Math 142 Practice for Test 1

You must show all of your work and reasoning to receive full credit.

[15] **1.** Evaluate the definite integral.

$$\int_0^{\frac{1}{2}} \sin^{-1} x \, \mathrm{d}x$$

Solution: This can be done by integration by parts:

$$\begin{aligned} u &= \sin^{-1} x \qquad & \mathrm{d}v &= \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \qquad & v &= x \end{aligned} \tag{1}$$

$$x \sin^{-1} x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x. \tag{2}$$

The new integral can now be done by a u-substitution (however, since u is already being used, we will call this new variable w):

$$\begin{aligned} w &= 1 - x^2 \qquad & x = 0 &\Rightarrow w = 1 \\ \mathrm{d}w &= -2x \, \mathrm{d}x \qquad & x = \tfrac{1}{2} &\Rightarrow w = \tfrac{3}{4} \end{aligned} \tag{3}$$

$$\begin{aligned} x \sin^{-1} x \Big|_0^{\frac{1}{2}} + \int_1^{\frac{3}{4}} \frac{1}{2\sqrt{w}} \, \mathrm{d}w &= x \sin^{-1} x \Big|_0^{\frac{1}{2}} + \frac{1}{2} \int_1^{\frac{3}{4}} w^{-\frac{1}{2}} \, \mathrm{d}w \\ = x \sin^{-1} x \Big|_0^{\frac{1}{2}} + \sqrt{w} \Big|_1^{\frac{3}{4}} &= \left(\frac{1}{2} \frac{\pi}{6} - 0 \right) + \left(\frac{\sqrt{3}}{2} - 1 \right) = \boxed{\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1}. \end{aligned} \tag{4}$$

[15] **2.** Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} \tan^3 t \; \mathrm{d}t$$

Solution: We can use the trigonometric identity $\sec^2 t - \tan^2 = 1$ to re-write this integral as

$$\int_0^{\frac{\pi}{4}} \left(\sec^2 t - 1\right) \tan t \; \mathrm{d}t = \int_0^{\frac{\pi}{4}} \tan t \sec^2 t \; \mathrm{d}t - \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t} \; \mathrm{d}t \tag{5}$$

Both of these can be handled by u-substitution, although the definitions of u will be different for each separate integral:

$$\begin{matrix} u_1 = \tan t & t = 0 \Rightarrow u_1 = 0 \\ \mathrm{d}u_1 = \sec^2 t \; \mathrm{d}t & t = \frac{\pi}{4} \Rightarrow u_1 = 1 \end{matrix}, \qquad \begin{matrix} u_2 = \cos t & t = 0 \Rightarrow u_2 = 1 \\ \mathrm{d}u_2 = -\sin t \; \mathrm{d}t & t = \frac{\pi}{4} \Rightarrow u_2 = \frac{1}{\sqrt{2}} \end{matrix} \tag{6}$$

$$\int_0^1 u_1 \; \mathrm{d}u + \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u_2} \; \mathrm{d}u_2 = \frac{1}{2} {u_1}^2 \Big|_0^1 + \ln |u_2| \Big|_1^{\frac{1}{\sqrt{2}}} = \boxed{\frac{1 - \ln 2}{2}}. \tag{7}$$

[15] **3.** Evaluate the indefinite integral.

$$\int x^2 \left(9 - x^2\right)^{-\frac{3}{2}} \, \mathrm{d}x$$

Solution: We proceed by trigonometric substitution:

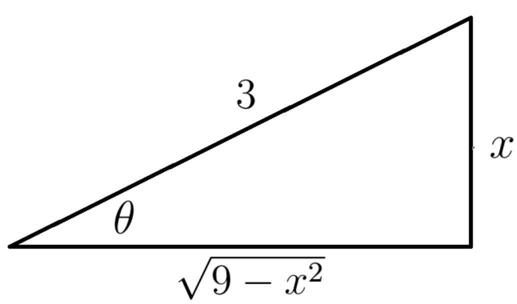
$$\begin{aligned} x &= 3\sin\theta \\ \mathrm{d}x &= 3\cos\theta \, \mathrm{d}\theta \end{aligned} \tag{8}$$

$$\int \frac{9\sin^2\theta}{\left(9 - 9\sin^2\theta\right)^{\frac{3}{2}}} 3\cos\theta \, \mathrm{d}\theta = \int \frac{9\sin^2\theta}{\left(9\left(1 - \sin^2\theta\right)\right)^{\frac{3}{2}}} 3\cos\theta \, \mathrm{d}\theta. \tag{9}$$

By the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$,

$$\begin{aligned} \int \frac{9\sin^2\theta}{(9\cos^2\theta)^{\frac{3}{2}}} 3\cos\theta \, \mathrm{d}\theta &= \int \frac{9\sin^2\theta}{27\cos^3\theta} 3\cos\theta \, \mathrm{d}\theta = \int \frac{\sin^2\theta}{\cos^2\theta} \, \mathrm{d}\theta \\ &= \int \frac{1 - \cos^2\theta}{\cos^2\theta} \, \mathrm{d}\theta = \int \sec^2\theta - 1 \, \mathrm{d}\theta = \tan\theta - \theta + C. \end{aligned} \tag{10}$$

In order to put this into terms of x, we construct a right triangle diagram based on the original substitution $\sin\theta = \frac{x}{3}$:



The diagram indicates that $\tan\theta = \frac{x}{\sqrt{9-x^2}}$, so this becomes:

$$\boxed{\frac{x}{\sqrt{9-x^2}} - \sin^{-1}\left(\frac{x}{3}\right) + C}. \tag{11}$$

[15] **4.** Evaluate the indefinite integral.

$$\int \frac{2x^3+2x^2-2x+2}{x^4-2x^2+1}\,\mathrm{d}x$$

Solution:

$$\frac{2x^3+2x^2-2x+2}{x^4-2x^2+1} = \frac{2x^3+2x^2-2x+2}{(x-1)^2(x+1)^2}. \tag{12}$$

We now proceed by partial fraction decomposition:

$$\frac{2x^3+2x^2-2x+2}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \tag{13}$$

$$\begin{aligned} 2x^3+2x^2-3x+3 &= A\left(x-1\right)\left(x+1\right)^2+B(x+1)^2+C\left(x+1\right)\left(x-1\right)^2+D(x-1)^2 \\ &= \left(A+C\right)x^3+\left(A+B-C+D\right)x^2+\left(-A+2B-C-2D\right)x+\left(-A+B+C+D\right). \end{aligned} \tag{14}$$

By equating corresponding coefficients, we get the system of equations

$$\begin{aligned} A+C&=2 \\ A+B-C+D&=2 \\ -A+2B-C-2D&=-2 \\ -A+B+C+D&=2 \end{aligned} \tag{15}$$

By adding together the second and fourth equations, we get $2B+2D=4$, or $B+D=2$. In that case, the fourth equation can be rewritten as $-A+C+2=2$, or $A=C$. The first equation now indicates that $A=C=1$, and so the equation becomes $-1+2B-1-2D=-2$, or $B-D=0$, thus $B=D$. As $B+D=2$, this indicates that $B=D=1$.

$$\int \frac{1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x+1}+\frac{1}{(x+1)^2}\,\mathrm{d}x = \boxed{\ln|x-1| - \tfrac{1}{x-1} + \ln|x+1| - \tfrac{1}{x+1} + C}. \tag{16}$$

[20] **5.** Evaluate the indefinite integral.

$$\int x^3 \ln\left(x^2+1\right)\ \mathrm{d}x$$

Solution: We begin with a u-substitution:

$$\begin{aligned} u &= x^2+1 \\ \mathrm{d}u &= 2x\ \mathrm{d}x \end{aligned} \tag{17}$$

This indicates that $u-1=x^2$, so the integral becomes

$$\frac{1}{2}\int (u-1)\ln u\ \mathrm{d}u. \tag{18}$$

Now we do integration by parts (since u is already being used, we will refer to the parts as w and $\mathrm{d}v$):

$$\begin{array}{cc} w=\ln u & \mathrm{d}v = u-1\ \mathrm{d}u \\ \mathrm{d}w = \frac{1}{u}\ \mathrm{d}u & v=\frac{1}{2}u^2-u \end{array} \tag{19}$$

$$\frac{1}{2}\left(\frac{u\left(u-1\right)}{2}\ln u - \int \frac{1}{2}u - 1\ \mathrm{d}u\right) = \frac{1}{2}\left(\frac{u\left(u-1\right)}{2}\ln u - \frac{1}{4}u^2+u+C\right) \tag{20}$$

We now put this into terms of x:

$$\boxed{\tfrac{x^2\left(x^2+1\right)}{4}\ln\left(x^2+1\right)-\tfrac{1}{4}\left(x^2+1\right)^2+x^2+1+C}. \tag{21}$$

[20] **6.** Find all real values of p such that the integral

$$\int_1^2 \frac{1}{x(\ln x)^p}\,\mathrm{d}x$$

is convergent.

Solution: First, we note that the integral is improper, since $\ln 1 = 0$, and so the integrand is not continuous at $x = 1$. By definition, then, this is

$$\lim_{s\to 1^+} \int_s^2 \frac{1}{x(\ln x)^p}\,\mathrm{d}x. \tag{22}$$

To evaluate this integral, we use the following u-substitution:

$$\begin{aligned} u &= \ln x & x = s &\Rightarrow u = \ln s \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x & x = 2 &\Rightarrow u = \ln 2 \end{aligned} \tag{23}$$

$$\lim_{s\to 1^+} \int_{\ln s}^{\ln 2} \frac{1}{u^p}\,\mathrm{d}u = \lim_{s\to 1^+} \int_{\ln s}^{\ln 2} u^{-p}\,\mathrm{d}u. \tag{24}$$

If $p = 1$, this becomes

$$\lim_{s\to 1^+} \ln u \Big|_{\ln s}^{\ln 2} = \lim_{s\to 1^+} \ln(\ln 2) - \ln(\ln s) = \infty, \tag{25}$$

so the integral is divergent if $p = 1$.

If $p \neq 1$, the integral becomes

$$\lim_{s\to 1^+} \frac{u^{1-p}}{1-p}\Big|_{\ln s}^{\ln 2} = \frac{1}{1-p} \lim_{s\to 1^+} \left((\ln 2)^{1-p} - (\ln s)^{1-p}\right). \tag{26}$$

If $1 - p < 0$, then $\lim\limits_{s\to 1^+} (\ln s)^{1-p} = \infty$, in which case the integral diverges. If $1 - p > 0$, then $\lim\limits_{s\to 1^+} (\ln s)^{1-p} = 0$, in which case the integral converges to $\frac{(\ln 2)^{1-p}}{1-p}$. Thus, the integral is convergent if and only if $p < 1$.