Math 142V Practice for Test 2

You must show all of your work and reasoning to receive full credit.

[10] **1.** Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}}$$

Solution: This is a geometric series:

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{2}{3}\right)^n. \tag{1}$$

Since $\left|-\frac{2}{3}\right| < 1$, the characterization theorem for geometric series indicates that the series is $\boxed{\text{convergent}}$, and

$$\sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{2}{3}\right)^n = \frac{\left(\frac{1}{3}\right)}{1-\left(-\frac{2}{3}\right)} = \boxed{\frac{1}{5}}. \tag{2}$$

□

[15] **2.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Solution: There are multiple ways to handle this one; the easiest is by noting that

$$\frac{1}{n^2+1} < \frac{1}{n^2}. \tag{3}$$

Additionally, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Thus, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is also $\boxed{\text{convergent}}$. $\square$

[15] **3.** Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{10}\right)}{10^n}$$

Solution: The terms of this series are not all positive, but this not an alternating series. The easiest way to handle this is by checking for absolute convergence, which requires studying the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{10}\right)}{10^n} \right| = \sum_{n=1}^{\infty} \frac{\left|\sin\left(\frac{n\pi}{10}\right)\right|}{10^n}. \tag{4}$$

We note that

$$\frac{\left|\sin\left(\frac{n\pi}{10}\right)\right|}{10^n} \leq \frac{1}{10^n} = \left(\frac{1}{10}\right)^n. \tag{5}$$

Further, $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ is a convergent geometric series. By the comparison test, this indicates that $\sum_{n=1}^{\infty} \left|\frac{\sin\left(\frac{n\pi}{10}\right)}{10^n}\right|$ is convergent, and so $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{10}\right)}{10^n}$ is absolutely convergent, which implies that it is $\boxed{\text{convergent}}$. □

[15] **4.** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$$

Solution: Concerning absolute convergence, we check the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[5]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{5}}}. \tag{6}$$

This is a divergent p-series. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$ cannot be absolutely convergent. However, if we consider the absolute values of the terms,

$$\frac{1}{\sqrt[5]{n}} > \frac{1}{\sqrt[5]{n+1}}, \tag{7}$$

and $\lim_{n\to\infty} \frac{1}{\sqrt[5]{n}} = 0$. These two facts indicate that the alternating series test applies, revealing that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$ is convergent. Ergo, it is $\boxed{\text{conditionally convergent}}$. $\square$

[15] **5.** Determine the radius and interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}}$$

Solution: As usual, we need the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{(2x-1)^{n+1}}{\sqrt{n+1}} \right)}{\left(\frac{(2x-1)^n}{\sqrt{n}} \right)} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |2x-1| = |2x-1|. \tag{8}$$

The ratio test guarantees absolute convergence if $|2x-1| < 1$, or in other words, if $\left| x - \frac{1}{2} \right| < \frac{1}{2}$. This shows that the radius of convergence of the power series is $\boxed{\frac{1}{2}}$. We know, therefore, that $(0, 1)$ is at least part of the interval of convergence, but we must now check the endpoints. When $x = 0$:

$$\sum_{n=1}^{\infty} \frac{(2(0)-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}. \tag{9}$$

The alternating series test indicates that this series is convergent, so 0 is contained in the interval of convergence. When $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(2(1)-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}. \tag{10}$$

This is a divergent p-series, and so 1 is not contained in the interval of convergence. We deduce that the interval of convergence is $\boxed{[0, 1)}$. $\square$

[15] **6.** Find a power series representation for the function and determine its radius of convergence.

$$f(x) = \frac{x^2}{x^2+1}$$

Solution: We note that

$$f(x) = x^2\left(\frac{1}{x^2+1}\right) = x^2\left(\frac{1}{1-(-x^2)}\right) \tag{11}$$

By the characterization theorem for geometric series, if $|-x^2| < 1$, then

$$f(x) = x^2\sum_{n=0}^{\infty}\left(-x^2\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{2n+2}}. \tag{12}$$

Further, $|-x^2| < 1$ is equivalent to $|x| < 1$, which reveals that the radius of convergence is $\boxed{1}$. □

[15] **7.** Find the Taylor series for f centered at the given value of a, and find the associated radius of convergence.

$$f(x) = \ln(2x), \qquad a = 2$$

Solution: As always, the definition of the Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, and so we proceed to find $f^{(n)}(a)$:

$$\begin{array}{cc} f^{(0)}(x) = \ln(2x) & f^{(0)}(2) = \ln 4 \\ f^{(1)}(x) = \frac{1}{x} & f^{(1)}(2) = \frac{1}{2} \\ f^{(2)}(x) = -\frac{1}{x^2} & f^{(2)}(2) = -\frac{1}{2^2} \\ f^{(3)}(x) = \frac{2}{x^3} & f^{(3)}(2) = \frac{2}{2^3} \\ f^{(4)}(x) = -\frac{(2)(3)}{x^4} & f^{(4)}(2) = -\frac{(2)(3)}{2^4} \end{array}. \tag{13}$$

Based on this, we deduce the pattern

$$f^{(n)}(2) = \begin{cases} \ln 4 & \text{if } n = 0 \\ (-1)^{n+1}\frac{(n-1)!}{2^n} & \text{if } n > 0 \end{cases}. \tag{14}$$

Therefore, the Taylor series is

$$\ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!2^n}(x-2)^n = \boxed{\ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n}. \tag{15}$$

We can find the radius of convergence by the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(-1)^{n+1}(x-2)^n} \right| = \lim_{n\to\infty} \frac{n}{2(n+1)}|x-2| = \frac{1}{2}|x-1|. \tag{16}$$

The ratio test guarantees absolute convergence when $\frac{1}{2}|x-1| < 1$, or in other words, when $|x-1| < 2$. Thus, the radius of convergence is $\boxed{2}$. □