Math 142: College Calculus II

Mark Sullivan

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Calculus II is organized into four parts, corresponding to four goals:

1. To provide practical strategies for solving integrals.
2. To apply integration strategies to problems in geometry.
3. To extend calculus to curves that are not described by functions.
4. To introduce the theory of infinite series.

In Calculus I, we explained the theory of the solutions of the tangent line problem and the area problem. These were related by the fundamental theorem of calculus, which states:

Theorem 1.1 *(Fundamental theorem of calculus) If f is a continuous function on the interval $[a, b]$, then the following statements are true.*
(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\ \mathrm{d}t = f(x). \tag{1}$$

(ii) If F is an antiderivative of f, then

$$\int_a^b f(x)\ \mathrm{d}x = F(b) - F(a). \tag{2}$$

This tells us that integrals can be solved by using antiderivatives. However, finding antiderivatives is **much** harder than finding derivatives. Therefore, the first part of Calculus II will focus on strategies for finding antiderivatives. First, we review what we know from Calculus I.

Let k be a real-valued constant.

$$\begin{array}{cl}
\int 1\ \mathrm{d}x = x + C & \\
\int x^k\ \mathrm{d}x = \frac{x^{k+1}}{k+1} + C & \text{if } k \neq -1 \\
\int \frac{1}{x}\ \mathrm{d}x = \ln|x| + C & \\
\int e^{kx}\ \mathrm{d}x = \frac{e^{kx}}{k} + C & \\
\int \cos x\ \mathrm{d}x = \sin x + C & \\
\int \sin x\ \mathrm{d}x = -\cos x + C & \\
\int \sec^2 x\ \mathrm{d}x = \tan x + C & \\
\int \sec x \tan x\ \mathrm{d}x = \sec x + C & \\
\int \csc x \cot x\ \mathrm{d}x = -\csc x + C & \\
\int \csc^2 x\ \mathrm{d}x = -\cot x + C & \\
\int \frac{1}{x^2+1}\ \mathrm{d}x = \tan^{-1} x + C & \\
\int \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x = \sin^{-1} x + C & \\
\int \frac{1}{x\sqrt{x^2-1}}\ \mathrm{d}x = \sec^{-1} x + C &
\end{array} \tag{3}$$

We also know that

$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x \tag{4}$$

and

$$\int f(x)+g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x. \tag{5}$$

As for definite integrals, we also [hopefully] learned:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{6}$$

and if $a \leq t \leq b$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^t f(x)\,\mathrm{d}x + \int_t^b f(x)\,\mathrm{d}x. \tag{7}$$

The last thing we learned in Calculus I was u-substitution.

Example 1.2 *Evaluate the integral.*

$$\int \frac{x}{x^2+1}\,\mathrm{d}x \tag{8}$$

Set

$$\begin{aligned} u &= x^2+1 \\ \mathrm{d}u &= 2x\,\mathrm{d}x \end{aligned}. \tag{9}$$

Therefore, the integral becomes

$$\int \frac{1}{2}\frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C = \boxed{\tfrac{1}{2}\ln|x^2+1|+C}. \tag{10}$$

□

Example 1.3 *Evaluate the integral.*

$$\int_0^{\frac{\pi}{2}} e^{\sin x}\cos x\,\mathrm{d}x \tag{11}$$

Set

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x \end{aligned}. \tag{12}$$

The integral becomes

$$\int_0^1 e^u \, \mathrm{d}u = e^u \Big|_0^1 = e^1 - e^0 = \boxed{e - 1}. \tag{13}$$

□

Example 1.4 *Evaluate the integral.*

$$\int_{\frac{1}{2}}^1 \frac{e^{\left(\frac{1}{x}\right)}}{x^2} \, \mathrm{d}x \tag{14}$$

Set

$$\begin{aligned} u &= \tfrac{1}{x} \\ \mathrm{d}u &= -\tfrac{1}{x^2} \, \mathrm{d}x \end{aligned}. \tag{15}$$

The integral becomes

$$\int_2^1 -e^u \, \mathrm{d}u = -\int_2^1 e^u \, \mathrm{d}u = \int_1^2 e^u \, \mathrm{d}u = e^u \Big|_1^2 = e^2 - e = \boxed{e\,(e-1)}. \tag{16}$$

□

Example 1.5

$$\int \frac{\ln x}{x} \, \mathrm{d}x \tag{17}$$

Set

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x} \, \mathrm{d}x \end{aligned}. \tag{18}$$

The integral becomes

$$\int u \, \mathrm{d}u = \frac{1}{2}u^2 + C = \boxed{\tfrac{1}{2}(\ln x)^2 + C}. \tag{19}$$

□

Example 1.6

$$\int \cot x \, \mathrm{d}x \tag{20}$$

Notice that

$$\int \cot x \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x. \tag{21}$$

Set

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x \end{aligned} \tag{22}$$

The integral becomes

$$\int \frac{1}{u} \, \mathrm{d}u = \ln |u| + C = \boxed{\ln |\sin x| + C}. \tag{23}$$

Does u-substitution always work? **Absolutely not**.

Example 1.7 *Consider the integrals*

$$\int x \sin x \, \mathrm{d}x \tag{24}$$

$$\int x \ln x \, \mathrm{d}x \tag{25}$$

$$\int x^2 e^x \, \mathrm{d}x \tag{26}$$

$$\int \ln x \, \mathrm{d}x \tag{27}$$

$$\int e^x \cos x \, \mathrm{d}x \tag{28}$$

We need a new method to solve these integrals. □

Section 7.1: Integration by parts

The product rule for derivatives is:

$$\frac{\mathrm{d}}{\mathrm{d}x}uv = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{29}$$

What if we took the antiderivative of both sides?

$$uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x. \tag{30}$$

However, we often write $\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$ as $\mathrm{d}v$, and $\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$ as $\mathrm{d}u$. Therefore, we can re-arrange and write this as

$$\boxed{\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u}. \tag{31}$$

This is called "integration by parts." We will illustrate this method of integration with an example.

Example 1.8 *Evaluate the integral*

$$\int x\sin x\,\mathrm{d}x. \tag{32}$$

Integration by parts is most effective when we can write the integrand as a product of two familiar functions, as above. Select one of these functions and name it u*. Select the other function and* $\mathrm{d}x$ *and name that* $\mathrm{d}v$*.*

$$u = x \quad \mathrm{d}v = \sin x\,\mathrm{d}x. \tag{33}$$

Now, take the derivative of u *and the antiderivative of* $\mathrm{d}v$*.*

$$\mathrm{d}u = \mathrm{d}x \quad v = -\cos x. \tag{34}$$

Now plug these "parts" into the integration by parts formula.

$$\begin{gathered}\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \\ \int x\sin x\,\mathrm{d}x = -x\cos x - \int -\cos x\,\mathrm{d}x.\end{gathered} \tag{35}$$

Notice that we can solve the integral on the right:

$$\int x \sin x \, \mathrm{d}x = -x \cos x + \int \cos x \, \mathrm{d}x = \boxed{-x \cos x + \sin x + C}. \tag{36}$$

□

Example 1.9 *Evaluate the integral*

$$\int x \ln x \, \mathrm{d}x. \tag{37}$$

We select

$$\begin{aligned} u &= \ln x \qquad \mathrm{d}v = x \, \mathrm{d}x \\ \mathrm{d}u &= \tfrac{1}{x} \, \mathrm{d}x \qquad v = \tfrac{1}{2}x^2 \end{aligned} \tag{38}$$

Now put these into the integration by parts formula:

$$\begin{aligned} \textstyle\int u \, \mathrm{d}v &= uv - \textstyle\int v \, \mathrm{d}u \\ \textstyle\int x \ln x \, \mathrm{d}x &= \tfrac{1}{2}x^2 \ln x - \textstyle\int \tfrac{1}{2}x^2 \tfrac{1}{x} \, \mathrm{d}x \end{aligned} \tag{39}$$

We can solve the integral on the right:

$$\int x \ln x \, \mathrm{d}x = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, \mathrm{d}x = \boxed{\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C}. \tag{40}$$

□

Sometimes, you need to use integration by parts more than once.

Example 1.10 *Evaluate the integral*

$$\int x^2 e^x \, \mathrm{d}x. \tag{41}$$

We select

$$\begin{aligned} u &= x^2 \qquad \mathrm{d}v = e^x \, \mathrm{d}x \\ \mathrm{d}u &= 2x \, \mathrm{d}x \qquad v = e^x \end{aligned} \tag{42}$$

Now put these into the integration by parts formula:

$$\begin{gathered}\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \\ \int x^2 e^x\,\mathrm{d}x = x^2 e^x - \int 2xe^x\,\mathrm{d}x.\end{gathered} \tag{43}$$

Thus,

$$\int x^2 e^x\,\mathrm{d}x = x^2 e^x - 2\int xe^x\,\mathrm{d}x. \tag{44}$$

To solve the integral on the right, we use integration by parts yet again:

$$\int xe^x\,\mathrm{d}x \tag{45}$$

We select

$$\begin{matrix} u = x & \mathrm{d}v = e^x\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = e^x \end{matrix}. \tag{46}$$

Now,

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x = xe^x - e^x + C. \tag{47}$$

We plug this result in to find

$$\begin{aligned}\int x^2 e^x\,\mathrm{d}x &= x^2 e^x - 2\int xe^x\,\mathrm{d}x \\ &= x^2 e^x - 2\left(xe^x - e^x + C\right) = \boxed{x^2 e^x - 2xe^x + 2e^x - 2C}.\end{aligned} \tag{48}$$

In finding antiderivatives, we are uninterested in the exact value of the constant C. Therefore, whether it is C or $-2C$ makes no difference to us; either way, it is a constant. We call it by the name C_1:

$$\boxed{x^2 e^x - 2xe^x + 2e^x + C_1} \tag{49}$$

□

Sometimes, integration by parts can be weird.

Example 1.11 *Evaluate the integral*

$$\int \ln x \, \mathrm{d}x. \tag{50}$$

We select

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & v = x \end{array}. \tag{51}$$

Now put these into the integration by parts formula:

$$\begin{gathered} \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \\ \int \ln x \, \mathrm{d}x = x \ln x - \int x \tfrac{1}{x} \, \mathrm{d}x \end{gathered} \tag{52}$$

We can solve the integral on the right:

$$\int \ln x \, \mathrm{d}x = x \ln x - \int 1 \, \mathrm{d}x = \boxed{x \ln x - x + C}. \tag{53}$$

Example 1.12 *Evaluate the integral*

$$\int e^x \cos x \, \mathrm{d}x. \tag{54}$$

We select

$$\begin{array}{cc} u = \cos x & \mathrm{d}v = e^x \, \mathrm{d}x \\ \mathrm{d}u = -\sin x \, \mathrm{d}x & v = e^x \end{array} \tag{55}$$

Putting this into the integration by parts formula,

$$\begin{gathered} \int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \\ \int e^x \cos x \, \mathrm{d}x = e^x \cos x + \int e^x \sin x \, \mathrm{d}x \end{gathered} \tag{56}$$

We still cannot solve the integral on the right, so we try integration by parts again:

$$\begin{array}{cc} u = \sin x & \mathrm{d}v = e^x \, \mathrm{d}x \\ \mathrm{d}u = \cos x \, \mathrm{d}x & v = e^x \end{array}. \tag{57}$$

Now we get

$$\int e^x \cos x \, \mathrm{d}x = e^x \cos x + \left(e^x \sin x - \int e^x \cos x \, \mathrm{d}x \right) \tag{58}$$

Ergo,

$$2 \int e^x \cos x \, \mathrm{d}x = e^x \cos x + e^x \sin x + C, \tag{59}$$

and so

$$\int e^x \cos x \, \mathrm{d}x = \boxed{\tfrac{1}{2} e^x \left(\cos x + \sin x \right) + C_1}. \tag{60}$$

□

For a definite integral, the integration by parts formula becomes

$$\int_a^b u \, \mathrm{d}v = uv \Big|_a^b - \int_a^b v \, \mathrm{d}u. \tag{61}$$

Example 1.13 *(Problem 7.1.26) Evaluate the integral.*

$$\int_1^2 x^2 \ln x \, \mathrm{d}x \tag{62}$$

We select

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = x^2 \, \mathrm{d}x \\ \mathrm{d}u = \frac{1}{x} \, \mathrm{d}x & v = \frac{1}{3} x^3. \end{array} \tag{63}$$

Now we put these into the integration by parts formula:

$$\begin{array}{c} \int_a^b u \, \mathrm{d}v = uv\big|_a^b - \int_a^b v \, \mathrm{d}u \\ \int_1^2 x^2 \ln x \, \mathrm{d}x = \frac{1}{3} x^3 \ln x \big|_1^2 - \int_1^2 \frac{1}{3} x^3 \frac{1}{x} \, \mathrm{d}x. \end{array} \tag{64}$$

We can solve the integral on the right:

$$\int_1^2 x^2 \ln x \,\mathrm{d}x = \frac{1}{3}x^3 \ln x\Big|_1^2 - \frac{1}{3}\int_1^2 x^2 \,\mathrm{d}x$$
$$= \frac{1}{3}x^3 \ln x\Big|_1^2 - \frac{1}{3}\left(\frac{1}{3}x^3\Big|_1^2\right) = \left(\frac{8}{3}\ln 2 - \frac{1}{3}\ln 1\right) - \frac{1}{3}\left(\frac{1}{3}2^3 - \frac{1}{3}1^3\right)$$
$$= \frac{8}{3}\ln 2 - \frac{7}{9} \quad (65)$$

□

Does integration by parts always work? **NO**.

Example 1.14

$$\int \sin^2 x \cos^2 x \,\mathrm{d}x. \quad (66)$$

This cannot be done by parts. □

Section 7.2: Trigonometric integrals

We need some strategies for integrals involving trigonometric functions.

Example 1.15 *Evaluate the integral.*

$$\int \sec x \, \mathrm{d}x \tag{67}$$

In order to do this, we first multiply by $\frac{\sec x + \tan x}{\tan x + \sec x}$*:*

$$\int \sec x \left(\frac{\sec x + \tan x}{\tan x + \sec x} \right) \, \mathrm{d}x = \int \frac{\sec^2 x + \sec x \, \tan x}{\tan x + \sec x} \, \mathrm{d}x. \tag{68}$$

Now we use u*-substitution:*

$$\begin{aligned} u &= \tan x + \sec x \\ \mathrm{d}u &= \sec^2 x + \sec x \, \tan x \end{aligned}. \tag{69}$$

The integral now becomes

$$\int \frac{1}{u} \, \mathrm{d}u = \ln |u| + C = \boxed{\ln \left| \sec x + \tan x \right| + C}. \tag{70}$$

□

How should we integrate

$$\int \sin^m x \cos^n x \, \mathrm{d}x \tag{71}$$

where m and n are non-negative integers? It depends on which of m and n are even, and which are odd.

Strategy Consider the integral above. We have three cases to consider.

Case 1: n is odd. In that case, $n = 2k + 1$ for some integer k. Therefore, we have

$$\int \sin^m x \cos^{2k+1} x \, \mathrm{d}x = \int \sin^m x \, \left(\cos^2 x \right)^k \, \cos x \, \mathrm{d}x. \tag{72}$$

At this point, we can use the trigonometric identity $\sin^2 x + \cos^2 x = 1$. This identity implies that $\cos^2 x = 1 - \sin^2 x$. Therefore,

$$\int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, \mathrm{d}x. \tag{73}$$

At this point, we can use u-substitution.

Case 2: n is even, and m is odd. In that case, $m = 2l + 1$ for some integer l. Therefore, we have

$$\int \sin^{2l+1} x \cos^n x \, \mathrm{d}x = \int \left(\sin^2 x\right)^l \cos^n x \ \sin x \, \mathrm{d}x. \tag{74}$$

Again, we can use $\cos^2 x + \sin^2 x = 1$, so we get

$$\int \left(1 - \cos^2 x\right)^l \cos^n x \ \sin x \, \mathrm{d}x. \tag{75}$$

At this point, we can use u-substitution.

Case 3: m and n are both even. In this case, $m = 2l$ and $n = 2k$ for some integers k and l. Therefore, we have

$$\int \left(\sin^2 x\right)^l \left(\cos^2 x\right)^k \mathrm{d}x. \tag{76}$$

We need to use the trigonometric identities

$$\begin{aligned} \sin^2 x &= \tfrac{1}{2}\left(1 - \cos\left(2x\right)\right) \\ \cos^2 x &= \tfrac{1}{2}\left(1 + \cos\left(2x\right)\right). \\ \sin x \cos x &= \tfrac{1}{2}\sin\left(2x\right) \end{aligned} \tag{77}$$

Distribute, and reduce further if necessary.

Example 1.16 *Evaluate the integral*

$$\int \sin^4 x \cos^5 x \, \mathrm{d}x. \tag{78}$$

This is Case 1. Therefore, we separate one factor of cosine:

$$\int \sin^4 x \cos^4 x \; \cos x \, \mathrm{d}x \tag{79}$$

Now we use the identity $\sin^2 x + \cos^2 x = 1$*:*

$$\int \sin^4 x \, \left(1 - \sin^2 x\right)^2 \; \cos x \, \mathrm{d}x. \tag{80}$$

We now use u*-substitution:*

$$\begin{aligned} u &= \sin x \\ \mathrm{d}u &= \cos x \, \mathrm{d}x, \end{aligned} \tag{81}$$

so this becomes

$$\int u^4 \left(1 - u^2\right)^2 \mathrm{d}u. \tag{82}$$

We distribute:

$$\int u^4 \left(1 - 2u^2 + u^4\right) \; \mathrm{d}u = \int u^4 - 2u^6 + u^8 \, \mathrm{d}u, \tag{83}$$

and take the antiderivative:

$$\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C. \tag{84}$$

In terms of x*, this becomes:*

$$\boxed{\tfrac{1}{5}\sin^5 x - \tfrac{2}{7}\sin^7 x + \tfrac{1}{9}\sin^9 x + C}. \tag{85}$$

□

Example 1.17 *(Problem 7.2.2) Evaluate the integral.*

$$\int \sin^3 \theta \cos^4 \theta \, \mathrm{d}\theta. \tag{86}$$

This is Case 2. Now we can separate one factor of sine:

$$\int \sin^2\theta \, \cos^4\theta \; \sin\theta \, \mathrm{d}\theta. \tag{87}$$

We apply the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ *to get*

$$\int \left(1 - \cos^2\theta\right) \cos^4\theta \; \sin\theta \, \mathrm{d}\theta. \tag{88}$$

Now we use u-substitution:

$$\begin{aligned} u &= \cos\theta \\ \mathrm{d}u &= -\sin\theta \, \mathrm{d}\theta\text{'} \end{aligned} \tag{89}$$

so this becomes

$$-\int \left(1 - u^2\right) u^4 \, \mathrm{d}u = \int u^6 - u^4 \, \mathrm{d}u = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C. \tag{90}$$

In terms of x, this is

$$\boxed{\tfrac{1}{7}\cos^7\theta - \tfrac{1}{5}\cos^5\theta + C}. \tag{91}$$

□

Example 1.18 *(Problem 7.2.10) Evaluate the integral.*

$$\int_0^\pi \sin^2 t \, \cos^4 t \, \mathrm{d}t \tag{92}$$

This is Case 3. We notice that $\cos t$ *has a larger power, so we will begin by separating those extra factors from the rest of the integrand:*

$$\int_0^\pi (\sin t \cos t)^2 \cos^2 t \, \mathrm{d}t. \tag{93}$$

We use the fact that $\sin t \cos t = \frac{1}{2}\sin(2t)$ *and* $\cos^2 t = \frac{1}{2}(1 + \cos(2t))$.

$$\int_0^\pi \left(\frac{1}{2}\sin(2t)\right)^2 \frac{1}{2}\left(1 + \cos(2t)\right) \, \mathrm{d}t. \tag{94}$$

We distribute:

$$\frac{1}{8}\int_0^{\pi} \sin^2(2t) + \sin^2(2t)\cos(2t)\ \mathrm{d}t. \tag{95}$$

We can now separate the integral into two terms:

$$\frac{1}{8}\int_0^{\pi} \sin^2(2t)\ \mathrm{d}t + \frac{1}{8}\int_0^{\pi} \sin^2(2t)\cos(2t)\ \mathrm{d}t. \tag{96}$$

For the left integral, we will use the trigonometric identity $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$*:*

$$\frac{1}{8}\int_0^{\pi} \frac{1}{2}(1-\cos(4t))\ \mathrm{d}t + \frac{1}{8}\int_0^{\pi} \sin^2(2t)\cos(2t)\ \mathrm{d}t \tag{97}$$

The right integral can be done by u*-substitution:*

$$\begin{aligned} u &= \sin(2t) \\ \mathrm{d}u &= 2\cos(2t)\,\mathrm{d}t\text{'} \end{aligned} \tag{98}$$

giving us

$$\frac{1}{16}\int_0^{\pi} 1-\cos(4t)\ \mathrm{d}t + \frac{1}{16}\int_0^{0} u^2\ \mathrm{d}u \tag{99}$$

Any definite integral with equal bounds is zero, so this becomes

$$\frac{1}{16}\int_0^{\pi} 1-\cos(4t)\ \mathrm{d}t \tag{100}$$

Taking the antiderivative, we get

$$\frac{1}{16}\left(t - \frac{1}{4}\sin(4t)\right)\Bigg|_0^{\pi} = \boxed{\tfrac{\pi}{16}}. \tag{101}$$

$\square$

2 Wednesday, May 29

How should we integrate

$$\int \tan^m x \sec^n x \, \mathrm{d}x, \tag{102}$$

where m and n are non-negative integers? This also depends on which of m and n are even, and which are odd.

Strategy Consider the integral above. We have two important cases to consider.

Case 1: n is even. In that case, $n = 2k$ for some integer k. Therefore, we have

$$\int \tan^m x \sec^{2k} x \, \mathrm{d}x = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \, \mathrm{d}x \tag{103}$$

Now we can use the trigonometric identity $\tan^2 x + 1 = \sec^2 x$. This now becomes

$$\int \tan^m x \left(\tan^2 x + 1\right)^{k-1} \sec^2 x \, \mathrm{d}x. \tag{104}$$

At this point, we can use u-substitution.

Case 2: m is odd, and $n \geq 1$. In that case, $m = 2l + 1$ for some integer l. Therefore, we have

$$\int \left(\tan^2 x\right)^l \sec^{n-1} x \ \sec x \tan x \, \mathrm{d}x. \tag{105}$$

Again, we use the trigonometric identity $\tan^2 x + 1 = \sec^2 x$. This becomes

$$\int \left(\sec^2 x - 1\right)^l \sec^{n-1} x \ \sec x \tan x \, \mathrm{d}x. \tag{106}$$

Now we can use u-substitution.

Example 2.1 *(Problem 7.2.22) Evaluate the integral*

$$\int \tan^2 \theta \sec^4 \theta \, \mathrm{d}\theta \tag{107}$$

This is Case 1. We can write this as

$$\int \tan^2\theta \sec^2\theta \sec^2\theta \, \mathrm{d}\theta. \tag{108}$$

Using the identity $\tan^2\theta + 1 = \sec^2\theta$*, this becomes*

$$\int \tan^2\theta \left(\tan^2\theta + 1\right) \sec^2\theta \, \mathrm{d}\theta. \tag{109}$$

We now proceed by u*-substitution:*

$$u = \tan\theta \mathrm{d}u = \sec^2\theta \, \mathrm{d}\theta, \tag{110}$$

which gives

$$\int u^2 \left(u^2 + 1\right) \, \mathrm{d}u = \int u^4 + u^2 \, \mathrm{d}u = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C, \tag{111}$$

which, in terms of x*, is*

$$\boxed{\tfrac{1}{5}\tan^5\theta + \tfrac{1}{3}\tan^3\theta + C}. \tag{112}$$

□

Example 2.2 *(Problem 7.2.28) Evaluate the integral*

$$\int \tan^5 x \sec^3 x \, \mathrm{d}x \tag{113}$$

This is Case 2. We can write this as

$$\int \left(\tan^2 x\right)^2 \sec^2 x \sec x \tan x \, \mathrm{d}x \tag{114}$$

We use the identity $\tan^2 x + 1 = \sec^2 x$ *to get*

$$\int \left(\sec^2 x - 1\right)^2 \sec^2 x \ \sec x \tan x \, \mathrm{d}x. \tag{115}$$

Now we use u-substitution:

$$\begin{aligned} u &= \sec x \\ \mathrm{d}u &= \sec x \tan x \, \mathrm{d}x \end{aligned}, \tag{116}$$

to get

$$\int \left(u^2 - 1\right)^2 u^2 \, \mathrm{d}u = \int \left(u^4 - 2u^2 + 1\right) u^2 \, \mathrm{d}u = \int u^6 - 2u^4 + u^2 \, \mathrm{d}u. \tag{117}$$

Taking the antiderivative,

$$\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C. \tag{118}$$

In terms of x, this becomes

$$\boxed{\tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C}. \tag{119}$$

□

Integrals that fall into neither case will require creativity to solve.

Example 2.3 *Find*

$$\int \sec^3 x \, \mathrm{d}x. \tag{120}$$

This falls into neither case. In order to do this, we need to use integration by parts:

$$\begin{array}{cc} u = \sec x & \mathrm{d}v = \sec^2 x \, \mathrm{d}x \\ \mathrm{d}u = \sec x \tan x \, \mathrm{d}x & v = \tan x \end{array}. \tag{121}$$

The integral becomes

$$\int \sec^3 x \, \mathrm{d}x = \sec x \tan x - \int \tan^2 x \sec x \, \mathrm{d}x. \tag{122}$$

We apply the trigonometric identity $\tan^2 x + 1 = \sec^2 x$*:*

$$\begin{aligned}\int \sec^3 x \, \mathrm{d}x &= \sec x \tan x - \int \left(\sec^2 x - 1\right) \sec x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x - \sec x \, \mathrm{d}x \\ &= \sec x \tan x - \int \sec^3 x \, \mathrm{d}x + \int \sec x \, \mathrm{d}x. \end{aligned} \tag{123}$$

Now we add $\int \sec^3 x \, \mathrm{d}x$ *to both sides:*

$$2 \int \sec^3 x \, \mathrm{d}x = \sec x \tan x + \int \sec x \, \mathrm{d}x. \tag{124}$$

As done in a previous example, $\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C_1$*, so this becomes*

$$\int \sec^3 x \, \mathrm{d}x = \boxed{\tfrac{1}{2} \sec x \tan x + \tfrac{1}{2} \ln|\sec x + \tan x| + C_2}. \tag{125}$$

□

Section 7.3: Trigonometric substitution

When an integral involves one of the following expressions:

$$\begin{aligned} &x^2 - a^2 \\ &x^2 + a^2 \\ &a^2 - x^2 \end{aligned} \tag{126}$$

(where a is any real valued constant), we can use the following strategy to find the integral: substitute $x = f(\theta)$, where f is some trigonometric function.

$$\begin{array}{lc} x^2 - a^2 & x = a\sec\theta \text{ for } 0 \le \theta < \frac{\pi}{2} \\ x^2 + a^2 & x = a\tan\theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \\ a^2 - x^2 & x = a\sin\theta \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{array} \tag{127}$$

The strategy is best illustrated by some examples.

Example 2.4 *(Problem 7.3.8) Evaluate the integral*

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-16}}. \tag{128}$$

This involves the form $x^2 - a^2$, so we substitute:

$$\begin{aligned} x &= 4\sec\theta \\ \mathrm{d}x &= 4\sec\theta\tan\theta\,\mathrm{d}\theta, \end{aligned} \tag{129}$$

where $0 \le \theta < \frac{\pi}{2}$. The integral now becomes

$$\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}\,\mathrm{d}\theta = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\left(\sec^2\theta - 1\right)}}. \tag{130}$$

We use the identity $\sec^2\theta - 1 = \tan^2\theta$*:*

$$\begin{aligned}\int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\,\mathrm{d}\theta &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\;4\tan\theta}\,\mathrm{d}\theta \\ &= \int \frac{1}{16\,\sec\theta}\,\mathrm{d}\theta = \frac{1}{16}\int \cos\theta\,\mathrm{d}\theta = \frac{1}{16}\sin\theta + C. \quad (131)\end{aligned}$$

Now we must put this answer back in terms of x*. By definition, we know that* $x = 4\sec\theta$*. Therefore,* $\cos\theta = \frac{4}{x}$*. We construct a right triangle to illustrate this fact. [Draw diagram]*

This triangle gives us that $\sin\theta = \frac{\sqrt{x^2-16}}{x}$*. Therefore, our answer is*

$$\boxed{\tfrac{1}{16}\tfrac{\sqrt{x^2-16}}{x} + C}. \quad (132)$$

□

Example 2.5 *Evaluate the integral.*

$$\int \sqrt{4 - x^2}\,\mathrm{d}x \quad (133)$$

This involves the form $a^2 - x^2$*, so we make the substitution:*

$$\begin{aligned} x &= 2\sin\theta \\ \mathrm{d}x &= 2\cos\theta\,\mathrm{d}\theta, \end{aligned} \quad (134)$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*. The integral now becomes*

$$\begin{aligned}\int \sqrt{4 - (2\sin\theta)^2}\,2\cos\theta\,\mathrm{d}\theta &= \int \sqrt{4 - 4\sin^2\theta}\,2\cos\theta\,\mathrm{d}\theta \\ &= \int \sqrt{4\left(1 - \sin^2\theta\right)}\,2\cos\theta\,\mathrm{d}\theta. \quad (135)\end{aligned}$$

We use the identity $1 - \sin^2\theta = \cos^2\theta$ *to write this as*

$$\int \sqrt{4\cos^2\theta}\,2\cos\theta\,\mathrm{d}\theta = \int 2\cos\theta\,2\cos\theta\,\mathrm{d}\theta = 4\int \cos^2\theta\,\mathrm{d}\theta. \quad (136)$$

In order to integrate this, we use the identity $\cos^2\theta = \frac{1}{2}\left(1+\cos\left(2\theta\right)\right)$*:*

$$4\int \frac{1}{2}\left(1+\cos\left(2\theta\right)\right)\,\mathrm{d}\theta = 2\int 1+\cos\left(2\theta\right)\,\mathrm{d}\theta$$
$$= 2\left(\theta + \frac{1}{2}\sin\left(2\theta\right)\right) + C = 2\theta + \sin\left(2\theta\right) + C. \quad (137)$$

Again, we must write this in terms of x*. By definition, we know that* $x = 2\sin\theta$*. Therefore,* $\sin\theta = \frac{x}{2}$*. We construct a right triangle to illustrate this fact. [Draw diagram]*

At first, this doesn't seem to help us with $\sin\left(2\theta\right)$*. However, we can write* $\sin\left(2\theta\right) = 2\sin\theta\cos\theta$*, so that now our antiderivative is*

$$2\theta + 2\sin\theta\cos\theta + C. \quad (138)$$

Now, from our triangle, we know that $\sin\theta = \frac{x}{2}$ *and* $\cos\theta = \frac{\sqrt{4-x^2}}{2}$*. At the same time, if* $\sin\theta = \frac{x}{2}$ *and* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*, then* $\theta = \sin^{-1}\left(\frac{x}{2}\right)$*. Therefore, our final answer is*

$$\boxed{2\sin^{-1}\left(\frac{x}{2}\right) + x\frac{\sqrt{4-x^2}}{2} + C}. \quad (139)$$

□

Example 2.6 *Evaluate the integral.*

$$\int \frac{1}{\sqrt{x^2+9}}\,\mathrm{d}x \quad (140)$$

We recognize the form $x^2 + a^2$ *and make the substitution:*

$$\begin{aligned} x &= 3\tan\theta \\ \mathrm{d}x &= 3\sec^2\theta\,\mathrm{d}\theta\text{'} \end{aligned} \quad (141)$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$*. The integral now becomes*

$$\int \frac{3\sec^2\theta}{\sqrt{\left(3\tan\theta\right)^2+9}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{\sqrt{9\left(\tan^2\theta+1\right)}}\,\mathrm{d}\theta. \quad (142)$$

We use the identity $\tan^2\theta + 1 = \sec^2\theta$ *to write this as*

$$\int \frac{3\sec^2\theta}{\sqrt{9\sec^2\theta}}\,\mathrm{d}\theta = \int \frac{3\sec^2\theta}{3\,\sec\theta}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta. \tag{143}$$

We know from a previous example how to integrate $\sec\theta$*:*

$$\int \sec\theta\left(\frac{\sec\theta + \tan\theta}{\tan\theta + \sec\theta}\right)\,\mathrm{d}\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\tan\theta + \sec\theta}\,\mathrm{d}\theta. \tag{144}$$

We can now use u*-substitution:*

$$\begin{aligned} u &= \tan\theta + \sec\theta \\ \mathrm{d}u &= \sec^2\theta + \sec\theta\tan\theta' \end{aligned} \tag{145}$$

to get

$$\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C = \ln\big|\tan\theta + \sec\theta\big| + C. \tag{146}$$

Again, we need to write this in terms of x*. By definition,* $x = 3\tan\theta$*, so we can draw a right triangle to illustrate this fact: [Draw diagram]*

The diagram indicates that $\sec\theta = \frac{\sqrt{x^2+9}}{3}$*, so our final answer becomes*

$$\boxed{\ln\left|\tfrac{x}{3} + \tfrac{\sqrt{x^2+9}}{3}\right| + C} \tag{147}$$

□

Example 2.7 *(Problem 7.3.14) Evaluate the integral*

$$\int_0^1 \frac{1}{(x^2+1)^2}\,\mathrm{d}x. \tag{148}$$

We recognize the form $x^2 + a^2$ *and make the substitution:*

$$\begin{aligned} x &= \tan\theta \\ \mathrm{d}x &= \sec^2\theta\,\mathrm{d}\theta, \end{aligned} \tag{149}$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *The integral now becomes*

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(\tan^2\theta + 1\right)^2}\, \mathrm{d}\theta. \tag{150}$$

(We have changed the bounds by recognizing that $x = 0$ *exactly when* $\theta = 0$ *and* $x = 1$ *exactly when* $\theta = \frac{\pi}{4}$*.) We use the identity* $\tan^2\theta + 1 = \sec^2\theta$ *to write this as*

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\left(\sec^2\theta\right)^2}\, \mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\, \mathrm{d}\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\, \mathrm{d}\theta. \tag{151}$$

In order to integrate this, we use the identity $\cos^2\theta = \frac{1}{2}\left(1 + \cos\left(2\theta\right)\right)$*:*

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1 + \cos\left(2\theta\right)\right)\, \mathrm{d}\theta &= \frac{1}{2}\left(\theta + \frac{1}{2}\sin\left(2\theta\right)\right)\Bigg|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\left(2\frac{\pi}{4}\right)\right) - \frac{1}{2}\left(0 + \frac{1}{2}\sin\left(0\right)\right) \\ &= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{8} + \frac{1}{4} = \boxed{\tfrac{\pi+2}{8}}. \end{aligned} \tag{152}$$

(For this definite integral, no triangle diagram is needed, because we have changed the bounds.) □

3 Thursday, May 30

Section 7.4: Integration of rational functions by partial fractions

Definition 3.1 *A rational function is a ratio $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.*

There is a method of integrating rational functions by factoring the denominator and then writing the function as a sum of fractions. This method is called "integration by partial fraction decomposition." It has three cases:

Case 1: the denominator $Q(x)$ is a product of distinct linear factors.

Case 2: the denominator $Q(x)$ has repeated linear factors.

Case 3: the denominator $Q(x)$ contains an irreducible quadratic factor.

We'll go through each of these cases individually.

Case 1: the denominator $Q(x)$ is a product of distinct linear factors.

In this case, we can write

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \ldots (a_k x + b_k) \tag{153}$$

for some appropriate constants $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$. In this case, we can find constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \ldots + \frac{A_k}{a_k x + b_k}. \tag{154}$$

This will make the function easier to integrate.

Example 3.2 *Evaluate the integral.*

$$\int \frac{x+1}{x^2 + x - 2} \, \mathrm{d}x \tag{155}$$

First, we can factor the denominator as $x^2 + x - 2 = (x-1)(x+2)$. This is a product of distinct linear factors, so we are working in Case 1.

We can find constants A and B such that

$$\frac{x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}. \tag{156}$$

We proceed by multiplying both sides by the denominator on the left to get:

$$x+1=A\left(x+2\right)+B\left(x-1\right). \tag{157}$$

Now we distribute, and combine like terms in powers of x to get:

$$1x+1=Ax+2A+Bx-B=\left(A+B\right)x+\left(2A-B\right). \tag{158}$$

We can now equate the corresponding coefficients of the polynomials on the left and right sides of this equation to get a system of equations:

$$\begin{aligned}1&=A+B\\1&=2A-B\end{aligned}. \tag{159}$$

There are many ways to solve this system of equations. My favorite is to add the two equations together to cancel the B terms: $2=3A$. This gives $A=\frac{2}{3}$, and so $B=\frac{1}{3}$. Our equation now becomes

$$\frac{x+1}{(x-1)(x+2)}=\frac{\frac{2}{3}}{x-1}+\frac{\frac{1}{3}}{x+2}. \tag{160}$$

This we can now integrate:

$$\begin{aligned}\int\frac{x+1}{x^2+x-2}\,\mathrm{d}x&=\int\frac{\frac{2}{3}}{x-1}+\frac{\frac{1}{3}}{x+2}\,\mathrm{d}x\\&=\frac{2}{3}\int\frac{1}{x-1}\,\mathrm{d}x+\frac{1}{3}\int\frac{1}{x+2}\,\mathrm{d}x\\&=\boxed{\tfrac{2}{3}\ln\left|x-1\right|+\tfrac{1}{3}\ln\left|x+2\right|+C}.\end{aligned} \tag{161}$$

Case 2: the denominator $Q(x)$ has repeated linear factors. In this case, the factor $\left(a_i+xb_i\right)^r$ occurs in the factorization of $Q(x)$. This results in additional terms in the partial fraction decomposition:

$$\frac{A_1}{\left(a_i+xb_i\right)^1}+\frac{A_2}{\left(a_i+xb_i\right)^2}+...+\frac{A_r}{\left(a_i+xb_i\right)^r}. \tag{162}$$

We illustrate with an example.

Example 3.3 *Evaluate the integral.*

$$\int \frac{4x}{(x-1)(x^2-1)}. \tag{163}$$

We can write the denominator as $(x+1)(x-1)^2$. This has a repeated linear factor of $x-1$, so this is Case 2.

We seek constants A, B and C such that

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \tag{164}$$

In order to find these constants, we multiply each side of the equation by the denominator on the left:

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1). \tag{165}$$

We distribute:

$$4x = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C, \tag{166}$$

and we combine like terms in powers of x:

$$0x^2 + 4x + 0 = (A+B)x^2 + (C-2A)x + (A-B+C). \tag{167}$$

Now we can equate the corresponding coefficients of these polynomials to obtain a system of three equations:

$$\begin{aligned} 0 &= A+B \\ 4 &= C-2A \\ 0 &= A-B+C \end{aligned}. \tag{168}$$

The first equation tells us that $-B = A$, and so by the third equation, $0 = 2A + C$, hence $C = -2A$. By the second equation, we deduce that $-4A = 4$, and so

$A = -1$. *This implies that* $B = 1$ *and* $C = 2$. *Thus, we have*

$$\frac{4x}{(x+1)(x-1)^2} = \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}. \tag{169}$$

This is a function that we can integrate:

$$\begin{aligned}\int \frac{4x}{(x+1)(x-1)^2}\,\mathrm{d}x &= \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,\mathrm{d}x \\ &= -\int \frac{1}{x+1}\,\mathrm{d}x + \int \frac{1}{x-1}\,\mathrm{d}x + 2\int \frac{1}{(x-1)^2}\,\mathrm{d}x \\ &= -\ln|x+1| + \ln|x-1| + 2\int u^{-2}\,\mathrm{d}u, \end{aligned} \tag{170}$$

where in the last term we have used the u*-substitution*

$$\begin{aligned} u &= x-1 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned}. \tag{171}$$

Our antiderivative now becomes

$$-\ln|x+1| + \ln|x-1| - 2u^{-1} + C = \boxed{\ln\left|\tfrac{x-1}{x+1}\right| - \tfrac{2}{x-1} + C}. \tag{172}$$

□

Case 3 The denominator $Q(x)$ contains an irreducible quadratic factor. In this case, ax^2+bx+c will appear in the factorization of $Q(x)$. This factor will contribute a term of the form

$$\frac{Ax+B}{ax^2+bx+c}. \tag{173}$$

We illustrate with an example.

Example 3.4 *Evaluate the integral.*

$$\int \frac{2x^2-x+4}{x^3+4x}\,\mathrm{d}x. \tag{174}$$

We factor the denominator into $x\,(x^2+4)$. *This cannot be factored further (over* *the real numbers), so this is Case 3.*

We seek constants A, B and C such that

$$\frac{2x^2 - x + 4}{x\left(x^2+4\right)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}. \tag{175}$$

To find these constants, we'll use the same procedure we used before: multiply by the denominator on the left:

$$2x^2 - x + 4 = A\left(x^2 + 4\right) + \left(Bx + C\right)x, \tag{176}$$

distribute:

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx, \tag{177}$$

then combine like terms in powers of x:

$$2x^2 - x + 4 = \left(A + B\right)x^2 + Cx + 4A. \tag{178}$$

Equating coefficients of these polynomials, we receive a system of three equations:

$$\begin{aligned} 2 &= A + B \\ -1 &= C \\ 4 &= 4A \end{aligned} \quad . \tag{179}$$

This tells us that $A = 1$, $C = -1$, and $B = 1$. Thus, we have

$$\frac{2x^2 - x + 4}{x\left(x^2+4\right)} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}. \tag{180}$$

This is a function that we can integrate:

$$\begin{aligned} \int \frac{2x^2 - x + 4}{x^3 + 4x}\,\mathrm{d}x = \int \frac{1}{x} + \frac{x-1}{x^2+4}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{x-1}{x^2+4}\,\mathrm{d}x \\ &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{x}{x^2+4}\,\mathrm{d}x - \int \frac{1}{x^2+4}\,\mathrm{d}x \end{aligned} \tag{181}$$

The first term is just $\ln|x|$*. The second term can be done by a u-susbstitution with*

$u = x^2 + 4$. *The third term can be done by trigonometric substitution, or it can be rewritten as follows:*

$$\int \frac{1}{x}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u - \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\,\mathrm{d}x \tag{182}$$

Now we get

$$\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\,\mathrm{d}x \tag{183}$$

By doing another u*-substitution with* $u = \frac{x}{2}$*, this becomes*

$$\ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}x \tag{184}$$

But this is familiar to us:

$$\boxed{\ln|x| + \tfrac{1}{2}\ln|x^2+4| - \tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right) + C}. \tag{185}$$

□

Example 3.5 *(Problem 7.4.12) Evaluate the integral.*

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x \tag{186}$$

We factor the denominator as $(x-2)(x-3)$*, which reveals that we are dealing with Case 1: distinct linear factors. Thus,*

$$\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}. \tag{187}$$

We multiply both sides by the denominator on the left to get

$$x - 4 = A(x-3) + B(x-2). \tag{188}$$

Combining like terms in powers of x*:*

$$x - 4 = (A+B)x + (-3A-2B). \tag{189}$$

Equating corresponding coefficients:

$$\begin{aligned} A + B &= 1 \\ -3A - 2B &= -4 \end{aligned}. \tag{190}$$

We add twice the first equation to the second equation to get $-A = -2$. *Thus,* $A = 2$, *and so* $B = -1$, *by the first equation. Thus,*

$$\begin{aligned} \int_0^1 \frac{x-4}{x^2-5x+6}\,\mathrm{d}x &= \int_0^1 \frac{2}{x-2} - \frac{1}{x-3}\,\mathrm{d}x \\ &= 2\ln|x-2| - \ln|x-3|\Big|_0^1 \\ &= (2\ln|1-2| - \ln|1-3|) - (2\ln|0-2| - \ln|0-3|) \\ &= 2\ln 1 - \ln 2 - 2\ln 2 + \ln 3 = -3\ln 2 + \ln 3 \\ &= \ln 2^{-3} + \ln 3 = \ln\frac{1}{8} + \ln 3 = \boxed{\ln\tfrac{3}{8}}. \end{aligned} \tag{191}$$

□

Example 3.6 *Evaluate the integral.*

$$\int \frac{x(3-5x)}{(3x-1)(x-1)^2}\,\mathrm{d}x. \tag{192}$$

This is Case 2, since $x-1$ *is a repeated linear factor of the denominator. Therefore, we can write this integrand as*

$$\frac{-5x^2+3x}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \tag{193}$$

We multiply by the denominator on the left to get:

$$-5x^2 + 3x = A(x-1)^2 + B\,(3x-1)\,(x-1) + C\,(3x-1)\,. \tag{194}$$

We distribute:

$$-5x^2 + 3x = A\left(x^2 - 2x + 1\right) + B\left(3x^2 - 4x + 1\right) + C\,(3x-1)\,. \tag{195}$$

Combining like terms in powers of x:

$$-5x^2 + 3x = (A + 3B)\, x^2 + (-2A - 4B + 3C)\, x + (A + B - C)\,. \qquad (196)$$

Equating corresponding coefficients gives the system of equations

$$\begin{aligned} A + 3B &= -5 \\ -2A - 4B + 3C &= 3. \\ A + B - C &= 0 \end{aligned} \qquad (197)$$

Multiply the last equation by 3*:*

$$\begin{aligned} A + 3B &= -5 \\ -2A - 4B + 3C &= 3 \\ 3A + 3B - 3C &= 0 \end{aligned} \qquad (198)$$

Now add the second and third equations to get $A - B = 3$. Therefore, $A = 3 + B$, and so the first equation becomes $3 + B + 3B = -5$, hence $B = -2$. This tells us that $A = 3 + B = 3 + (-2) = 1$. Finally, the equation $A + B - C = 0$ now becomes $C = A + B = 1 + (-2) = -1$. We deduce

$$\begin{aligned} \int \frac{x(3-5x)}{(3x-1)(x-1)^2}\, \mathrm{d}x &= \int \frac{1}{3x-1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\, \mathrm{d}x \\ &= \int \frac{1}{3x-1}\, \mathrm{d}x - 2\int \frac{1}{x-1}\, \mathrm{d}x - \int (x-1)^{-2}\, \mathrm{d}x. \end{aligned} \qquad (199)$$

We can do each of these three integrals separately. The first integral can be done by u-substitution with $u = 3x - 1$. The second can be done by u-substitution with $u = x - 1$. The third can be done by u-substitution with $u = x - 1$. In the end, we get

$$\int \frac{x(3-5x)}{(3x-1)(x-1)^2}\, \mathrm{d}x = \boxed{\tfrac{1}{3}\ln|3x-1| - 2\ln|x-1| + \tfrac{1}{x-1} + C}. \qquad (200)$$

□

Example 3.7 *(Problem 7.4.28) Evaluate the integral.*

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x \tag{201}$$

We factor the denominator as $x^2\left(x^2+6\right)$. This is a hybrid of Cases 2 and 3; we have both a repeated linear factor of x and an irreducible quadratic factor of x^2+6. Thus, we use both guidelines.

We seek constants A, B, C and D such that

$$\frac{x^3+6x-2}{x^2\left(x^2+6\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+6}. \tag{202}$$

As before, we multiply by the denominator on the left:

$$x^3+6x-2 = Ax\left(x^2+6\right) + B\left(x^2+6\right) + \left(Cx+D\right)x^2. \tag{203}$$

Now we distribute and combine like terms in powers of x:

$$x^3+0x^2+6x-2 = (A+C)x^3 + (B+D)x^2 + (6A)x + (6B). \tag{204}$$

We equate corresponding coefficients of these two polynomials to get a system of four equations:

$$\begin{aligned} A+C &= 1 \\ B+D &= 0 \\ 6A &= 6 \\ 6B &= -2 \end{aligned}. \tag{205}$$

We immediately deduce from the third equation that $A=1$ and from the fourth equation that $B=-\frac{1}{3}$. The first equation now implies that $C=0$. The second equation implies that $D=\frac{1}{3}$. Thus,

$$\frac{x^3+6x-2}{x^2\left(x^2+6\right)} = \frac{1}{x} + \frac{-\frac{1}{3}}{x^2} + \frac{0x+\frac{1}{3}}{x^2+6}. \tag{206}$$

This is a function that we can integrate:

$$\begin{aligned}\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x &= \int \frac{1}{x} - \frac{1}{3x^2} + \frac{1}{3\left(x^2+6\right)}\,\mathrm{d}x \\ &= \int \frac{1}{x}\,\mathrm{d}x - \frac{1}{3}\int x^{-2}\,\mathrm{d}x + \frac{1}{3}\int \frac{1}{x^2+6}\,\mathrm{d}x \\ &\boxed{= \ln|x| + \tfrac{1}{3x} + \tfrac{\sqrt{6}}{18}\tan^{-1}\left(\tfrac{x}{\sqrt{6}}\right) + C}. \qquad (207)\end{aligned}$$

□

4 Monday, June 3

Section 7.5: Strategy for integration (?)

Sometimes, more than one method for finding antiderivatives must be used to solve a problem. Especially, sometimes a clever u-substitution will reveal a separate method for evaluating an integral.

Example 4.1 *Evaluate the integral.*

$$\int \frac{e^x}{e^{2x} + 3e^x + 2} \, \mathrm{d}x \tag{208}$$

We begin with a u-substitution:

$$\begin{aligned} u &= e^x \\ \mathrm{d}u &= e^x \, \mathrm{d}x \end{aligned} \tag{209}$$

The integral now becomes

$$\int \frac{1}{u^2 + 3u + 2} \, \mathrm{d}u = \int \frac{1}{(u+1)(u+2)} \, \mathrm{d}u. \tag{210}$$

We can now use partial fraction decomposition:

$$\frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}. \tag{211}$$

Multiplying both sides by the denominator on the left gives:

$$1 = A(u+2) + B(u+1) = (A+B)u + (2A+B). \tag{212}$$

Equating coefficients of u on the left and right gives the system of equations

$$A + B = 0 2A + B = 1 \tag{213}$$

From the first equation, we gather that $B = -A$, so the second equation reads

$2A - A = 1$, hence $A = 1$, and so $B = -1$:

$$\begin{aligned}\int \frac{1}{u^2+3u+2}\,\mathrm{d}u &= \int \frac{1}{u+1}\,\mathrm{d}u - \int \frac{1}{u+2}\,\mathrm{d}u \\ &= \ln|u+1| - \ln|u+2| + C = \ln\left|\frac{u+1}{u+2}\right| + C = \boxed{\ln\left|\tfrac{e^x+1}{e^x+2}\right| + C}.\end{aligned} \tag{214}$$

□

Example 4.2 *Evaluate the integral.*

$$\int \sin\sqrt{2t}\,\mathrm{d}t \tag{215}$$

We begin with a u-substitution:

$$\begin{aligned} u &= \sqrt{2t} \\ \mathrm{du} &= \tfrac{1}{\sqrt{2t}}\,\mathrm{d}t \end{aligned}. \tag{216}$$

From this, we deduce that $u\,\mathrm{d}u = \sqrt{2t}\,\mathrm{d}u = \mathrm{d}t$. Therefore, the integral becomes

$$\int u\sin u\,\mathrm{d}u. \tag{217}$$

From here we can use integration by parts.

$$\begin{array}{cc} u_1 = u & \mathrm{d}v_1 = \sin u\,\mathrm{d}u \\ \mathrm{d}u_1 = \mathrm{d}u & v_1 = -\cos u \end{array}. \tag{218}$$

Now,

$$\begin{array}{c} \int u_1\,\mathrm{d}v_1 = u_1v_1 - \int v_1\,\mathrm{d}u_1 \\ \int u\sin u\,\mathrm{d}u = -u\cos u + \int \cos u\,\mathrm{d}u = -u\cos u + \sin u + C. \end{array} \tag{219}$$

This becomes

$$\int \sin\sqrt{2t}\,\mathrm{d}t = \sin u - u\cos u = \boxed{\sin\sqrt{2t} - \sqrt{2t}\cos\sqrt{2t} + C}. \tag{220}$$

□

Example 4.3 *(Problem 7.5.10) Evaluate the integral.*

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\,\mathrm{d}x \tag{221}$$

We begin with a u-substitution:

$$\begin{aligned} u &= \tfrac{1}{x} \\ \mathrm{d}u &= -\tfrac{1}{x^2}\,\mathrm{d}x \end{aligned}. \tag{222}$$

Now $\frac{1}{x^3}\,\mathrm{d}x = -\frac{1}{x}\,\mathrm{d}u = -u\,\mathrm{d}u$, so the integral becomes

$$-\int u\cos u\,\mathrm{d}u. \tag{223}$$

We can now use integration by parts:

$$\begin{array}{cc} u_1 = u & \mathrm{d}v_1 = \cos u\,\mathrm{d}u \\ \mathrm{d}u_1 = \mathrm{d}u & v_1 = \sin u \end{array}. \tag{224}$$

The integral now becomes

$$-\int u\cos u\,\mathrm{d}u = -u\sin u + \int \sin u\,\mathrm{d}u = -u\sin u - \cos u + C. \tag{225}$$

This gives us

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^3}\,\mathrm{d}x = \boxed{-\frac{\sin\left(\frac{1}{x}\right)}{x} - \cos\left(\frac{1}{x}\right) + C}. \tag{226}$$

□

Example 4.4 *(Problem 7.5.42) Evaluate the integral.*

$$\int \frac{\tan^{-1}x}{x^2}\,\mathrm{d}x \tag{227}$$

We begin with integration by parts:

$$\begin{array}{cc} u = \tan^{-1}x & \mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x \\ \mathrm{d}u = \frac{1}{x^2+1}\,\mathrm{d}x & v = -\frac{1}{x} \end{array}. \tag{228}$$

The integral becomes

$$\int \frac{\tan^{-1}x}{x^2}\,\mathrm{d}x = -\frac{\tan^{-1}x}{x} + \int \frac{1}{x\left(x^2+1\right)}\,\mathrm{d}x. \tag{229}$$

We can now proceed by partial fraction decomposition:

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}. \tag{230}$$

Multiplying both sides by the denominator on the left, we get:

$$1 = A\left(x^2+1\right) + \left(Bx+C\right)x = \left(A+B\right)x^2 + Cx + A. \tag{231}$$

Equating coefficients in powers of x, we get the following system of equations:

$$\begin{array}{r} A+B=0 \\ C=0 \\ A=1 \end{array}. \tag{232}$$

From this, it's clear that $A=1$, $B=-1$ and $C=0$, so

$$\begin{aligned} \int \frac{1}{x\left(x^2+1\right)}\,\mathrm{d}x = \int \frac{1}{x} - \frac{x}{x^2+1}\,\mathrm{d}x &= \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + C \\ &= \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C \end{aligned} \tag{233}$$

Thus,

$$\int \frac{\tan^{-1}x}{x^2}\,\mathrm{d}x = \boxed{-\tfrac{\tan^{-1}x}{x} + \ln\left|\tfrac{x}{\sqrt{x^2+1}}\right| + C}. \tag{234}$$

□

Moral of the story: antiderivatives are harder than derivatives.

Section 7.8: Improper integrals

Integrals find the signed area between a curve and the x-axis. Usually, this is the signed area of a bounded region. However, sometimes, it is possible to find a finite signed area of an unbounded region, using limits. Integrals over unbounded regions are called improper integrals.

There are two types of improper integrals: integrals over infinite intervals, and integrals of functions with vertical asymptotes in a finite interval. We'll discuss the first type first.

Definition 4.5 *Let f be a function, and let a be a real value.*
(i) The improper integral $\int_a^\infty f(x)\,\mathrm{d}x$ is defined as the limit

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x. \tag{235}$$

(ii) The improper integral $\int_{-\infty}^a f(x)\,\mathrm{d}x$ is defined as the limit

$$\int_{-\infty}^a f(x)\,\mathrm{d}x = \lim_{s\to-\infty}\int_s^a f(x)\,\mathrm{d}x. \tag{236}$$

(iii) We say that an improper integral is convergent if the limit has a real value, and divergent if it does not have a real value.
(iv) If $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x. \tag{237}$$

Example 4.6 *(Problem 7.8.8) Evaluate the improper integral.*

$$\int_1^\infty \frac{1}{(2x+1)^3}\,\mathrm{d}x \tag{238}$$

We write this as

$$\lim_{t\to\infty}\int_1^t (2x+1)^{-3}\,\mathrm{d}x, \tag{239}$$

and use u-substitution:

$$\begin{aligned} u &= 2x+1 \\ \mathrm{d}u &= 2\,\mathrm{d}x \end{aligned} \tag{240}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to\infty}\int_3^{2t+1}\frac{1}{2}u^{-3}\,\mathrm{d}u &= \lim_{t\to\infty} -\frac{1}{4}u^{-2}\Big|_3^{2t+1} \\ &= \lim_{t\to\infty}\frac{1}{4}\left(\frac{1}{3^2}-\frac{1}{(2t+1)^2}\right) = \frac{1}{4}\left(\frac{1}{9}-0\right) = \boxed{\tfrac{1}{36}}. \end{aligned} \tag{241}$$

□

Example 4.7 *(Problem 7.8.6) Evaluate the improper integral.*

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,\mathrm{d}x. \tag{242}$$

We write this as

$$\lim_{t\to\infty}\int_0^t (1+x)^{-\frac{1}{4}}\,\mathrm{d}x. \tag{243}$$

We use u-substitution:

$$\begin{aligned} u &= x+1 \\ \mathrm{d}u &= \mathrm{d}x \end{aligned} \tag{244}$$

so this integral becomes

$$\lim_{t\to\infty}\int_1^{t+1} u^{-\frac{1}{4}}\,\mathrm{d}u = \lim_{t\to\infty}\frac{4}{3}u^{\frac{3}{4}}\Big|_1^{t+1} = \lim_{t\to\infty}\frac{4}{3}\left((t+1)^{\frac{3}{4}}-1^{\frac{3}{4}}\right) = \infty. \tag{245}$$

Thus, the integral is divergent. □

Example 4.8 *(Problem 7.8.16) Evaluate the improper integral.*

$$\int_0^\infty \sin\theta\, e^{\cos\theta}\,\mathrm{d}\theta \tag{246}$$

We write this as

$$\lim_{t\to\infty}\int_0^t \sin\theta\, e^{\cos\theta}\,\mathrm{d}\theta. \tag{247}$$

We now use u-substitution:

$$\begin{aligned} u &= \cos\theta \\ \mathrm{d}u &= -\sin\theta \,\mathrm{d}\theta \end{aligned}. \tag{248}$$

The integral now becomes

$$\lim_{t\to\infty}\int_1^{\cos t} -e^u \,\mathrm{d}u = \lim_{t\to\infty} -e^u\Big|_1^{\cos t} = \lim_{t\to\infty} e^1 - e^{\cos t}. \tag{249}$$

However, as $t\to\infty$, $\cos t$ oscillates between -1 and 1. Therefore, $e^{\cos t}$ oscillates between e^{-1} and e^1. The limit has no real value, so the integral is divergent. □

Example 4.9 *(Problem 7.8.12) Evaluate the improper integral.*

$$\int_{-\infty}^{\infty} y^3 - 3y^2 \,\mathrm{d}y. \tag{250}$$

We first write this as

$$\int_{-\infty}^{\infty} y^3 - 3y^2 \,\mathrm{d}y = \int_{-\infty}^{0} y^3 - 3y^2 \,\mathrm{d}y + \int_{0}^{\infty} y^3 - 3y^2 \,\mathrm{d}y. \tag{251}$$

(The number 0 was chosen arbitrarily; if this integral exists, then any number can be used.) We consider first

$$\begin{aligned} \int_{-\infty}^{0} y^3 - 3y^2 \,\mathrm{d}y &= \lim_{s\to-\infty}\int_s^0 y^3 - 3y^2 \,\mathrm{d}y \\ &= \lim_{s\to-\infty} \frac{1}{4}y^4 - y^3\Big|_s^0 = \lim_{s\to-\infty} -\frac{1}{4}s^4 + s^3 = -\infty, \end{aligned} \tag{252}$$

and so the integral is divergent. □

Example 4.10 *(Problem 7.8.22) Evaluate the improper integral*

$$\int_1^{\infty} \frac{\ln x}{x^2}\,\mathrm{d}x \tag{253}$$

We first write this as

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x. \tag{254}$$

We use integration by parts:

$$\begin{array}{cc} u = \ln x & \mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & v = -\frac{1}{x} \end{array}. \tag{255}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x &= \lim_{t\to\infty}\left(-\frac{\ln x}{x}\bigg|_1^t + \int_1^t \frac{1}{x^2}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\bigg|_1^t \\ &= \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t}\right) - \left(-\frac{\ln 1}{1} - \frac{1}{1}\right) = \lim_{t\to\infty} -\frac{\ln t}{t} - \frac{1}{t} + 1 \end{aligned} \tag{256}$$

Now, by L'Hopital's rule,

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} -\frac{\frac{1}{t}}{1} - \frac{1}{t} + 1 = \boxed{1}. \tag{257}$$

□

Now for the second type.

Definition 4.11 *Let a and b be real values such that $a \le b$, and let f be a function. (i) If f is defined and continuous on the interval $[a, b)$, and has a vertical asymptote at $x = b$, then we define the improper integral*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x. \tag{258}$$

(ii) If f is defined and continuous on the interval $(a, b]$, and has a vertical asymptote at $x = a$, then we defined the improper integral

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s\to a^+}\int_s^b f(x)\,\mathrm{d}x. \tag{259}$$

(iii) Given a real value c such that $a < c < b$, if f has a vertical asymptote at $x = c$

and both $\int_a^c f(x)\,\mathrm{d}x$ *and* $\int_c^b f(x)\,\mathrm{d}x$ *exist, then we define the improper integral*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{260}$$

Example 4.12 *(Problem 7.8.28) Evaluate the improper integral.*

$$\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,\mathrm{d}x \tag{261}$$

We write this as

$$\lim_{t\to 5^-}\int_0^t (5-x)^{-\frac{1}{3}}\,\mathrm{d}x. \tag{262}$$

We use u*-substitution:*

$$\begin{aligned} u &= 5-x \\ \mathrm{d}u &= -\mathrm{d}x \end{aligned}. \tag{263}$$

The integral now becomes

$$\begin{aligned} \lim_{t\to 5^-}\int_5^{5-t} -u^{-\frac{1}{3}}\,\mathrm{d}u &= \lim_{t\to 5^-} -\frac{3}{2}u^{\frac{2}{3}}\Big|_5^{5-t} \\ &= \lim_{t\to 5^-}\frac{3}{2}\left(5^{\frac{2}{3}} - (5-t)^{\frac{2}{3}}\right) = \frac{3}{2}\left(5^{\frac{2}{3}} - 0^{\frac{2}{3}}\right) = \boxed{\tfrac{3}{2}5^{\frac{2}{3}}}. \end{aligned} \tag{264}$$

□

Example 4.13 *(Problem 7.8.32) Evaluate the improper integral*

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x. \tag{265}$$

We write this as

$$\lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \lim_{t\to 1^-}\sin^{-1}x\Big|_0^t = \lim_{t\to 1^-}\sin^{-1}t - \sin^{-1}0 = \boxed{\tfrac{\pi}{2}}. \tag{266}$$

□

Example 4.14 *(Problem 7.8.34) Evaluate the improper integral.*

$$\int_0^5 \frac{x}{x-2}\,\mathrm{d}x \tag{267}$$

We write this as

$$\lim_{t\to 2^-}\int_0^t \frac{x}{x-2}\,\mathrm{d}x + \lim_{s\to 2^+}\int_s^5 \frac{x}{x-2}\,\mathrm{d}x \tag{268}$$

We can do this antiderivative either by a u-substitution or by recognizing:

$$\frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}. \tag{269}$$

The integral now becomes

$$\lim_{t\to 2^-}\int_0^t 1 + \frac{2}{x-2}\,\mathrm{d}x + \lim_{s\to 2^+}\int_s^5 1 + \frac{2}{x-2}\,\mathrm{d}x. \tag{270}$$

Let's focus first on the second term.

$$\begin{aligned}\lim_{s\to 2^+}\int_s^5 1 + \frac{2}{x-2}\,\mathrm{d}x &= \lim_{s\to 2^+} x + 2\ln|x-2|\Big|_s^5 \\ &= \lim_{s\to 2^+} (5 + 2\ln 3) - (s + 2\ln|s-2|) = \infty \end{aligned} \tag{271}$$

The integral is divergent. □

5 Thursday, June 6

Section 6.1: Areas between curves

The definite integral of f over the interval $[a, b]$ is defined as the signed area under the curve.

Suppose we have two functions, f and g, defined over an interval $[a, b]$, such that $f(x) \geq g(x)$ for all x in the interval. In that case, the area enclosed between the curves is

$$A = \int_a^b f(x) - g(x)\,\mathrm{d}x. \tag{272}$$

Example 5.1 *(Problem 6.1.6) Find the area of the region enclosed by the curves $y = \sin x$, $y = x$, $x = \frac{\pi}{2}$ and $x = \pi$.*

[Draw diagram]

The area is

$$\begin{aligned} A = \int_{\frac{\pi}{2}}^{\pi} x - \sin x\,\mathrm{d}x = \frac{1}{2}x^2 + \cos x\bigg|_{\frac{\pi}{2}}^{\pi} \\ = \left(\frac{1}{2}\pi^2 + \cos\pi\right) - \left(\frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \cos\left(\frac{\pi}{2}\right)\right) \\ = \frac{1}{2}\pi^2 - 1 - \frac{1}{8}\pi^2 = \boxed{\tfrac{3}{8}\pi^2 - 1}. \end{aligned} \tag{273}$$

□

Example 5.2 *(Problem 6.1.8) Find the area of the region enclosed by the curves $y = x^2 - 4x$ and $y = 2x$.*

[Draw diagram]

The area is

$$A = \int_a^b 2x - \left(x^2 - 4x\right)\,\mathrm{d}x. \tag{274}$$

What are a and b? In order to find the upper and lower bounds, we need to find the x-coordinates of the points of intersection of the curves. To do this, we set the functions equal to each other and solve for x:

$$\begin{aligned} 2x &= x^2 - 4x \\ 0 &= x^2 - 6x = x\left(x - 6\right) \end{aligned} \tag{275}$$

This gives $x = 0$ and $x = 6$, so the area is

$$A = \int_0^6 2x - \left(x^2 - 4x\right)\,\mathrm{d}x = \int_0^6 -x^2 + 6x\,\mathrm{d}x = -\frac{1}{3}x^3 + 3x^2\bigg|_0^6 = \boxed{36}. \tag{276}$$

□

Example 5.3 *(Problem 6.1.14) Find the area enclosed by the curves $y = x^2$ and $y = 4x - x^2$.*

[Draw diagram]

The area is

$$A = \int_a^b \left(4x - x^2\right) - x^2\,\mathrm{d}x. \tag{277}$$

What are a and b? We set the functions equal to each other and solve for x:

$$\begin{aligned} x^2 &= 4x - x^2 \\ 2x^2 - 4x &= 0\ . \\ 2x\left(x - 2\right) &= 0 \end{aligned} \tag{278}$$

This gives $x = 0$ and $x = 2$, so the area is

$$\int_0^2 \left(4x - x^2\right) - x^2\,\mathrm{d}x = \int_0^2 4x - 2x^2\,\mathrm{d}x = 2x^2 - \frac{2}{3}x^3\bigg|_0^2 = 8 - \frac{16}{3} = \boxed{\tfrac{8}{3}}. \tag{279}$$

□

Example 5.4 *(Problem 6.1.20) Find the area enclosed by the curves $x = y^4$,*

$y = \sqrt{2-x}$ *and* $y = 0$.

[Draw diagram]

Here the roles of x *and* y *are reversed. We write* $y = \sqrt{2-x}$ *as* $x = 2 - y^2$. *The area is*

$$A = \int_a^b \left(2 - y^2\right) - y^4 \, \mathrm{d}y. \tag{280}$$

Again, we need a *and* b. *To do this, we set the functions equal to each other and solve for* y*:*

$$\begin{gathered} y^4 = 2 - y^2 \\ y^4 + y^2 - 2 = 0 \\ \left(y^2+2\right)\left(y^2-1\right) = 0 \\ \left(y^2+2\right)\left(y-1\right)\left(y+1\right) \end{gathered}. \tag{281}$$

The lower bound is $y = 0$. *The upper bound is* $y = 1$. *Thus, the area is*

$$\begin{aligned} A = \int_0^1 \left(2 - y^2\right) - y^4 \, \mathrm{d}y = \int_0^1 2 - y^2 - y^4 \, \mathrm{d}y \\ = 2y - \frac{1}{3}y^3 - \frac{1}{5}y^5 \Big|_0^1 = 2 - \frac{1}{3} - \frac{1}{5} = \boxed{\tfrac{22}{15}}. \end{aligned} \tag{282}$$

□

Section 6.2: Volumes

Definition 5.5 *Let S be a solid region that is bounded by the lines $x = a$ and $x = b$. For each x such that $a \le x \le b$, define $A(x)$ as the cross-sectional area of S. The volume of S is the integral*

$$V = \int_a^b A(x)\,\mathrm{d}x. \tag{283}$$

Given a region in the xy-plane bounded by a function, we can rotate that region around a (horizontal or vertical) axis to form a 3-dimensional shape. The shape that results is called a "solid of revolution." In this section, we discuss the "disk-and-washer method" of finding the volume of a solid of revolution.

Example 5.6 *(Problem 6.2.2) Find the volume of the solid obtained by rotating the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$ and $x = 4$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a disk. The area of that disk is

$$A(x) = \pi r^2 = \pi \left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}. \tag{284}$$

This is the cross-sectional area of the solid, so by definition, the volume is

$$V = \int_1^4 \frac{\pi}{x^2}\,\mathrm{d}x = \pi \int_1^4 x^{-2}\,\mathrm{d}x = -\frac{\pi}{x}\bigg|_1^4 = -\frac{\pi}{4} - \left(-\frac{\pi}{1}\right) = \boxed{\tfrac{3\pi}{4}}. \tag{285}$$

□

Example 5.7 *(Problem 6.2.4) Find the volume of the solid obtained by rotating the region bounded by $y = e^x$, $y = 0$, $x = -1$ and $x = 1$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a disk. The area of that disk is

$$A(x) = \pi r^2 = \pi (e^x)^2 = \pi e^{2x}. \tag{286}$$

Now, the volume is

$$V = \int_{-1}^{1} \pi e^{2x} \, \mathrm{d}x = \pi \int_{-1}^{1} e^{2x} \, \mathrm{d}x = \frac{\pi}{2} e^{2x} \Big|_{-1}^{1} = \boxed{\tfrac{\pi}{2} \left(e^2 - e^{-2}\right)}. \tag{287}$$

□

Example 5.8 *Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{2x}$, $x = 0$ and $y = 4$ about the x-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the x-axis, forms a washer. The area of that washer is

$$A(x) = \pi R^2 - \pi r^2 = \pi (4)^2 - \pi \left(\sqrt{2x}\right)^2 = \pi \left(16 - 2x\right). \tag{288}$$

The volume is the integral of this, but what are the bounds? The lower bound is certainly $x = 0$, but what about the upper bound? To find this, we need to set $y = \sqrt{2x}$ and $y = 4$ equal to each other and solve for x:

$$\sqrt{2x} = 4, \tag{289}$$

and so the upper bound is $x = 8$. Thus,

$$V = \int_{0}^{8} \pi \left(16 - 2x\right) \, \mathrm{d}x = \pi \left(16x - x^2\right) \Big|_{0}^{8} = \boxed{64\pi}. \tag{290}$$

□

Example 5.9 *(Problem 6.2.12) Find the volume of the solid obtained by rotating the region bounded by rotating the region bounded by $y = x^3$, $y = 1$ and $x = 2$ about the line $y = -3$.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about $y = -3$, forms a washer. The area of that washer is

$$\begin{aligned} A(x) = \pi R^2 - \pi r^2 = \pi\big(x^3 - (-3)\big)^2 - \pi(1-(-3))^2 \\ = \pi\left(\left(x^3+3\right)^2 - 4^2\right) = \pi\left(x^6 + 6x^3 - 7\right). \end{aligned} \tag{291}$$

The lower bound is the x-coordinate of the intersection of $y = x^3$ and $y = 1$:

$$x^3 = 1, \tag{292}$$

so $x = 1$ is the lower bound. The upper bound is $x = 2$. Thus,

$$V = \int_1^2 \pi\left(x^6 + 6x^3 - 7\right)\,\mathrm{d}x = \pi\left(\frac{1}{7}x^7 + \frac{3}{2}x^4 - 7x\right)\Bigg|_1^2 = \boxed{\frac{471\pi}{14}}. \tag{293}$$

□

Example 5.10 *(Problem 6.2.10) Find the volume of the solid obtained by rotating the region bounded by $x = 2 - y^2$ and $x = y^4$ about the y-axis.*

[Draw diagram]

To do this, we take a typical slice of the region perpendicular to the axis of rotation. This slice, when rotated about the y-axis, forms a washer. The area of that washer is

$$A(y) = \pi R^2 - \pi r^2 = \pi\big(2 - y^2\big)^2 - \pi\big(y^4\big)^2 = \pi\left(-y^8 + y^4 - 4y^2 + 4\right). \tag{294}$$

What are the bounds of the region? They are the y-coordinates where $x = 2 - y^2$ and $x = y^4$ cross:

$$\begin{aligned} 2 - y^2 &= y^4 \\ 0 &= y^4 + y^2 - 2 \\ 0 &= \left(y^2 + 2\right)\left(y^2 - 1\right) \\ 0 &= \left(y^2 + 2\right)(y + 1)(y - 1)\,. \end{aligned} \tag{295}$$

This gives $y = -1$ and $y = 1$ as bounds. Thus,

$$\begin{aligned} V = \int_{-1}^{1} \pi \left(-y^8 + y^4 - 4y^2 + 4\right)\,\mathrm{d}y \\ = \pi\left(-\frac{1}{9}y^9 + \frac{1}{5}y^5 - \frac{4}{3}y^3 + 4y\right)\Bigg|_{-1}^{1} \\ = \pi\left(-\frac{2}{9} + \frac{2}{5} - \frac{8}{3} + 8\right) = \boxed{\tfrac{248\pi}{45}}. \end{aligned} \tag{296}$$

□

6 Monday, June 10

Section 6.3: Volumes by cylindrical shells

There is another method for finding the volume of a solid of revolution: the "method of shells."

In the method of disks and washers, we selected a typical slice that was perpendicular to the axis of rotation to produce a disk or washer.

In the method of shells, we will select slices that are parallel to the axis of rotation to produce a cylindrical shell.

Example 6.1 *(Problem 6.3.4) Find the volume generated by rotating the region bounded by the curves $y = x^3$, $y = 0$, $x = 1$ and $x = 2$ about the y-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the y-axis, forms a cylindrical shell. The area of that shell is

$$A = 2\pi r h = 2\pi x \left(x^3\right) = 2\pi x^4. \tag{297}$$

Now, the volume is

$$V = \int_1^2 2\pi x^4 \, \mathrm{d}x = \frac{2\pi}{5} x^5 \bigg|_1^2 = \frac{2\pi}{5}(32 - 1) = \boxed{\tfrac{62\pi}{5}}. \tag{298}$$

□

Example 6.2 *(Problem 6.3.6) Find the volume generated by rotating the region bounded by the curves $y = 4x - x^2$ and $y = x$ about the y-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the y-axis, forms a cylindrical shell. The area of that shell is

$$A = 2\pi rh = 2\pi x\left(\left(4x - x^2\right) - (x)\right) = 2\pi\left(3x^2 - x^3\right) \tag{299}$$

What will be the bounds? They are the x-coordinates where $y = 4x - x^2$ and $y = x$ cross:

$$\begin{aligned} 4x - x^2 &= x \\ 3x - x^2 &= 0 \ . \\ x(3 - x) &= 0 \end{aligned} \tag{300}$$

This gives $x = 0$ and $x = 3$ as bounds. Thus,

$$\begin{aligned} V = \int_0^3 2\pi\left(3x^2 - x^3\right)\,\mathrm{d}x = 2\pi\int_0^3 3x^2 - x^3\,\mathrm{d}x \\ = 2\pi\left(x^3 - \frac{1}{4}x^4\right)\Bigg|_0^3 = 2\pi\left(27 - \frac{1}{4}81\right) = \boxed{\tfrac{27}{2}\pi}. \end{aligned} \tag{301}$$

□

Example 6.3 *(Problem 6.3.10) Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$ and $y = 2$ about the x-axis.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the x-axis, forms a cylindrical shell. The area of that shell is

$$A = 2\pi rh = 2\pi y\left(y^2 - 0\right) = 2\pi y^3. \tag{302}$$

Now, the volume is

$$V = \int_0^2 2\pi y^3\,\mathrm{d}y = \frac{\pi}{2}y^4\Bigg|_0^2 = \boxed{8\pi}. \tag{303}$$

□

Example 6.4 *(Problem 6.3.16) Find the volume generated by rotating the region* *bounded by the curves $y = 4 - 2x$, $y = 0$ and $x = 0$ about the line $x = -1$.*

[Draw diagram]

We take a typical slice of the region parallel to the axis of rotation. This slice, when rotated about the line $x = -1$, forms a cylindrical shell. The area of that shell is

$$A = 2\pi rh = 2\pi(x+1)(4-2x) = -4\pi\left(x^2 - x - 2\right). \tag{304}$$

Now, the volume is

$$\begin{aligned} V = \int_0^2 -4\pi\left(x^2 - x - 2\right)\,\mathrm{d}x &= -4\pi\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right)\Bigg|_0^2 \\ &= -4\pi\left(\frac{8}{3} - 2 - 4\right) = \boxed{\tfrac{40\pi}{3}}. \end{aligned} \tag{305}$$

□

Section 6.5: Average value of a function

Definition 6.5 *Let f be a function defined on the interval $[a,b]$. We define the average value of f over $[a,b]$ as the value*

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \tag{306}$$

Example 6.6 *(Problem 6.5.8) Let $h(x) = \frac{\ln x}{x}$. Find the average value of the function on the interval $[1,5]$.*
This is a straightforward application of the definition:

$$h_{ave} = \frac{1}{5-1}\int_1^5 h(x)\,\mathrm{d}x = \frac{1}{4}\int_1^5 \frac{\ln x}{x}\,\mathrm{d}x. \tag{307}$$

We use a u-substitution:

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x}\,\mathrm{d}x \end{aligned} \tag{308}$$

This becomes

$$h_{ave} = \frac{1}{4}\int_0^{\ln 5} u\,\mathrm{d}u = \frac{1}{8}u^2\Big|_0^{\ln 5} = \boxed{\tfrac{1}{8}(\ln 5)^2}. \tag{309}$$

□

Recall the mean value theorem: if f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a real value c such that $a \le c \le b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$. We now have a mean value theorem for integrals.

Theorem 6.7 *Let f be continuous on $[a,b]$. There exists a real value c such that $a \le c \le b$ and $f(c) = f_{ave}$.*

7 Tuesday, June 11

Section 8.1: Arc length

Definition 7.1 *Let f be a continuous function defined on an interval $[a, b]$. The arc length of f from a to b is the length of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$.*

The arc length of a continuous function f from $x = a$ to $x = b$ is given by the following formula:

$$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x. \tag{310}$$

The arc length function beginning at $x = a$ is defined as

$$s(x) = \int_a^x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x. \tag{311}$$

Example 7.2 *Find the length of the arc of $y = x^{\frac{3}{2}}$ between the points $(1, 1)$ and $(4, 8)$.*

First, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}}$. Now,

$$s = \int_1^4 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,\mathrm{d}x = \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x \tag{312}$$

We use u-substitution:

$$\begin{aligned} u &= 1 + \tfrac{9}{4}x \\ \mathrm{d}u &= \tfrac{9}{4}\,\mathrm{d}x. \end{aligned} \tag{313}$$

The integral becomes

$$\frac{4}{9}\int_{\frac{13}{4}}^{10} u^{\frac{1}{2}}\,\mathrm{d}u = \frac{4}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\Bigg|_{\frac{13}{4}}^{10}\right) = \boxed{\tfrac{8}{27}\left(10^{\frac{3}{2}} - \left(\tfrac{13}{4}\right)^{\frac{3}{2}}\right)}. \tag{314}$$

□

Example 7.3 *Find the exact length of the curve $y = \frac{1}{6}(x^2 - 4)^{\frac{3}{2}}$ from $x = 2$ to $x = 3$.*

First, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x\sqrt{x^2-4}$. *Now,*

$$\begin{aligned}
s = \int_2^3 \sqrt{1+\frac{1}{4}x^2\left(x^2-4\right)}\,\mathrm{d}x &= \int_2^3 \sqrt{1+\frac{1}{4}x^4-x^2}\,\mathrm{d}x \\
&= \int_2^3 \frac{1}{2}\sqrt{x^4-4x^2+4}\,\mathrm{d}x = \frac{1}{2}\int_2^3 \sqrt{\left(x^2-2\right)^2}\,\mathrm{d}x \\
&= \frac{1}{2}\int_2^3 x^2-2\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{3}x^3-2x\Big|_2^3\right) \\
&= \frac{1}{2}(9-6)-\frac{1}{2}\left(\frac{8}{3}-4\right) = \frac{3}{2}+\frac{4}{6} = \boxed{\tfrac{13}{6}}. \qquad (315)
\end{aligned}$$

□

Example 7.4 *(Problem 8.1.14) Find the exact length of the curve* $y = \ln(\cos x)$ *from* $x=0$ *to* $x=\frac{\pi}{3}$.

First, we find $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan x$. *Now,*

$$\begin{aligned}
s = \int_0^{\frac{\pi}{3}} \sqrt{1+(-\tan x)^2}\,\mathrm{d}x &= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\,\mathrm{d}x \\
&= \int_0^{\frac{\pi}{3}} \sec x\,\mathrm{d}x = \ln\left|\sec x+\tan x\right|\Big|_0^{\frac{\pi}{3}} \\
&= \ln\left|\sec\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right| - \ln\left|\sec 0+\tan 0\right| \\
&= \ln\left|2+\sqrt{3}\right| - \ln\left|1+0\right| = \boxed{\ln\left|2+\sqrt{3}\right|}. \qquad (316)
\end{aligned}$$

□

Section 8.2: Area of a surface of revolution

To find the surface area of a surface of revolution created by revolving $y = f(x)$ about the x-axis, we take a typical point on the curve:

[Draw diagram]

When revolved about the x-axis, this point forms a circle with circumference $2\pi f(x)$. To find the surface area, we integrate this circumference with respect to arc length:

$$A = \int_a^b 2\pi f(x)\,\mathrm{d}s. \tag{317}$$

Since $s(x) = \int_a^x \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$, we know, by the fundamental theorem of calculus, that $\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (f'(x))^2}$. Thus, $\mathrm{d}s = \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$, and so the surface area is given by:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,\mathrm{d}x. \tag{318}$$

Example 7.5 *Find the exact area of the surface obtained by rotating the curve $y = \sqrt{5 - x}$, for $3 \le x \le 5$, about the x-axis.*

First, we find that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{2\sqrt{5-x}}$. Thus, the surface area is

$$\begin{aligned} A = \int_3^5 2\pi\sqrt{5-x}\sqrt{1 + \left(\frac{-1}{2\sqrt{5-x}}\right)^2}\,\mathrm{d}x \\ &= 2\pi \int_3^5 \sqrt{5-x}\sqrt{1 + \frac{1}{4(5-x)}}\,\mathrm{d}x \\ &= 2\pi \int_3^5 \sqrt{5 - x + \frac{1}{4}}\,\mathrm{d}x = 2\pi \int_3^5 \sqrt{\frac{21}{4} - x}\,\mathrm{d}x. \end{aligned} \tag{319}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \tfrac{21}{4} - x \\ \mathrm{d}u &= -\,\mathrm{d}x \end{aligned} \tag{320}$$

The integral becomes

$$-2\pi \int_{\frac{9}{4}}^{\frac{1}{4}} u^{\frac{1}{2}} \, \mathrm{d}u = \frac{4}{3}\pi u^{\frac{3}{2}} \Big|_{\frac{1}{4}}^{\frac{9}{4}} = \boxed{\frac{13\pi}{4}}. \tag{321}$$

□

Example 7.6 *Find the area of the surface generated by rotating the curve $y = e^x$ for $0 \leq x \leq 1$ about the x-axis.*

First, we notice that $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$. Therefore,

$$A = \int_0^1 2\pi e^x \sqrt{1 + e^{2x}} \, \mathrm{d}x. \tag{322}$$

We proceed by u-substitution:

$$\begin{aligned} u &= e^x \\ \mathrm{d}u &= e^x \, \mathrm{d}x \end{aligned}. \tag{323}$$

The integral becomes

$$2\pi \int_1^e \sqrt{1 + u^2} \, \mathrm{d}u. \tag{324}$$

We now proceed by trigonometric substitution:

$$\begin{aligned} u &= \tan\theta \\ \mathrm{d}u &= \sec^2\theta \, \mathrm{d}\theta, \end{aligned} \tag{325}$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Define a real value α in this interval such that $\tan\alpha = e$. Now, the integral becomes

$$2\pi \int_{\frac{\pi}{4}}^{\alpha} \sqrt{1 + \tan^2\theta} \sec^2\theta \, \mathrm{d}\theta = 2\pi \int_{\frac{\pi}{4}}^{\alpha} \sec^3\theta \, \mathrm{d}\theta. \tag{326}$$

From a previous example, this is

$$\begin{aligned} &\pi \left(\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\right) \Big|_{\frac{\pi}{4}}^{\alpha} \\ &\qquad = \pi \left(\sec\alpha \tan\alpha + \ln|\sec\alpha + \tan\alpha| - \sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right). \end{aligned} \tag{327}$$

Since $\tan\alpha = e$, $e^2+1 = \sec^2\alpha$ *and so* $\sqrt{e^2+1} = \sec\alpha$*:*

$$\boxed{\pi\left(e\sqrt{e^2+1} + \ln\left|e+\sqrt{e^2+1}\right| - \sqrt{2} - \ln\left(\sqrt{2}+1\right)\right)} \tag{328}$$

□

Example 7.7 *Find the surface area of a sphere with a radius of* R*.*

We can understand a sphere centered at the origin with a radius of R *as a revolution of the upper half of the circle* $x^2+y^2=R^2$ *about the* x*-axis. This upper half is given by* $y=\sqrt{R^2-x^2}$*. We compute*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(R^2-x^2\right)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{R^2-x^2}}. \tag{329}$$

Now, the surface area is

$$\begin{aligned} A = \int_{-R}^{R} 2\pi\sqrt{R^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{R^2-x^2}}\right)^2}\,\mathrm{d}x \\ = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\sqrt{1+\frac{x^2}{R^2-x^2}}\,\mathrm{d}x \\ = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2+x^2}\,\mathrm{d}x = 2\pi\int_{-R}^{R}R\,\mathrm{d}x = 2\pi R\int_{-R}^{R}\mathrm{d}x \\ = 2\pi R\,(R-(-R)) = 2\pi R\,(2R) = \boxed{4\pi R^2}. \end{aligned} \tag{330}$$

□

8 Monday, June 17

Section 10.1: Curves defined by parametric equations

Some curves cannot be defined by functions, since all functions must pass a vertical line test. In order to describe curves, one can understand the curve as a path of a particle taken over a given time period. We describe the x- and y- coordinates of the particle's position separately as functions of time. These functions are called "parametric equations."

Example 8.1 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 4\sin t \\ y(t) &= 5\cos t \end{aligned} \tag{331}$$

for $0 \leq t \leq 2\pi$.

We can write $\frac{x}{4} = \sin t$ *and* $\frac{y}{5} = \cos t$*, so*

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = \sin^2 t + \cos^2 t = 1. \tag{332}$$

Thus, the equation can be written in the Cartesian form as

$$\frac{x^2}{16} + \frac{y^2}{5} = 1. \tag{333}$$

(This is called "eliminating the parameter.") This is an ellipse with x*-intercepts* $x = \pm 4$ *and* y*-intercepts* $y = \pm 5$*:*

[Draw diagram]

The parameter t *begins at* 0*, corresponding to the point* $(0, 5)$*. As* t *increases, the* x*-coordinate increases and the* y*-coordinate decreases, creating a clockwise motion.* $\square$

Example 8.2 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= 1 - t \\ y(t) &= \sqrt{t} \end{aligned} \tag{334}$$

for $t \geq 0$.

We notice that $y^2 = t$, so $x = 1 - y^2$. This is a parabola:

[Draw diagram]

The parameter t begins at 0, corresponding to the point $(1, 0)$. As t increases, the x-coordinate decreases while the y-coordinate increases, creating the upper half of the parabola. □

Example 8.3 *Given a curve which is the graph of a function $y = f(x)$ defined on the entire real line, we can describe the curve parametrically as*

$$\begin{aligned} x(t) &= t \\ y(t) &= f(t) \end{aligned} \tag{335}$$

for all real values of t. □

Example 8.4 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= t^2 + 3 \\ y(t) &= t^2 + 4 \end{aligned} \tag{336}$$

for all real values of t.

We notice that $x - 3 = t^2 = y - 4$, so $y = x + 1$. This is a line.

[Draw diagram]

□

Example 8.5 *Describe the curve given by the following parametric equations:*

$$\begin{aligned} x(t) &= a + r\cos t \\ y(t) &= b + r\sin t \end{aligned}, \tag{337}$$

where a, b, and r are real constants, $r > 0$, and $0 \leq t \leq 2\pi$.

We notice that $x - a = r\cos t$ and $y - b = r\sin t$, so

$$(x-a)^2 + (y-b)^2 = r^2\cos^2 t + r^2\sin^2 t = r^2\left(\cos^2 t + \sin^2 t\right) = r^2. \tag{338}$$

This is the equation of a circle with radius r centered at the point (a, b).

[Draw diagram]

The parameter t begins at 0, describing the point $(a + r, b)$. As t increases, the x-coordinate decreases while the y-coordinate increases, creating a counterclockwise motion. $\square$

Section 10.2: Calculus with parametric curves

In this section, we have two goals: to describe the tangent lines to parametric curves and to find the arc lengths of parametric curves.

The slope of the tangent line is always given by $\frac{\mathrm{d}y}{\mathrm{d}x}$. In order to find this quantity for a parametric curve, use the following formula:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}. \tag{339}$$

The concavity is always given by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. To find this quantity for a parametric curve, use the following formula:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}t}}. \tag{340}$$

Example 8.6 *(Problem 10.2.2) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$:*

$$\begin{aligned} x(t) &= te^t \\ y(t) &= t + \sin t \end{aligned} \tag{341}$$

We know that $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \cos t$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = te^t + e^t = (1+t)e^t$, so the slope of the tangent line at any given value of t is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \boxed{\frac{1+\cos t}{(1+t)e^t}}. \tag{342}$$

□

Example 8.7 *(Problem 10.2.4) Find an equation of the tangent line to the curve*

$$\begin{aligned} x(t) &= \sqrt{t} \\ y(t) &= t^2 - 2t \end{aligned} \tag{343}$$

at the point corresponding to $t = 4$.

First, we need the slope of the tangent line:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-2}{\frac{1}{2}t^{-\frac{1}{2}}} = 4\left(t-1\right)\sqrt{t}. \tag{344}$$

At $t = 4$*,*

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=4} = 4\left(4-1\right)\sqrt{4} = 24. \tag{345}$$

Now we need a point on the line. We notice that

$$\begin{aligned} x(4) &= \sqrt{4} = 2 \\ y(4) &= (4)^2 - 2(4) = 8 \end{aligned}. \tag{346}$$

This gives the point $(2, 8)$*. Thus, the equation of the tangent line is* $y - 8 = 24(x - 2)$*, or* $\boxed{y = 24x - 40}$. $\square$

Example 8.8 *For which values of* t *is the curve*

$$\begin{aligned} x(t) &= t^3 + 1 \\ y(t) &= t^2 - t \end{aligned} \tag{347}$$

concave upward?

We need $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$*. First, we find* $\frac{\mathrm{d}y}{\mathrm{d}x}$*:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t-1}{3t^2} \tag{348}$$

Next, we need $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}$*:*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(3t^2\right)\left(2\right) - \left(2t-1\right)\left(6t\right)}{\left(3t^2\right)^2} \\ &= \frac{6t^2 - 12t^2 + 6t}{9t^4} = \frac{6t\left(1-t\right)}{9t^4} = \frac{2\left(1-t\right)}{3t^3}. \end{aligned} \tag{349}$$

Finally, we need to divide by $\frac{\mathrm{d}x}{\mathrm{d}t}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}x}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{2(1-t)}{3t^3}\frac{1}{3t^2} = \frac{2(1-t)}{9t^5}. \tag{350}$$

Now we need to find where this is greater than 0. We produce a number line.

[Draw diagram]

We mark the points $t = 0$ and $t = 1$, since those are where $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ could change sign. For $t < 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$. For $0 < t < 1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$. For $t > 1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$. Thus, the curve is concave up when $0 < t < 1$. $\square$

The arc length of a parametric curve for $\alpha \leq t \leq \beta$ is given by

$$s = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t. \tag{351}$$

Example 8.9 *(Problem 10.2.42) Find the exact length of the curve:*

$$\begin{aligned} x(t) &= e^t - t \\ y(t) &= 4e^{\frac{t}{2}} \end{aligned}, \tag{352}$$

for $0 \leq t \leq 2$.

First, we note that

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= e^t - 1 \\ \tfrac{\mathrm{d}y}{\mathrm{d}t} &= 2e^{\frac{t}{2}} \end{aligned}. \tag{353}$$

Now,

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \left(e^t - 1\right)^2 + \left(2e^{\frac{t}{2}}\right)^2 \\ &= e^{2t} - 2e^2 + 1 + 4e^t \\ &= e^{2t} + 2e^t + 1 = \left(e^t + 1\right)^2 \end{aligned} \tag{354}$$

Therefore,

$$\begin{aligned} s = \int_0^2 \sqrt{\left(e^t+1\right)^2}\,\mathrm{d}t = \int_0^2 e^t + 1\,\mathrm{d}t = e^t + t\Big|_0^2 \\ = \left(e^2+2\right) - \left(e^0+0\right) = \boxed{e^2+1}. \end{aligned} \tag{355}$$

□

9 Tuesday, June 18

Section 10.3: Polar coordinates

A "coordinate system" represents a point as a sequence of numbers.
In the "Cartesian coordinate system," a point in the plane is represented by an ordered pair (x, y) of distances from two perpendicular axes.
In the "polar coordinate system," a point in the plane is represented by an ordered pair (r, θ), where r is the length of a line segment pointing from the origin to the point and θ is the angle that the line segment makes with the positive x-axis.

We can convert from polar coordinates to Cartesian coordinates by using a triangle:

[Draw diagram]

As the diagram illustrates, $x = r\cos\theta$ and $y = r\sin\theta$.
Similarly, in converting from Cartesian coordinates to polar coordinates, we can write $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, provided that $x \neq 0$.

Example 9.1 *(Problem 10.3.4) Plot and find the Cartesian coordinates of the point whose polar coordinates are:*

$$\begin{aligned} &(a)\ \left(4, \tfrac{4\pi}{3}\right) \\ &(b)\ \left(-2, \tfrac{3\pi}{4}\right) \\ &(c)\ \left(-3, -\tfrac{\pi}{3}\right) \end{aligned} \tag{356}$$

(a) *[Draw diagram]*

Now $x = 4\cos\left(\frac{8\pi}{6}\right) = 4\left(-\frac{1}{2}\right) = -2$, $y = 4\sin\left(\frac{8\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$.
Therefore, the Cartesian coordinates are $\boxed{\left(-2, -2\sqrt{3}\right)}$

(b) *A negative value of* r *means the following:* $(-r, \theta) = (r, \theta + \pi)$. *[Draw diagram]*

Now $x = 2\cos\left(-\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$, $y = 2\sin\left(-\frac{\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$. *Therefore, the Cartesian coordinates are* $\boxed{\left(\sqrt{2}, -\sqrt{2}\right)}$.

(c) *[Draw diagram]*

Now $x = 3\cos\left(\frac{4\pi}{6}\right) = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$, $y = 3\sin\left(\frac{4\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$. *Therefore, the Cartesian coordinates are* $\boxed{\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)}$. $\square$

In the same way that we can write functions $y = f(x)$ in Cartesian coordinates, we can also write functions like $r = f(\theta)$ in polar coordinates. We are therefore interested in the tangent line to a curve defined by polar coordinates.

As mentioned before, the slope of the tangent line is always $\frac{\mathrm{d}y}{\mathrm{d}x}$. Now, given that $y = r\sin\theta$ and $x = r\cos\theta$, this becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}r\sin\theta}{\frac{\mathrm{d}}{\mathrm{d}\theta}r\cos\theta} = \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta}. \tag{357}$$

Given $r = f(\theta)$, this is possible to compute.

Example 9.2 *(Problem 10.3.56) Find the slope of the tangent line to the curve* $r = 2 + \sin(3\theta)$ *at the point where* $\theta = \frac{\pi}{4}$.

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 3\cos(3\theta). \tag{358}$$

Therefore, at $\theta = \frac{\pi}{4}$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} \\ &= \frac{\left(2+\sin\left(3\frac{\pi}{4}\right)\right)\cos\left(\frac{\pi}{4}\right) + 3\cos\left(3\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{-\left(2+\sin\left(3\frac{\pi}{4}\right)\right)\sin\left(\frac{\pi}{4}\right) + 3\cos\left(3\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)} \\ &= \frac{\left(2+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}{-\left(2+\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - 3\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}+\frac{1}{2}-\frac{3}{2}}{-\sqrt{2}-\frac{1}{2}-\frac{3}{2}} \\ &= \frac{\sqrt{2}-1}{-\sqrt{2}-2} = \frac{-2+2\sqrt{2}+\sqrt{2}-2}{2-4} = \boxed{2-\frac{3}{2}\sqrt{2}}. \end{aligned} \tag{359}$$

Example 9.3 *(Problem 10.3.56) Find the points on the curve $r = e^{\theta}$ where the tangent line is horizontal or vertical.*

Here we seek the points where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is either 0 or undefined. We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = e^{\theta}. \tag{360}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} = \frac{e^{\theta}\cos\theta + e^{\theta}\sin\theta}{-e^{\theta}\sin\theta + e^{\theta}\cos\theta} \\ &= \frac{e^{\theta}\left(\cos\theta+\sin\theta\right)}{e^{\theta}\left(\cos\theta-\sin\theta\right)} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}. \end{aligned} \tag{361}$$

Now, if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, then $\cos\theta + \sin\theta = 0$. This means that $\cos\theta = -\sin\theta$. This horizontal tangents occur at $\boxed{\theta = \frac{3\pi}{4} + n\pi}$, where n is any integer. If $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined, then $\cos\theta - \sin\theta = 0$, in which case $\cos\theta = \sin\theta$. These vertical tangents occur at $\boxed{\theta = \frac{\pi}{4} + n\pi}$, where n is any integer. □

Example 9.4 *Find the points on the curve $r = 1 - \sin\theta$ where the tangent line is vertical.*

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\cos\theta. \tag{362}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{r\cos\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta}{-r\sin\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta} &= \frac{(1-\sin\theta)\cos\theta + (-\cos\theta)\sin\theta}{-(1-\sin\theta)\sin\theta + (-\cos\theta)\cos\theta} \\ = \frac{(1-2\sin\theta)\cos\theta}{-\sin\theta + \sin^2\theta - \cos^2\theta} &= \frac{(1-2\sin\theta)\cos\theta}{-\sin\theta + \sin^2\theta - \left(1-\sin^2\theta\right)} \\ &= \frac{(1-2\sin\theta)\cos\theta}{2\sin^2\theta - \sin\theta - 1}. \end{aligned} \tag{363}$$

The tangent is vertical when the denominator equals zero: $2\sin^2\theta - \sin\theta - 1 = 0$. *This is a quadratic equation in* $\sin\theta$*:*

$$\sin\theta = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1\pm 3}{4} = 1, -\frac{1}{2}. \tag{364}$$

This gives $\theta = \frac{\pi}{2} + 2\pi n$, $\theta = \frac{7\pi}{6} + 2\pi n$, *and* $\theta = \frac{11\pi}{6} + 2\pi n$. *These correspond to the points whose polar coordinates are* $\boxed{\left(0, \frac{\pi}{2}\right), \left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)}$. □

Section 10.4: Areas and lengths in polar coordinates

When a region is bounded between two rays pointing from the origin and a polar curve, the area of the region is given by

$$A = \int_a^b \frac{1}{2} r^2 \, \mathrm{d}\theta, \tag{365}$$

where a and b are the angles between the rays and the x-axis.

Example 9.5 *(Problem 10.4.2) Find the area of the region bounded by the curve $r = \cos\theta$ and the rays $\theta = 0$ and $\theta = \frac{\pi}{6}$.*

This is

$$\begin{aligned} A = \int_0^{\frac{\pi}{6}} \frac{1}{2} \cos^2\theta \, \mathrm{d}\theta &= \frac{1}{4} \int_0^{\frac{\pi}{6}} 1 + \cos(2\theta) \, \mathrm{d}\theta \\ &= \frac{1}{4} \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Bigg|_0^{\frac{\pi}{6}} = \frac{1}{4} \left(\frac{\pi}{6} + \frac{1}{2} \frac{\sqrt{3}}{2} \right) = \boxed{\tfrac{\pi}{24} + \tfrac{\sqrt{3}}{16}} \end{aligned} \tag{366}$$

□

Example 9.6 *(Problem 10.4.4) Find the area of the region bounded by the curve $r = \frac{1}{\theta}$ and the rays $\theta = \frac{\pi}{2}$ and $\theta = 2\pi$.*

This is

$$A = \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2} \frac{1}{\theta^2} \, \mathrm{d}\theta = -\frac{1}{2} \frac{1}{\theta} \Bigg|_{\frac{\pi}{2}}^{2\pi} = \frac{1}{2} \left(\frac{2}{\pi} - \frac{1}{2\pi} \right) = \frac{1}{2} \left(\frac{4}{2\pi} - \frac{1}{2\pi} \right) = \boxed{\tfrac{3}{4\pi}}. \tag{367}$$

□

Example 9.7 *(Problem 10.4.6) Find the area in the third quadrant enclosed by $r = 2 + \cos\theta$.*

[Draw diagram]

The third quadrant is the region $\frac{\pi}{2} \leq \theta \leq \pi$. Therefore,

$$\begin{aligned} A = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(2+\cos\theta)^2 \,\mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} 4 + 4\cos\theta + \cos^2\theta \,\mathrm{d}\theta \\ = \int_{\frac{\pi}{2}}^{\pi} \frac{9}{4} + 2\cos\theta + \frac{1}{4}\cos(2\theta) \,\mathrm{d}\theta \\ = \frac{9}{4}\theta + 2\sin\theta + \frac{1}{8}\sin(2\theta)\bigg|_{\frac{\pi}{2}}^{\pi} = \boxed{\tfrac{9}{8}\pi - 2}. \end{aligned} \tag{368}$$

□

Now we examine the matter of the arc length of a curve given by a polar equation $r = f(\theta)$ for some interval $\alpha \leq \theta \leq \beta$. Suppose we regard θ as a parameter, and write parametric equations describing such a curve. These would be

$$\begin{aligned} x(\theta) &= r\cos\theta = f(\theta)\cos\theta \\ y(\theta) &= r\sin\theta = f(\theta)\sin\theta \end{aligned}. \tag{369}$$

We know that the arc length of a parametric curve is

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \,\mathrm{d}\theta. \tag{370}$$

Doing the calculation gives the following:

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \,\mathrm{d}\theta. \tag{371}$$

Example 9.8 *(Problem 10.4.46) Find the exact length of the curve $r = 5^{\theta}$ for $0 \leq \theta \leq 2\pi$.*

We know that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 5^\theta \ln 5$. *Therefore,*

$$s = \int_0^{2\pi} \sqrt{5^{2\theta} + 5^{2\theta}(\ln 5)^2}\, \mathrm{d}\theta = \int_0^{2\pi} 5^\theta \sqrt{1 + (\ln 5)^2}\, \mathrm{d}\theta$$
$$= \frac{5^\theta}{\ln 5}\sqrt{1 + (\ln 5)^2}\Bigg|_0^{2\pi} = \frac{\sqrt{1 + (\ln 5)^2}}{\ln 5}\left(5^{2\pi} - 1\right). \quad (372)$$

□

Example 9.9 *(Problem 10.4.48) Find the exact length of the curve defined by the polar equation* $r = 2\left(1 + \cos\theta\right)$ *for* $0 \le \theta \le 2\pi$.

We note that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2\sin\theta. \quad (373)$$

Therefore,

$$s = \int_0^{2\pi} \sqrt{\left(2\left(1 + \cos\theta\right)\right)^2 + \left(-2\sin\theta\right)^2}\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} \sqrt{4\left(1 + 2\cos\theta + \cos^2\theta\right) + 4\sin^2\theta}\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} \sqrt{4\left(1 + 2\cos\theta\right) + 4\cos^2\theta + 4\sin^2\theta}\, \mathrm{d}\theta = \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} 4\sqrt{\frac{1}{2}\left(1 + \cos\theta\right)}\, \mathrm{d}\theta = 4\int_0^{2\pi} \sqrt{\cos^2\left(\frac{1}{2}\theta\right)}\, \mathrm{d}\theta$$
$$= 4\int_0^{2\pi} \left|\cos\left(\frac{1}{2}\theta\right)\right| \mathrm{d}\theta = 8\int_0^{\pi} \cos\left(\frac{1}{2}\theta\right)\, \mathrm{d}\theta$$
$$= 16\sin\left(\frac{1}{2}\theta\right)\Bigg|_0^{\pi} = 16\left(1 - 0\right) = \boxed{16}. \quad (374)$$

□

10 Monday, June 24

Section 11.1: Sequences

Definition 10.1 *A sequence of real numbers is an infinite list of real numbers with a defined order.*

Example 10.2 *Find a formula for the general term a_n of the sequence whose terms are $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 16$, $a_5 = 32$, ... (In other words, find a formula for a_n in terms of n.)*

Here the formula is $a_n = 2^n$. $\square$

Example 10.3 *(Problem 11.1.14) Find a formula for the general term a_n of the sequence whose terms are $a_1 = 4$, $a_2 = -1$, $a_3 = \frac{1}{4}$, $a_4 = -\frac{1}{16}$, $a_5 = \frac{1}{64}$, ...*

Here the formula is $a_n = (-1)^{n+1}4^{2-n}$. $\square$

We will be largely concerned with limits of sequences.

Definition 10.4 *Let (a_n) be a sequence of real numbers. Given a real number L, we say that L is the limit of (a_n) as n approaches infinity, or that (a_n) converges to L provided that the following is true: by taking n to be sufficiently large, we can make the distance $|a_n - L|$ arbitrarily small.*

Notation: $\lim\limits_{n\to\infty} a_n = L$.

If a sequence has a real-valued limit, we say that the sequence is "convergent." If it has no real-valued limit, we say that it is "divergent."

We take limits of sequences in the same way we took limits of functions in Calculus I.

Example 10.5 *Determine whether the sequence $a_n = \frac{3n^4+5}{4n^4-7n^2+9}$ converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{3n^4+5}{4n^4-7n^2+9} = \lim_{n\to\infty} \frac{3+\frac{5}{n^4}}{4-\frac{7n^2}{n^4}+\frac{9}{n^4}} = \frac{3+0}{4-0+0} = \boxed{\frac{3}{4}}. \tag{375}$$

□

Example 10.6 *(Problem 11.1.28) Determine whether the sequence* $a_n = \frac{3\sqrt{n}}{\sqrt{n}+2}$ *converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{3}{1+\frac{2}{\sqrt{n}}} = \frac{3}{1+0} = \boxed{3}. \tag{376}$$

□

Example 10.7 *Determine whether the sequence* $a_n = \frac{n^2-1}{n}$ *converges or diverges. If it converges, find the limit.*

We know that

$$\lim_{n\to\infty} \frac{n^2-1}{n} = \infty, \tag{377}$$

so the sequence is divergent. □

Example 10.8 *Determine whether the sequence* $a_n = 1 + (-1)^n$ *converges or diverges. If it converges, find the limit.*

The terms in the sequence are $0, 2, 0, 2, 0, 2, ...$*, so the sequence has no limit; it is divergent.* □

Theorem 10.9 *If* $\lim_{n\to\infty} a_n = L$ *and* f *is continuous at* L*, then* $\lim_{n\to\infty} f(a_n) = f(L)$*.*

Example 10.10 *(Problem 11.1.32) Determine whether the sequence* $a_n = \cos\left(\frac{n\pi}{n+1}\right)$ *converges or diverges. If it converges, find the limit.*

We take

$$\lim_{n\to\infty} \frac{n\pi}{n+1} = \lim_{n\to\infty} \frac{\pi}{1+\frac{1}{n}} = \pi. \tag{378}$$

Now $\cos(\pi) = -1$*, so by the theorem,* $\lim_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right) = \boxed{-1}$. □

Definition 10.11 *A geometric sequence is a sequence* (a_n) *in which* $a_n = br^n$ *for some real values* b *and* r*. We call* r *the common ratio of* a_n*.*

Theorem 10.12 *The geometric sequence* (br^n) *is:*
(i) Convergent to 0 *for* $-1 < r < 1$*.*
(ii) Convergent to b *for* $r = 1$*.*
(iii) Divergent for $r \leq -1$ *or* $r > 1$*.*

Definition 10.13 *A sequence a_n of real numbers is called:*
(i) decreasing if $a_{n+1} \leq a_n$ for all n.
(ii) increasing if $a_{n+1} \geq a_n$ for all n.
(iii) monotone if it is either decreasing or increasing.
(iv) eventually decreasing if $a_{n+1} \leq a_n$ for all n greater than a certain value.
(v) eventually increasing if $a_{n+1} \geq a_n$ for all n greater than a certain value.

Example 10.14 *Determine whether the sequence $a_n = \frac{1}{2n+3}$ is increasing, decreasing, or not monotone.*

We know that $2(n+1)+3 > 2n+3$ for any positive integer n. Therefore, $a_n = \frac{1}{2n+3} > \frac{1}{2(n+1)+3} = a_{n+1}$, so the sequence is $\boxed{\textit{decreasing}}$. □

Example 10.15 *Determine whether the sequence $a_n = \frac{n}{n^2+1}$ is increasing, decreasing, or not monotone.*

Consider the function

$$f(x) = \frac{x}{x^2+1}. \tag{379}$$

Taking the derivative,

$$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}. \tag{380}$$

This is negative when $x > 1$, so f is decreasing on $(1, \infty)$. This tells us that $a_{n+1} = f(n+1) < f(n) = a_n$ for any positive integer n. Ergo, a_n is $\boxed{\textit{decreasing}}$.
□

Definition 10.16 *The harmonic sequence is the sequence $a_n = \frac{1}{n}$.*

The harmonic sequence is a decreasing sequence which converges to 0.

Section 11.2: Series

It makes sense to discuss $a+b$, because addition is defined for two real numbers. You can extend this to discussing $a_1 + a_2 + ... + a_n$, for any positive integer n. What would be the meaning of a sum of infinitely many numbers?

Definition 10.17 *Let a_n be a sequence of real numbers. Given a positive integer k, the kth partial sum of a_n is the value $s_k = \sum_{n=1}^{k} a_n$.*

An "infinite series" is the limit of the partial sums of a sequence.

Definition 10.18 *Let a_n be a sequence of real numbers, and let s_n be the nth partial sum of a_n. If the sequence (s_n) converges to a real number s, then we say that the series $\sum_{n=1}^{\infty} a_n$ is convergent to s, and that s is the sum of the series. If no such real number exists, we say that the series $\sum_{n=1}^{\infty} a_n$ is divergent.*

We begin by discussing one of the most important examples.

Definition 10.19 *A geometric series is a series $\sum_{n=1}^{\infty} a_n$, where a_n is a geometric sequence.*

Do geometric series converge? It depends.

Consider the geometric series $\sum_{n=1}^{\infty} br^{n-1}$.
If the common ratio $r = 1$, then the kth partial sum of $a_n = br^{n-1}$ is

$$s_k = \sum_{n=1}^{k} a_n = \sum_{n=1}^{k} br^{n-1} = \sum_{n=1}^{k} b = kb. \tag{381}$$

Now, $\lim_{k\to\infty} kb = \infty$, so the series $\sum_{n=1}^{\infty} br^{n-1}$ is divergent in this case.
If the common ratio $r \neq 1$, then the kth partial sum is

$$s_k = b + br + br^2 + ... + br^{k-1}. \tag{382}$$

We notice:

$$rs_k = br + br^2 + br^3 + ... + br^k = s_k - b + br^k. \tag{383}$$

Therefore,

$$\begin{aligned} rs_k - s_k &= -b + br^k \\ (r-1)\, s_k &= b\left(-1 + r^k\right). \\ s_k &= \tfrac{b\left(r^k - 1\right)}{r-1} = \tfrac{b\left(1-r^k\right)}{1-r}. \end{aligned} \tag{384}$$

Now, if $-1 < r < 1$, then

$$\sum_{n=1}^{\infty} br^{n-1} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{b\left(1-r^k\right)}{1-r} = \frac{b\,(1-0)}{1-r} = \frac{b}{1-r}. \tag{385}$$

On the other hand, if $r \le -1$ or $r > 1$, then $\lim\limits_{k\to\infty} r^k$ diverges, so therefore $\sum\limits_{n=1}^{\infty} br^{n-1}$ also diverges. To summarize:

Theorem 10.20 *Let b and r be real numbers.*
(i) The geometric series $\sum\limits_{n=1}^{\infty} br^{n-1}$ diverges if $|r| \ge 1$.
(ii) The geometric series $\sum\limits_{n=1}^{\infty} br^{n-1} = \frac{b}{1-r}$ if $-1 < r < 1$.

Note: $\sum\limits_{n=1}^{\infty} br^{n-1} = \sum\limits_{n=0}^{\infty} br^{n}$.

What can we say about non-geometric series?

Some series are called "telescoping series," and have terms that cancel pairwise in the partial sums.

Example 10.21 *Determine whether the series $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ is convergent or divergent. If it is convergent, find its sum.*

We notice that the kth partial sum of this series is

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{k} - \frac{1}{k+1}\right). \tag{386}$$

Most of these terms cancel to give us:

$$s_k = 1 - \frac{1}{k+1}. \tag{387}$$

Now,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n\to\infty} s_k = \lim_{k\to\infty} 1 - \frac{1}{k+1} = \boxed{1}. \tag{388}$$

□

Theorem 10.22 *(Test for divergence) If* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

This theorem proposes the following test: if $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ cannot be convergent. The converse is untrue:

Theorem 10.23 *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Why? We'll see in the next section.

Example 10.24 *(Problem 11.2.30) Determine whether the series* $\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$ *is convergent or divergent. If it is convergent, find its sum.*

We notice that

$$\lim_{n\to\infty} \frac{n^2}{n^2-2n+5} = \lim_{n\to\infty} \frac{1}{1-\frac{2}{n}+\frac{5}{n^2}} = \frac{1}{1-0+0} = 1. \tag{389}$$

Since the limit of the terms is nonzero, the test for divergence dictates that the series cannot converge. □

11 Tuesday, June 25

Section 11.3: The integral test

Finding the exact sum of a series is a difficult problem in general. For the next few sections, we'll concentrate on testing whether a series is convergent or divergent without finding the sum.

Theorem 11.1 *(The integral test) Let (a_n) be a positive, eventually decreasing sequence of real numbers. Given a continuous function f defined on $[1, \infty)$ such that for each positive integer n, $f(n) = a_n$, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^\infty f(x)\, \mathrm{d}x$ is convergent.*

Example 11.2 *Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is convergent or divergent.*

In order to use the integral test, we must verify that the function $f(x) = \frac{\ln x}{x}$ is eventually decreasing. We do so by examining the derivative:

$$f'(x) = \frac{1 - \ln x}{x^2} \tag{390}$$

This is negative when $1 - \ln x < 0$, or in other words, when $\ln x > 1$, hence $x > e$. Thus, f is eventually decreasing, and so the integral test can be used.

$$\int_1^\infty \frac{\ln x}{x}\, \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, \mathrm{d}x \tag{391}$$

We proceed by u-substitution:

$$\begin{aligned} u &= \ln x \\ \mathrm{d}u &= \tfrac{1}{x}\, \mathrm{d}x \end{aligned}. \tag{392}$$

Now,

$$\int_1^\infty \frac{\ln x}{x}\, \mathrm{d}x = \lim_{t\to\infty} \int_0^{\ln t} u\, \mathrm{d}u = \lim_{t\to\infty} \frac{1}{2}u^2 \bigg|_0^{\ln t} = \lim_{t\to\infty} \frac{1}{2}(\ln t)^2 = \infty. \tag{393}$$

This shows that the series is $\boxed{\textit{divergent}}$. □

Example 11.3 *(Problem 11.3.8) Determine whether the series* $\sum\limits_{n=1}^{\infty} n^2e^{-n^3}$ *is convergent or divergent.*

First we need to verify that $f(x) = x^2e^{-x^3}$ *is eventually decreasing. We examine the derivative:*

$$f'(x) = -3x^4e^{-x^3} + 2xe^{-x^3} = xe^{-x^3}\left(2 - 3x^3\right). \tag{394}$$

For $x > \sqrt[3]{\frac{2}{3}}$*, this is negative, so the sequence of terms is eventually decreasing. We can now use the integral test:*

$$\int_1^{\infty} x^2e^{-x^3}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t x^2e^{-x^3}. \tag{395}$$

We proceed by u*-substitution:*

$$\begin{aligned} u &= x^3 \\ \mathrm{d}u &= 3x^2\,\mathrm{d}x \end{aligned}. \tag{396}$$

Now,

$$\begin{aligned} \int_1^{\infty} x^2e^{-x^3}\,\mathrm{d}x &= \lim_{t\to\infty}\int_1^{t^3}\frac{1}{3}e^{-u}\,\mathrm{d}u \\ &= \lim_{t\to\infty} -\frac{1}{3}e^{-u}\Big|_1^{t^3} = \lim_{t\to\infty}\frac{1}{3}e^{-1} - \frac{1}{3}e^{-t^3} = \frac{1}{3e}. \end{aligned} \tag{397}$$

This shows that the series is $\boxed{\textit{convergent}}$*. (It does not tell us the sum.)* □

Example 11.4 *(Problem 11.3.6) Determine whether the series* $\sum\limits_{n=1}^{\infty}\frac{1}{(3n-1)^4}$ *is convergent or divergent.*

First we verify that $f(x) = \frac{1}{(3x-1)^4}$ *is eventually decreasing:*

$$f'(x) = -4(3x-1)^{-5}(3) = \frac{-12}{(3x-1)^5}. \tag{398}$$

This is negative for $x > \frac{1}{3}$*, so the sequence of terms is eventually decreasing. We* *can now use the integral test:*

$$\int_1^\infty \frac{1}{(3x-1)^4}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t (3x-1)^{-4}\,\mathrm{d}x. \tag{399}$$

We proceed by u-substitution:

$$\begin{aligned} u &= 3x-1 \\ \mathrm{d}u &= 3\,\mathrm{d}x \end{aligned}, \tag{400}$$

so

$$\begin{aligned} \int_1^\infty \frac{1}{(3x-1)^4}\,\mathrm{d}x &= \lim_{t\to\infty}\int_2^{3t-1}\frac{1}{3}u^{-4}\,\mathrm{d}u \\ &= \lim_{t\to\infty} -\frac{1}{9}u^{-3}\Big|_2^{3t-1} = \lim_{t\to\infty}\frac{1}{9}\left(\frac{1}{8}-\frac{1}{(3t-1)^3}\right) = \frac{1}{9}\frac{1}{8} = \frac{1}{72}. \end{aligned} \tag{401}$$

This shows that the series is $\boxed{\textit{convergent}}$. □

Example 11.5 *The series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ for an integer p is called the "p-series." For what values of p does the p-series converge?*

If $p<0$, then $\lim\limits_{n\to\infty}\frac{1}{n^p}=\infty$, in which case the test for divergence reveals that the p-series is divergent.

If $p=0$, then $\lim\limits_{n\to\infty}\frac{1}{n^p}=\lim\limits_{n\to\infty}1=1$, so again the test for divergence dictates that the p-series is divergent.

If $p=1$, then this is the harmonic series. We will now show that the harmonic series is divergent, using the integral test. We know that $n<n+1$, so $\frac{1}{n+1}<\frac{1}{n}$ for any positive integer n. Thus, the terms of the harmonic series form a decreasing sequence. Consider the function $f(x)=\frac{1}{x}$. Now use the integral test:

$$\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty}\ln x\Big|_1^t = \lim_{t\to\infty}\ln t - \ln 1 = \lim_{t\to\infty}\ln t = \infty. \tag{402}$$

As this is divergent, we see that the harmonic series is also divergent. Thus, the p-series is divergent for $p=1$.

If $p>1$, then we notice that $n^p<(n+1)^p$, so $\frac{1}{(n+1)^p}<\frac{1}{n^p}$ for any positive integer n. This shows that the terms of the p-series form a decreasing sequence. We *consider the function $f(x) = \frac{1}{x^p}$. Now, we use the integral test:*

$$\begin{aligned}\int_1^\infty f(x)\,\mathrm{d}x = \int_1^\infty x^{-p}\,\mathrm{d}x &= \lim_{t\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^t \\ &= \lim_{t\to\infty} \frac{1}{1-p}\left(t^{1-p}-1\right) = \lim_{t\to\infty} \frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right) \end{aligned} \tag{403}$$

This improper integral is convergent if and only if $p-1>0$. This shows that the p-series is convergent exactly when $\boxed{p>1}$. □

From the previous example, we can state the following theorem.

Theorem 11.6 *(p-series test) Let p be a real number. The p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p>1$.*

Example 11.7 *(Problem 11.3.4) Determine whether the series $\sum\limits_{n=1}^{\infty} n^{-0.3}$ is convergent or divergent.*

This is a p-series:

$$\sum_{n=1}^{\infty} n^{-0.3} = \sum_{n=1}^{\infty} \frac{1}{n^{0.3}}. \tag{404}$$

Since $0.3 \le 1$, the series is $\boxed{\textit{divergent}}$. □

Section 11.4: The comparison tests

If you know that a given series with positive terms is convergent or divergent, you can sometimes use that information to deduce whether another series with positive terms is convergent or divergent.

Theorem 11.8 *(Comparison test) Let* $\sum_{n=1}^{\infty} a_n$ *be a series with positive terms.*
(i) Suppose that for all positive integers n, $a_n \leq b_n$. *If* $\sum_{n=1}^{\infty} b_n$ *is convergent, then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*
(ii) Suppose that for all positive integers n, $c_n \leq a_n$. *If* $\sum_{n=1}^{\infty} c_n$ *is divergent, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example 11.9 *Determine whether the series* $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ *converges or diverges.*

We notice that

$$\frac{1}{2^n+1} < \frac{1}{2^n}, \tag{405}$$

so since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ *is a convergent geometric series, the series is* $\boxed{\textit{convergent}}$.
□

Example 11.10 *(Problem 11.4.4) Determine whether the series* $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ *converges or diverges.*

We notice that

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}, \tag{406}$$

so since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ *is a divergent* p*-series, the series is* $\boxed{\textit{divergent}}$. □

Example 11.11 *(Problem 11.4.6) Determine whether the series* $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ *is convergent or divergent.*

We notice that

$$\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}. \tag{407}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the series is* $\boxed{\textit{convergent}}$. $\square$

Example 11.12 *(Problem 11.4.8) Determine whether the series* $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ *is convergent or divergent.*

We notice that

$$\frac{6^n}{5^n - 1} > \frac{6^n}{5^n}. \tag{408}$$

As $\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ *is a divergent geometric series, the series is also* $\boxed{\textit{divergent}}$.
$\square$

Example 11.13 *(Problem 11.4.14) Determine whether the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$ *is convergent or divergent.*

We notice that

$$\frac{1}{\sqrt[3]{3n^4 + 1}} < \frac{1}{\sqrt[3]{3n^4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{\frac{4}{3}}}. \tag{409}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$ *is a convergent* p*-series, the series is* $\boxed{\textit{convergent}}$. $\square$

Theorem 11.14 *(Limit comparison test) Suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with positive terms. If* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *is positive and finite, then either both converge series or both diverge.*

Example 11.15 *(Problem 11.4.22) Determine whether the series* $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ *converges or diverges.*

We use the limit comparison test with $\sum_{n=3}^{\infty} \frac{1}{n^2}$*:*

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{n+2}{(n+1)^3}\right)} = \lim_{n\to\infty} \frac{(n+1)^3}{n^2\,(n+2)} = \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1}{n^3 + 2n^2} = 1. \tag{410}$$

Since 1 *is positive and finite, and since* $\sum_{n=3}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the limit comparison test indicates that* $\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ *is also* $\boxed{\textit{convergent}}$.

Example 11.16 *Determine whether the series* $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ *converges or diverges.*

We use the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$*:*

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{2n+1}\right)} = \lim_{n\to\infty} \frac{2n+1}{n} = 2. \tag{411}$$

Since 2 *is positive and finite, and since* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent, the limit comparison test indicates that* $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ *is also* $\boxed{\textit{divergent}}$*.*

12 Wednesday, June 26

Section 11.5: Alternating series

Definition 12.1 *An alternating series is a series whose successive terms are alternately positive and negative.*

Example 12.2 *The following are alternating series:*

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+... \tag{412}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}n^2 = 1-4+9-16+25-... \tag{413}$$

$$\sum_{n=1}^{\infty}\cos(n\pi) = (-1)+1+(-1)+1+(-1)+... \tag{414}$$

□

Theorem 12.3 *(Alternating series test) Let $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ be an alternating series, where for each positive integer n, $b_n > 0$. If the sequence (b_n) is decreasing and $\lim_{n\to\infty} b_n = 0$, then the series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ is convergent.*

Example 12.4 *(Problem 11.5.2) Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{2n+1}$ is convergent or divergent.*

We note that

$$\frac{2}{2n+1} > \frac{2}{2(n+1)+1}. \tag{415}$$

Additionally,

$$\lim_{n\to\infty}\frac{2}{2n+1} = 0. \tag{416}$$

Therefore, by the alternating series test, the series is $\boxed{\textit{convergent}}$. □

Example 12.5 *(Problem 11.5.4) Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(2n+1)}$ is convergent or divergent.*

We note that

$$\frac{1}{\ln(2n+1)} > \frac{1}{\ln(2(n+1)+1)}. \tag{417}$$

Additionally,

$$\lim_{n\to\infty} \frac{1}{\ln(2n+1)} = 0. \tag{418}$$

Therefore, by the alternating series test, the series is $\boxed{\textit{convergent}}$. □

Example 12.6 *(Problem 11.5.8) Determine whether the series* $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$ *is convergent or divergent.*

We note that

$$\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1. \tag{419}$$

Therefore, $\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+n+1}$ *does not exist. By the test for divergence, the series is* $\boxed{\textit{divergent}}$. □

Example 12.7 *Determine whether the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *is convergent or divergent.*

We note that

$$\frac{1}{n} > \frac{1}{n+1}. \tag{420}$$

Further,

$$\lim_{n\to\infty} \frac{1}{n} = 0. \tag{421}$$

Therefore, by the alternating series test, the series is $\underline{\textit{convergent}}$. □

Example 12.8 *Determine whether the series* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$ *is convergent or divergent.*

We note that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1, \tag{422}$$

so the limit $\lim_{n\to\infty} (-1)^{n+1} \frac{n}{n+1}$ *does not exist. Therefore, by the test for divergence, the series is* $\boxed{\textit{divergent}}$. □

Section 11.6: Absolute convergence and the ratio and root tests

Definition 12.9 *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. We say that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent provided that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Theorem 12.10 *If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

Definition 12.11 *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. We say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent, but not absolutely convergent.*

Example 12.12 *Determine whether $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$ converges absolutely, converges conditionally, or is divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right|. \tag{423}$$

We notice that $\left|\sin\left(\frac{n\pi}{3}\right)\right| \leq 1$, so

$$\left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right| \leq \frac{1}{n^2}. \tag{424}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, so by the comparison test, $\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \right|$ converges. Ergo, $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2}$ is $\boxed{\textit{absolutely convergent}}$. $\square$

Example 12.13 *(Problem 11.6.36) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}}$ is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right| \tag{425}$$

Since $\left|\sin\left(\frac{n\pi}{6}\right)\right| \leq 1$,

$$\left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right| \leq \frac{1}{1+n\sqrt{n}} < \frac{1}{n^{\frac{3}{2}}}. \tag{426}$$

Now $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ *is a convergent* p*-series, so by the comparison test,* $\sum\limits_{n=1}^{\infty} \left|\frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}}\right|$ *converges. Ergo,* $\sum\limits_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}}$ *is* $\boxed{\textit{absolutely convergent}}$. □

Example 12.14 *(Problem 11.6.4) Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ *is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}. \tag{427}$$

We note that

$$\frac{1}{n^3+1} < \frac{1}{n^3}. \tag{428}$$

Now $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ *is a convergent* p*-series, so by the comparison test,* $\sum\limits_{n=1}^{\infty} \frac{1}{n^3+1}$ *is convergent. Therefore, the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ *is* $\boxed{\textit{absolutely convergent}}$. □

Example 12.15 *(Problem 11.6.2) Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *is absolutely convergent, conditionally convergent, or divergent.*

We consider

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}. \tag{429}$$

This is a divergent p*-series, so the series is not absolutely convergent. However,*

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}, \tag{430}$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0, \tag{431}$$

so by the alternating series test, the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *is convergent; the series is* $\boxed{\textit{conditionally convergent}}$. □

Theorem 12.16 *(Ratio test) Let* $\sum\limits_{n=1}^{\infty} a_n$ *be an infinite series.*

(i) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$*, then the series converges absolutely.*

(ii) If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$*, then the series diverges.*

Notice that we have no conclusion if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Example 12.17 *(Problem 11.6.12) Determine whether the series* $\sum\limits_{n=1}^{\infty} ne^{-n}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{e} = \frac{1}{e} < 1. \tag{432}$$

By the ratio test, the series is $\boxed{\textit{absolutely convergent}}$. □

Example 12.18 *(Problem 11.6.14) Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{n!}{100^n}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{100^{n+1}}\frac{100^n}{n!} = \lim_{n\to\infty}\frac{n+1}{100} = \infty \tag{433}$$

By the ratio test, the series is $\boxed{\textit{divergent}}$. □

Example 12.19 *(Problem 11.6.8) Determine whether the series* $\sum\limits_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^2}\frac{n^2}{2^n} = \lim_{n\to\infty}\frac{2n^2}{n^2+2n+1} = 2 > 1. \tag{434}$$

By the ratio test, the series is $\boxed{\textit{divergent}}$. □

Theorem 12.20 *(Root test) Let* $\sum\limits_{n=1}^{\infty} a_n$ *be an infinite series.*

(i) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$*, then the series converges absolutely.*

(ii) If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$*, then the series diverges.*

Again, notice that no conclusion can be drawn if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Example 12.21 *(Problem 10.6.32) Determine whether the series* $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n}\right)^n$ *is absolutely convergent, conditionally convergent, or divergent.*

We use the root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{1-n}{2+3n}\right| = \frac{1}{3} < 1. \tag{435}$$

Therefore, by the root test, the series is $\boxed{\textit{absolutely convergent}}$. □

13 Thursday, June 27

<u>Section 11.7</u>: Strategy for testing series (?)

Example 13.1 *(Problem 11.7.2) Test the series* $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ *for convergence or divergence.*

We note that

$$\frac{n-1}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}. \tag{436}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent* p*-series, the series* $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ $\boxed{\textit{converges}}$ *by the comparison test.* $\square$

Example 13.2 *(Problem 11.7.4) Test the series* $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1. \tag{437}$$

Therefore, $\lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1}$ *does not exist. By the test for divergence, the series* $\boxed{\textit{diverges}}$. $\square$

Example 13.3 *(Problem 11.7.6) Test the series* $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{n^2}{(1+n)^3} = 0 < 1, \tag{438}$$

so by the root test, the series $\boxed{\textit{converges}}$. $\square$

Example 13.4 *(Problen 11.7.8) Test the series* $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$ *for convergence or divergence.*

We note that

$$\lim_{n\to\infty} \frac{(n+1)^4}{4^{n+1}} \frac{4^n}{n^4} = \lim_{n\to\infty} \frac{(n+1)^4}{n^4} \frac{1}{4} = \frac{1}{4} < 1. \tag{439}$$

Thus, by the ratio test, the series $\boxed{\textit{converges}}$. $\square$

Example 13.5 *(Problem 11.7.18) Test the series* $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ *for convergence or divergence.*

We note that

$$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n+1}-1} \tag{440}$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0. \tag{441}$$

Thus, by the alternating series test, the series $\boxed{\textit{converges}}$. □

Section 11.8: Power series

Definition 13.6 *Let a be a real value. A power series about $x = a$ is a function f defined via $f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n$ for some sequence of real numbers (c_n).*

Theorem 13.7 *Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Exactly one of the following statements is true.*
(i) $f(x)$ converges only when $x = a$.
(ii) $f(x)$ converges for all real values x.
(iii) There exists a positive number R (called the "radius of convergence") such that $f(x)$ converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(If $f(x)$ converges for all real x, we say that "the radius of convergence is ∞.")

Definition 13.8 *Let $f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n$ be a power series. The interval of convergence of f is the interval of real numbers x such that $f(x)$ converges.*

To find the interval of convergence, use the ratio test to determine the x-values for which $f(x)$ converges absolutely. This will give the radius of convergence, and some open interval of values of x. Next, test the endpoints of this interval.

Example 13.9 *Find the radius of convergence and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} x^n$.*

We note that the series converges absolutely when the ratio test gives a limit less than 1:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{n+3} \frac{n+2}{(-1)^n x^n} \right| = \lim_{n\to\infty} \frac{n+2}{n+3}|x| = |x| < 1. \tag{442}$$

This gives us a radius of convergence $\boxed{R = 1}$. We deduce that the power series converges for $-1 < x < 1$. What about for $x = \pm 1$? We check these separately. First, for $x = -1$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}(-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+2} = \sum_{n=0}^{\infty} \frac{1}{n+2}. \tag{443}$$

This series is

$$\sum_{n=0}^{\infty} \frac{1}{n+2} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... = \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) - 1, \tag{444}$$

which diverges, since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ *diverges. Next, for* $x = 1$*:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2} 1^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}. \tag{445}$$

This converges by the alternating series test. Therefore, the power series converges only for $-1 < x \leq 1$*, and so the interval of convergence is* $\boxed{(-1, 1]}$*.* $\square$

Example 13.10 *Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=0}^{\infty} \frac{(x-2)^n}{2^n}$*.*

We use the ratio test:

$$\lim_{n \to \infty} \left| \frac{(x-2)^{n+1}}{2^{n+1}} \frac{2^n}{(x-2)^n} \right| = \frac{|x-2|}{2} < 1. \tag{446}$$

This tells us that $|x - 2| < 2$*, so the radius of convergence is* $\boxed{R = 2}$*. We deduce that the power series converges for* $0 < x < 4$*. We test* $x = 0$*:*

$$\sum_{n=0}^{\infty} \frac{(0-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n. \tag{447}$$

This is divergent, by the test for divergence. We test $x = 4$*:*

$$\sum_{n=0}^{\infty} \frac{(4-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1, \tag{448}$$

which is also divergent, by the test for divergence. Therefore, the power series converges only for $0 < x < 4$*, and so the interval of convergence is* $\boxed{(0, 4)}$*.* $\square$

Example 13.11 *Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$*.*

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0. \tag{449}$$

This is less than 1 *for all* x*, so the power series converges for all* x*; "the radius of convergence is* $\boxed{R=\infty}$*" and the interval of convergence is* $\boxed{(-\infty,\infty)}$. $\square$

Example 13.12 *(Problem 11.8.4) Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=1}^{\infty}\frac{(-1)^n x^n}{\sqrt[3]{n}}$.

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{\sqrt[3]{n}}{\sqrt[3]{n+1}}|x| = |x| < 1 \tag{450}$$

The radius of convergence is $\boxed{R=1}$*. We deduce that the power series converges for* $-1<x<1$*. We test* $x=-1$*:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{3}}}. \tag{451}$$

This is a divergent p*-series. We test* $x=1$*:*

$$\sum_{n=1}^{\infty}\frac{(-1)^n 1^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}. \tag{452}$$

This is convergent, by the alternating series test. Therefore, the power series converges only for $-1<x\leq 1$*, and so the interval of convergence is* $\boxed{(-1,1]}$. $\square$

Example 13.13 *(Problem 11.8.6) Find the radius of convergence and interval of convergence of the power series* $\sum\limits_{n=1}^{\infty}\frac{(-1)^n x^n}{n^2}$.

We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)^2}\frac{n^2}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1}|x| = |x| < 1. \tag{453}$$

The radius of convergence is $\boxed{R=1}$*. We deduce that the power series converges* *for $-1 < x < 1$. We test $x = -1$:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{454}$$

This is a convergent p-series. We test $x = 1$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}. \tag{455}$$

This converges by the alternating series test. Therefore, the power series converges only for $-1 \leq x \leq 1$, and so the interval of convergence is $\boxed{[-1, 1]}$. $\square$

Example 13.14 *(Problem 11.8.20) Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$.*

We use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}} \frac{5^n\sqrt{n}}{(2x-1)^n} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{5\sqrt{n+1}} |2x-1| = \frac{1}{5}|2x-1| < 1. \tag{456}$$

This gives $|2x-1| < 5$, so $\left|x - \frac{1}{2}\right| < \frac{5}{2}$, so the radius of convergence is $\boxed{R = \frac{5}{2}}$. We deduce that the power series converges for $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2}$, or in other words, $-2 < x < 3$.
We test $x = -2$:

$$\sum_{n=1}^{\infty} \frac{(2(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}. \tag{457}$$

This converges by the alternating series test. We test $x = 3$:

$$\sum_{n=1}^{\infty} \frac{2(3) - 1^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}. \tag{458}$$

This is a divergent p-series. Thus, the power series converges only for $-2 \leq x < 3$, and so the interval of convergence is $\boxed{[-2, 3)}$. $\square$

14 Monday, July 1

Section 11.9: Representations of functions as power series

Certain functions can be expressed as power series.

Recall that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $-1 < r < 1$. Therefore,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \tag{459}$$

whose interval of convergence is $(-1, 1)$.

Example 14.1 *(Problem 11.9.4) Find a power series representation of the function $f(x) = \frac{5}{1-4x^2}$ and determine the interval of convergence.*

We note that

$$\frac{5}{1-4x^2} = 5\sum_{n=0}^{\infty} \left(4x^2\right)^n = \boxed{\sum_{n=0}^{\infty} 5\left(4^n\right)x^{2n}}. \tag{460}$$

As for the interval of convergence: the geometric series $\sum_{n=0}^{\infty} \left(4x^2\right)^n$ converges if and only if $|4x^2| < 1$. This means that $x^2 < \frac{1}{4}$, so $-\frac{1}{2} < x < \frac{1}{2}$. The interval of convergence is $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$. $\square$

Example 14.2 *(Problem 11.9.8) Find a power series representation of the function $f(x) = \frac{x}{2x^2+1}$ and determine the interval of convergence.*

We note that

$$\frac{x}{2x^2+1} = x\left(\frac{1}{1-(-2x^2)}\right) = x\sum_{n=0}^{\infty} \left(-2x^2\right)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^n x^{2n+1}}. \tag{461}$$

As for the interval of convergence: the geometric series $\sum_{n=0}^{\infty} \left(-2x^2\right)^n$ converges if and only if $|-2x^2| < 1$, or in other words if $x^2 < \frac{1}{2}$. Ergo, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, so the *interval of convergence is* $\boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}$. $\square$

Theorem 14.3 *Let* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *be a power series with a radius of convergence* $R > 0$*. This function is differentiable on* $(a-R, a+R)$ *and:*
(i)

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} \tag{462}$$

with radius of convergence R*.*
(ii)

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \tag{463}$$

with radius of convergence R*.*

Example 14.4 *Find a power series representation for* $f(x) = \ln(1+x)$ *and its radius of convergence.*

We know that

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n. \tag{464}$$

This is convergent if and only if $|-x| < 1$*, so* $|x| < 1$ *and thus the radius of convergence is* $\boxed{R = 1}$*. Further,*

$$\ln(1+x) = \int f'(x)\,\mathrm{d}x = \int \sum_{n=0}^{\infty}(-1)^n x^n\,\mathrm{d}x = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \tag{465}$$

To find C*, we substitute* $x = 0$*:*

$$C = C + \sum_{n=0}^{\infty}(-1)^n \frac{0^{n+1}}{n+1} = \ln(1+0) = \ln 1 = 0. \tag{466}$$

Therefore,

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}}. \tag{467}$$

□

Example 14.5 *Find a power series representation for $f(x) = \tan^{-1}x$ and find its radius of convergence.*

We know that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = (-1)^n x^{2n}. \tag{468}$$

This is convergent if and only if $|-x^2| < 1$, so $|x| < 1$ and thus the radius of convergence is $\boxed{R=1}$. Further,

$$\tan^{-1}x = \int f'(x)\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{469}$$

To find the value of C, we substitute $x = 0$:

$$C = C + \sum_{n=0}^{\infty} (-1)^n \frac{0^{2n+1}}{2n+1} = \tan^{-1}0 = 0. \tag{470}$$

Thus,

$$\boxed{\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}. \tag{471}$$

□

Section 11.10: Taylor and Maclaurin series

Definition 14.6 *Let f be a function, and let a be a real number. The Taylor series of f centered at $x = a$ is the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{472}$$

The Maclaurin series of f is the Taylor series of f centered at $x = 0$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{473}$$

Theorem 14.7 *If f has a power series representation about $x = a$, then f is equal to its Taylor series about $x = a$.*

Example 14.8 *Find the Maclaurin series for $f(x) = e^x$.*

We note that for any positive integer n, $f^{(n)}(x) = e^x$. Therefore, $f^{(n)}(0) = 1$. Ergo, the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}. \tag{474}$$

□

Example 14.9 *Find the Maclaurin series for $f(x) = \sin x$.*

We note that

$$\begin{array}{cc} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \end{array}. \tag{475}$$

Ergo, the Maclaurin series is

$$\begin{aligned}\frac{x^0}{0!}f(0)+\frac{x^1}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f^{(3)}(0)+\frac{x^4}{4!}f^{(4)}(0)+...\\ =0+\frac{x^1}{1!}+0-\frac{x^3}{3!}+0+\frac{x^5}{5!}+0-\frac{x^7}{7!}+0+...\\ =\boxed{\sum_{n=0}^{\infty}(-1)^n\tfrac{x^{2n+1}}{(2n+1)!}}.\end{aligned} \tag{476}$$

□

Example 14.10 *Let k be a real number. Find the Maclaurin series for the function $f(x)=(1+x)^k$.*

We note that

$$\begin{array}{cc} f(x)=(1+x)^k & f(0)=1\\ f'(x)=k(1+x)^{k-1} & f'(0)=k\\ f''(x)=k(k-1)(1+x)^{k-2} & f''(0)=k\,(k-1)\\ f^{(3)}(x)=k(k-1)(k-2)(1+x)^{k-3} & f^{(3)}(0)=k(k-1)(k-2)\\ \vdots & \vdots\\ f^{(n)}(x)=k(k-1)(k-2)...(k-n+1)(1+x)^{k-n} & f^{(n)}=k(k-1)(k-2)...(k-n+1)\end{array}. \tag{477}$$

Therefore, the Maclaurin series is

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(0)=\boxed{\sum_{n=0}^{\infty}\tfrac{k(k-1)(k-2)...(k-n+1)}{n!}x^n}=\sum_{n=0}^{\infty}\binom{k}{n}x^n. \tag{478}$$

□

Example 14.11 *(Problem 11.10.22) Find the Taylor series for $f(x)=\frac{1}{x}$ centered at $a=-3$.*

We note that

$$\begin{array}{ll} f(x) = x^{-1} & f(a) = -\frac{1}{3} \\ f'(x) = -x^{-2} & f'(a) = -\frac{1}{9} \\ f''(x) = 2x^{-3} & f''(a) = -\frac{2}{27} \\ f^{(3)}(x) = -6x^{-4} & f^{(3)}(a) = -\frac{6}{81} \\ & \vdots \\ & f^{(n)}(a) = -\frac{n!}{3^{n+1}} \end{array} \tag{479}$$

Thus, the Taylor series about $a = -3$ *is*

$$\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) = \sum_{n=0}^{\infty} -\frac{1}{n!} \frac{n!}{3^{n+1}} (x+3)^n = \boxed{\sum_{n=0}^{\infty} -\frac{(x+3)^n}{3^{n+1}}}. \tag{480}$$

□