vertical tangent line at the points $\boxed{(2,-4) \textit{ and } (-2,2)}$.

*At first, this may seem extremely perplexing: doesn't this mean that the graph has a horizontal tangent line **and** a vertical tangent line at $(2, -4)$? Indeed, it does, which may sound rather impossible until you look at the graph:*



As you can see, the curve actually passes through $(2, -4)$ twice: the first time is when $t = -1$, at which time the tangent line is vertical, and the second time is when $t = 2$, at which time the tangent line is horizontal. □

Example 10.16 *For which values of t is the curve*

$$\begin{aligned} x(t) &= t^3 + 1 \\ y(t) &= t^2 - t \end{aligned} \tag{524}$$

concave upward?

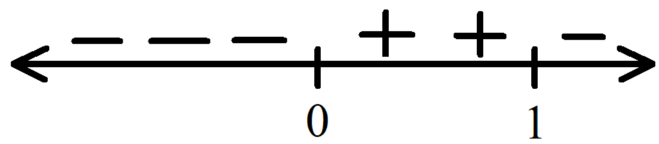
First, we find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t - 1}{3t^2} \tag{525}$$

From this, we need $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. To find this, we evaluate

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2t-1}{3t^2}\right)}{\frac{\mathrm{d}}{\mathrm{d}t}\left(t^3 + 1\right)} = \frac{\left(\frac{3t^2(2)-(2t-1)(6t)}{9t^4}\right)}{3t^2} = \frac{2\left(1 - t\right)}{9t^5} \tag{526}$$

Now we need to find where this is greater than 0. *We produce a number line.*



We mark the points $t = 0$ *and* $t = 1$*, since those are the locations where* $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ *could change sign. For* $t < 0$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$. *For* $0 < t < 1$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$. *For* $t > 1$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$. *Thus, the curve is concave up when* $0 < t < 1$. □

The arc length of a parametric curve for $\alpha \le t \le \beta$ is given by

$$s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t. \tag{527}$$

Example 10.17 *(Problem 10.2.42) Find the exact length of the curve:*

$$\begin{aligned} x(t) &= e^t - t \\ y(t) &= 4e^{\frac{t}{2}} \end{aligned}, \tag{528}$$

for $0 \le t \le 2$.

First, we note that

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= e^t - 1 \\ \tfrac{\mathrm{d}y}{\mathrm{d}t} &= 2e^{\frac{t}{2}} \end{aligned}. \tag{529}$$

Now,

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \left(e^t - 1\right)^2 + \left(2e^{\frac{t}{2}}\right)^2 \\ &= e^{2t} - 2e^2 + 1 + 4e^t \\ &= e^{2t} + 2e^t + 1 = \left(e^t + 1\right)^2 \end{aligned} \tag{530}$$

Therefore,

$$s = \int_0^2 \sqrt{\left(e^t + 1\right)^2}\,\mathrm{d}t = \int_0^2 e^t + 1\,\mathrm{d}t = e^t + t\Big|_0^2$$
$$= \left(e^2 + 2\right) - \left(e^0 + 0\right) = \boxed{e^2 + 1}. \quad (531)$$

□

Example 10.18 *Find the exact length of the curve:*

$$\begin{aligned} x(t) &= \cos t + \ln\left(\tan\left(\tfrac{1}{2}t\right)\right) \\ y(t) &= \sin t \end{aligned} \quad (532)$$

for $\frac{\pi}{6} \leq t \leq \frac{\pi}{3}$.

We note that

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= -\sin t + \tfrac{1}{\tan\left(\frac{1}{2}t\right)}\sec^2\left(\tfrac{1}{2}t\right) \\ \tfrac{\mathrm{d}y}{\mathrm{d}t} &= \cos t \end{aligned}. \quad (533)$$

Further, we can simplify $\frac{\mathrm{d}x}{\mathrm{d}t}$ *by writing*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t + \frac{1}{2\sin\left(\frac{1}{2}t\right)\cos\left(\frac{1}{2}t\right)} = -\sin t + \frac{1}{\sin t} = \csc t - \sin t. \quad (534)$$

Therefore,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (\csc t - \sin t)^2 + \cos^2 t$$
$$= \csc^2 t - 2 + \sin^2 t + \cos^2 t = \csc^2 t - 1 = \cot^2 t \quad (535)$$

Over the region $\frac{\pi}{6} \leq t \leq \frac{\pi}{3}$, $\cot t > 0$*, so*

$$s = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\cot^2 t}\,\mathrm{d}t = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot t\,\mathrm{d}t = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos t}{\sin t}\,\mathrm{d}t. \quad (536)$$

This can be most easily handled by a u-substitution:

$$\begin{aligned} u &= \sin t & x = \tfrac{\pi}{6} \Rightarrow u = \tfrac{1}{2} \\ \mathrm{d}u &= \cos t \; \mathrm{d}t & x = \tfrac{\pi}{3} \Rightarrow u = \tfrac{\sqrt{3}}{2} \end{aligned} \tag{537}$$

$$s = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u} \, \mathrm{d}u = \ln|u| \Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) = \ln\left(\sqrt{3}\right) = \boxed{\tfrac{1}{2}\ln 3}. \tag{538}$$

□

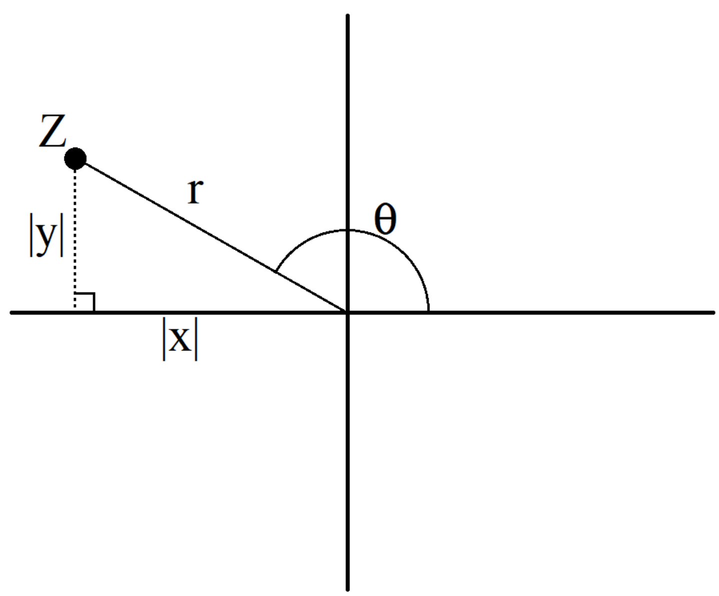
10.3 Polar coordinates

A "coordinate system" represents a point as a sequence of numbers.

In the "Cartesian coordinate system," a point Z in the plane is represented by an ordered pair $Z = (x, y)$ of signed distances from two perpendicular lines: y-coordinate of Z is the signed distance from Z to the x-axis, and the x-coordinate of Z is the signed distance from Z to the y-axis.

In the "polar coordinate system," a point Z in the plane is represented by an ordered pair $Z = (r, \theta)$, where r is the length of a line segment pointing from the origin to the point and θ is the counter-clockwise angle that the line segment makes with the positive x-axis.

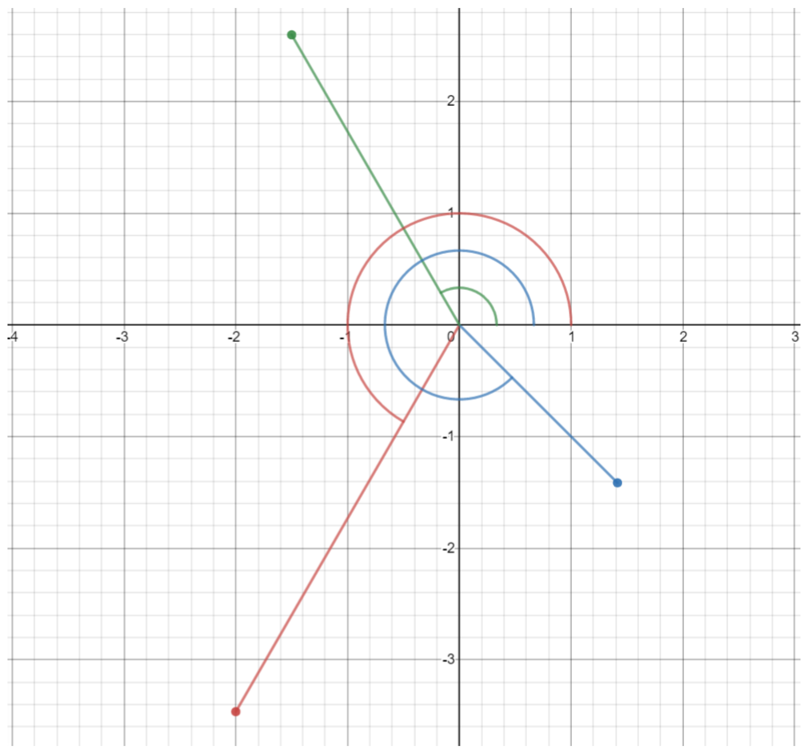
We can convert from polar coordinates to Cartesian coordinates by using a right triangle:



As the diagram illustrates, $x = r\cos\theta$ and $y = r\sin\theta$. (Similarly, in converting from Cartesian coordinates to polar coordinates, we can write $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, provided that $x \neq 0$.)

Example 10.19 *(Problem 10.3.4) Plot and find the Cartesian coordinates of the point whose polar coordinates are:*

$$\begin{array}{ll} (a) & \left(4, \frac{4\pi}{3}\right) \\ (b) & \left(-2, \frac{3\pi}{4}\right) \\ (c) & \left(-3, -\frac{\pi}{3}\right) \end{array} \tag{539}$$



(a) *We need only apply* $x = r\cos\theta$ *and* $y = r\sin\theta$*:*

$$\begin{aligned} x &= 4\cos\left(\tfrac{4\pi}{3}\right) = -2 \\ y &= 4\sin\left(\tfrac{4\pi}{3}\right) = -2\sqrt{3} \end{aligned}. \tag{540}$$

Therefore, in Cartesian coordinates, the point is $\boxed{\left(-2, -2\sqrt{3}\right)}$.

(b) *This notation does not make literal sense, since* r *is the length of a line segment, which cannot be negative. Truly, negative values of* r *should be understood* *using the following equality for polar coordinates:*

$$(-r, \theta) = (r, \theta + \pi). \tag{541}$$

Now we can more intuitively state the point as $\left(2, \frac{3\pi}{4} + \pi\right) = \left(2, \frac{7\pi}{4}\right)$ *in polar coordinates. This gives us*

$$\begin{array}{c} x = 2\cos\left(\frac{7\pi}{4}\right) = \sqrt{2} \\ y = 2\sin\left(\frac{7\pi}{4}\right) = -\sqrt{2} \end{array}. \tag{542}$$

Therefore, in Cartesian coordinates, the point is $\boxed{\left(\sqrt{2}, -\sqrt{2}\right)}$

(c) *In this case as well, we can write that* $\left(-3, -\frac{\pi}{3}\right) = \left(3, \frac{2\pi}{3}\right)$ *in polar coordinates:*

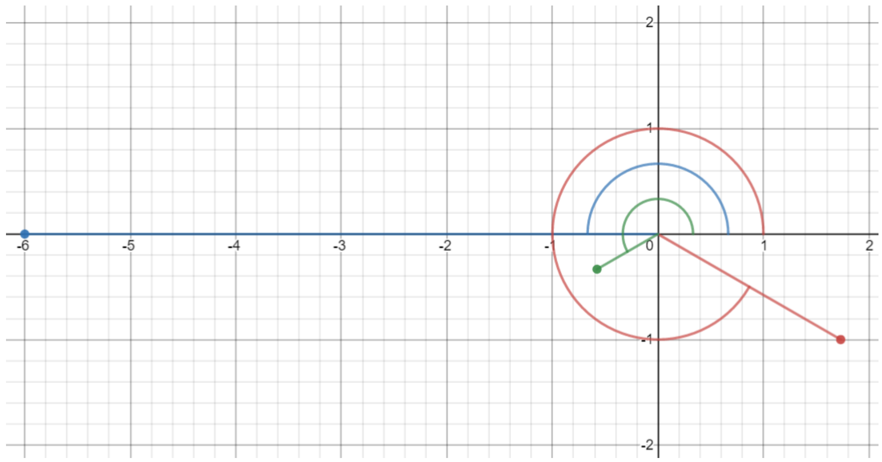
$$\begin{array}{c} x = 3\cos\left(\frac{2\pi}{3}\right) = -\frac{3}{2} \\ y = 3\sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2} \end{array}. \tag{543}$$

Thus, in Cartesian coordinates, the point is $\boxed{\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)}$. $\square$

Converting from Cartesian coordinates to polar coordinates can, at times, be a bit trickier. This comes down to solving a system of two equations for r and θ. Finding r just requires squaring both equations and adding them together, but finding θ requires more geometric thinking.

Example 10.20 *The Cartesian coordinates of a point are given. Find polar coordinates* (r, θ) *of the point.*

$$\begin{array}{cc} (a) & \left(\sqrt{3}, -1\right) \\ (b) & (-6, 0) \\ (c) & \left(-\frac{1}{\sqrt{3}}, -\frac{1}{3}\right) \end{array} \tag{544}$$



(a) *In this case, we must solve the system of equations*

$$\begin{aligned} \sqrt{3} &= r\cos\theta \\ -1 &= r\sin\theta \end{aligned}. \tag{545}$$

To get started, we can square both equations and add them together:

$$4 = \left(\sqrt{3}\right)^2 + (-1)^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2. \tag{546}$$

This reveals that $r = \pm 2$. *In particular, if we want a non-negative value for* r, *then we need to take* $r = 2$. *Now, the system becomes*

$$\begin{aligned} \sqrt{3} &= 2\cos\theta \\ -1 &= 2\sin\theta \end{aligned}. \tag{547}$$

In other words, $\cos\theta = \frac{\sqrt{3}}{2}$ *and* $\sin\theta = -\frac{1}{2}$. *There are infinitely many real values of* θ *which satisfy both of these equations, but for now, we'll just take the only solution that satisfies* $0 \le \theta \le 2\pi$*: specifically,* $\theta = \frac{11\pi}{6}$. *Thus, the polar coordinates* $\boxed{\left(2, \frac{11\pi}{6}\right)}$ *will represent the point in question. It's worth noting that this is not the only set of polar coordinates that could represent the point;* $\boxed{\left(-2, \frac{5\pi}{6}\right)}$ *and* $\boxed{\left(2, -\frac{\pi}{6}\right)}$ *are just two of many more different ways to represent the point in polar coordinates.*

(b) *This problem can also be solved in terms of a system of equations, but it's* *far easier to just realize that the point is a distance of* 6 *units away from the origin, along the negative* x*-axis. Thus, the polar coordinates* $\boxed{(6,\pi)}$ *will represent the point.*

(c) *This one is fairly difficult unless we set up the system of equations:*

$$\begin{aligned} -\tfrac{1}{\sqrt{3}} &= r\cos\theta \\ -\tfrac{1}{3} &= r\sin\theta \end{aligned} \tag{548}$$

Again, squaring both equations and then adding them together gives

$$\frac{4}{9} = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{3}\right)^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2. \tag{549}$$

This reveals that $r = \pm\frac{2}{3}$*, so we'll take* $r = \frac{2}{3}$*. The system now becomes*

$$\begin{aligned} -\tfrac{1}{\sqrt{3}} &= \tfrac{2}{3}\cos\theta \\ -\tfrac{1}{3} &= \tfrac{2}{3}\sin\theta \end{aligned} \tag{550}$$

This means that $\cos\theta = -\frac{\sqrt{3}}{2}$ *and* $\sin\theta = -\frac{1}{2}$*. The solution for* θ *that satisfies* $0 \leq \theta \leq 2\pi$ *is* $\theta = \frac{7\pi}{6}$*. Thus, the polar coordinates* $\boxed{\left(\frac{2}{3}, \frac{7\pi}{6}\right)}$ *will represent the point.*

(Note: at some point during this process, it's possible that you might have noticed that $\tan\theta = \sqrt{3}$*. This is true. However, if we try to use the arc-tangent function, we get* $\theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$*, which is decidedly **wrong** for the current question.)* $\square$

In the same way that we can write functions $y = f(x)$ in Cartesian coordinates, we can also write functions like $r = f(\theta)$ in polar coordinates. So, how can one find the tangent line to a curve defined by a polar function?

As mentioned in the previous section, the slope of the tangent line is always $\frac{\mathrm{d}y}{\mathrm{d}x}$. Now, given that $y = r\sin\theta$ and $x = r\cos\theta$, this becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}r\sin\theta}{\frac{\mathrm{d}}{\mathrm{d}\theta}r\cos\theta} = \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta}. \tag{551}$$

Given $r = f(\theta)$, this is possible to compute.

Example 10.21 *(Problem 10.3.56) Find the slope of the tangent line to the curve $r = 2 + \sin(3\theta)$ at the point where $\theta = \frac{\pi}{4}$.*

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 3\cos(3\theta). \tag{552}$$

Therefore, at $\theta = \frac{\pi}{4}$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} \\ &= \frac{\left(2+\sin\left(3\frac{\pi}{4}\right)\right)\cos\left(\frac{\pi}{4}\right) + 3\cos\left(3\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{-\left(2+\sin\left(3\frac{\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right) + 3\cos\left(3\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)} \\ &= \frac{\left(2+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{-\left(2+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}+\frac{1}{2}-\frac{3}{2}}{-\sqrt{2}-\frac{1}{2}-\frac{3}{2}} \\ &= \frac{\sqrt{2}-1}{-\sqrt{2}-2} = \frac{-2+2\sqrt{2}+\sqrt{2}-2}{2-4} = \boxed{2-\frac{3}{2}\sqrt{2}}.\end{aligned} \tag{553}$$

□

Example 10.22 *(Problem 10.3.56) Find the points on the curve $r = e^{\theta}$ where the tangent line is horizontal or vertical.*

Here we seek the points where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is either 0 or undefined. We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = e^{\theta}. \tag{554}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} = \frac{e^{\theta}\cos\theta + e^{\theta}\sin\theta}{-e^{\theta}\sin\theta + e^{\theta}\cos\theta} \\ &= \frac{e^{\theta}(\cos\theta + \sin\theta)}{e^{\theta}(\cos\theta - \sin\theta)} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}.\end{aligned} \tag{555}$$

Now, if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, then $\cos\theta + \sin\theta = 0$. This means that $\cos\theta = -\sin\theta$. This horizontal tangents occur at $\boxed{\theta = \frac{3\pi}{4} + n\pi}$, where n is any integer. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined, *then* $\cos\theta - \sin\theta = 0$*, in which case* $\cos\theta = \sin\theta$*. These vertical tangents occur at* $\boxed{\theta = \frac{\pi}{4} + n\pi}$*, where* n *is any integer.* □

Example 10.23 *Find the points on the curve* $r = 1 - \sin\theta$ *where the tangent line is vertical.*

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\cos\theta. \tag{556}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} &= \frac{(1-\sin\theta)\cos\theta + (-\cos\theta)\sin\theta}{-(1-\sin\theta)\sin\theta + (-\cos\theta)\cos\theta} \\ = \frac{(1-2\sin\theta)\cos\theta}{-\sin\theta + \sin^2\theta - \cos^2\theta} &= \frac{(1-2\sin\theta)\cos\theta}{-\sin\theta + \sin^2\theta - \left(1 - \sin^2\theta\right)} \\ &= \frac{(1-2\sin\theta)\cos\theta}{2\sin^2\theta - \sin\theta - 1}.\end{aligned} \tag{557}$$

The tangent is vertical when the denominator equals zero: $2\sin^2\theta - \sin\theta - 1 = 0$*. This is a quadratic equation in* $\sin\theta$*:*

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1 \pm 3}{4} = 1, -\frac{1}{2}. \tag{558}$$

This gives $\theta = \frac{\pi}{2} + 2\pi n$*,* $\theta = \frac{7\pi}{6} + 2\pi n$*, and* $\theta = \frac{11\pi}{6} + 2\pi n$*. These correspond to the points whose polar coordinates are* $\boxed{\left(0, \frac{\pi}{2}\right), \left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)}$*.* □

10.4 Areas and lengths in polar coordinates

When a region is bounded between two rays pointing from the origin and a polar curve, the area of the region is given by

$$A = \int_a^b \frac{1}{2} r^2 \, \mathrm{d}\theta, \tag{559}$$

where a and b are the angles between the rays and the x-axis.

Example 10.24 *(Problem 10.4.2) Find the area of the region bounded by the curve $r = \cos\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{6}$.*

This is

$$\begin{aligned} A = \int_0^{\frac{\pi}{6}} \frac{1}{2} \cos^2\theta \, \mathrm{d}\theta &= \frac{1}{4} \int_0^{\frac{\pi}{6}} 1 + \cos(2\theta) \, \mathrm{d}\theta \\ &= \frac{1}{4} \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Bigg|_0^{\frac{\pi}{6}} = \frac{1}{4} \left(\frac{\pi}{6} + \frac{1}{2} \frac{\sqrt{3}}{2} \right) = \boxed{\tfrac{\pi}{24} + \tfrac{\sqrt{3}}{16}} \end{aligned} \tag{560}$$

□

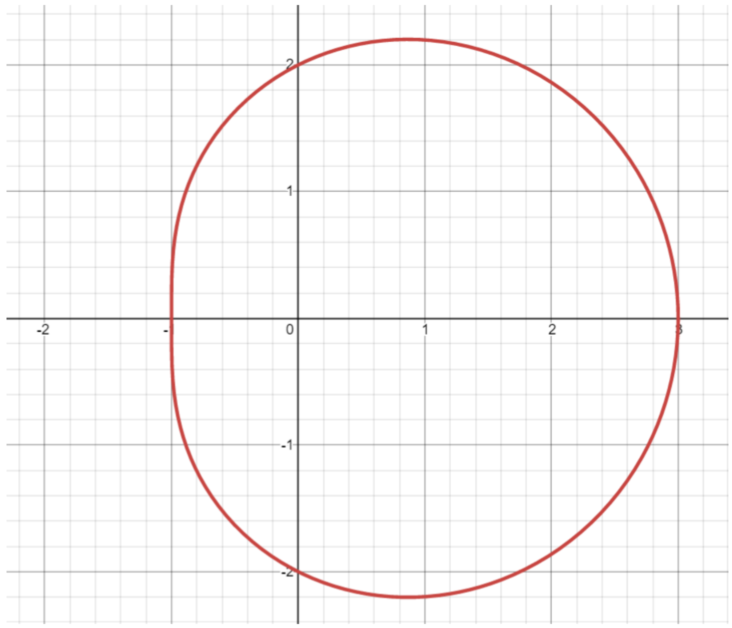
Example 10.25 *(Problem 10.4.4) Find the area of the region bounded by the curve $r = \frac{1}{\theta}$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = 2\pi$.*

This is

$$A = \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2} \frac{1}{\theta^2} \, \mathrm{d}\theta = -\frac{1}{2} \frac{1}{\theta} \Bigg|_{\frac{\pi}{2}}^{2\pi} = \frac{1}{2} \left(\frac{2}{\pi} - \frac{1}{2\pi} \right) = \frac{1}{2} \left(\frac{4}{2\pi} - \frac{1}{2\pi} \right) = \boxed{\tfrac{3}{4\pi}}. \tag{561}$$

□

Example 10.26 *(Problem 10.4.6) Find the area in the third quadrant enclosed by $r = 2 + \cos\theta$.*



The third quadrant is the region $\pi \leq \theta \leq \frac{3\pi}{2}$. Therefore,

$$
\begin{aligned}
A = \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2}(2+\cos\theta)^2 \, \mathrm{d}\theta &= \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} 4 + 4\cos\theta + \cos^2\theta \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} 4 + 4\cos\theta + \frac{1}{2}\left(1+\cos\left(2\theta\right)\right) \, \mathrm{d}\theta \\
&= \int_{\pi}^{\frac{3\pi}{2}} \frac{9}{4} + 2\cos\theta + \frac{1}{4}\cos\left(2\theta\right) \, \mathrm{d}\theta \\
&= \frac{9}{4}\theta + 2\sin\theta + \frac{1}{8}\sin\left(2\theta\right)\Big|_{\pi}^{\frac{3\pi}{2}} \\
&= \frac{9}{4}\left(\frac{3\pi}{2} - \pi\right) + 2\left(\sin\left(\frac{3\pi}{2}\right) - \sin\pi\right) + \frac{1}{8}\left(\sin\left(3\pi\right) - \sin\left(2\pi\right)\right) \\
&= \boxed{\tfrac{9}{8}\pi - 2}. \qquad (562)
\end{aligned}
$$

□

Now we examine the matter of the arc length of a curve given by a polar equation $r = f\left(\theta\right)$ for some interval $\alpha \leq \theta \leq \beta$. Suppose we regard θ as a parameter, and write parametric equations describing such a curve. These would be

$$\begin{aligned} x(\theta) &= r\cos\theta = f(\theta)\cos\theta \\ y(\theta) &= r\sin\theta = f(\theta)\sin\theta \end{aligned}. \tag{563}$$

We know that the arc length of a parametric curve is

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta. \tag{564}$$

Doing the calculation gives the following:

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\, \mathrm{d}\theta. \tag{565}$$

Example 10.27 *(Problem 10.4.46) Find the exact length of the curve $r = 5^\theta$ for $0 \le \theta \le 2\pi$.*

We know that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 5^\theta \ln 5$. Therefore,

$$\begin{aligned} s = \int_0^{2\pi} \sqrt{5^{2\theta} + 5^{2\theta}(\ln 5)^2}\, \mathrm{d}\theta &= \int_0^{2\pi} 5^\theta \sqrt{1 + (\ln 5)^2}\, \mathrm{d}\theta \\ &= \left.\frac{5^\theta}{\ln 5}\sqrt{1+(\ln 5)^2}\right|_0^{2\pi} = \frac{\sqrt{1+(\ln 5)^2}}{\ln 5}\left(5^{2\pi} - 1\right). \end{aligned} \tag{566}$$

□

Example 10.28 *(Problem 10.4.48) Find the exact length of the curve defined by the polar equation $r = 2(1+\cos\theta)$ for $0 \le \theta \le 2\pi$.*

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2\sin\theta. \tag{567}$$

Therefore,

$$\begin{aligned} s &= \int_0^{2\pi} \sqrt{\left(2\left(1+\cos\theta\right)\right)^2 + \left(-2\sin\theta\right)^2}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{4\left(1+2\cos\theta+\cos^2\theta\right)+4\sin^2\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{4\left(1+2\cos\theta\right)+4\cos^2\theta+4\sin^2\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \sqrt{8+8\cos\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} 4\sqrt{\frac{1}{2}\left(1+\cos\theta\right)}\,\mathrm{d}\theta = 4\int_0^{2\pi} \sqrt{\cos^2\left(\frac{1}{2}\theta\right)}\,\mathrm{d}\theta \\ &= 4\int_0^{2\pi} \left|\cos\left(\frac{1}{2}\theta\right)\right|\mathrm{d}\theta = 8\int_0^{\pi} \cos\left(\frac{1}{2}\theta\right)\,\mathrm{d}\theta \\ &= 16\sin\left(\frac{1}{2}\theta\right)\Big|_0^{\pi} = 16\left(1-0\right) = \boxed{16}. \end{aligned} \tag{568}$$

□

6 Option: Applications of integration

6.1 Option: Areas between curves

The definite integral of f over the interval $[a, b]$ is defined as the signed area under the curve.

Suppose we have two functions, f and g, defined over an interval $[a, b]$, such that $f(x) \geq g(x)$ for all x in the interval. In that case, the area enclosed between the curves is

$$A = \int_a^b f(x) - g(x)\, \mathrm{d}x. \tag{569}$$

Example 6.1 *(Problem 6.1.6) Find the area of the region enclosed by the curves $y = \sin x$, $y = x$, $x = \frac{\pi}{2}$ and $x = \pi$.*

[Draw diagram]

The area is

$$\begin{aligned} A = \int_{\frac{\pi}{2}}^{\pi} x - \sin x\, \mathrm{d}x = \frac{1}{2}x^2 + \cos x \Big|_{\frac{\pi}{2}}^{\pi} \\ = \left(\frac{1}{2}\pi^2 + \cos\pi\right) - \left(\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \cos\left(\frac{\pi}{2}\right)\right) \\ = \frac{1}{2}\pi^2 - 1 - \frac{1}{8}\pi^2 = \boxed{\tfrac{3}{8}\pi^2 - 1}. \end{aligned} \tag{570}$$

□

Example 6.2 *(Problem 6.1.8) Find the area of the region enclosed by the curves $y = x^2 - 4x$ and $y = 2x$.*

[Draw diagram]

The area is

$$A = \int_a^b 2x - \left(x^2 - 4x\right)\, \mathrm{d}x. \tag{571}$$

What are a and b? In order to find the upper and lower bounds, we need to find the x-coordinates of the points of intersection of the curves. To do this, we set the functions equal to each other and solve for x:

$$\begin{aligned} 2x &= x^2 - 4x \\ 0 &= x^2 - 6x = x\left(x - 6\right) \end{aligned} \tag{572}$$

This gives $x = 0$ and $x = 6$, so the area is

$$A = \int_0^6 2x - \left(x^2 - 4x\right) \, \mathrm{d}x = \int_0^6 -x^2 + 6x \, \mathrm{d}x = -\frac{1}{3}x^3 + 3x^2 \Big|_0^6 = \boxed{36}. \tag{573}$$

□

Example 6.3 *(Problem 6.1.14) Find the area enclosed by the curves $y = x^2$ and $y = 4x - x^2$.*

[Draw diagram]

The area is

$$A = \int_a^b \left(4x - x^2\right) - x^2 \, \mathrm{d}x. \tag{574}$$

What are a and b? We set the functions equal to each other and solve for x:

$$\begin{aligned} x^2 &= 4x - x^2 \\ 2x^2 - 4x &= 0 \; . \\ 2x\left(x - 2\right) &= 0 \end{aligned} \tag{575}$$

This gives $x = 0$ and $x = 2$, so the area is

$$\int_0^2 \left(4x - x^2\right) - x^2 \, \mathrm{d}x = \int_0^2 4x - 2x^2 \, \mathrm{d}x = 2x^2 - \frac{2}{3}x^3 \Big|_0^2 = 8 - \frac{16}{3} = \boxed{\tfrac{8}{3}}. \tag{576}$$

□

Example 6.4 *(Problem 6.1.20) Find the area enclosed by the curves $x = y^4$,*

$y = \sqrt{2-x}$ *and* $y = 0$.

[Draw diagram]

Here the roles of x *and* y *are reversed. We write* $y = \sqrt{2-x}$ *as* $x = 2 - y^2$. *The area is*

$$A = \int_a^b \left(2 - y^2\right) - y^4 \,\mathrm{d}y. \tag{577}$$

Again, we need a *and* b. *To do this, we set the functions equal to each other and solve for* y:

$$\begin{gathered} y^4 = 2 - y^2 \\ y^4 + y^2 - 2 = 0 \\ \left(y^2 + 2\right)\left(y^2 - 1\right) = 0 \\ \left(y^2 + 2\right)(y - 1)(y + 1) \end{gathered}. \tag{578}$$

The lower bound is $y = 0$. *The upper bound is* $y = 1$. *Thus, the area is*

$$\begin{aligned} A = \int_0^1 \left(2 - y^2\right) - y^4 \,\mathrm{d}y = \int_0^1 2 - y^2 - y^4 \,\mathrm{d}y \\ = 2y - \frac{1}{3}y^3 - \frac{1}{5}y^5\bigg|_0^1 = 2 - \frac{1}{3} - \frac{1}{5} = \boxed{\tfrac{22}{15}}. \end{aligned} \tag{579}$$

□

6.2 Option: Volumes

Definition 6.5 *Let S be a solid region that is bounded by the lines $x = a$ and $x = b$. For each x such that $a \leq x \leq b$, define $A(x)$ as the cross-sectional area of S. The* *volume of S* *is the integral*

$$V = \int_a^b A(x)\, \mathrm{d}x. \tag{580}$$

Given a region in the xy-plane bounded by a function, we can rotate that region around a (horizontal or vertical) axis to form a 3-dimensional shape. The shape that results is called a "solid of revolution." In this section, we discuss the "disk-and-washer method" of finding the volume of a solid of revolution.

Example 6.6 *(Problem 6.2.2) Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 4$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a disk. The area of that disk is

$$A(x) = \pi r^2 = \pi \left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}. \tag{581}$$

This is the cross-sectional area of the solid, so by definition, the volume is

$$V = \int_1^4 \frac{\pi}{x^2}\, \mathrm{d}x = \pi \int_1^4 x^{-2}\, \mathrm{d}x = -\frac{\pi}{x}\bigg|_1^4 = -\frac{\pi}{4} - \left(-\frac{\pi}{1}\right) = \boxed{\tfrac{3\pi}{4}}. \tag{582}$$

□

Example 6.7 *(Problem 6.2.4) Find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = 0$, $x = -1$ and $x = 1$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a disk. The area of that disk is

$$A(x) = \pi r^2 = \pi (e^x)^2 = \pi e^{2x}. \tag{583}$$

Now, the volume is

$$V = \int_{-1}^{1} \pi e^{2x}\,\mathrm{d}x = \pi \int_{-1}^{1} e^{2x}\,\mathrm{d}x = \frac{\pi}{2} e^{2x}\Big|_{-1}^{1} = \boxed{\tfrac{\pi}{2}\left(e^2 - e^{-2}\right)}. \tag{584}$$

□

Example 6.8 *Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{2x}$, $x = 0$ and $y = 4$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a washer. The area of that washer is

$$A(x) = \pi R^2 - \pi r^2 = \pi (4)^2 - \pi \left(\sqrt{2x}\right)^2 = \pi\left(16 - 2x\right). \tag{585}$$

The volume is the integral of this, but what are the bounds? The lower bound is certainly $x = 0$, but what about the upper bound? To find this, we need to set $y = \sqrt{2x}$ and $y = 4$ equal to each other and solve for x:

$$\sqrt{2x} = 4, \tag{586}$$

and so the upper bound is $x = 8$. Thus,

$$V = \int_0^8 \pi\left(16 - 2x\right)\,\mathrm{d}x = \pi\left(16x - x^2\right)\Big|_0^8 = \boxed{64\pi}. \tag{587}$$

□

Example 6.9 *(Problem 6.2.12) Find the volume of the solid obtained by rotating the region bounded by rotating the region bounded by $y = x^3$, $y = 1$ and $x = 2$ about the line $y = -3$.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about $y = -3$, forms a washer. The area of that washer is

$$\begin{aligned} A(x) = \pi R^2 - \pi r^2 = \pi\big(x^3 - (-3)\big)^2 - \pi(1 - (-3))^2 \\ = \pi\left(\left(x^3 + 3\right)^2 - 4^2\right) = \pi\left(x^6 + 6x^3 - 7\right). \end{aligned} \tag{588}$$

The lower bound is the x-coordinate of the intersection of $y = x^3$ and $y = 1$:

$$x^3 = 1, \tag{589}$$

so $x = 1$ is the lower bound. The upper bound is $x = 2$. Thus,

$$V = \int_1^2 \pi\left(x^6 + 6x^3 - 7\right)\, \mathrm{d}x = \pi\left(\frac{1}{7}x^7 + \frac{3}{2}x^4 - 7x\right)\Bigg|_1^2 = \boxed{\tfrac{471\pi}{14}}. \tag{590}$$

□

Example 6.10 *(Problem 6.2.10) Find the volume of the solid obtained by rotating the region bounded by $x = 2 - y^2$ and $x = y^4$ about the y-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the y-axis, forms a washer. The area of that washer is

$$A(y) = \pi R^2 - \pi r^2 = \pi\big(2 - y^2\big)^2 - \pi\big(y^4\big)^2 = \pi\left(-y^8 + y^4 - 4y^2 + 4\right). \tag{591}$$

What are the bounds of the region? They are the y-coordinates where $x = 2 - y^2$ and $x = y^4$ cross:

$$\begin{aligned} 2 - y^2 &= y^4 \\ 0 &= y^4 + y^2 - 2 \\ 0 &= \left(y^2 + 2\right)\left(y^2 - 1\right) \\ 0 &= \left(y^2 + 2\right)\left(y + 1\right)\left(y - 1\right). \end{aligned} \tag{592}$$

This gives $y = -1$ and $y = 1$ as bounds. Thus,

$$\begin{aligned} V = \int_{-1}^{1} \pi \left(-y^8 + y^4 - 4y^2 + 4\right)\, \mathrm{d}y \\ = \pi \left(-\frac{1}{9}y^9 + \frac{1}{5}y^5 - \frac{4}{3}y^3 + 4y\right)\Big|_{-1}^{1} \\ = \pi \left(-\frac{2}{9} + \frac{2}{5} - \frac{8}{3} + 8\right) = \boxed{\frac{248\pi}{45}}. \end{aligned} \tag{593}$$

□

6.3 Option: Volumes by cylindrical shells

There is another method for finding the volume of a solid of revolution: the "method of shells."

In the method of disks and washers, we selected a typical slice that was perpendicular to the axis of rotation to produce a disk or washer.

In the method of shells, we will select slices that are parallel to the axis of rotation to produce a cylindrical shell.

Example 6.11 *(Problem 6.3.4) Find the volume generated by rotating the region bounded by the curves $y = x^3$, $y = 0$, $x = 1$ and $x = 2$ about the y-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the y-axis, forms a cylindrical shell. The area of that shell is

$$A = 2\pi r h = 2\pi x \left(x^3\right) = 2\pi x^4. \tag{594}$$

Now, the volume is

$$V = \int_1^2 2\pi x^4 \, \mathrm{d}x = \frac{2\pi}{5} x^5 \bigg|_1^2 = \frac{2\pi}{5} (32 - 1) = \boxed{\tfrac{62\pi}{5}}. \tag{595}$$

□

Example 6.12 *(Problem 6.3.6) Find the volume generated by rotating the region bounded by the curves $y = 4x - x^2$ and $y = x$ about the y-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, *when rotated about the y-axis, forms a cylindrical shell. The area of that shell is*

$$A = 2\pi rh = 2\pi x\left(\left(4x - x^2\right) - (x)\right) = 2\pi\left(3x^2 - x^3\right) \tag{596}$$

What will be the bounds? They are the x-coordinates where $y = 4x - x^2$ and $y = x$ cross:

$$\begin{aligned} 4x - x^2 &= x \\ 3x - x^2 &= 0\ . \\ x\left(3 - x\right) &= 0 \end{aligned} \tag{597}$$

This gives $x = 0$ and $x = 3$ as bounds. Thus,

$$\begin{aligned} V = \int_0^3 2\pi\left(3x^2 - x^3\right)\,\mathrm{d}x &= 2\pi\int_0^3 3x^2 - x^3\,\mathrm{d}x \\ &= 2\pi\left(x^3 - \frac{1}{4}x^4\right)\Bigg|_0^3 = 2\pi\left(27 - \frac{1}{4}81\right) = \boxed{\tfrac{27}{2}\pi}. \end{aligned} \tag{598}$$

□

Example 6.13 *(Problem 6.3.10) Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the x-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the x-axis, forms a cylindrical shell. The area of that shell is

$$A = 2\pi rh = 2\pi y\left(y^2 - 0\right) = 2\pi y^3. \tag{599}$$

Now, the volume is

$$V = \int_0^2 2\pi y^3\,\mathrm{d}y = \frac{\pi}{2}y^4\Bigg|_0^2 = \boxed{8\pi}. \tag{600}$$

□

Example 6.14 *(Problem 6.3.16) Find the volume generated by rotating the region bounded by the curves $y = 4 - 2x$, $y = 0$ and $x = 0$ about the line $x = -1$.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the line $x = -1$, forms a cylindrical shell. The area of that shell is

$$A = 2\pi r h = 2\pi(x+1)(4-2x) = -4\pi\left(x^2 - x - 2\right). \tag{601}$$

Now, the volume is

$$\begin{aligned} V = \int_0^2 -4\pi\left(x^2 - x - 2\right)\,\mathrm{d}x &= -4\pi\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right)\Bigg|_0^2 \\ &= -4\pi\left(\frac{8}{3} - 2 - 4\right) = \boxed{\tfrac{40\pi}{3}}. \end{aligned} \tag{602}$$

□

6.4 Option: Average value of a function

Definition 6.15 *Let f be a function defined on the interval $[a, b]$. We define the* *<u>average value of f over $[a, b]$</u>* *as the value*

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x. \tag{603}$$

Example 6.16 *(Problem 6.5.8) Let $h(x) = \frac{\ln x}{x}$. Find the average value of the function on the interval $[1, 5]$.*
This is a straightforward application of the definition:

$$h_{ave} = \frac{1}{5-1} \int_1^5 h(x)\, \mathrm{d}x = \frac{1}{4} \int_1^5 \frac{\ln x}{x}\, \mathrm{d}x. \tag{604}$$

We use a u-substitution:

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x}\, \mathrm{d}x \end{aligned} \tag{605}$$

This becomes

$$h_{ave} = \frac{1}{4} \int_0^{\ln 5} u\, \mathrm{d}u = \frac{1}{8} u^2 \Big|_0^{\ln 5} = \boxed{\tfrac{1}{8}(\ln 5)^2}. \tag{606}$$

□

Recall the mean value theorem: if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a real value c such that $a \le c \le b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$. We now have a mean value theorem for integrals.

Theorem 6.17 *Let f be continuous on $[a, b]$. There exists a real value c such that $a \le c \le b$ and $f(c) = f_{ave}$.*

Appendix: Exponents and logarithms

Exponential functions

Theorem 6.18 *Let a be a real number. The following statements are true.*
(i) For any real numbers x and y, $a^{x+y} = a^x a^y$.
(ii) For any real numbers x and y, $a^{xy} = (a^x)^y$.
(iii) If $a \neq 0$, then for any real number x, $a^{-x} = \frac{1}{a^x}$.
(iv) Given a positive integer n, $a^{\frac{1}{n}} = \sqrt[n]{a}$.
(v) If $a \neq 0$, then $a^0 = 1$.

Note: most sources agree that 0^0 should be defined as 1. This choice is made for notational convenience, not due to any important mathematical truth.

Theorem 6.19 *Let a and b be real numbers greater than 1. Given a real number x, $(ab)^x = a^x b^x$.*

Theorem 6.20 *(Binomial theorem) Let a and b be real numbers. Given a positive integer n,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \tag{607}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, the binomial coefficient.

Theorem 6.21 *Let a be a real number. If $a > 1$, then the following statements are true.*
(i) $\lim\limits_{x \to \infty} a^x = \infty$.
(ii) $\lim\limits_{x \to -\infty} a^x = 0$.

Logarithmic functions

Definition 6.22 *Let f and g be real-valued functions of a real variable. We say that $\underline{f \text{ and } g \text{ are inverse functions}}$, or that $\underline{g \text{ is the inverse function of } f}$, or that $\underline{f \text{ is the inverse function of } g}$ provided that for any real value x, $f(g(x)) = x$ and $g(f(x)) = x$.*

Definition 6.23 *Let a be a real number greater than 1. The $\underline{\text{base } a \text{ logarithm}}$ is the inverse function $g(x) = \log_a x$ of the function $f(x) = a^x$.*

As a direct result of the definition of $\log_a$, for any real value x, $\log_a(a^x) = x$, and $a^{\log_a x} = x$ if $x > 0$. Additionally, since $a^x > 0$ for any $a > 0$ and any real number x, $\log_a(y)$ is not defined for any non-positive real number y.

Theorem 6.24 *Let a be a real number such that $a > 1$. The following statements are true.*
(i) For any positive real numbers x and y, $\log_a(xy) = \log_a(x) + \log_a(y)$.
(ii) For any real numbers x and y such that $x > 0$, $\log_a(x^y) = y\log_a(x)$.
(iii) For any positive real numbers x and y, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.

Theorem 6.25 *Let a be a real number such that $a > 1$. The following statements are true.*
(i) $\lim\limits_{x\to\infty} \log_a x = \infty$.
(ii) $\lim\limits_{x\to 0^+} \log_a x = -\infty$.

Calculus of exponential and logarithmic functions

Definition 6.26 *The natural exponential function is the real-valued function f of a real variable such that $\frac{\mathrm{d}}{\mathrm{d}x}f(x) = f(x)$ and $f(0) = 1$.*

Definition 6.27 *Euler's number is the real value e such that the function $f(x) = e^x$ is the natural exponential function.*

Definition 6.28 *The natural logarithm is the base e logarithm $\ln x = \log_e x$.*

Theorem 6.29 *The following statements are true.*
(i) If $a > 1$, then $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \ln a$.
(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\ln|x| = \frac{1}{x}$.

Appendix: Properties of trigonometric functions

List of trigonometric identities

Theorem 6.30 *(Pythagorean identity) Given a real number θ,*

$$\cos^2\theta + \sin^2\theta = 1.$$

Theorem 6.31 *(Angle sum formulas) Let θ and ϕ be real numbers. The following statements are true.*

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

Theorem 6.32 *(Double angle and half angle formulas) Let θ be a real number. The following statements are true.*

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta))$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta))$$

Definition 6.33 *Let θ be a real number.*
(i) If $\cos\theta \neq 0$, then $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
(ii) If $\cos\theta \neq 0$, then $\sec\theta = \frac{1}{\cos\theta}$.
(iii) If $\sin\theta \neq 0$, then $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
(iv) If $\sin\theta \neq 0$, then $\csc\theta = \frac{1}{\sin\theta}$.

Theorem 6.34 *Let θ be a real number. The following statements are true.*

$$\sec^2\theta - \tan^2\theta = 1$$
$$\csc^2\theta - \cot^2\theta = 1$$

Inverse trigonometric functions

Definition 6.35 *(i) The* <u>*arccosine function*</u> *is the function* $\cos^{-1}$ *whose domain is* $[0, \pi]$ *such that for any real value* x*, if* $-1 \leq x \leq 1$*, then* $\cos\left(\cos^{-1}(x)\right) = x$*, and for any real value* θ*, if* $0 \leq \theta \leq \pi$*, then* $\cos^{-1}(\cos(\theta)) = \theta$*.*
(ii) The <u>*arcsine function*</u> *is the function* $\sin^{-1}$ *whose domain is* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *such that for any real value* y*, if* $-1 \leq y \leq 1$*, then* $\sin\left(\sin^{-1}(y)\right) = y$*, and for any real value* θ*, if* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*, then* $\sin^{-1}(\sin(\theta)) = \theta$*.*
(iii) The <u>*arctangent function*</u> *is the function* $\tan^{-1}$*, whose domain is all real numbers, such that for any real value* z*, if* $\tan\left(\tan^{-1}(z)\right) = z$*, and for any real value* θ*, if* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$*, then* $\tan^{-1}(\tan(\theta)) = \theta$*.*
(iv) The <u>*arcsecant function*</u> *is the function* $\sec^{-1} z = \cos^{-1}\left(\frac{1}{z}\right)$*.*
(v) The <u>*arccosecant function*</u> *is the function* $\csc^{-1} z = \sin^{-1}\left(\frac{1}{z}\right)$*.*
(vi) The <u>*arccotangent function*</u> *is the function*

$$\cot^{-1} z = \begin{cases} \tan^{-1}\left(\frac{1}{z}\right) & \textit{if } z > 0 \\ \pi + \tan^{-1}\left(\frac{1}{z}\right) & \textit{if } z < 0 \end{cases}.$$

Derivatives of trigonometric functions and inverse trigonometric functions

Theorem 6.36 *The following statements are true.*

$$\begin{array}{cc}
\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \cos\theta & \frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta = -\sin\theta \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta = \sec\theta\tan\theta & \frac{\mathrm{d}}{\mathrm{d}\theta}\csc\theta = -\csc\theta\cot\theta \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta = \sec^2\theta & \frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta = -\csc^2\theta
\end{array}$$

Theorem 6.37 *The following statements are true.*

$$\begin{array}{cc}
\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} & \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} & \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}} \\
\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{x^2+1} & \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{x^2+1}
\end{array}$$

Appendix: Theorems on infinite series

Constant series

Theorem 6.38 *(Characterization theorem for geometric series) Let b and r be real numbers.*
(i) The geometric series $\sum_{n=1}^{\infty} br^{n-1}$ diverges if $|r| \geq 1$.
(ii) The geometric series $\sum_{n=1}^{\infty} br^{n-1} = \frac{b}{1-r}$ if $-1 < r < 1$.

Theorem 6.39 *(Test for divergence) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.*

Theorem 6.40 *(The integral test) Let (a_n) be a positive, eventually decreasing sequence of real numbers. Given a continuous function f defined on $[1, \infty)$ such that for each positive integer n, $f(n) = a_n$, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.*

Theorem 6.41 *(p-series test) Let p be a real number. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.*

Theorem 6.42 *(The comparison tests for infinite series) Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.*
(i) Suppose that for all positive integers n, $a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) Suppose that for all positive integers n, $c_n \leq a_n$. If $\sum_{n=1}^{\infty} c_n$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 6.43 *(Limit comparison test) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is positive and finite, then either both converge series or both diverge.*

Theorem 6.44 *(Alternating series test) Let* $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *be an alternating series, where for each positive integer* n, $b_n > 0$. *If the sequence* (b_n) *is eventually decreasing and* $\lim_{n\to\infty} b_n = 0$, *then the series* $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *is convergent.*

Theorem 6.45 *(Alternating series estimation theorem) Let* $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *be an alternating series, where for each positive integer* n, $b_n > 0$. *Assume that the sequence* (b_n) *is decreasing and that* $\lim_{n\to\infty} b_n = 0$. *Let* s *be the real number to which the series converges, and for each positive integer* k, *define*

$$R_k = s - \sum_{n=1}^{k}(-1)^{n+1}b_n. \tag{608}$$

(In other words, R_k *is the difference between the sum of the series and the* k*th partial sum.) In that case,* $|R_k| \leq b_{k+1}$.

Theorem 6.46 *If a series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

Theorem 6.47 *(Ratio test) Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series.*
(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, *then the series converges absolutely.*
(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, *then the series diverges.*

Theorem 6.48 *(Root test) Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series.*
(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, *then the series converges absolutely.*
(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$, *then the series diverges.*

Power series

Theorem 6.49 *Let* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *be a power series. Exactly one of the following statements is true.*
(i) $f(x)$ *converges only when* $x = a$*.*
(ii) $f(x)$ *converges for all real values* x*.*
(iii) There exists a positive real number R *(called the "radius of convergence") such that* $f(x)$ *converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Theorem 6.50 *If* f *has a power series representation about* $x = a$*, then* f *is equal to its Taylor series about* $x = a$ *(which is* $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$*).*

Common power series representations

Theorem 6.51 *The following statements are true.*
(i) If $|x| < 1$, then

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \tag{609}$$

(ii) If $|x| < 1$, then

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n. \tag{610}$$

(iii) If $|x| < 1$, then

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{611}$$

(iv) If $|x| < 1$ and k is any real number, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \tag{612}$$

where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-(n-1))}{n!}$, the binomial coefficient.
(v) For all real values of x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{613}$$

(vi) For all real values of x,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{614}$$

(vii) For all real values of x,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{615}$$