Math 141: College Calculus I

Mark Sullivan

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1 Tuesday, May 29

Calculus is the mathematical study of change.
There are two fundamental questions that have driven the development of calculus.

1. The tangent line problem: Given a function f, how can one find the tangent line to f at a given x value?
The field of mathematics that was developed to answer this question is called "differential calculus."

2. The area problem: Given a function f, how can one find the area between the graph of f and the x-axis from one given x value to another?
The field of mathematics that was developed to answer this question is called "integral calculus."

In order to address these problems, we'll need a concept known as a "limit."

Chapter 2: Limits and Derivatives

Section 2.2: The Limit of a Function

Definition 1.1 *Let a be a real number, and suppose that $f(x)$ is a function that is defined near a.*
(i) Given a real number L_1, we say that L₁ is the limit of f as x approaches a from the left provided that the distance between $f(x)$ and L_1 can be made arbitrarily small by selecting x-values that are close to a and less than a.
(ii) Given a real number L_2, we say that L_2 is the limit of f as x approaches a from the right provided that the distance between $f(x)$ and L_2 can be made arbitrarily small by selecting x-values that are close to a and greater than a.

Notation:
(1) $\lim_{x\to a^-} f(x) = L_1$ means "L_1 is the limit of f as x approaches a from the left."
(2) $\lim_{x\to a^+} f(x) = L_2$ means "L_2 is the limit of f as x approaches a from the right."

Example 1.2 *Take*

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } x > \pi \end{cases} . \tag{1}$$

(@ Draw graph) In this case:

$$\begin{array}{cc} \lim_{x\to\pi^-} f(x) = -1 & \lim_{x\to\pi^+} f(x) = 0 \\ \lim_{x\to 0^-} f(x) = 1 & \lim_{x\to 0^+} f(x) = 1 \end{array} \tag{2}$$

□

Definition 1.3 *Given a function f and a real number a:*
(i) If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, we call this value the limit of f as x approaches a.
(ii) If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, we say that the limit of f as x approaches a does not exist.

Notation: $\lim_{x\to a} f(x) = L$ means "L is the limit of f as x approaches a."

In Example 1.2, $\lim_{x\to\pi} f(x)$ does not exist, but $\lim_{x\to 0} f(x) = 1$.

Example 1.4 *Take*

$$f(x) = \begin{cases} x & \text{if } x \leq 2 \\ (x-2)^2 + 1 & \text{if } x > 2 \end{cases}. \tag{3}$$

(@ Draw graph) In this case,

$$\lim_{x\to 2^-} f(x) = 2 \quad \lim_{x\to 2^+} f(x) = 1. \tag{4}$$

Therefore, $\lim_{x\to 2} f(x)$ does not exist. □

Example 1.5 *Take*

$$f(x) = \frac{(x-3)(x-2)}{x-3}. \tag{5}$$

(@ Draw graph) $f(1)$ is undefined.
At the same time, $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 1$.
Therefore, $\lim_{x\to 1} f(x) = 1$. □

Example 1.6 *Take*

$$f(x) = \begin{cases} x(x-2) & \text{if } x < 2 \\ 1 & \text{if } x = 2 \\ \frac{4}{x} & \text{if } x > 2 \end{cases} \tag{6}$$

(@ Draw graph) Here $\lim_{x\to 2^-} f(x) = 0$ and $\lim_{x\to 2^+} f(x) = 2$,
so $\lim_{x\to 2} f(x)$ does not exist. □

Example 1.7 *Take*

$$f(x) = \begin{cases} \frac{x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}. \tag{7}$$

(@ Draw graph) Now $\lim_{x\to 0} f(x) = 1$, despite that $f(0) = 0$. □

Example 1.8 *Draw the graph of a function satisfying the following conditions:*

$$\begin{array}{cc} \lim_{x\to 0^-} f(x) = 2 & \lim_{x\to 0^+} f(x) = 0 \\ \lim_{x\to 4^-} f(x) = 3 & \lim_{x\to 4^+} f(x) = 0. \\ f(0) = 2 & f(4) = 1 \end{array} \tag{8}$$

(@ Draw graph) □

Definition 1.9 *Let $f(x)$ be a function defined near a real number a.*
(i) We say that the limit as f approaches a is infinity provided that $f(x)$ can be made arbitrarily large by selecting x-values close to a.
(ii) We say that the limit as f approaches a is negative infinity provided that $f(x)$ can be made arbitrarily small by selecting x-values close to a.

Notation: $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} f(x) = -\infty$.

Example 1.10 *Take*

$$f(x) = \frac{1}{x^2} \tag{9}$$

(@ Draw graph) Here $\lim_{x\to 0} f(x) = \infty$. □

Example 1.11 *Take*

$$f(x) = \ln x \tag{10}$$

(@ Draw graph) Here $\lim_{x\to 0^+} f(x) = -\infty$, but $\lim_{x\to 0^-} f(x)$ is not defined, since $f(x)$ is not defined for negative values of x. □

Example 1.12 *Take*

$$f(x) = \frac{1}{x}. \tag{11}$$

(@ Draw graph) Here $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$.
Thus, $\lim_{x\to 0} f(x)$ does not exist. □

Definition 1.13 *Let $f(x)$ be a function, and let a be a real number. We say that f has a vertical asymptote at $x = a$ provided that either $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.*

Example 1.14 *Take*

$$f(x) = \frac{1}{x^2 - 4} \tag{12}$$

(@ Draw graph) This has vertical asymptotes at $x = 2$ and $x = -2$. $\square$

Example 1.15 *Take*

$$f(x) = \begin{cases} e^{\frac{1}{x}} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases} \tag{13}$$

(@ Draw graph) This has a vertical asymptote at $x = 0$. $\square$

Example 1.16 *Take*

$$f(x) = \frac{x+1}{x-5}. \tag{14}$$

What is $\lim_{x \to 5^-} f(x)$?
If x is close to 5 and less than 5, then $x + 1 > 0$ and $x - 5 < 0$. As the denominator gets smaller, $f(x)$ gets larger. Thus, $\lim_{x \to 5^-} f(x) = -\infty$. $\square$

Two trivial examples:

Example 1.17 *Let c and a be constant real numbers. $\lim_{x \to a} c = c$.* $\square$

Example 1.18 *Let a be a constant real number. $\lim_{x \to a} x = a$.* $\square$

Section 2.3: Calculating Limits Using the Limit Laws

Theorem 1.19 *Given functions $f(x)$ and $g(x)$ defined near a real number a:*
(i)

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x). \tag{15}$$

(ii) If c is any real number,

$$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x). \tag{16}$$

(iii)

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right). \tag{17}$$

(iv) $\boxed{\text{If } \lim_{x\to a} g(x) \neq 0}$*, then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}. \tag{18}$$

Example 1.20 *Let*

$$f(x) = \left(x^4-3x\right)\left(x^2+5x+3\right). \tag{19}$$

What is $\lim_{x\to -1} f(x)$?
By the multiplication limit law, we can write this as

$$\lim_{x\to -1}\left(x^4-3x\right)\left(x^2+5x+3\right) = \left(\lim_{x\to -1} x^4-3x\right)\left(\lim_{x\to -1} x^2+5x+3\right) \tag{20}$$

By the addition limit law, this is

$$= \left(\lim_{x\to -1} x^4 - \lim_{x\to -1} 3x\right)\left(\lim_{x\to -1} x^2 + \lim_{x\to -1} 5x + \lim_{x\to -1} 3\right)$$
$$= (1-(-3))\,(1+(-5)+3) = 4. \tag{21}$$

□

Example 1.21 *Let*

$$f(x) = \frac{x^2 - x - 6}{x - 2}. \tag{22}$$

What is $\lim_{x \to 2} f(x)$*?*
We can write this as

$$\lim_{x \to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x \to 2} \left(\frac{x-2}{x-2}(x+3) \right) \tag{23}$$

By the multiplication limit law,

$$= \left(\lim_{x \to 2} \frac{x-2}{x-2} \right) \left(\lim_{x \to 2} x + 3 \right) = (1) \left(\lim_{x \to 2} x + \lim_{x \to 2} 3 \right) = (1)(2+3) = 5. \tag{24}$$

□

Example 1.22 *Let*

$$f(h) = \frac{(2+h)^3 - 8}{h}. \tag{25}$$

What is $\lim_{h \to 0} f(h)$*?*

$$\begin{aligned} \lim_{h \to 0} f(h) &= \lim_{h \to 0} \frac{(2+h)\left(4 + 4h + h^2\right) - 8}{h} \\ &= \lim_{h \to 0} \frac{8 + 8h + 2h^2 + 4h + 4h^2 + h^3 - 8}{h} \\ &= \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} \\ &= \lim_{h \to 0} 12 + 6h + h^2 \\ &= \lim_{h \to 0} 12 + 6 \lim_{h \to 0} h + \lim_{h \to 0} h^2 = 12. \end{aligned} \tag{26}$$

□

2 Wednesday, May 30

What about $f(x) = x\cos\left(\frac{1}{x}\right)$? How could one find $\lim_{x\to 0} f(x)$?

Theorem 2.1 *(Squeeze theorem) Let $f(x)$, $g(x)$ and $h(x)$ be functions defined near a real number a. Suppose that for every x-value, $f(x) \le g(x) \le h(x)$. If L is a real number such that $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = L$ as well.*

Example 2.2 *Let*

$$g(x) = x\cos\left(\frac{1}{x}\right). \tag{27}$$

What is $\lim_{x\to 0} g(x)$? We know that for all nonzero x-values, $-1 \le \cos\left(\frac{1}{x}\right) \le 1$. Therefore,

$$-|x| \le x\cos\left(\frac{1}{x}\right) \le |x|. \tag{28}$$

In other words, $-|x| \le g(x) \le |x|$. We know that

$$\lim_{x\to 0} -|x| = 0 \quad \textit{and} \quad \lim_{x\to 0} |x| = 0. \tag{29}$$

Thus, by the squeeze theorem, $\lim_{x\to 0} g(x) = 0$. $\square$

Example 2.3 *Let*

$$g(x) = x^2 e^{\sin\left(\frac{\pi}{x}\right)}. \tag{30}$$

What is $\lim_{x\to 0} g(x)$?
We know that for any nonzero x-value, $-1 \le \sin\left(\frac{\pi}{x}\right) \le 1$. Therefore,

$$e^{-1} \le e^{\sin\left(\frac{\pi}{x}\right)} \le e^1. \tag{31}$$

Thus,

$$x^2e^{-1} \le x^2 e^{\sin\left(\frac{\pi}{x}\right)} \le x^2 e, \tag{32}$$

or in other words, $x^2e^{-1} \le g(x) \le x^2e$. We know that

$$\lim_{x\to 0} x^2e^{-1} = 0 \quad \textit{and} \quad \lim_{x\to 0} x^2e = 0. \tag{33}$$

Thus, by the squeeze theorem, $\lim_{x\to 0} g(x) = 0$.

Section 2.5: Continuity

Example 2.4 *Let*

$$f(x) = x^2 + x - 1. \tag{34}$$

What is $\lim_{x\to 1} f(x)$*? By the limit laws,*

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x^2 + x - 1 = \lim_{x\to 1} x^2 + \lim_{x\to 1} x - \lim_{x\to 1} 1 = 1 + 1 - 1 = 1. \tag{35}$$

□

Notice in Example 2.4 that $\lim_{x\to 1} f(x) = f(1)$. In fact, for any real number a, $\lim_{x\to a} f(x) = f(a)$ for this function. This is not true in general, so functions that satisfy this property are given a special name.

Definition 2.5 *Let* $f(x)$ *be a function defined near a real number* a*. We say that* *f is continuous at* a *provided that the following conditions are true:*
(i) $f(a)$ *exists.*
(ii) $\lim_{x\to a} f(x)$ *exists.*
(iii) $f(a) = \lim_{x\to a} f(x)$.
If f *is not continuous at* a*, we say that* f *is discontinuous at* a*. If* f *is continuous at every real number in its domain, then we say that* f *is a continuous function.*

Example 2.6 *Let*

$$f(x) = \begin{cases} |x| & \text{if } x < 1 \\ 0 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x = 2 \\ (x-2)^2 & \text{if } 2 < x < 3 \\ 0 & \text{if } x = 3 \\ x - 4 & \text{if } x > 3 \end{cases}. \tag{36}$$

(@ Draw graph) f *is discontinuous at* 1*,* 2*, and* 3*.* f *is continuous everywhere else.*
□

Which functions are continuous?

Theorem 2.7 *The following functions are continuous at each point in their domain.*
(i) Every polynomial (functions like $f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_2 x^2 + c_1 x + c_0$*).*
(ii) Every rational function (functions like $f(x) = \frac{p(x)}{q(x)}$*, where* p *and* q *are polynomials).*
(iii) Every root (functions like $f(x) = \sqrt[n]{x}$*, where* n *is an integer).*
(iv) Every trigonometric function ($\sin$*,* $\cos$*,* $\tan$*,* $\sec$*,* $\csc$*,* $\cot$*).*
(v) Every arc-trigonometric function ($\sin^{-1}$*,* $\cos^{-1}$*,* $\tan^{-1}$*, etc.).*
(vi) Every exponential function (functions like $f(x) = a^x$ *for some* $a > 1$*).*
(vii) Every logarithm (functions like $f(x) = \log_a(x)$ *for some* $a > 1$*).*

Theorem 2.8 *If* $f(x)$ *and* $g(x)$ *are continuous, then:*
(i) $f(x) + g(x)$ *is continuous.*
(ii) $f(x)g(x)$ *is continuous.*
(iii) $f(g(x))$ *is continuous on its domain.*
(iv) $\frac{f(x)}{g(x)}$ *is continuous on its domain.*

Example 2.9 *All of the following functions are continuous on their domains:*

$$\begin{array}{c} f_1(x) = \frac{\ln x}{\frac{1}{x} + x^2 \sin x} \\ f_2(x) = \sin^2 x + 3 \sin x + 2 \\ f_3(x) = \sin^{-1} x + e^{\frac{1}{x}} \\ f_4(x) = \sqrt{x^2 + \sin(\ln x)} \end{array}. \tag{37}$$

□

It's easy to come up with continuous functions, but it's also easy to come up with functions that are not continuous. Moreover, sometimes it's hard to tell whether a function is continuous or not.

Example 2.10 *Let*

$$f(x) = \begin{cases} \sin x & \textit{if } x < \frac{\pi}{4} \\ \cos x & \textit{if } x \geq \frac{\pi}{4} \end{cases}. \tag{38}$$

Is f *continuous?*
First of all, if $a > \frac{\pi}{4}$*, then* $\lim_{x \to a} f(x) = \lim_{x \to a} \sin x = \sin a$*, since* $\sin x$ *is continuous.*

Similarly, If $a < \frac{\pi}{4}$*, then* $\lim_{x\to a} f(x) = \lim_{x\to a} \cos x = \cos a$*, since* $\cos x$ *is continuous.*
It remains to determine whether f *is continuous at* $a = \frac{\pi}{4}$*. We note that*

$$\lim_{x\to \frac{\pi}{4}^-} f(x) = \lim_{x\to \frac{\pi}{4}^-} \sin x = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}. \tag{39}$$

At the same time,

$$\lim_{x\to \frac{\pi}{4}^+} f(x) = \lim_{x\to \frac{\pi}{4}^+} \cos x = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}. \tag{40}$$

Therefore, $\lim_{x\to \frac{\pi}{4}} f(x) = \frac{1}{\sqrt{2}}$*. Additionally,*

$$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \lim_{x\to \frac{\pi}{4}} f(x). \tag{41}$$

Therefore, f *is continuous at* $\frac{\pi}{4}$*, and so* f *is a continuous function. 1*□

In what ways can the concept of continuity help us? Limits can be "passed through" continuous functions:

Theorem 2.11 *If* f *and* g *are continuous, then*

$$\lim_{x\to a} f\left(g\left(x\right)\right) = f\left(\lim_{x\to a} g(x)\right). \tag{42}$$

Example 2.12 *Let*

$$f(x) = \ln\left(\sqrt{1+x^2}\right). \tag{43}$$

What is $\lim_{x\to 0} f(x)$*?*
Since $\ln x$ *is continuous on its domain,*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \ln\left(\sqrt{1+x^2}\right) = \ln\left(\lim_{x\to 0}\sqrt{1+x^2}\right). \tag{44}$$

Since $\sqrt{x}$ *is continuous on its domain,*

$$= \ln\left(\sqrt{\lim_{x\to 0}\left(1+x^2\right)}\right) = \ln\sqrt{1} = 0. \tag{45}$$

□

Example 2.13 *Let*

$$f(\theta) = \sin\left(\theta + \sin\theta\right). \tag{46}$$

What is $\lim_{\theta\to\pi} f(\theta)$*?*

$$\lim_{\theta\to\pi} f\left(\theta\right) = \lim_{\theta\to\pi} \sin\left(\theta + \sin\theta\right). \tag{47}$$

Since $\sin x$ *is continuous,*

$$\begin{aligned} &= \sin\left(\lim_{\theta\to\pi}\left(\theta + \sin\theta\right)\right) = \sin\left(\lim_{\theta\to\pi}\theta + \lim_{\theta\to\pi}\sin\theta\right) \\ &= \sin\left(\pi + \sin\left(\pi\right)\right) = \sin\left(\pi + 0\right) = \sin\pi = 0. \end{aligned} \tag{48}$$

□

Example 2.14 *Let*

$$f(t) = \frac{\sqrt{t^2+9}-3}{t^2}. \tag{49}$$

What is $\lim_{t\to 0} f(t)$*?*
We multiply the numerator and denominator by the "conjugate," $\sqrt{t^2+9}+3$*:*

$$\begin{aligned} \lim_{t\to 0} f(t) &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right) \\ &= \lim_{t\to 0}\frac{t^2+9-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)} \\ &= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{\lim_{t\to 0} 1}{\lim_{t\to 0}\sqrt{t^2+9}+\lim_{t\to 0} 3} \end{aligned} \tag{50}$$

Now, since $\sqrt{x}$ *is continuous,* $\lim_{t\to 3}\sqrt{t^2+9} = \sqrt{\lim_{t\to 0}\left(t^2+9\right)}$*, so*

$$= \frac{1}{\sqrt{\lim_{t\to 0}\left(t^2+9\right)}+\lim_{t\to 0} 3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}. \tag{51}$$

□

Section 2.6: Limits at Infinity and Horizontal Asymptotes

Definition 2.15 *Let f be a function defined on the real line.*
(i) Given a real number L_1, we say that L_1 is the limit of f as x approaches infinity provided that the distance between $f(x)$ and L_1 can be made arbitrarily small by selecting x-values that are sufficiently large.
(ii) Given a real number L_2, we say that L_2 is the limit of f as x approaches negative infinity provided that the distance between $f(x)$ and L_2 can be made arbitrarily small by selecting x-values that are sufficiently small.

Reminder: "very small" means "very negative."

Notation: $\lim_{x\to\infty} f(x) = L_1$, and $\lim_{x\to-\infty} f(x) = L_2$.

Example 2.16 *Let*

$$f(x) = \tan^{-1}x \tag{52}$$

(@ Draw graph) Here $\lim_{x\to\infty} f(x) = \frac{\pi}{2}$ *and* $\lim_{x\to-\infty} f(x) = -\frac{\pi}{2}$. $\square$

Example 2.17 *Let*

$$f(x) = e^x \tag{53}$$

(@ Draw graph) Here $\lim_{x\to-\infty} f(x) = 0$. $\square$

Example 2.18 *Let*

$$f(x) = e^{-x^2} \tag{54}$$

(@ Draw graph) Here $\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to\infty} f(x)$.

Definition 2.19 *Let f be a function defined on the real line. Given a real number L, we say that f has a horizontal asymptote at $y = L$ provided that* $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$.

Example 2.20 *Suppose*

$$f(x) = \frac{1}{x^r}, \tag{55}$$

where $r > 0$. What is $\lim\limits_{x\to\infty} f(x)$?

Case 1: r is a positive integer. In that case,

$$f(x) = \frac{1}{x^r} = \left(\frac{1}{x}\right)^r = \underbrace{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)...\left(\frac{1}{x}\right)}_{r\ factors}. \tag{56}$$

Therefore, by the properties of limits,

$$\begin{aligned}\lim_{x\to\infty} f(x) = \lim_{x\to\infty}&\left(\underbrace{\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)...\left(\frac{1}{x}\right)}_{r\ factors}\right)\\ &= \underbrace{\left(\lim_{x\to\infty}\frac{1}{x}\right)\left(\lim_{x\to\infty}\frac{1}{x}\right)...\left(\lim_{x\to\infty}\frac{1}{x}\right)}_{r\ factors}\\ &= \underbrace{(0)\,(0)\,...\,(0)}_{r\ factors} = 0.\end{aligned} \tag{57}$$

What if r is not an integer?

Case 2: r may not be an integer, but r is rational.
Suppose $r = \frac{m}{n}$, where m and n are positive integers. In that case,

$$\lim_{x\to\infty}\frac{1}{x^r} = \lim_{x\to\infty}\frac{1}{x^{\frac{m}{n}}} = \lim_{x\to\infty}\frac{1}{\sqrt[n]{x^m}} = \lim_{x\to\infty}\sqrt[n]{\frac{1}{x^m}} = \sqrt[n]{\lim_{x\to\infty}\frac{1}{x^m}} = \sqrt[n]{0} = 0. \tag{58}$$

Case 3: r is irrational.
Select two rational numbers, p and q, such that $0 < p < r < q$. For $x > 1$, this implies that $x^p < x^r < x^q$. Therefore,

$$\frac{1}{x^q} < \frac{1}{x^r} < \frac{1}{x^p}. \tag{59}$$

By Case 2, $\lim\limits_{x\to\infty}\frac{1}{x^q} = 0$ and $\lim\limits_{x\to\infty}\frac{1}{x^q} = 0$. Thus, we have that $\lim\limits_{x\to\infty}\frac{1}{x^r} = 0$, by the squeeze theorem.

3 Thursday, May 31

The fact that $\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$ will be greatly useful to us!

Example 3.1 *Let*

$$f(x) = \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}. \tag{60}$$

What are the horizontal asymptotes of f?

Divide the numerator and denominator by the highest power of x that appears in the denominator:

$$f(x) = \frac{4 + \frac{6}{x} - \frac{2}{x^3}}{2 - \frac{4}{x^2} + \frac{5}{x^3}} \text{ for } x \neq 0. \tag{61}$$

By the example, we know that $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^3} = 0$. Therefore,

$$\lim_{x\to\infty} f(x) = \frac{4 + 0 - 0}{2 - 0 + 0} = 2. \tag{62}$$

Similarly,

$$\lim_{x\to-\infty} f(x) = \frac{4 + 0 - 0}{2 - 0 + 0} = 2. \tag{63}$$

Thus, f has a horizontal asymptote at $y = 2$, on both right and left. $\square$

Example 3.2 *Let*

$$f(x) = \frac{\sqrt{1 + 4x^6}}{2 - x^3}. \tag{64}$$

What are the horizontal asymptotes of f?

Divide by the highest power of x that appears in the denominator:

$$f(x) = \frac{\sqrt{\frac{1}{x^6} + 4}}{\frac{2}{x^3} - 1} \text{ for } x \neq 0. \tag{65}$$

We know that $\lim_{x\to\infty} \frac{1}{x^3} = \lim_{x\to\infty} = 0$, so

$$\lim_{x\to\infty} f(x) = \frac{\sqrt{0 + 4}}{0 - 1} = -2. \tag{66}$$

Similarly,

$$\lim_{x\to-\infty} f(x) = \frac{\sqrt{0+4}}{0-1} = -2. \tag{67}$$

Thus, f has a horizontal asymptote at $y = 2$, on both right and left. $\square$

Definition 3.3 *Let f be a function defined on the real numbers.*
(i) We say that <u>the limit of f as x approaches infinity is infinity</u> provided that $f(x)$ can be made arbitrarily large by selecting x-values that are sufficiently large.
(ii) We say that <u>the limit of f as x approaches negative infinity is infinity</u> provided that $f(x)$ can be made arbitrarily large by selecting x-values that are sufficiently small.

Notation: $\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$.

Example 3.4 *For any of the following functions, $\lim\limits_{x\to\infty} f(x) = \infty$.*

$$\begin{gathered} f(x) = x \\ f(x) = x^2 - 9 \\ f(x) = 2^x \\ f(x) = \sqrt{x} \\ f(x) = \ln x \end{gathered} \quad . \tag{68}$$

$\square$

CAUTION: ∞ is not a real number. Therefore, it does not make sense to add, subtract, multiply or divide things with ∞.

Example 3.5 *Let*

$$f(x) = \sqrt{4x^2 + 3x} + 2x. \tag{69}$$

What are the horizontal asymptotes of f?
First of all, by taking x sufficiently large, $f(x)$ can be made arbitrarily large. Thus, $\lim\limits_{x\to\infty} f(x) = \infty$, and so there is no asymptote on the right.

It remains to find $\lim_{x \to -\infty} f(x)$*. In order to do this, we'll multiply this by the conjugate divided by itself:*

$$\left(\sqrt{4x^2+3x}+2x\right)\left(\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right)=\frac{3x}{\sqrt{4x^2+3x}-2x} \tag{70}$$

Now, as before, we'll divide by the highest power of x *appearing in the denominator. We see* x^2*, but since it is under a square root,* x^1 *will suffice:*

$$\lim_{x \to -\infty} \frac{3}{-\sqrt{4+\frac{3}{x}}-2} = \frac{3}{-\sqrt{4+0}-2} = -\frac{3}{4}. \tag{71}$$

$\boxed{\textit{Difficult question: where did that negative sign in the denominator come from?}}$ *Therefore,* f *has a horizontal asymptote at* $y = -\frac{3}{4}$ *on the left.* $\square$

Section 2.7: Derivatives and Rates of Change, and

Section 2.8: The Derivative as a Function

Now that we have a concept of limit we can solve the tangent line problem, at least theoretically.

Definition 3.6 *Let f be a function defined on the real line. A secant line of f is a line that contains two distinct points on the graph of f.*

Given a function f and any distinct real numbers a and b on the x-axis, we can draw the secant line to f that contains the points $(a, f(a))$ and $(b, f(b))$.
(@ Draw graph)
The slope of such a line would be

$$m = \frac{f(b) - f(a)}{b - a}. \tag{72}$$

However, as a and b become close, the secant line begins to approximate a tangent line. Using limits, we can now find the slope of the tangent line.

Definition 3.7 *Let f be a function defined around a real number a. The derivative of f at a is the slope of the tangent line to f at a:*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{73}$$

Example 3.8 *Let*

$$f(x) = x^2. \tag{74}$$

What is derivative of f at $x = 4$?
Directly from the definition:

$$\lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{x^2 - 4^2}{x - 4} = \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = 8. \tag{75}$$

□

Notation: $f'(a)$, $Df(a)$, $\dot{f}(a)$ all mean "the derivative of f at a."

Example 3.9 *Let*

$$f(x) = \frac{1}{x}. \tag{76}$$

What is $f'(2)$?

$$\begin{aligned} f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2} = \lim_{x\to 2} \frac{\frac{2-x}{2x}}{x-2} \\ = \lim_{x\to 2} \frac{2-x}{2x(x-2)} = \lim_{x\to 2} \frac{-1}{2x} = -\frac{1}{4}. \end{aligned} \tag{77}$$

□

There is another way of evaluating the derivative. If we define $h = x - a$, then the definition becomes

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}. \tag{78}$$

Example 3.10 *Let*

$$f(x) = x^3 - 3x + 1. \tag{79}$$

What is $f'(1)$?

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{\left((1+h)^3 - 3\,(1+h) + 1\right) - \left(1^3 - 3\,(1) + 1\right)}{h} \\ &= \lim_{h\to 0} \frac{(1 + 3h + 3h^2 + h^3 - 3 - 3h + 1) - (-1)}{h} \\ &= \lim_{h\to 0} \frac{(h^3 + 3h^2 - 1) + 1}{h} = \lim_{h\to 0} \frac{h^3 + 3h^2}{h} \\ &= \lim_{h\to 0} \frac{h\,(h^2 + 3h)}{h} = \lim_{h\to 0} h^2 + 3h = 0. \end{aligned} \tag{80}$$

□

We can do even better than this. In each of the previous examples, we found the derivative $f'(a)$ at only one particular value of a. We can, in fact, find the derivative at *every* value of a at once.

Example 3.11 *Let*

$$f(x) = x^2 + 10. \tag{81}$$

Find $f'(a)$ at every real value of a.
Directly from the definition,

$$\begin{aligned} f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{(x^2 + 10) - (a^2 + 10)}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} \\ &= \lim_{x \to a} x + a = a + a = 2a. \end{aligned} \tag{82}$$

Now, whatever a is, we know that $f'(a) = 2a$. $\square$

Definition 3.12 *Let f be a function defined around a real number a. The <u>derivative function of f</u> is the function whose value at each x is the derivative of f at x. In other words,*

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \tag{83}$$

Notation: f', $\dot{f}$, $\frac{\mathrm{d}}{\mathrm{d}x} f$, $\frac{\mathrm{d}f}{\mathrm{d}x}$ all mean “the derivative function of f.”

Example 3.13 *Let*

$$f(x) = \sqrt{x + 1}. \tag{84}$$

Find $\frac{\mathrm{d}f}{\mathrm{d}x}$.

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h} \\
= \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \left(\frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}\right) \\
= \lim_{h\to 0} \frac{x+h+1-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} \\
= \lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} \\
= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} \\
= \frac{1}{\sqrt{x+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}.
\end{aligned} \tag{85}$$

□

Is it possible for the derivative to fail to exist at a point?

Definition 3.14 *Let f be a function defined around a real number a. We say that* <u>*f is differentiable at a*</u> *provided that $f'(a)$ is a real number. If f is differentiable at every x-value in its domain, then we say that* <u>*f is a differentiable function*</u>*.*

4 Monday, June 3

Theorem 4.1 *Let f be a function defined around a real number a. If f is differentiable at a, then f must be continuous at a.*

In other words, if f is not continuous at a, then f cannot be differentiable at a. Is there any other way a function could fail to be differentiable?

Example 4.2 *Let*

$$f(x) = |x|. \tag{86}$$

Determine whether f is differentiable at $x = 0$.
We note that

$$f(x) = \begin{cases} -x & \textit{if } x < 0 \\ x & \textit{if } x \geq 0 \end{cases}. \tag{87}$$

Therefore,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = \lim_{x \to 0^+} 1 = 1. \tag{88}$$

However,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = \lim_{x \to 0^-} -1 = -1. \tag{89}$$

Thus, $f'(0)$ does not exist; f is not differentiable at $x = 0$. $\square$

Reasons that a function f may fail to be differentiable at $x = a$:
(i) f is not continuous at a. For example,

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}. \tag{90}$$

(@ Draw graph)
(ii) f has a corner at a. For example, $f(x) = |x|$. (@ Draw graph)
(iii) f has a cusp at a. For example, $f(x) = |\cos x|$. (@ Draw graph)
(iv) f is not defined on both sides of a. For example, $f(x) = \sqrt{1 - x^2}$ (@ Draw graph)

(v) f has a vertical tangent line at a. For example, $f(x) = \sqrt[3]{x}$. (@ Draw graph)

In general, when one says "find the derivative of f" without specifying a point, this means to find the derivative function of f at all of the points where it is defined.

Example 4.3 *Let*

$$f(x) = x^{\frac{3}{2}}. \tag{91}$$

Find the derivative of f.
We note that f is not defined for $x < 0$, since $f(x) = \sqrt{x^3}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{\frac{3}{2}} - x^{\frac{3}{2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} \left(\frac{\sqrt{(x+h)^3} + \sqrt{x^3}}{\sqrt{(x+h)^3} + \sqrt{x^3}} \right) \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)} \\
&= \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h\left(\sqrt{(x+h)^3} + \sqrt{x^3}\right)} = \lim_{h\to 0} \frac{3x^2 + 3xh + h^2}{\sqrt{(x+h)^3} + \sqrt{x^3}} \\
&= \frac{3x^2 + 0 + 0}{\sqrt{(x+0)^3} + \sqrt{x^3}} = \frac{3x^2}{2x^{\frac{3}{2}}} = \frac{3}{2}\sqrt{x} \textit{ for } x \neq 0.
\end{aligned} \tag{92}$$

Thus, $f'(x)$ is not defined at $x = 0$. $\square$

5 Tuesday, June 5

(Test 1 was given on this day.)

6 Wednesday, June 6

Chapter 3: Differentiation Rules
Section 3.1: Derivatives of Polynomials and Exponential Functions

The derivative is one of the most important theoretical accomplishments of mathematics. However, computing the derivative has been difficult so far. In this chapter, we'll work on some theorems that will assist us in finding the derivatives of a large variety of functions.

If f and g are two functions defined around a point a, then we can consider the derivative of $f+g$ at a:

$$\begin{aligned}(f+g)'(a) &= \lim_{x\to a}\frac{(f(x)+g(x))-(f(a)+g(a))}{x-a}\\ &= \lim_{x\to a}\frac{(f(x)-f(a))+(g(x)-g(a))}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a}\right)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a)+g'(a). \qquad (93)\end{aligned}$$

Therefore, the derivative of a sum of functions is the sum of the derivatives:

Theorem 6.1 *Given functions f and g defined on the real line,*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x}+\frac{\mathrm{d}g}{\mathrm{d}x}. \qquad (94)$$

If f is defined around a point a, and c is a constant, then similarly, we can consider the derivative of cf at a:

$$\begin{aligned}(cf)'(a) &= \lim_{x\to a}\frac{cf(x)-cf(a)}{x-a} = \lim_{x\to a}\frac{c\,(f(x)-f(a))}{x-a}\\ &= \lim_{x\to a}c\frac{f(x)-f(a)}{x-a} = c\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = cf'(a). \qquad (95)\end{aligned}$$

Therefore, a constant can just be "pulled out" of a derivative:

Theorem 6.2 *Given a function f defined on the real line and a constant c,*

$$\frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}. \tag{96}$$

Now let's look at some specific types of functions.

$\boxed{\text{1. Constant functions}}$: suppose $f(x) = c$ for some real number c. In that case,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0. \tag{97}$$

Thus,

Theorem 6.3 *Given any constant c,*

$$\frac{\mathrm{d}}{\mathrm{d}x}c = 0. \tag{98}$$

$\boxed{\text{2. Powers of } x}$

Theorem 6.4 *(Power rule) Given any nonzero real number,*

$$\frac{\mathrm{d}}{\mathrm{d}x}x^r = rx^{r-1}. \tag{99}$$

Example 6.5 *Let*

$$f(x) = 3x^{\frac{5}{3}} - x^3 + 1. \tag{100}$$

Find $f'(x)$.

We need not resort to the definition of the derivative here. Due to the theorems we've discussed,

$$\begin{aligned} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^{\frac{5}{3}} - x^3 + 1\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^{\frac{5}{3}}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(-x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}(1) \\ &= 3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{5}{3}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}(1) \\ &= 3\left(\frac{5}{3}x^{\frac{2}{3}}\right) - \left(3x^2\right) + 0 \\ &= 5x^{\frac{2}{3}} - 3x^2. \end{aligned} \tag{101}$$

□

Example 6.6 *Let*

$$H(u) = (3u - 1)(u + 2). \tag{102}$$

Find $H'(u)$.

$$H'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\left(3u^2 + 5u - 2\right) = 6u + 5. \tag{103}$$

□

Example 6.7 *Let*

$$G(t) = \sqrt{5t} + \frac{\sqrt{7}}{t}. \tag{104}$$

Find $G'(t)$.

$$\begin{aligned} G'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{5t} + \frac{\sqrt{7}}{t}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{5t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sqrt{7}}{t}\right) \\ &= \sqrt{5}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t} + \sqrt{7}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{t} = \sqrt{5}\frac{\mathrm{d}}{\mathrm{d}t}t^{\frac{1}{2}} + \sqrt{7}\frac{\mathrm{d}}{\mathrm{d}t}t^{-1} \\ &= \sqrt{5}\left(\frac{1}{2}t^{-\frac{1}{2}}\right) + \sqrt{7}\left(-t^{-2}\right) = \frac{1}{2}\sqrt{\frac{5}{t}} - \frac{\sqrt{7}}{t^2}. \end{aligned} \tag{105}$$

□

Example 6.8 *Let*

$$y = \frac{\sqrt{x} + x}{x^2}. \tag{106}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x} + x}{x^2}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\frac{1}{2}} + x^1}{x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{3}{2}} + x^{-1}\right) \\ &= -\frac{3}{2}x^{-\frac{5}{2}} - x^{-2} = -\frac{3}{2x^{\frac{5}{2}}} - \frac{1}{x^2}. \end{aligned} \tag{107}$$

What is e?

Definition 6.9 *Euler's number is the real number* e *such that*

$$\lim_{h\to 0}\frac{e^h-1}{h}=1. \tag{108}$$

$\boxed{\text{3. The natural exponential function}}$: suppose $f(x)=e^x$. Let's observe that

$$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}=e^x\lim_{h\to 0}\frac{e^h-1}{h}=e^x(1)=e^x. \tag{109}$$

Therefore,

Theorem 6.10

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x=e^x. \tag{110}$$

This seemingly useless fact will become important soon.

Section 3.3: Derivatives of Trigonometric Functions, and
Section 3.2: The Product and Quotient Rules

Theorem 6.11

$$\lim_{x\to 0} \tfrac{\sin\theta}{\theta} = 1 \quad \textit{and} \quad \lim_{x\to 0} \tfrac{\cos\theta - 1}{\theta} = 0. \tag{111}$$

These are interesting by themselves, but more importantly:
Suppose $f(x) = \sin x$. In that case,

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right) \\ &= \sin x(0) + \cos x(1) = \cos x. \end{aligned} \tag{112}$$

Using similar methods, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x. \tag{113}$$

How could we find the derivatives of the other trigonometric functions?

$$\tan x = \tfrac{\sin x}{\cos x} \quad \sec x = \tfrac{1}{\cos x} \quad \csc x = \tfrac{1}{\sin x} \quad \cot x = \tfrac{\cos x}{\sin x}. \tag{114}$$

In order to find these, we a rule for the derivative of a quotient of two functions.

Suppose that f and g are functions defined around $x = a$. Let's look at the derivative of fg at a:

$$\begin{aligned}
(fg)'(a) &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}\\
&= \lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\
&= \lim_{x\to a}\frac{f(x)\left(g(x)-g(a)\right)+g(a)\left(f(x)-f(a)\right)}{x-a}\\
&= \lim_{x\to a}\left(\frac{f(x)\left(g(x)-g(a)\right)}{x-a}+\frac{g(a)\left(f(x)-f(a)\right)}{x-a}\right)\\
&= \lim_{x\to a} f(x)\frac{g(x)-g(a)}{x-a}+\lim_{x\to a} g(a)\frac{f(x)-f(a)}{x-a}\\
&= f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}+g(a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&= f(a)g'(a)+g(a)f'(a). \qquad (115)
\end{aligned}$$

Therefore,

Theorem 6.12 *(Product rule) If f and g are functions defined on the real line, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg)=f\frac{\mathrm{d}g}{\mathrm{d}x}+g\frac{\mathrm{d}f}{\mathrm{d}x}=fg'+gf'. \qquad (116)$$

Example 6.13 *Let*

$$h(x)=\left(x+2\sqrt{x}\right)e^x. \qquad (117)$$

Find $h'(x)$.

$$\begin{aligned}
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x+2x^{\frac{1}{2}}\right)e^x=\left(x+2x^{\frac{1}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}e^x+e^x\frac{\mathrm{d}}{\mathrm{d}x}\left(x+2x^{\frac{1}{2}}\right)\\
&= \left(x+2x^{\frac{1}{2}}\right)e^x+e^x\left(1+x^{-\frac{1}{2}}\right)=\left(x+2x^{\frac{1}{2}}+1+x^{-\frac{1}{2}}\right)e^x. \qquad (118)
\end{aligned}$$

□

Example 6.14 *Let*

$$f(x)=\sqrt[3]{x}\sin x. \qquad (119)$$

Find $f'(x)$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\sqrt[3]{x}\sin x = x^{\frac{1}{3}}\frac{\mathrm{d}}{\mathrm{d}x}\sin x + \sin x\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{3}}$$
$$= x^{\frac{1}{3}}(\cos x) + \sin x\left(\frac{1}{3}x^{-\frac{2}{3}}\right). \quad (120)$$

□

Next, what about $\frac{f}{g}$?

Theorem 6.15 *(Quotient rule) If* f *and* g *are functions defined on the real line and* $g \neq 0$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right) = \frac{g\frac{\mathrm{d}f}{\mathrm{d}x} - f\frac{\mathrm{d}g}{\mathrm{d}x}}{g^2} = \frac{gf' - fg'}{g^2}. \quad (121)$$

Example 6.16 *Let*

$$G(x) = \frac{x^2 - 2}{2x + 1}. \quad (122)$$

Find $G'(x)$.

$$G'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2 - 2}{2x + 1} = \frac{(2x+1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 2) - (x^2 - 2)\frac{\mathrm{d}}{\mathrm{d}x}(2x + 1)}{(2x+1)^2}$$
$$= \frac{(2x+1)(2x) - (x^2 - 2)(2)}{(2x+1)^2}$$
$$= \frac{4x^2 + 2x - 2x^2 + 4}{(2x+1)^2} = \frac{2x^2 + 2x + 4}{(2x+1)^2}. \quad (123)$$

□

Example 6.17 *Let*

$$y = \frac{e^x}{1 - e^x}. \quad (124)$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x}{1-e^x} &= \frac{(1-e^x)\frac{\mathrm{d}}{\mathrm{d}x}e^x - e^x\frac{\mathrm{d}}{\mathrm{d}x}(1-e^x)}{(1-e^x)^2} \\ &= \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} \\ &= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2} = \frac{e^x}{(1-e^x)^2}.\end{aligned} \tag{125}$$

□

$\boxed{\text{4. Trigonometric functions}}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} &= \frac{(\cos x)\frac{\mathrm{d}}{\mathrm{d}x}\sin x - (\sin x)\frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned} \tag{126}$$

Using similar methods, one can prove the other parts of the following theorem.

Theorem 6.18

$$\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x & \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x \\ \frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x & \frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x \\ \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\tan x & \frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x\cot x \end{array}. \tag{127}$$

Example 6.19 *Let*

$$g(\theta) = e^\theta(\tan\theta - \theta). \tag{128}$$

Find $g'(\theta)$.

$$\begin{aligned}g'(\theta) &= e^\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta - \theta) + (\tan\theta - \theta)\frac{\mathrm{d}}{\mathrm{d}\theta}e^\theta \\ &= e^\theta\left(\sec^2\theta - 1\right) + (\tan\theta - \theta)e^\theta = e^\theta\tan^2\theta + e^\theta\tan\theta - e^\theta\theta \\ &= e^\theta\left(\tan^2\theta + \tan\theta - \theta\right).\end{aligned} \tag{129}$$

□

7 Thursday, June 7

Section 3.4: The Chain Rule

We already have theorems regarding functions being put together by addition, multiplication and division. There is one other important way to put functions together.

Definition 7.1 *Let f and g be functions defined on the real line. The composite function $f \circ g$ is the function $f(g(x))$.*

Example 7.2 *(i) If $f(x) = x^2$ and $g(x) = x + 1$, then*

$$\begin{array}{r} f(g(x)) = (x+1)^2 \\ g(f(x)) = x^2 + 1 \end{array}. \tag{130}$$

(ii) If $f(x) = \sqrt{x}$ and $g(x) = 4x + 1$, then

$$\begin{array}{r} f(g(x)) = \sqrt{4x+1} \\ g(f(x)) = 4\sqrt{x} + 1 \end{array}. \tag{131}$$

(iii) If $f(x) = \frac{1}{1+x}$ and $g(x) = \tan^{-1}x$, then

$$\begin{array}{r} f(g(x)) = \frac{1}{1+\tan^{-1}x} \\ g(f(x)) = \tan^{-1}\left(\frac{1}{1+x}\right) \end{array}. \tag{132}$$

$\square$

Theorem 7.3 *(Chain rule) Let f and g be functions defined on the real line. Define $h = f \circ g$. Given a real value x, if g is differentiable at x and f is differentiable at $g(x)$, then*

$$h'(x) = f'(g(x))\, g'(x)\,. \tag{133}$$

Example 7.4 *Let*

$$h(x) = \sin\sqrt{x}. \tag{134}$$

Find $h'(x)$.
Let $f(x) = \sin x$ and $g(x) = \sqrt{x}$. We note that $f'(x) = \cos x$ and $g'(x) = \frac{1}{2\sqrt{x}}$. Therefore, by the chain rule,

$$h'(x) = f'\left(g\left(x\right)\right) g'\left(x\right) = \cos\left(\sqrt{x}\right) \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}. \tag{135}$$

□

Example 7.5 *Let*

$$h(x) = \frac{1}{x^3 + 2x^2 + 3x + 1}. \tag{136}$$

Find $h'(x)$.
Let $f(x) = \frac{1}{x}$ and $g(x) = x^3 + 2x^2 + 3x + 1$. We know that $f'(x) = -\frac{1}{x^2}$ and $g'(x) = 3x^2 + 4x + 3$. Therefore, by the chain rule,

$$\begin{aligned} h'(x) = f'\left(g\left(x\right)\right) g'\left(x\right) &= \frac{-1}{\left(x^3 + 2x^2 + 3x + 1\right)^2} \left(3x^2 + 4x + 3\right) \\ &= \frac{3x^2 + 4x + 3}{\left(x^3 + 2x^2 + 3x + 1\right)^2} \end{aligned} \tag{137}$$

□

Example 7.6 *Let*

$$f(x) = \sin\left(\cot x\right). \tag{138}$$

Find $f'(x)$.
By the chain rule,

$$f'(x) = \cos\left(\cot x\right)\left(-\csc^2 x\right). \tag{139}$$

□

Example 7.7 *Let*

$$y = \sqrt{2 - e^x}. \tag{140}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
We note that

$$y = \left(2 - e^x\right)^{\frac{1}{2}}. \tag{141}$$

Thus, by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(2-e^x)^{-\frac{1}{2}}\left(-e^x\right) = \frac{-e^x}{2\sqrt{2-e^x}}. \tag{142}$$

□

Example 7.8 *Let*

$$F(x) = \left(1+x+x^2\right)^{99}. \tag{143}$$

Find $F'(x)$.

$$F'(x) = 99\left(1+x+x^2\right)^{98}(1+2x). \tag{144}$$

□

Example 7.9 *Let*

$$g(\theta) = \cos^2\theta. \tag{145}$$

Find $g'(\theta)$.
We note that $g(\theta) = (\cos\theta)^2$, so

$$g'(\theta) = 2(\cos\theta)(\sin\theta) = \sin(2\theta). \tag{146}$$

□

Example 7.10 *Let*

$$g(x) = e^{x^2-x}. \tag{147}$$

Find $g'(x)$.

$$g'(x) = e^{x^2-x}(2x-1). \tag{148}$$

□

$\boxed{\text{5. Exponential functions}}$ Let $a > 0$, and suppose that $f(x) = a^x$. In that case,

$$f(x) = \left(e^{\ln a}\right)^x = e^{(\ln a)x}. \tag{149}$$

Therefore, by the chain rule,

$$f'(x) = e^{(\ln a)x} \frac{\mathrm{d}}{\mathrm{d}x} ((\ln a)\, x) = e^{(\ln a)x} \ln a = a^x \ln a. \tag{150}$$

Theorem 7.11 *If $a > 0$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a. \tag{151}$$

Example 7.12 *Let*

$$f(t) = 2^{\left(t^3\right)}. \tag{152}$$

Find $f'(t)$.

$$f'(t) = \left(2^{\left(t^3\right)} \ln 2\right) \frac{\mathrm{d}}{\mathrm{d}t} t^3 = 2^{\left(t^3\right)} \ln 2 \left(3t^2\right) = 3\,(\ln 2)\, 2^{\left(t^3\right)} t^2. \tag{153}$$

□

Example 7.13 *Let*

$$s(t) = \sqrt{\frac{1+\sin t}{1+\cos t}}. \tag{154}$$

Find $s'(t)$.
We note that $s(t) = \left(\frac{1+\sin t}{1+\cos t}\right)^{\frac{1}{2}}$, so

$$\begin{aligned} s'(t) &= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1+\sin t}{1+\cos t}\right) \\ &= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \left(\frac{(1+\cos t)(\cos t) - (1+\sin t)(-\sin t)}{(1+\cos t)^2}\right) \\ &= \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-\frac{1}{2}} \frac{\left(\cos t + \cos^2 t + \sin t + \sin^2 t\right)}{(1+\cos t)^2} \\ &= \frac{\cos t + \sin t}{2(1+\cos t)^2}\sqrt{\frac{1+\cos t}{1+\sin t}}. \end{aligned} \tag{155}$$

□

Example 7.14 *Let*

$$y = xe^{-x^2}. \tag{156}$$

Find an equation of the tangent line to the curve at the point $(0, 0)$.
We know that the tangent line will have the equation $y = mx + b$ *for some real values* m *and* b.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^{-x^2}\right) = x\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2} + e^{-x^2}\frac{\mathrm{d}}{\mathrm{d}x}x \\ = x\left(e^{-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^2\right)\right) + e^{-x^2}(1) = xe^{-x^2}(-2x) + e^{-x^2} \\ = \left(1 - 2x^2\right)e^{-x^2}.\end{aligned} \tag{157}$$

Here $m = y'(0) = \left(1 - 2(0)^2\right)e^{-(0)^2} = 1$. *Therefore, the tangent line has the equation* $y = x + b$ *for some value of* b. *Since the tangent line contains the point* $(0, 0)$, *we have that* $0 = 0 + b$, *and so* $b = 0$. *Thus, the tangent line has the equation* $y = x$. □

Example 7.15 *At what point on the curve* $y = \sqrt{1 + 2x}$ *is the tangent line perpendicular to the line* $6x + 2y = 1$*?*
First, we find $\frac{\mathrm{d}y}{\mathrm{d}x}$*:*

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(1 + 2x)^{\frac{1}{2}} = \frac{1}{2}(1 + 2x)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(1 + 2x) \\ = \frac{1}{2}(1 + 2x)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{1 + 2x}}.\end{aligned} \tag{158}$$

Next, we note that the line in question is $y = -3x + \frac{1}{2}$. *Therefore, we must solve the equation* $y' = \frac{1}{3}$*:*

$$\frac{1}{\sqrt{1 + 2x}} = \frac{1}{3}, \tag{159}$$

and so $x = 4$. *This implies that* $y = \sqrt{1 + 2(4)} = 3$, *and so the point we seek is* $(4, 3)$. □

Example 7.16 *Let*

$$y = \sqrt{x + \sqrt{x + \sqrt{x}}}. \tag{160}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
First, we write

$$y = \left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \tag{161}$$

Using the chain rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{\mathrm{d}}{\mathrm{d}x}\left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}x}\left(x + x^{\frac{1}{2}}\right)\right) \\ & \frac{1}{2}\left(x + \left(x + x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right)\right) \\ &= \frac{1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}. \end{aligned} \tag{162}$$

□

Example 7.17 *Let*

$$f(z) = e^{\frac{z}{z-1}}. \tag{163}$$

Find $f'(z)$.

$$\begin{aligned} f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}e^{\frac{z}{z-1}} = e^{\frac{z}{z-1}}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{z-1} &= e^{\frac{z}{z-1}}\frac{(z-1)\frac{\mathrm{d}}{\mathrm{d}z}(z) - (z)\frac{\mathrm{d}}{\mathrm{d}z}(z-1)}{(z-1)^2} \\ &= e^{\frac{z}{z-1}}\frac{(z-1)(1) - (z)(1)}{(z-1)^2} = \frac{-e^{\frac{z}{z-1}}}{(z-1)^2}. \end{aligned} \tag{164}$$

□

Example 7.18 *Let*

$$y = e^{\sin 2x} + \sin\left(e^{2x}\right). \tag{165}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\sin 2x} + \sin\left(e^{2x}\right)\right) = \frac{\mathrm{d}}{\mathrm{d}x}e^{\sin 2x} + \frac{\mathrm{d}}{\mathrm{d}x}\sin\left(e^{2x}\right)\\ = e^{\sin 2x}\frac{\mathrm{d}}{\mathrm{d}x}\sin 2x + \cos\left(e^{2x}\right)\frac{\mathrm{d}}{\mathrm{d}x}e^{2x}\\ = e^{\sin 2x}\cos(2x)\frac{\mathrm{d}}{\mathrm{d}x}(2x) + \cos\left(e^{2x}\right)e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}(2x)\\ = 2e^{\sin(2x)}\cos(2x) + 2e^{2x}\cos\left(e^{2x}\right). \end{aligned} \tag{166}$$

□

Example 7.19 *Let*

$$y = \frac{1}{(1+\tan x)^2}. \tag{167}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
We can write

$$y = (1+\tan x)^{-2}. \tag{168}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = -2(1+\tan x)^{-3}\frac{\mathrm{d}}{\mathrm{d}x}(1+\tan x)\\ = -2(1+\tan x)^{-3}\left(\sec^2 x\right) = \frac{-2\sec^2 x}{(1+\tan x)^3}.\end{aligned} \tag{169}$$

□

Example 7.20 *Let*

$$y = 2^{\left(3^{\left(4^x\right)}\right)}. \tag{170}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}2^{\left(3^{(4^x)}\right)} &= 2^{\left(3^{(4^x)}\right)}(\ln 2)\left(\frac{\mathrm{d}}{\mathrm{d}x}3^{(4^x)}\right) \\
&= 2^{\left(3^{(4^x)}\right)}(\ln 2)\left(3^{(4^x)}(\ln 3)\left(\frac{\mathrm{d}}{\mathrm{d}x}4^x\right)\right) \\
&= 2^{\left(3^{(4^x)}\right)}(\ln 2)\left(3^{(4^x)}(\ln 3)\left(4^x(\ln 4)\right)\right) \\
&= (\ln 2)(\ln 3)(\ln 4)\,4^x 3^{(4^x)} 2^{\left(3^{(4^x)}\right)}. \qquad (171)
\end{aligned}
$$

□

Example 7.21 *Let*

$$
F(t) = \frac{t^2}{\sqrt{t^3+1}}. \qquad (172)
$$

Find $F'(t)$.

$$
\begin{aligned}
F'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{t^2}{\sqrt{t^3+1}} &= \frac{(t^3+1)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}t^2 - t^2\frac{\mathrm{d}}{\mathrm{d}t}(t^3+1)^{\frac{1}{2}}}{t^3+1} \\
&= \frac{(t^3+1)^{\frac{1}{2}}(2t) - t^2\left(\frac{1}{2}(t^3+1)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}(t^3+1)\right)}{t^3+1} \\
&= \frac{2t\sqrt{t^3+1} - t^2\left(\frac{1}{2\sqrt{t^3+1}}(3t^2)\right)}{t^3+1} = \frac{2t\sqrt{t^3+1} - \frac{3t^4}{\sqrt{t^3+1}}}{t^3+1} \\
&= \frac{2t(t^3+1) - 3t^4}{(t^3+1)^{\frac{3}{2}}} = \frac{t(2-t^3)}{(t^3+1)^{\frac{3}{2}}}. \qquad (173)
\end{aligned}
$$

8 Monday, June 11

Section 3.5: Implicit Differentiation

Some curves are defined by equations that are not functions. In these cases, we need to use a technique called "implicit differentiation:" taking the derivative of both sides of the equation.

Example 8.1 *Find the slope of the tangent line to the circle* $x^2 + y^2 = 25$ *at the point* $(4, 3)$.
We note that the slope of the tangent line to the curve is $\frac{\mathrm{d}y}{\mathrm{d}x}$. *We differentiate implicitly:*

$$\begin{gathered} \tfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + y^2\right) = \tfrac{\mathrm{d}}{\mathrm{d}x}\left(25\right) \\ \tfrac{\mathrm{d}}{\mathrm{d}x}x^2 + \tfrac{\mathrm{d}}{\mathrm{d}x}y^2 = \tfrac{\mathrm{d}}{\mathrm{d}x}25 \\ 2x + 2y\tfrac{\mathrm{d}y}{\mathrm{d}x} = 0. \end{gathered} \tag{174}$$

We now solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}. \tag{175}$$

Thus, the slope of the tangent line at $(4, 3)$ *is* $-\frac{4}{3}$. $\square$

Example 8.2 *Find the slope of the tangent line to the ellipse defined by the equation*

$$x^2 + 2xy + 4y^2 = 12, \tag{176}$$

at the point $(2, 1)$.
We seek $\frac{\mathrm{d}y}{\mathrm{d}x}$. *Differentiating implicitly,*

$$\begin{gathered} \tfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 2xy + 4y^2\right) = \tfrac{\mathrm{d}}{\mathrm{d}x}\left(12\right) \\ \tfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) + \tfrac{\mathrm{d}}{\mathrm{d}x}\left(2xy\right) + \tfrac{\mathrm{d}}{\mathrm{d}x}\left(4y^2\right) = 0 \\ 2x + 2\left(x\tfrac{\mathrm{d}}{\mathrm{d}x}y + y\tfrac{\mathrm{d}}{\mathrm{d}x}x\right) + 8y\tfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ 2x + 2x\tfrac{\mathrm{d}y}{\mathrm{d}x} + 2y + 8y\tfrac{\mathrm{d}y}{\mathrm{d}x} = 0. \end{gathered} \tag{177}$$

Solving for $\frac{dy}{dx}$:

$$\begin{aligned} (2x+2y) + (2x+8y)\frac{dy}{dx} &= 0 \\ (2x+8y)\frac{dy}{dx} &= -(2x+2y) \\ \frac{dy}{dx} &= -\frac{2x+2y}{2x+8y} = -\frac{x+y}{x+4y}. \end{aligned} \tag{178}$$

Therefore, the slope of the tangent line to the ellipse at $(2,1)$ is

$$-\frac{(2)+(1)}{(2)+4(1)} = -\frac{3}{6} = -\frac{1}{2}. \tag{179}$$

□

Example 8.3 *Given the curve*

$$x^3 - xy^2 + y^3 = 1, \tag{180}$$

find $\frac{dy}{dx}$.
Differentiating implicitly,

$$\begin{aligned} \frac{d}{dx}\left(x^3 - xy^2 + y^3\right) &= \frac{d}{dx}(1) \\ \frac{d}{dx}\left(x^3\right) - \frac{d}{dx}\left(xy^2\right) + \frac{d}{dx}\left(y^3\right) &= 0 \\ 3x^2 - \left(x\frac{d}{dx}y^2 + y^2\frac{d}{dx}x\right) + 3y^2\frac{dy}{dx} &= 0 \\ 3x^2 - 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} &= 0 \end{aligned} \tag{181}$$

We now solve for $\frac{dy}{dx}$:

$$\begin{aligned} 3x^2 + y^2 + (3y^2 - 2xy)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{3x^2+y^2}{2xy-3y^2}. \end{aligned} \tag{182}$$

□

Example 8.4 *Given the curve*

$$xe^y = x - y, \tag{183}$$

find $\frac{dy}{dx}$.

Differentiating implicitly,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(xe^y\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x-y\right) \\
x\frac{\mathrm{d}}{\mathrm{d}x}e^y + e^y\frac{\mathrm{d}}{\mathrm{d}x}x &= \frac{\mathrm{d}}{\mathrm{d}x}x - \frac{\mathrm{d}}{\mathrm{d}x}y. \\
xe^y\frac{\mathrm{d}y}{\mathrm{d}x} + e^y &= 1 - \frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
\tag{184}
$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$*:*

$$
\begin{aligned}
xe^y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - e^y \\
\left(xe^y + 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - e^y. \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1-e^y}{xe^y+1}
\end{aligned}
\tag{185}
$$

□

Example 8.5 *Given the curve*

$$
\cos\left(xy\right) = 1 + \sin y,
\tag{186}
$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Differentiating implicitly,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(xy\right)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1+\sin y\right) \\
-\sin\left(xy\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(xy\right) &= \frac{\mathrm{d}}{\mathrm{d}x}1 + \frac{\mathrm{d}}{\mathrm{d}x}\sin y. \\
-\sin\left(xy\right)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) &= \cos y\frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
\tag{187}
$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$*:*

$$
\begin{aligned}
-x\sin\left(xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} - y\sin\left(xy\right) &= \cos y\frac{\mathrm{d}y}{\mathrm{d}x} \\
-y\sin\left(xy\right) &= \cos y\frac{\mathrm{d}y}{\mathrm{d}x} + x\sin\left(xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} \\
-y\sin\left(xy\right) &= \left(\cos y + x\sin\left(xy\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{-y\sin(xy)}{\cos y + x\sin(xy)} &= \frac{\mathrm{d}y}{\mathrm{d}x}.
\end{aligned}
\tag{188}
$$

□

6. Arc trigonometric functions

Sine and cosine take angles and give us a coordinate of a point on the unit circle. Arcsine and arccosine take a coordinate of a point on the unit circle and give us an angle, within some particular range.

The following are always true:

$$\sin\left(\sin^{-1}x\right) = x \qquad\qquad \cos\left(\cos^{-1}x\right) = x. \tag{189}$$

However, the following are *not* always true:

$$\sin^{-1}(\sin\theta) = \theta \qquad\qquad \cos^{-1}(\cos\theta) = \theta. \tag{190}$$

This is because the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the range of $\cos^{-1}$ is $[0, \pi]$.

Example 8.6

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right) = \sin^{-1}(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}. \tag{191}$$

$$\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \cos^{-1}(0) = \frac{\pi}{2} \neq \frac{3\pi}{2}. \tag{192}$$

□

What happens if we differentiate $x = \sin\left(\sin^{-1}x\right)$ implicitly?

$$\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}x}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\sin^{-1}x\right)\right) \\ 1 = \cos\left(\sin^{-1}x\right)\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x \end{array}. \tag{193}$$

However, $\cos\left(\sin^{-1}x\right) = \sqrt{1-x^2}$. (?!) (@ Draw triangle)
Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}. \tag{194}$$

Using similar methods, one can prove the rest of the following theorem.

Theorem 8.7

$$\begin{array}{ll} \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} & \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} & \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{x^2+1} & \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = \frac{-1}{x^2+1} \end{array}. \tag{195}$$

Example 8.8 *Let*

$$R(t) = \sin^{-1}\left(\frac{1}{t}\right). \tag{196}$$

Find $R'(t)$.

$$R'(t) = \frac{1}{\sqrt{1-\left(\frac{1}{t}\right)^2}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{t} = \frac{1}{\sqrt{1-\left(\frac{1}{t}\right)^2}}\left(-t^{-2}\right) = \frac{-1}{t^2\sqrt{1-\frac{1}{t^2}}}. \tag{197}$$

□

Section 3.6: Derivatives of Logarithmic Functions

$\boxed{\text{7. The natural logarithm}}$ The function $\log|x|$ is an extension of the function $\log x$, in the sense that it is defined for all real nonzero x.
We know that $e^{\ln|x|} = |x|$. What if we differentiate this implicitly?

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} e^{\ln|x|} &= \tfrac{\mathrm{d}}{\mathrm{d}x}|x| \\
\tfrac{\mathrm{d}}{\mathrm{d}x} e^{\ln|x|} &= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\
e^{\ln|x|} \tfrac{\mathrm{d}}{\mathrm{d}x} \ln|x| &= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\
|x| \tfrac{\mathrm{d}}{\mathrm{d}x} \ln|x| &= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\
\tfrac{\mathrm{d}}{\mathrm{d}x} \ln|x| &= \begin{cases} \frac{1}{|x|} & \text{if } x > 0 \\ \frac{1}{-|x|} & \text{if } x < 0 \end{cases} \\
\tfrac{\mathrm{d}}{\mathrm{d}x} \ln|x| &= \tfrac{1}{x}.
\end{aligned} \tag{198}$$

Theorem 8.9

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = \frac{1}{x}. \tag{199}$$

Example 8.10 *Let*

$$f(x) = \ln\left(\sin^2 x\right). \tag{200}$$

Find $f'(x)$.

$$\begin{aligned}
f'(x) = \frac{1}{\sin^2 x} \frac{\mathrm{d}}{\mathrm{d}x} \sin^2 x = \frac{1}{\sin^2 x} 2\sin x \frac{\mathrm{d}}{\mathrm{d}x} \sin x& \\
= \frac{1}{\sin^2 x} (2\sin x)(\cos x) = 2\cot x.&
\end{aligned} \tag{201}$$

□

Logarithms have some useful properties for simplifying expressions.

Theorem 8.11 *(i) Given positive real numbers a and b,*

$$\ln a + \ln b = \ln (ab) . \tag{202}$$

(ii) Given a positive real number a and a real number r,

$$\ln (a^r) = r \ln a. \tag{203}$$

From this theorem, you can also see that

$$\ln a - \ln b = \ln a + \ln \left(b^{-1}\right) = \ln a + \ln \left(\frac{1}{b}\right) = \ln \left(\frac{a}{b}\right) . \tag{204}$$

By taking the logarithm of both sides of an equation, one can make some derivatives easier.

Example 8.12 *Let*

$$y = \frac{e^{-x}\cos^2 x}{x^2 + x + 1}. \tag{205}$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
We could use the quotient rule, but it would be annoying. What if we take the logarithm of both sides first?

$$\begin{aligned} \ln y = \ln \left(\frac{e^{-x}\cos^2 x}{x^2 + x + 1}\right) &= \ln \left(e^{-x}\cos^2 x\right) - \ln \left(x^2 + x + 1\right) \\ &= \ln \left(e^{-x}\right) + \ln \left(\cos^2 x\right) - \ln \left(x^2 + x + 1\right) \\ &= -x + 2 \ln (\cos x) - \ln \left(x^2 + x + 1\right) . \end{aligned} \tag{206}$$

Differentiating implicitly,

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x} \left(-x + 2 \ln (\cos x) - \ln \left(x^2 + x + 1\right)\right) \\ &= -1 + 2\frac{1}{\cos x} (\sin x) - \frac{1}{x^2 + x + 1} (2x + 1) \\ &= -1 + 2 \tan x - \frac{2x + 1}{x^2 + x + 1}. \end{aligned} \tag{207}$$

Therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(-1 + 2\tan x - \frac{2x+1}{x^2+x+1}\right)$$
$$= \frac{e^{-x}\cos^2 x}{x^2+x+1}\left(-1 + 2\tan x - \frac{2x+1}{x^2+x+1}\right). \quad (208)$$

□

What about $f(x) = x^x$?

$\boxed{\text{8. Variable bases raised to variable powers}}$: in order to take the derivative of $f(x)^{g(x)}$, use logarithmic differentiation.

Example 8.13 *Let*

$$f(x) = x^x. \quad (209)$$

Find $f'(x)$.
Define $y = f(x)$. *We use logarithmic differentiation:*

$$\ln y = \ln\left(x^x\right) = x \ln x. \quad (210)$$

Now,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = (x)\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) + (\ln x)\frac{\mathrm{d}}{\mathrm{d}x}(x) = (x)\left(\frac{1}{x}\right) + \ln x = 1 + \ln x. \quad (211)$$

Thus,

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = y\left(1 + \ln x\right) = x^x\left(1 + \ln x\right). \quad (212)$$

□

Example 8.14 *Let*

$$y = (\sin x)^{\ln x}. \quad (213)$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Using logarithmic differentiation:

$$\ln y = \ln\left((\sin x)^{\ln x}\right) = (\ln x)\left(\ln\left(\sin x\right)\right). \tag{214}$$

Differentiating implicitly,

$$\begin{aligned}
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= (\ln x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\sin x\right)\right) + \left(\ln\left(\sin x\right)\right)\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) \\
&= (\ln x)\frac{1}{\sin x}\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) + \left(\ln\left(\sin x\right)\right)\frac{1}{x} \\
&= (\ln x)\frac{1}{\sin x}(\cos x) + \frac{\ln\left(\sin x\right)}{x} \\
&= (\ln x)\cot x + \frac{\ln\left(\sin x\right)}{x}.
\end{aligned} \tag{215}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left((\ln x)\cot x + \frac{\ln\left(\sin x\right)}{x}\right) \\
&= (\sin x)^{\ln x}\left((\ln x)\cot x + \frac{\ln\left(\sin x\right)}{x}\right).
\end{aligned} \tag{216}$$

□

Example 8.15 *Given the curve*

$$x^y = y^x, \tag{217}$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
We take the logarithm of both sides:

$$y\ln x = x\ln y. \tag{218}$$

Differentiating implicitly,

$$\begin{array}{c}
(y)\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) + (\ln x)\frac{\mathrm{d}}{\mathrm{d}x}y = (x)\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) + (\ln y)\frac{\mathrm{d}}{\mathrm{d}x}(x) \\
y\left(\frac{1}{x}\right) + (\ln x)\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \ln y \\
\frac{y}{x} + (\ln x)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \ln y
\end{array}. \tag{219}$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{array}{c}(\ln x)\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\ln y-\frac{y}{x}\\ \left(\ln x-\frac{x}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\ln y-\frac{y}{x}\\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\ln y-\frac{y}{x}}{\ln x-\frac{x}{y}}=\frac{x\ln y-y}{y\ln x-x}\end{array}. \tag{220}$$

□

Example 8.16 *Given the curve*

$$y=\sqrt{x}e^{x^2-x}(x+1)^{\frac{2}{3}}, \tag{221}$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Taking the logarithm,

$$\begin{aligned}\ln y=\ln\left(\sqrt{x}e^{x^2-x}(x+1)^{\frac{2}{3}}\right)=\ln\sqrt{x}+\ln\left(e^{x^2-x}\right)+\ln\left((x+1)^{\frac{2}{3}}\right)\\ =\frac{1}{2}\ln x+\left(x^2-x\right)+\frac{2}{3}\ln(x+1).\end{aligned} \tag{222}$$

differentiating implicitly,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2x}+2x-1+\frac{2}{3(x+1)}. \tag{223}$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{x}e^{x^2-x}(x+1)^{\frac{2}{3}}\left(\frac{1}{2x}+2x-1+\frac{2}{3(x+1)}\right). \tag{224}$$

□

9 Tuesday, June 12

(The test review was given on this day)

10 Wednesday, June 13

(Test 2 was given on this day)

11 Thursday, June 14

Chapter 4: Applications of Derivatives
Section 4.2: The Mean Value Theorem

Question: What sorts of functions have derivatives that are constantly zero?

Theorem 11.1 *(Rolle's theorem) Let f be a differentiable function. Given real values a and b such that $a < b$, if $f(a) = f(b)$, then there exists a real value c such that $a < c < b$ and $f'(c) = 0$.*

Example 11.2 *Consider the function $f(x) = -x^2 + x$. (@ Draw graph.) We notice that $f(0) = f(1)$. Therefore, by Rolle's theorem, there exists a real value c such that $0 < c < 1$ and $f'(c) = 0$.* $\square$

Let f be a differentiable function. Suppose that a and b are real values such that $a < b$. Consider the new function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a). \tag{225}$$

We have that

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}. \tag{226}$$

Notice that

$$\begin{array}{c} g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(0) = f(a) \\ g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) = f(a) \end{array}. \tag{227}$$

Now $g(a) = g(b)$, so Rolle's theorem applies; there must exist some c such that $a < c < b$ and $g'(c) = 0$. But this would mean that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0. \tag{228}$$

Thus, we have the following theorem.

Theorem 11.3 *Let f be a differentiable function. Given real values a and b such that $a < b$, there exists a real value c such that $a < c < b$ and*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{229}$$

Example 11.4 *Suppose that f is a differentiable function and $f'(x) = 0$ for all real x-values. What sort of function is f?*
Suppose a and b are two different real values. Suppose $a < b$. By the mean value theorem, there must exist a real value c such that $a < c < b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$. However, f' is constantly zero, so $f'(c) = 0$. This means that $0 = \frac{f(b)-f(a)}{b-a}$. Multiplying both sides by $b - a$, this gives us that $0 = f(b) - f(a)$, and so $f(a) = f(b)$. Thus, all the values of f are the same; f is a constant function. $\square$

Section 3.7: Rates of Change in the Natural and Social Sciences

Question: given a function $y = f(x)$, how sensitive is y to changes in x? Or: by *how much* does y depend on x?

A few interpretations:

1. How quickly does the position of an object change as time goes on?
2. By how much does a company's profit change if the price of a unit changes?
3. By how much does a substance's melting point change as the pressure decreases?
4. How quickly does the fish population of a body of water decline as the pollution level of the water increases?
5. By how much do a building's maintenance costs change as the outside temperature changes?

As we've shown, the mean value theorem implies that only constant functions can have derivatives that are constantly zero. Thus, the derivative expresses the *rate of change* of y as x varies.

Definition 11.5 *(i) If $y = f(t)$, where y is the position of a particle with respect to a chosen reference point (measured in meters) and t is the time since a particular moment (measured in seconds), then $v(t) = \frac{\mathrm{d}y}{\mathrm{d}t}$ is the velocity of the particle, measured in meters per second. The speed of the particle is the absolute value of the velocity of the particle.*
(ii) If $M = f(t)$, where M is the concentration of a chemical (measured in atoms per cubic centimeter) and t is the time since a chemical reaction began (measured in seconds), then $\frac{\mathrm{d}M}{\mathrm{d}t}$ is the rate of reaction of the chemical reaction, measured in atoms per cubic centimeter per second.
(iii) If $P = f(t)$, where P is the number of organisms born in a closed environment, and t is the time since a particular moment (measured in days), then $\frac{\mathrm{d}P}{\mathrm{d}t}$ is the birth rate of the population, measured in individuals per day.
(iv) If $C = f(x)$, where C is the cost of creating x units of a particular product (measured in dollars), then $\frac{\mathrm{d}C}{\mathrm{d}x}$ is the marginal cost of the product, measured in dollars per unit.

Example 11.6 *If a rock is thrown upward on a certain planet with a velocity of* 10 *m/s, then its height,* t *seconds after being thrown, can be described by the function*

$$y(t) = 10t - 2t^2. \tag{230}$$

(a) Find the velocity of the rock after 1 *second.*
(b) When will the rock hit the surface?
(c) What will be the rock's velocity as it hits the surface?

(a) We know that

$$v(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = 10 - 4t. \tag{231}$$

Therefore, $v(1) = 10 - 4 = 6$ *m/s.*

(b) We set $y(t) = 0$*:*

$$0 = 10t - 2t^2 = t\,(10 - 2t)\,. \tag{232}$$

This has two solutions: $t = 0$ *s and* $t = 5$ *s. Since the rock was thrown at* $t = 0$*, we must have that the rock hits the surface at* $t = 5$*.*

(c) We know that $v(t) = 10 - 4t$*, and the rock will hit the surface at* $t = 5$ *s. Therefore, the velocity of the rock as it hits the surface will be* $v(5) = -10$ *m/s.* □

Question: How quickly is an object's velocity changing?

Definition 11.7 *Let* f *be a differentiable function defined on the real line.*
(i) The first derivative of f *is the derivative of* f*.*
(ii) The second derivative of f *is the derivative of the first derivative.*
(iii) The n*th derivative of* f *is the derivative of the* $(n-1)$*th derivative of* f*.*

Notation:
(i) First derivative: f', or $\frac{\mathrm{d}f}{\mathrm{d}x}$.
(ii) Second derivative: f'', or $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.
(iii) Third derivative: f''', or $\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$.
(iv) nth derivative: $f^{(n)}$, or $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$.

Definition 11.8 *Let $y = f(t)$ describe the position of an object (measured in meters) as a function of time (measured in seconds). The* <u>*acceleration*</u> *of the object is the second derivative $a(t) = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = y''(t)$ of the position function, measured in (meters per second) per second.*

12 Monday, June 18

Example 12.1 *Suppose a particle moves along the x-axis with a position function*

$$x(t) = t^3 - 8t^2 + 24, \tag{233}$$

where t is measured in seconds and x is measured in feet. The particle's motion begins at $t = 0$.

(i) Find the velocity of the particle.

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 16t. \tag{234}$$

(ii) Find the velocity of the particle after 1 second.

$$v(1) = 3(1)^2 - 16(1) = -7\ m/s \tag{235}$$

(iii) When is the particle motionless?
We must solve equation $v(t) = 0$:

$$0 = v(t) = 3t^2 - 16t = (3t - 16)\,t. \tag{236}$$

The particle is motionless at $t = 0$ s and at $t = \frac{16}{3}$ s.

(iv) When is the particle moving in the positive direction?
We must solve the inequality $v(t) > 0$:

$$v(t) = (3t - 16)\,t > 0 \tag{237}$$

This occurs when: ($3t - 16 > 0$ and $t > 0$), and when ($3t - 16 < 0$ and $t < 0$). We are not concerned with the latter situation, since the particle begins moving at $t = 0$. Thus, the particle is moving in the positive direction when $t > \frac{16}{3}$ s.

(v) Find the acceleration of the particle.

$$a(t) = x''(t) = v'(t) = 6t - 16. \tag{238}$$

(vi) Find the acceleration of the particle after 1 *second.*

$$a(1) = 6(1) - 16 = -10\, m/s/s \tag{239}$$

(vii) Graph the position, velocity and acceleration for $0 \leq t \leq 6$*.*
(@ Draw graphs)

(iix) When is the particle speeding up?

The particle is speeding up in the positive direction when both v *and* a *are positive. We know that* $v(t) > 0$ *when* $t > \frac{16}{6}$*. Thus, we need to know when* $a(t) > 0$*:*

$$a(t) = 6t - 16 > 0. \tag{240}$$

This occurs when $t > \frac{16}{6} = \frac{8}{3}$ *s. Therefore, the particle is speeding up in the positive direction when both* $t > \frac{16}{3}$ *s and* $t > \frac{8}{3}$ *s, or in other words, when* $t > \frac{16}{3}$ *s.*

The particle is speeding up in the negative direction when both v *and* a *are negative. Therefore, we must solve the following system of inequalities:*

$$\begin{array}{c} v(t) = 3t^2 - 16t < 0 \\ a(t) = 6t - 16 < 0 \end{array}. \tag{241}$$

The first inequality holds when $t < \frac{16}{3}$ *s. The second holds when* $t < \frac{16}{6} = \frac{8}{3}$ *s. Thus, the particle is speeding up in the negative direction when both* $t < \frac{16}{3}$ *s and* $t < \frac{8}{3}$ *s, or in other words, when* $t < \frac{8}{3}$ *s.*

(ix) When is the particle slowing down?
This can only occur when the particle is not speeding up. Thus, this can only occur when $t \geq \frac{8}{3}$ *s and* $t \leq \frac{16}{3}$ *s. At* $t = \frac{8}{3}$ *s,* $a(t) = 0$*, so the particle is neither speeding nor slowing at* $t = \frac{8}{3}$ *s. At* $t = \frac{16}{3}$ *s,* $a(t) \neq 0$*, so the particle must be either speeding up or slowing down. Since it is not speeding up at* $t = \frac{16}{3}$ *s, it must be slowing down at* $t = \frac{16}{3}$ *s. Thus, the particle is slowing down in the time interval* $\frac{8}{3} < t \leq \frac{16}{3}$*.* □

<u>Question</u>: Does an object have to stop in order to change its direction of motion?

Theorem 12.2 *(Intermediate value theorem) Let* f *be a continuous function defined on the real line. Given real values* a *and* b *and a real value* k*, if* $f(a) \leq k \leq f(b)$

or $f(b) \leq k \leq f(a)$, then there exists a real value c between a and b such that $f(c) = k$.

Applied mathematicians usually assume that every function they deal with is continuous and differentiable.

Example 12.3 *A flying device is launched on earth. Its height as a function of time is*

$$y(t) = t^5 + t^2 - 3t \tag{242}$$

Is there any time during its flight at which it is neither rising nor falling?
We want to know whether $v(t) = 0$ has any solutions. First, we note that

$$v(t) = y'(t) = 5t^4 + 2t - 3. \tag{243}$$

We notice that $v(0) = -3$ and $v(1) = 4$. Therefore, since $v(0) \leq 0 \leq v(1)$, the intermediate value theorem indicates that there exists a value t between 0 and 1 such that $v(t) = 0$. The answer, therefore, is yes.

Section 4.1: Maximum and Minimum Values

Definition 12.4 *Let f be a function, and let c be a real value in the domain of f.*
(i) We say that $f(c)$ is an absolute or global maximum value of f provided that for all x-values in the domain of f, $f(x) \leq f(c)$.
(ii) We say that $f(c)$ is an absolute or global minimum value of f provided that for all x-values in the domain of f, $f(x) \geq f(c)$.

Definition 12.5 *Let f be a function, and let c be a real value in the domain of f.*
(i) We say that $f(c)$ is a relative or local maximum value of f provided that for every x-value in an open interval containing c , $f(x) \leq f(c)$.
(ii) We say that $f(c)$ is a relative or local minimum value of f provided that for every x-value in an open interval containing c, $f(x) \geq f(c)$.

Example 12.6 *(@ Draw a graph, label its interesting points, and ad lib which ones are local and global extrema)* $\square$

Question: How can one find the local and absolute extreme values of a function?

Theorem 12.7 *(Extreme value theorem) Let f be a continuous function defined on the real line. On any closed interval $[a, b]$, f has at least one absolute maximum value and at least one absolute minimum value.*

Example 12.8 *Consider $f(x) = x^3$. (@ Draw graph.) We know that there exists no absolute maximum or minimum value of f over the entire real line. However, the extreme value theorem indicates that there must be both an absolute maximum and an absolute minimum of f over the closed interval $[0, 1]$.* $\square$

Example 12.9 *Consider $f(x) = 1 - x$. (@ Draw graph.) On the interval $(0, 1]$, f has an absolute minimum value of 0, but no absolute maximum value. (The extreme value theorem does not apply, because $(0, 1]$ is not a closed interval.)* $\square$

Theorem 12.10 *(Fermat's theorem on local extrema) Let f be a function defined on the real line. Given a real value c, if f has a local maximum or local minimum at c, then either f is not differentiable at c or $f'(c) = 0$.*

Is the converse to Fermat's theorem true? No.

Example 12.11 *Consider $f(x) = x^3$. We know that $f'(x) = 3x^2$, and so $f'(0) = 0$. However, the point $(0, 0)$ is neither a local maximum nor a local minimum of f.* $\square$

Definition 12.12 *Let f be a function defined on the real line. Given a real value c in the domain of f, we say that c is a critical number of f provided that either $f'(c) = 0$ or $f'(c)$ does not exist.*

Theorem 12.13 *Let f be a function defined on the real line. If c is a real value such that $f(c)$ is an absolute extreme value of f over a closed interval $[a, b]$, then either c is a critical number of f, or $c = a$ or $c = b$.*

Example 12.14 *Let*

$$f(x) = x^3 - 6x^2 + 5. \tag{244}$$

Find the absolute maximum and absolute minimum values of f on the closed interval $[-3, 5]$.
We must find the critical numbers of f.

$$f'(x) = 3x^2 - 12x = 3x\,(x - 4)\,. \tag{245}$$

This has critical numbers at $x = 0$ and $x = 4$. Now we know that the absolute maximum must occur at $x = -3$, $x = 0$, $x = 4$ or $x = 5$. We inspect the y-values at each of these:

$$\begin{array}{c} f\,(-3) = (-3)^3 - 6(-3)^2 + 5 = -74 \\ f\,(0) = (0)^3 - 6(0)^2 + 5 = 5 \\ f\,(4) = (4)^3 - 6(4)^2 + 5 = -27 \\ f\,(5) = (5)^3 - 6(5)^2 + 5 = -20 \end{array}. \tag{246}$$

Thus, the absolute maximum value of f is 5 and the absolute minimum value of f is -74. $\square$

Example 12.15 *Let*

$$f(t) = t + \cot\left(\frac{t}{2}\right). \tag{247}$$

Find the critical numbers of f on $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

First, we must find the critical numbers of f.

$$f'(t) = 1 - \frac{1}{2}\csc^2\left(\frac{t}{2}\right). \tag{248}$$

We seek values of t for which $f'(t)$ is either zero or undefined.

In seeking values for t such that $f'(t) = 0$:

$$\begin{gathered} 0 = 1 - \tfrac{1}{2}\csc^2\left(\tfrac{t}{2}\right) \\ \csc^2\left(\tfrac{t}{2}\right) = 2 \\ \tfrac{1}{\sin\left(\frac{t}{2}\right)} = \csc\left(\tfrac{t}{2}\right) = \pm\sqrt{2} \\ \sin\left(\tfrac{t}{2}\right) = \pm\tfrac{1}{\sqrt{2}} \end{gathered}. \tag{249}$$

This has solutions when $\frac{t}{2} = n\frac{\pi}{4}$, where n is an odd integer. Thus, $t = n\frac{\pi}{2}$. Since we are only considering values for t such that $\frac{\pi}{4} \leq t \leq \frac{7\pi}{4}$, this gives us critical numbers at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

In seeking values for t such that $f'(t)$ is undefined, we need to know when $\csc\left(\frac{t}{2}\right)$ is undefined. This occurs when $\sin\left(\frac{t}{2}\right) = 0$, or in other words, $\frac{t}{2} = m\pi$, where m is any integer. Thus, we seek values $t = 2m\pi$, where m is an integer, in the interval $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$. No such values exist, so the only critical numbers are $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. $\square$

Section 4.3: How Derivatives Affect the Shape of a Graph

If $f'(x) > 0$ on some interval, then f is increasing on that interval. If $f'(x) < 0$ on some interval, then f is decreasing on that interval.

At a local maximum, a function transitions from increasing to decreasing.
At a local minimum, a function transitions from decreasing to increasing.

13 Tuesday, June 19

Example 13.1 *Let*

$$f(x) = 2x^3 - 9x^2 + 12x - 3. \tag{250}$$

(i) Find the intervals on which f is increasing and decreasing.

We need to find where $f'(x) > 0$. In order to do this, we'll first find the critical numbers of f.

$$f'(x) = 6x^2 - 18x + 12 = 6\left(x^2 - 3x + 2\right) = 6\,(x-1)\,(x-2)\,. \tag{251}$$

This yields the critical numbers $x = 1$ and $x = 2$. (@ Draw number line)
For $x < 1$, $f'(x) > 0$.
For $1 < x < 2$, $f'(x) < 0$.
For $x > 2$, $f'(x) > 0$.
Therefore, f is increasing on the intervals $(-\infty, 1)$ and $(2, \infty)$. Also, f is decreasing on the interval $(1, 2)$.
(ii) Find the local maximum points of f.

Local maxima occur at the critical numbers where f changes from increasing to decreasing: the point $(1, 2)$.
(iii) Find the local minimum points of f.

Local minima occur at the critical numbers where f changes from decreasing to increasing: the point $(2, 1)$.

Issue: there are multiple different ways in which a graph could increase. The sign of the first derivative cannot distinguish them. (@ Draw graphs)

Definition 13.2 *Let f be a function that is differentiable on an interval I.*
(i) We say that f is <u>concave up on I</u> provided that the slope of the tangent line is increasing on I.
(ii) We say that f is <u>concave down</u> provided that the slope of the tangent line is decreasing on I.

A function is concave up if it curves upward. (@ Draw graph) This occurs when the second derivative is positive.

A function is concave down if it curves downward. (@ Draw graph) This occurs when the second derivative is negative.

Definition 13.3 *Let f be a function that is differentiable on an interval I. Given a point (x, y) on the graph of f, we say that (x, y) is an* <u>*inflection point*</u> *of f provided that f is continuous at x and f changes from concave upward to concave downward, or from concave downward to concave upward, at (x, y).*

Example 13.4 *Let*

$$f(x) = \frac{x^2 - 4}{x^2 + 4} \tag{252}$$

(i) Find the vertical and horizontal asymptotes.
(ii) Find the intervals on which f is increasing and decreasing.
(iii) find the local maxima and local minima.
(iv) Find the intervals of concavity, and inflection points.
(v) Sketch a graph of f.

(i) We note that $x^2+1 \neq 0$, so f has no vertical asymptotes. As for the horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 - 4}{x^2 + 4} = \lim_{x\to\infty} \frac{1 - \frac{4}{x^2}}{1 + \frac{4}{x^2}} = \frac{1 - 0}{1 + 0} = 1. \tag{253}$$

Thus, f has a horizontal asymptote at $y = 1$ on the right. As f is an even function, the same is true on the left.

(ii) We first find the critical numbers of f.

$$\begin{aligned} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2 - 4}{x^2 + 4} = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} \\ = \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}. \end{aligned} \tag{254}$$

This is never undefined. However, $f'(x) = 0$ at $x = 0$. Therefore, $x = 0$ is a critical number.
(@ Draw number line.)
For $x < 0$, $f'(x) < 0$, so f is decreasing on $(-\infty, 0)$.

For $x > 0$, $f'(x) > 0$, so f is increasing on $(0, \infty)$.

(iii) f has a local minimum at $x = 0$, since it changes from decreasing to increasing. Thus, $(0, -1)$ is the only local minimum of f, and f has no local maximum.

(iv) We first need the second derivative of f:

$$\begin{aligned} f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{16x}{\left(x^2+4\right)^2} &= \frac{\left(x^2+4\right)^2(16) - (16x)\,2\left(x^2+4\right)(2x)}{\left(x^2+4\right)^4} \\ &= \frac{\left(x^2+4\right)(16) - (16x)(2)(2x)}{\left(x^2+4\right)^3} = \frac{16x^2+64-64x^2}{\left(x^2+4\right)^3} \\ &= \frac{64-48x^2}{\left(x^2+4\right)^3} = \frac{16\left(4-3x^2\right)}{\left(x^2+4\right)^3}. \end{aligned} \tag{255}$$

This gives that $f''(x) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$.
(@ Draw number line.)
For $x < -\frac{2}{\sqrt{3}}$, $f''(x) < 0$, so f is concave down on the interval $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$.
For $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, $f''(x) > 0$, so f is concave up on the interval $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.
For $x > \frac{2}{\sqrt{3}}$, $f''(x) < 0$, so f is concave down on the interval $\left(\frac{2}{\sqrt{3}}, \infty\right)$.
We deduce that the points $\left(-\frac{2}{\sqrt{3}}, -\frac{1}{2}\right)$ and $\left(\frac{2}{\sqrt{3}}, -\frac{1}{2}\right)$ are inflection points of f.

(v) We note that the graph of f has x-intercepts at $(-2, 0)$ and $(2, 0)$. (@ Draw graph.) □

Example 13.5 *Sketch the graph of a function that satisfies all of the given conditions.*

$$\begin{array}{cc} f'(0) = 0 & f'(4) = 0 \\ f'(x) = 1 \textit{ on } (-\infty, -1) & \\ f'(x) > 0 \textit{ on } (0, 2) & f'(x) < 0 \textit{ on } (-1, 0) \cup (2, 4) \cup (4, \infty) \\ \lim_{x \to 2^-} f'(x) = \infty & \lim_{x \to 2^+} f'(x) = -\infty \\ f''(x) > 0 \textit{ on } (-1, 2) \cup (2, 4) & f''(x) < 0 \textit{ on } (4, \infty) \end{array} \tag{256}$$

(@ Draw graph) □

Wait a second: if $f'(x) = 0$ and $f''(x) > 0$, then x must be a local minimum!

Similarly, if $f'(x) = 0$ and $f''(x) < 0$, then x must be a local maximum!

Example 13.6 *Let*

$$f(x) = \frac{x^2}{x-1}. \tag{257}$$

Find the local maximum and local minimum values of f.

$$\begin{aligned} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{x-1} &= \frac{(x-1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2) - (x^2)\frac{\mathrm{d}}{\mathrm{d}x}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(2x) - (x^2)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2} \end{aligned} \tag{258}$$

This gives us critical numbers $x = 0$, $x = 1$ and $x = 2$. If we find $f''(x)$:

$$\begin{aligned} f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2 - 2x}{x^2 - 2x + 1} & \\ &= \frac{(x^2 - 2x + 1)(2x - 2) - (x^2 - 2x)(2x - 2)}{(x-1)^4} \\ &= \frac{2x - 2}{(x-1)^4} = \frac{2}{(x-1)^3}. \end{aligned} \tag{259}$$

Since $f''(0) < 0$, f must have a local maximum at $x = 0$. Since $f''(2) > 0$, f must have a local minimum at $x = 2$. However, $f''(1)$ is undefined. (@ Draw graph) □

Section 4.4: Indeterminate forms and l'Hôpital's rule

Definition 13.7 *Let f and g be functions defined on the real line such that g' is not constantly zero.*
(i) Given an (extended) real value a, if $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) = 0$, then we say that the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} \tag{260}$$

is an indeterminate form of type $\frac{0}{0}$.
(ii) Given an (extended) real value a, if $\lim_{x\to a} g(x) = \pm\infty$ and $\lim_{x\to a} f(x) = \pm\infty$, then we say that the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)} \tag{261}$$

is an indeterminate form of type $\frac{\infty}{\infty}$.

Theorem 13.8 *(l'Hôpital's rule) Let f and g be differentiable functions such that g' is not constantly zero. Given an (extended) real value a, if $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{262}$$

Example 13.9 *Find the limit*

$$\lim_{x\to -2} \frac{x^3+8}{x+2}. \tag{263}$$

We could factor $x+2$ out of x^3+8 and then cancel the factor of $x+2$, but that would be annoying. Since $\lim_{x\to -2} x+2 = 0$ and $\lim_{x\to -2} x^3+8 = 0$, l'Hôpital's rule implies that

$$\lim_{x\to -2} \frac{x^3+8}{x+2} = \lim_{x\to -2} \frac{3x^2}{1} = 12. \tag{264}$$

$\square$

14 Wednesday, June 20

Example 14.1 *Find the limit*

$$\lim_{t\to 0} \frac{8^t - 5^t}{t}. \tag{265}$$

Since $\lim_{t\to 0} 8^t - 5^t = 0$ *and* $\lim_{t\to 0} t = 0$, *l'Hôpital's rule implies that*

$$\lim_{t\to 0} \frac{8^t - 5^t}{t} = \lim_{t\to 0} \frac{8^t \ln 8 - 5^t \ln 5}{1} = \ln 8 - \ln 5 = \ln\left(\frac{8}{5}\right). \tag{266}$$

□

Example 14.2 *Find the limit*

$$\lim_{x\to 0} \frac{x^2}{1 - \cos x}. \tag{267}$$

Since $\lim_{x\to 0} x^2 = 0$ *and* $\lim_{x\to 0} 1 - \cos x = 0$, *l'Hôpital's rule implies that*

$$\lim_{x\to 0} \frac{x^2}{1 - \cos x} = \lim_{x\to 0} \frac{2x}{\sin x} = \lim_{x\to 0} \frac{2}{\cos x} = 2. \tag{268}$$

□

Example 14.3 *Find the limit*

$$\lim_{u\to\infty} \frac{u^3}{e^{\frac{u}{10}}}. \tag{269}$$

Since $\lim_{u\to\infty} e^{\frac{u}{10}} = \infty$ *and* $\lim \lim_{u\to\infty} u^3 = \infty$, *l'Hôpital's rule implies that*

$$\lim_{u\to\infty} \frac{u^3}{e^{\frac{u}{10}}} = \lim_{u\to\infty} \frac{3u^2}{\frac{1}{10}e^{\frac{u}{10}}} = \lim_{u\to\infty} \frac{6u}{\frac{1}{100}e^{\frac{u}{10}}} = \lim_{u\to\infty} \frac{6}{\frac{1}{1000}e^{\frac{u}{10}}} = 0. \tag{270}$$

□

Example 14.4 *Find the limit*

$$\lim_{\theta\to\pi} \frac{1 + \cos\theta}{1 - \cos\theta}. \tag{271}$$

CAUTION: $\lim_{\theta\to\pi} 1-\cos\theta = 2$*, so l'Hôpital's rule does not apply. The following is* ***not correct****:*

$$\lim_{\theta\to\pi}\frac{1+\cos\theta}{1-\cos\theta} = \lim_{\theta\to\pi}\frac{-\sin\theta}{\sin\theta} = -1. \tag{272}$$

In this case,

$$\lim_{\theta\to\pi}\frac{1+\cos\theta}{1-\cos\theta} = \frac{\lim_{\theta\to\pi} 1+\cos\theta}{\lim_{\theta\to\pi} 1-\cos\theta} = \frac{1+(-1)}{1-(-1)} = 0. \tag{273}$$

□

Example 14.5 *Find the limit*

$$\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2}. \tag{274}$$

We can write this as

$$\lim_{x\to\infty}\frac{\ln x}{2x^2}. \tag{275}$$

As $\lim_{x\to\infty}\ln x = \infty$ *and* $\lim_{x\to\infty} 2x^2 = \infty$*, l'Hôpital's rule implies that*

$$\lim_{x\to\infty}\frac{\ln x}{2x^2} = \lim_{x\to\infty}\frac{1}{4x^2} = 0. \tag{276}$$

□

Example 14.6 *Find the limit*

$$\lim_{x\to\infty}\sqrt{x}e^{-\frac{x}{2}}. \tag{277}$$

We can write this as

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^{\frac{x}{2}}}. \tag{278}$$

Since $\lim_{x\to\infty}\sqrt{x} = \infty$ *and* $\lim_{x\to\infty} e^{\frac{x}{2}} = \infty$*, l'Hôpital's rule can be applied:*

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^{\frac{x}{2}}} = \lim_{x\to\infty}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}e^{\frac{x}{2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}e^{\frac{x}{2}}} = 0. \tag{279}$$

□

Example 14.7 *Find the limit*

$$\lim_{x\to-\infty} x\ln\left(1-\frac{1}{x}\right). \tag{280}$$

We can write this as

$$\lim_{x\to-\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{x^{-1}}. \tag{281}$$

Since $\lim\limits_{x\to-\infty}\ln\left(1-\frac{1}{x}\right)=0$ *and* $\lim\limits_{x\to-\infty}\frac{1}{x}=0$*, l'Hôpital's rule implies that*

$$\begin{aligned}\lim_{x\to-\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{x^{-1}} &= \lim_{x\to-\infty}\frac{\frac{1}{1-\frac{1}{x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(1-\frac{1}{x}\right)}{-x^{-2}} = \lim_{x\to-\infty}\frac{\frac{1}{1-\frac{1}{x}}x^{-2}}{-x^{-2}} \\ &= \lim_{x\to-\infty}\frac{-1}{1-\frac{1}{x}} = \lim_{x\to-\infty}\frac{1}{\frac{1}{x}-1} = -1. \end{aligned} \tag{282}$$

□

Example 14.8 *Find the limit*

$$\lim_{x\to 0}\left(\csc x-\cot x\right). \tag{283}$$

We can write this as

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos x}{\sin x}. \tag{284}$$

Since $\lim\limits_{x\to 0}1-\cos x=0$ *and* $\lim\limits_{x\to 0}\sin x=0$*, l'Hôpital's rule implies that*

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{\sin x}{\cos x}=0. \tag{285}$$

□

Example 14.9 *Find the limit*

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right). \tag{286}$$

We can write this as

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)=\lim_{x\to 0}\frac{\tan^{-1}x-x}{x\tan^{-1}x}. \tag{287}$$

Since $\lim_{x\to 0}(\tan^{-1}x-x)=0$ *and* $\lim_{x\to 0}x\tan^{-1}x=0$*, l'Hôpital's rule implies that*

$$\lim_{x\to 0}\frac{\tan^{-1}x-x}{x\tan^{-1}x}=\lim_{x\to 0}\frac{\frac{1}{x^2+1}-1}{x\frac{1}{x^2+1}+\tan^{-1}x}=\lim_{x\to 0}\frac{-x^2}{x+(x^2+1)\tan^{-1}x}. \tag{288}$$

Since $\lim_{x\to 0}-x^2=0$ *and* $\lim_{x\to 0}(x+(x^2+1)\tan^{-1}x)=0$*, we can use l'Hôpital's rule again:*

$$\begin{aligned}\lim_{x\to 0}\frac{-x^2}{x+(x^2+1)\tan^{-1}x}&=\lim_{x\to 0}\frac{-2x}{1+(x^2+1)\frac{1}{x^2+1}+2x\tan^{-1}x}\\&=\lim_{x\to 0}\frac{-2x}{1+1+2x\tan^{-1}x}=\lim_{x\to 0}\frac{-x}{1+x\tan^{-1}x}=0.\end{aligned} \tag{289}$$

□

Section 4.7: Optimization problems

An optimization problem is a problem in which we want to make a certain parameter (called the "objective") as small (or as large) as possible.

Example 14.10 *What is the minimum vertical distance between the two parabolas $y = x^2 + 1$ and $y = x - x^2$?*
(@ Draw picture)
The vertical distance between the curves at some point x would be given by

$$d(x) = \left| \left(x^2 + 1\right) - \left(x - x^2\right) \right| = 2x^2 - x + 1. \tag{290}$$

We say that d is our "objective:" we want to make d as small as possible. In order to do this, we'll look for local minima of d.

$$d'(x) = 4x - 1. \tag{291}$$

This gives us a critical point at $x = \frac{1}{4}$. One can show that this is a local minimum of d. The vertical distance between the curves at this x-value is

$$d\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right) + 1 = \frac{1}{8} - \frac{1}{4} + 1 = \frac{1}{8} - \frac{2}{8} + \frac{8}{8} = \frac{7}{8}. \tag{292}$$

Thus, the minimum vertical distance between the curves is $\frac{7}{8}$. $\square$

Usually in optimization problems, we are given a condition that forces a certain equation (called the "constraint") to hold.

Example 14.11 *Find the dimensions of a rectangle whose area is $100\ m^2$ and whose perimeter is as small as possible.*
(@ Draw picture)
We note that the area of a rectangle is $A = xy$. By assumption, $A = 100$, so $xy = 100$. The perimeter is $P = 2x + 2y$. Therefore, we have

$$\begin{array}{cc} P(x,y) = 2x + 2y & \textit{(objective)} \\ xy = 100 & \textit{(constraint)} \end{array}. \tag{293}$$

Since $xy = 100$, we have that $x = \frac{100}{y}$. Therefore, we can rewrite the objective as

$$P\left(x\right) = x + \frac{100}{x}. \tag{294}$$

In order to make the sum as small as possible, we must find a local minimum of P. We first find the derivative of P:

$$P'(x) = 1 - \frac{100}{x^2}. \tag{295}$$

This grants us the critical numbers $x = 0$ and $x = \pm 10$. We cannot have $x = 0$ or $x = -10$, so we are only concerned with $x = 10$. One can show that this is a local minimum of P. Therefore, P is minimal when $x = 10$. Additionally, $y = \frac{100}{x} = \frac{100}{10} = 10$. $\square$

Example 14.12 *The sum of the lengths of the legs of a right triangle is* 16. *What is the smallest possible length of the hypotenuse?*
(@ Draw picture)
By assumption, $a + b = 16$. Additionally, we know that $c^2 = a^2 + b^2$, by the Pythagorean theorem. Thus, we have the two equations

$$\begin{array}{ll} c^2(a,b) = a^2 + b^2 & \textit{(objective)} \\ a + b = 16 & \textit{(constraint)} \end{array}. \tag{296}$$

We know that $b = 16 - a$, so we can rewrite the objective as

$$c^2\left(a\right) = a^2 + \left(16 - a\right)^2 = 2a^2 - 32a + 256 \tag{297}$$

We can now take the derivative:

$$\left(c^2\right)'\left(a\right) = 4a - 32. \tag{298}$$

This gives us a critical number at $a = 8$. One can show that this is a local minimum *of c^2. Therefore, the smallest possible length of the hypotenuse is*

$$c(8) = \sqrt{2(8)^2 - 32(8) + 256} = \sqrt{128} = 8\sqrt{2}. \tag{299}$$

□

Example 14.13 *We have a square sheet of cardboard that is 3 ft wide. We want to turn this into a box with an open top by cutting a square out from each of the four corners and then bending up the sides. What is the largest possible volume of a box constructed in this way?*
(@ Draw picture)
We want to maximize the volume of the box, which would be $V = w^2h$. We know that the width of the cardboard sheet is 3 ft, so $w + 2h = 3$. Therefore, we have

$$\begin{aligned} V(w,h) &= w^2h \quad \textit{(objective)} \\ w + 2h &= 3 \qquad \textit{(constraint)} \end{aligned}. \tag{300}$$

The constraint implies that $w = 3 - 2h$, so we can rewrite the objective as

$$V(h) = (3-2h)^2h. \tag{301}$$

We can now look for critical numbers of V.

$$\begin{aligned} V'(h) = (3-2h)^2\frac{\mathrm{d}}{\mathrm{d}h}h + h\frac{\mathrm{d}}{\mathrm{d}h}(3-2h)^2 \\ &= (3-2h)^2 - 4h(3-2h) \\ &= (3-2h)(3-2h-4h) = (3-2h)(3-6h) \end{aligned} \tag{302}$$

This gives the critical numbers $h = \frac{3}{2}$ and $h = \frac{1}{2}$. We cannot have $h = \frac{3}{2}$ (WHY?), so we are only interested in $h = \frac{1}{2}$. At this value of h,

$$V\left(\frac{1}{2}\right) = \left(3 - 2\left(\frac{1}{2}\right)\right)^2\left(\frac{1}{2}\right) = 2\ ft^3. \tag{303}$$

□

Example 14.14 *A box with a square base and open top must have a volume of* $32,000\ cm^3$*. Find the dimensions of the box that minimize that amount of material used to construct the box.*
(@ Draw picture)
The amount of material used to construct the box is proportional to the surface area of the box.

$$\begin{array}{cc} S\left(x,z\right)=x^2+4xz & \textit{(objective)} \\ x^2z=32000 & \textit{(constraint)} \end{array}. \tag{304}$$

The constraint implies that

$$S(x)=x^2+4x\left(\frac{32000}{x^2}\right)=x^2+\frac{128000}{x}. \tag{305}$$

We seek the critical numbers of S*.*

$$S'(x)=2x-\frac{128000}{x^2}. \tag{306}$$

If $S'(x)=0$*, then*

$$\begin{array}{c} 0=2x-\frac{128000}{x^2} \\ x=\frac{64000}{x^2} \\ x^3=64000=4^310^3 \\ x=40 \end{array}. \tag{307}$$

We also have a critical number at $x=0$*. One can show that a local minimum of* S *occurs at* $x=40$*. Additionally,* $z=\frac{32000}{40^2}=\frac{32000}{1600}=20$*. Therefore, the dimensions of the box that uses the least amount of material are* $40\ cm\times 40\ cm\times 20\ cm$*.* □

Example 14.15 *A poster is to have an area of* $180\ in^2$ *with* 1 *inch margins at the bottom and sides and a* 2 *inch margin at the top. What dimensions will give the largest interior area?*
(@ Draw picture)
The interior area is $A(x,y)=(x-2)(y-3)$*. We know that* $xy=180$*. Therefore,*

$$\begin{array}{cc} A(x,y)=(x-2)(y-3) & \textit{(objective)} \\ xy=180 & \textit{(constraint)} \end{array}. \tag{308}$$

The constraint indicates that $y = \frac{180}{x}$*, so we can rewrite the objective as*

$$A(x) = (x-2)\left(\frac{180}{x} - 3\right). \tag{309}$$

We now seek the critical numbers of A*.*

$$A'(x) = (x-2)\left(-\frac{180}{x^2}\right) + \left(\frac{180}{x} - 3\right)(1) = \frac{360}{x^2} - 3 = 3\left(\frac{120}{x^2} - 1\right). \tag{310}$$

This gives the critical numbers $x = 0$ *and* $x = \pm\sqrt{120} = \pm 2\sqrt{30}$*. We cannot have* $x = 0$ *or* $x = -2\sqrt{30}$*, so* $x = 2\sqrt{30}$ *is the width which maximizes the interior area. When* $x = 2\sqrt{30}$*,* $y = \frac{180}{2\sqrt{30}} = \frac{90}{\sqrt{30}} = 3\sqrt{30}$*.* $\square$

Example 14.16 *Find the point on the curve* $y = \sqrt{x}$ *that is closest to the point* $(3,0)$*.*
(@ Draw picture)

$$\begin{array}{ll} d^2(x,y) = (x-3)^2 + y^2 & \textit{(objective)} \\ y = \sqrt{x} & \textit{(constraint)} \end{array}. \tag{311}$$

The constraint implies that

$$d^2(x) = (x-3)^2 + x. \tag{312}$$

We seek the critical numbers of d^2*:*

$$\left(d^2\right)'(x) = 2(x-3) + 1 = 2x - 5. \tag{313}$$

This yields the critical number $x = \frac{5}{2}$*. One can show that a local minimum of* d^2 *occurs at* $x = \frac{5}{2}$*. At this point,* $y = \sqrt{\frac{5}{2}}$*, so* $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ *is the point on* $y = \sqrt{x}$ *that is closest to* $(3,0)$*.* $\square$

Example 14.17 *Find the area of the largest trapezoid that can be inscribed in a circle of radius* 1 *whose base is a diameter of the circle.*
(@ Draw picture)

We note that

$$\begin{aligned} A(x,y) &= (1+x)y \quad \textit{(objective)} \\ x^2+y^2 &= 1 \qquad\quad \textit{(constraint)} \end{aligned}. \tag{314}$$

The constraint implies that

$$y^2 = 1 - x^2. \tag{315}$$

We know that A will be maximal if A^2 is maximal, so this allows us to write

$$A^2(x) = (1+x)^2 y^2 = (1+x)^2 \left(1-x^2\right) \tag{316}$$

The derivative is

$$\begin{aligned} \left(A^2\right)'(x) &= (1+x)^2(-2x) + \left(1-x^2\right) 2(1+x)(1) \\ &= (1+x)\left((1+x)(-2x) + 2\left(1-x^2\right)\right) \\ &= (1+x)\left(-2x-2x^2+2-2x^2\right) \\ &= (1+x)\left(-4x^2-2x+2\right) \\ &= -2(1+x)\left(2x^2+x-1\right). \end{aligned} \tag{317}$$

This gives us critical points at $x=-1$ and

$$x = \frac{-1 \pm \sqrt{1+4(2)(1)}}{2(2)} = \frac{-1\pm\sqrt{9}}{4} = \frac{-1\pm 3}{4} = -1, \frac{1}{2}. \tag{318}$$

We cannot have $x=-1$, so $x=\frac{1}{2}$ is the only critical point in which we are interested. One can show that a local maximum occurs here. Now,

$$A\left(\frac{1}{2}\right) = \sqrt{\left(1+\frac{1}{2}\right)^2\left(1-\left(\frac{1}{2}\right)^2\right)} = \sqrt{\frac{9}{4}\frac{3}{4}} = \frac{3\sqrt{3}}{4}. \tag{319}$$

□

Example 14.18 *A painting in an art gallery has height h and is hung on a wall so that its lower edge is a distance d above the eye of an observer.*
(@ Draw picture)

How far away from the wall should the observer stand to get the best view? (That is, to maximize the angle θ subtended at the observer's eye by the painting?)

Example 14.19 *Let v_1 be the velocity of light in air and v_2 be the velocity of light in water. A ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken.*
(@ Diagram: label l as the horizontal distance between the two objects, a as the vertical distance between A and the interface, b as the vertical distance between B and the interface, and x as the horizontal distance between A and C.)
Prove Snell's law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}. \tag{320}$$

15 Monday, June 25

(Test 3 was given on this day.)

16 Tuesday, June 26

Section 5.1: Areas and Distances

We now know how to solve the tangent line problem for a lot of functions. Let's return to the area problem: this asks how to find the area between a curve and the x-axis from one chosen x-value to another. [@ Draw picture]

Sometimes, finding this is easy.

Example 16.1 *Consider the curve*

$$y = x - 1. \tag{321}$$

Find the area between the curve and the x-axis from 1 to 5.
We can see this as the area of a triangle: [@ Draw graph] As $A = \frac{1}{2}bh$, this is

$$A = \frac{1}{2}(5-1)(4) = 8. \tag{322}$$

□

Example 16.2 *Consider the curve*

$$y = \sqrt{4 - x^2}. \tag{323}$$

Find the area between the curve and the x-axis from -2 to 2.
We can see this as the area of a semicircle, since here $x^2 + y^2 = 4$: [@ Draw graph] As πr^2 is the area of a circle, this is

$$A = \frac{1}{2}\pi(2)^2 = 2\pi. \tag{324}$$

□

For functions that are not circles or lines, this cannot be solved so easily.

The approach to solving the area problem will involve the "Riemann sum:"
1. Divide the interval into n subintervals of equal width. [@ Draw number line] These each have the length $\Delta x = \frac{b-a}{n}$.
2. For each k between 1 and n, select a "sample point," called x_k, from the kth subinterval. [@ Draw picture]
3. For each subinterval, draw a rectangle whose width is Δx and whose height is the value $f(x_k)$.
4. Find the area of each of these rectangles (that is, $f(x_k)\,\Delta x$) and add them all up:

$$A = f(x_1)\,\Delta x + f(x_2)\,\Delta x + ... + f(x_n)\,\Delta x = \sum_{k=1}^{n} f(x_k)\,\Delta x. \tag{325}$$

This sum is called a "Riemann sum with n rectangles." It approximates the area under the curve.

Example 16.3 *Let*

$$f(x) = 1 + x^3. \tag{326}$$

Approximate the area between the graph of f and the x-axis from $x = -1$ to $x = 1$ using a Riemann sum with $n = 4$ rectangles whose sample points are:
(i) the right endpoints of the subintervals.
(ii) the left endpoints of the subintervals.
(iii) the midpoints of the subintervals.
First, we note that

$$\Delta x = \frac{(1) - (-1)}{(4)} = \frac{2}{4} = \frac{1}{2}. \tag{327}$$

(i) [@ Draw graph] We have sample points at $x = -\frac{1}{2}$, $x = 0$, $x = \frac{1}{2}$ and $x = 1$. At these points, the y-values are

$$\begin{array}{c} f\left(-\frac{1}{2}\right) = 1 + \left(-\frac{1}{2}\right)^3 = \frac{7}{8} \\ f(0) = 1 + (0)^3 = 1 \\ f\left(\frac{1}{2}\right) = 1 + \left(\frac{1}{2}\right)^3 = \frac{9}{8} \\ f(1) = 1 + (1)^3 = 2 \end{array}. \tag{328}$$

Therefore, the area is approximately

$$A = \left(\frac{7}{8} + 1 + \frac{9}{8} + 2\right)\left(\frac{1}{2}\right) = \frac{5}{2}. \tag{329}$$

(ii) [@ Draw graph] We have sample points at $x = -1$, $x = -\frac{1}{2}$, $x = 0$ *and* $x = \frac{1}{2}$. *At these points, the* y*-values are*

$$\begin{array}{c} f(-1) = 1 + (-1)^3 = 0 \\ f\left(-\frac{1}{2}\right) = 1 + \left(-\frac{1}{2}\right)^3 = \frac{7}{8} \\ f(0) = 1 + (0)^3 = 1 \\ f\left(\frac{1}{2}\right) = 1 + \left(\frac{1}{2}\right)^3 = \frac{9}{8} \end{array}. \tag{330}$$

Therefore, the area is approximately

$$A = \left(0 + \frac{7}{8} + 1 + \frac{9}{8}\right)\left(\frac{1}{2}\right) = \frac{3}{2}. \tag{331}$$

(iii) [@ Draw graph] We have sample points at $x = -\frac{3}{4}$, $x = -\frac{1}{4}$, $x = \frac{1}{4}$ *and* $x = \frac{3}{4}$. *At these points, the* y*-values are*

$$\begin{array}{c} f\left(-\frac{3}{4}\right) = 1 + \left(-\frac{3}{4}\right)^3 = \frac{37}{64} \\ f\left(-\frac{1}{4}\right) = 1 + \left(-\frac{1}{4}\right)^3 = \frac{63}{64} \\ f\left(\frac{1}{4}\right) = 1 + \left(\frac{1}{4}\right)^3 = \frac{65}{64} \\ f\left(\frac{3}{4}\right) = 1 + \left(\frac{3}{4}\right)^3 = \frac{91}{64} \end{array}. \tag{332}$$

Therefore, the area is approximately

$$A = \left(\frac{37}{64} + \frac{63}{64} + \frac{65}{64} + \frac{91}{64}\right)\left(\frac{1}{2}\right) = 2. \tag{333}$$

□

Example 16.4 *Let*

$$f(x) = 4x(1 - x) \tag{334}$$

Approximate the area between the graph of f *and the* x*-axis from* 0 *to* 1 *using a*

Riemann sum with right endpoints as the sample points and:

(i) $n = 1$ *rectangle.*

(ii) $n = 2$ *rectangles.*

(iii) $n = 4$ *rectangles.*

(iv) $n = 8$ *rectangles.*

(i) [@ Draw graph] We have one subinterval with a width of $\Delta x = \frac{1-0}{1}$*. This gives us one sample point, at* $x = 1$*. The* y*-value at this point is*

$$f\left(1\right) = 4\left(1\right)\left(1-\left(1\right)\right) = 0. \tag{335}$$

Therefore, the area is approximately

$$A = \left(0\right)\left(1\right) = 0. \tag{336}$$

(ii) [@ Draw graph] We have two subintervals with widths of $\Delta x = \frac{1-0}{2} = \frac{1}{2}$*. This gives us two sample points, at* $x = \frac{1}{2}$ *and* $x = 1$*. The* y*-values at these points are*

$$\begin{array}{c} f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = 1 \\ f\left(1\right) = 4\left(1\right)\left(1-1\right) = 0 \end{array}. \tag{337}$$

Therefore, the area is approximately

$$A = \left(1+0\right)\left(\frac{1}{2}\right) = \frac{1}{2}. \tag{338}$$

(iii) [@ Draw graph] We have four subintervals with widths of $\Delta x = \frac{1-0}{4} = \frac{1}{4}$*. This gives us four sample points, at* $x = \frac{1}{4}$*,* $x = \frac{1}{2}$*,* $x = \frac{3}{4}$ *and* $x = 1$*. The* y*-values at these points are*

$$\begin{array}{c} f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)\left(1-\frac{1}{4}\right) = \frac{3}{4} \\ f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = 1 \\ f\left(\frac{3}{4}\right) = 4\left(\frac{3}{4}\right)\left(1-\frac{3}{4}\right) = \frac{3}{4} \\ f\left(1\right) = 4\left(1\right)\left(1-1\right) = 0 \end{array}. \tag{339}$$

Therefore, the area is approximately

$$A = \left(\frac{3}{4} + 1 + \frac{3}{4} + 0\right)\left(\frac{1}{4}\right) = \frac{10}{16} = \frac{5}{8}. \tag{340}$$

(iv) [@ Draw graph] We have eight subintervals with widths of $\Delta x = \frac{1-0}{8} = \frac{1}{8}$. *This gives us eight sample points, at* $x = \frac{1}{8}$, $x = \frac{1}{4}$, $x = \frac{3}{8}$, $x = \frac{1}{2}$, $x = \frac{5}{8}$, $x = \frac{3}{4}$, $x = \frac{7}{8}$ *and* $x = 1$. *The* y*-values at these points are*

$$\begin{gathered}
f\left(\tfrac{1}{8}\right) = 4\left(\tfrac{1}{8}\right)\left(1 - \tfrac{1}{8}\right) = \tfrac{7}{16} \\
f\left(\tfrac{1}{4}\right) = \tfrac{3}{4} \\
f\left(\tfrac{3}{8}\right) = 4\left(\tfrac{3}{8}\right)\left(1 - \tfrac{3}{8}\right) = \tfrac{15}{16} \\
f\left(\tfrac{1}{2}\right) = 1 \\
f\left(\tfrac{5}{8}\right) = 4\left(\tfrac{5}{8}\right)\left(1 - \tfrac{5}{8}\right) = \tfrac{15}{16} \\
f\left(\tfrac{3}{4}\right) = \tfrac{3}{4} \\
f\left(\tfrac{7}{8}\right) = 4\left(\tfrac{7}{8}\right)\left(1 - \tfrac{7}{8}\right) = \tfrac{7}{16} \\
f\left(1\right) = 0
\end{gathered} \tag{341}$$

Therefore, the area is approximately

$$A = \left(\frac{7}{16} + \frac{3}{4} + \frac{15}{16} + 1 + \frac{15}{16} + \frac{3}{4} + \frac{7}{16} + 0\right)\left(\frac{1}{8}\right) = \frac{84}{128} = \frac{21}{32}. \tag{342}$$

□

1

Section 5.2: The Definite Integral

Observe: more rectangles give more accurate approximations. In order to get the true area, we'll take the limit as the number of rectangles increases without bound.

Definition 16.5 *Let f be a function defined on the real line. Given real values a and b, the definite integral of f over the interval $[a, b]$ is the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x, \tag{343}$$

where $\Delta x = \frac{b-a}{n}$, and for each k, x_k is any real value satisfying the inequalities $a + (k-1)\Delta x \leq x_k \leq a + k\Delta x$.

Notation: $\int_a^b f(x)\,\mathrm{d}x$ means "the definite integral of f over the interval $[a, b]$.

Notice: the definite integral could be negative. It gives a "signed area under the curve."

Theorem 16.6 *Let f be a function defined on the real line, and let a and b be real values. The following statements are true.*

(i)

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x. \tag{344}$$

(ii)

$$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x. \tag{345}$$

(iii) Given a real value c,

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x. \tag{346}$$

(iv) Given a real value c,

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x. \tag{347}$$

Example 16.7 *Let*

$$g(x) = \begin{cases} 4-2x & \textit{if } 0 \le x < 2 \\ -\sqrt{4-x^2} & \textit{if } 2 \le x < 6 \\ x-6 & \textit{if } 6 \le x < 7 \end{cases}. \tag{348}$$

[@ Draw graph]

(i) Find

$$\int_0^2 g(x)\,\mathrm{d}x. \tag{349}$$

This is

$$\int_0^2 g(x)\,\mathrm{d}x = A = \frac{1}{2}(2)(4) = 4. \tag{350}$$

(ii) Find

$$\int_2^6 g(x)\,\mathrm{d}x. \tag{351}$$

The area of the semicircle is

$$A = \frac{1}{2}\pi(2)^2 = 2\pi. \tag{352}$$

Since the graph is below the x-axis between 2 and 6, the integral is

$$\int_2^6 g(x)\,\mathrm{d}x = -2\pi. \tag{353}$$

(iii) Find

$$\int_0^7 g(x)\,\mathrm{d}x. \tag{354}$$

We know that

$$\int_0^7 g(x)\,\mathrm{d}x = \int_0^2 g(x)\,\mathrm{d}x + \int_2^6 g(x)\,\mathrm{d}x + \int_6^7 g(x)\,\mathrm{d}x$$
$$= 4 - 2\pi + \int_6^7 g(x)\,\mathrm{d}x. \quad (355)$$

Additionally,

$$\int_6^7 g(x)\,\mathrm{d}x = \frac{1}{2}\,(1)\,(1) = \frac{1}{2}. \quad (356)$$

Therefore,

$$\int_0^7 g(x)\,\mathrm{d}x = 4 - 2\pi + \frac{1}{2} = \frac{9}{2} - 2\pi. \quad (357)$$

□

Finding the definite integral without any geometric reasoning can prove rather difficult.

Example 16.8 *Evaluate the integral*

$$\int_2^5 4 - 2x\,\mathrm{d}x. \quad (358)$$

[@ Draw graph] By definition,

$$\int_2^5 4 - 2x\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} (4 - 2x_k)\,\Delta x. \quad (359)$$

If we choose the sample points to be the right endpoints of the intervals, then we have $x_k = a + k\Delta x$. [@ Draw picture] Thus,

$$\int_2^5 4 - 2x\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} (4 - 2\,(2 + k\Delta x))\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} (-2k\Delta x)\,\Delta x \quad (360)$$

In our case, $\Delta x = \frac{5-2}{n} = \frac{3}{n}$. Therefore, this becomes

$$\int_2^5 4 - 2x\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} \left(-2k\frac{3}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \left(-\frac{18}{n^2}\right) \sum_{k=1}^{n} k. \quad (361)$$

One can show that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$*, so this is*

$$\int_2^5 4 - 2x \, \mathrm{d}x = \lim_{n\to\infty} \left(-\frac{18}{n^2}\right) \frac{n(n+1)}{2}$$
$$= \lim_{n\to\infty} \frac{-9\,(n+1)}{n} = \lim_{n\to\infty} -9 + \frac{1}{n} = -9. \quad (362)$$

□

17 Wednesday, June 27

Section 5.3: The Fundamental Theorem of Calculus, and
Section 4.9: Antiderivatives

The fundamental theorem of calculus explains a relationship between the derivative and the definite integral.

In order to understand the statement, we will first examine functions of the form

$$g(x) = \int_a^x f(t)\ \mathrm{d}t, \tag{363}$$

where a is a constant and f is a function defined on the real line.

Example 17.1 *Let*

$$f(t) = t + 1. \tag{364}$$

Define

$$g(x) = \int_0^x f(t)\ \mathrm{d}t. \tag{365}$$

Let's examine some values of the function g. [@ Draw graph]

$$\begin{array}{c} g(0) = \int_0^0 t + 1\ \mathrm{d}t = 0 \\ g(1) = \int_0^1 t + 1\ \mathrm{d}t = (1)(1) + \frac{1}{2}(1)(1) = \frac{3}{2} \\ g(2) = \int_0^2 t + 1\ \mathrm{d}t = (2)(1) + \frac{1}{2}(2)(2) = 4 \end{array} \tag{366}$$

□

Theorem 17.2 *(The fundamental theorem of calculus) If f is continuous on an interval $[a, b]$, then*

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\ \mathrm{d}t = f(x). \tag{367}$$

Example 17.3 *Let*

$$g(x) = \int_1^x \ln\left(1 + t^2\right)\ \mathrm{d}t. \tag{368}$$

Find $g'(x)$.
By the fundamental theorem of calculus,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}t} \int_1^x \ln\left(1+t^2\right)\ \mathrm{d}t = \ln\left(1+x^2\right). \tag{369}$$

□

Example 17.4 *Let*

$$R(y) = \int_y^2 t^3 \sin t\ \mathrm{d}t. \tag{370}$$

Find $R'(y)$.
By the fundamental theorem of calculus,

$$R'(y) = \frac{\mathrm{d}}{\mathrm{d}y} \int_y^2 t^3 \sin t\ \mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{dy}} \int_2^y t^3 \sin t\ \mathrm{d}t = -y^3 \sin y. \tag{371}$$

□

Example 17.5 *Let*

$$h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\ \mathrm{d}z. \tag{372}$$

Find $h'(x)$.
We can understand $h(x) = g\left(f\left(x\right)\right)$, where

$$\begin{array}{c} f\left(x\right) = \sqrt{x} \\ g\left(x\right) = \int_1^x \frac{z^2}{z^4+1}\ \mathrm{d}z \end{array}. \tag{373}$$

By the fundamental theorem of calculus,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_1^x \frac{z^2}{z^4+1}\ \mathrm{d}z = \frac{x^2}{x^4+1}. \tag{374}$$

Now, by the chain rule,

$$h'(x) = g'\left(f\left(x\right)\right) f'\left(x\right) = \frac{\left(\sqrt{x}\right)^2}{\left(\sqrt{x}\right)^4+1} \left(\frac{1}{2} x^{-\frac{1}{2}}\right) = \frac{x}{2\left(x^2+1\right)\sqrt{x}}. \tag{375}$$

□

If we define $g(x) = \int_a^x f(t)\ \mathrm{d}t$, then $g'(x) = f(x)$. We say that g is an "antiderivative" of f.

Definition 17.6 *Let f and g be functions defined on the real line. We say that g is an antiderivative of f provided that $g' = f$.*

Question: can a function have more than one antiderivative? In other words, can there exist two functions that have the same derivative?

If $\frac{\mathrm{d}g}{\mathrm{d}x} = f$ and $\frac{\mathrm{d}h}{\mathrm{d}x} = f$, then

$$\begin{array}{c} \frac{\mathrm{d}g}{\mathrm{d}x} = f = \frac{\mathrm{d}h}{\mathrm{d}x} \\ \frac{\mathrm{d}g}{\mathrm{d}x} - \frac{\mathrm{d}h}{\mathrm{d}x} = 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(g - h) = 0 \end{array} . \tag{376}$$

Thus, $g - h$ is a constant function. Therefore, if g and h are antiderivatives of f, then they differ by a constant.

Example 17.7 *Consider*

$$f(x) = 2x. \tag{377}$$

One can show that $g(x) = x^2$ and $h(x) = x^2 + 4$ are both antiderivatives of f. In fact, for any fixed value C, $k(x) = x^2 + C$ is an antiderivative of f. □

If f is defined on the real line, and $g(x) = \int_a^x f(t)\ \mathrm{d}t$ for some real value a, then $g'(x) = f(x)$, by the fundamental theorem of calculus.

If F is any antiderivative of f, then F and g differ by a constant; suppose that $F(x) = g(x) + C$. Now, notice that

$$\begin{aligned} F(b) - F(a) = (g(b) + C) - (g(a) + C) = g(b) - g(a) \\ = \int_a^b f(t)\ \mathrm{d}t - \int_a^a f(t)\ \mathrm{d}t = \int_a^b f(t)\ \mathrm{d}t. \end{aligned} \tag{378}$$

Corollary 17.8 *If f is continuous on the interval $[a, b]$, and F is any antiderivative of f, then*

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a). \tag{379}$$

This makes evaluation of integrals much easier!

Example 17.9 *Evaluate the integral*

$$\int_2^5 4 - 2x\,\mathrm{d}x. \tag{380}$$

First, we need an antiderivative of $4 - 2x$. How about $F(x) = 4x - x^2$?

$$\begin{aligned}\int_2^5 4 - 2x\,\mathrm{d}x = F(5) - F(2) = 4x - x^2\Big|_2^5 \\ = \left(4\,(5) - (5)^2\right) - \left(4\,(2) - (2)^2\right) = (-5) - (4) = -9.\end{aligned} \tag{381}$$

□

$$\begin{array}{cc} \text{Function} & \text{Antiderivative} \\ a & ax \\ x^r \text{ for } r \neq -1 & \frac{x^{r+1}}{r+1} \\ \frac{1}{x} & \ln|x| \\ a^x \text{ for } a > 1 & \frac{a^x}{\ln a} \\ \sin x & -\cos x \\ \cos x & \sin x \\ \sec^2 x & \tan x \\ \sec x \tan x & \sec x \\ -\csc^2 x & \cot x \\ -\csc x \cot x & \csc x \\ \frac{1}{\sqrt{1-x^2}} & \sin^{-1} x \\ \frac{1}{x\sqrt{x^2-1}} & \sec^{-1} x \\ \frac{1}{x^2+1} & \tan^{-1} x \end{array} \tag{382}$$

Example 17.10 *Evaluate the integral*

$$\int_1^8 x^{-\frac{2}{3}}\ \mathrm{d}x. \tag{383}$$

$$\int_1^8 x^{-\frac{2}{3}}\ \mathrm{d}x = 3x^{\frac{1}{3}}\Big|_1^8 = 3(8)^{\frac{1}{3}} - 3(1)^{\frac{1}{3}} = 3(2) - 3(1) = 3. \tag{384}$$

□

Example 17.11 *Evaluate the integral*

$$\int_0^1 1 - 8v^3 + 16v^7\ \mathrm{d}v. \tag{385}$$

$$\int_0^1 1 - 8v^3 + 16v^7\ \mathrm{d}v = v - 2v^4 + 2v^8\Big|_0^1 = 1 - 2 + 2 = 1. \tag{386}$$

□

Example 17.12 *Evaluate the integral*

$$\int_1^{18} \sqrt{\frac{3}{z}}\ \mathrm{d}z. \tag{387}$$

$$\int_1^{18} \sqrt{3}z^{-\frac{1}{2}}\ \mathrm{d}z = 2\sqrt{3}z^{\frac{1}{2}}\Big|_1^{18} = 2\sqrt{3}\left(\sqrt{18} - \sqrt{1}\right) = 2\sqrt{3}\left(3\sqrt{2} - 1\right). \tag{388}$$

□

Example 17.13 *Evaluate the integral*

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta\ \mathrm{d}\theta \tag{389}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta\ \mathrm{d}\theta = -\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left(\frac{\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} - \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}\right) = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} - \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = 1 - \frac{1}{\sqrt{3}}. \tag{390}$$

□

Example 17.14 *Evaluate the integral*

$$\int_0^3 2\sin x - e^x \, \mathrm{d}x. \tag{391}$$

$$\begin{aligned} 2\int_0^3 \sin x \, \mathrm{d}x - \int_0^3 e^x \, \mathrm{d}x &= 2\left(-\cos x\Big|_0^3\right) - e^x\Big|_0^3 \\ &= 2\left(1-\cos 3\right) - \left(e^3 - 1\right) = 3 - 2\cos 3 - e^3. \end{aligned} \tag{392}$$

□

Example 17.15 *Evaluate the integral*

$$\int_1^3 \frac{y^3 - 2y^2 - y}{y^2} \, \mathrm{d}y. \tag{393}$$

$$\begin{aligned} \int_1^3 y - 2 - \frac{1}{y} \, \mathrm{d}y &= \frac{1}{2}y^2 - 2y - \ln y\Big|_1^3 \\ &= \left(\frac{1}{2}(3)^3 - 2\,(3) - \ln(3)\right) - \left(\frac{1}{2}(1)^3 - 2\,(1) - \ln(1)\right) \\ &= \frac{9}{2} - 6 - \ln 3 - \frac{1}{2} + 2 = -\ln 3 \end{aligned} \tag{394}$$

□

Example 17.16 *Let*

$$f(x) = \begin{cases} 2 & \textit{if } -2 \le x \le 0 \\ 4 - x^2 & \textit{if } 0 < x \le 2 \end{cases}. \tag{395}$$

Evaluate the integral

$$\int_{-2}^2 f(x) \, \mathrm{d}x. \tag{396}$$

$$
\begin{aligned}
\int_{-2}^{2} f(x)\ \mathrm{d}x &= \int_{-2}^{0} f(x)\ \mathrm{d}x + \int_{0}^{2} f(x)\ \mathrm{d}x \\
&= \int_{-2}^{0} 2\ \mathrm{d}x + \int_{0}^{2} 4 - x^2\ \mathrm{d}x = 2x\Big|_{-2}^{0} + \left(4x - \frac{1}{3}x^3\Big|_{0}^{2}\right) \\
&= \left(2\left(0\right) - 2\left(-2\right)\right) + \left(\left(4\left(2\right) - \frac{1}{3}\left(2\right)^3\right) - \left(4\left(0\right) - \frac{1}{3}\left(0\right)^3\right)\right) \\
&= 4 + \left(8 - \frac{8}{3}\right) - \left(0\right) = 12 - \frac{8}{3} = \frac{28}{3}.
\end{aligned} \tag{397}
$$

□

Section 5.4: Indefinite integrals and the Net Change Theorem

Definition 17.17 *Let f be a function defined on the real line. The indefinite integral of f is the set of antiderivatives of f.*

Example 17.18 *Let*

$$f(x) = e^x - 2x^2. \tag{398}$$

Find the indefinite integral of f.

$$\int f(x)\,\mathrm{d}x = \int e^x - 2x^2\,\mathrm{d}x = e^x - \frac{2}{3}x^3 + C, \tag{399}$$

where C is any real value. $\square$

Notice: the definite integral is a real value. The indefinite integral is a family of functions.

The corollary to the fundamental theorem of calculus says that if f is continuous and F is an antiderivative of f, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a). \tag{400}$$

If F is an antiderivative of f, then $F' = f$, so this can be written as

$$\int_a^b F'(t)\,\mathrm{d}t = F(b) - F(a). \tag{401}$$

Thus, in applications, we can think of $\int_a^b f(t)\,\mathrm{d}t$ as the "cumulative effect of a varying rate of change."

18 Thursday, June 28

Example 18.1 *Water flows from the bottom of a tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leq t \leq 50$. Find the amount of water that flows from the tank during the first 10 minutes.*

$$\int_0^{10} r(t)\,\mathrm{d}t = \int_0^{10} 200 - 4t\,\mathrm{d}t = 200t - 2t^2\Big|_0^{10} = (2000 - 200) - (0 - 0) = 1800\ L. \tag{402}$$

□

Definition 18.2 *If $v(t)$ is the velocity of a particle with respect to a chosen reference point (measured in meters per second) and t is the time since a particular moment (measured in seconds), then given times $t = a$ and $t = b$, the <u>displacement of the particle from $t = a$ to $t = b$</u> is the value $\int_a^b v(t)\ \mathrm{d}t$. The <u>distance traveled by the particle from $t = a$ to $t = b$</u> is the value $\int_a^b |v(t)|\ \mathrm{d}t$.*

Example 18.3 *The velocity (in meters per second) for a particle moving along a line is*

$$v(t) = t^2 - 2t - 3. \tag{403}$$

(i) Find the displacement of the particle from $t = 2$ to $t = 4$.

$$\begin{aligned}\int_2^4 v(t)\ \mathrm{d}t = \int_2^4 t^2 - 2t - 3\ \mathrm{d}t &= \frac{1}{3}t^3 - t^2 - 3t\Big|_2^4 \\ &= \left(\frac{1}{3}(4)^3 - (4)^2 - 3(4)\right) - \left(\frac{1}{3}(2)^3 - (2)^2 - 3(2)\right) \\ &= \frac{64}{3} - 16 - 12 - \frac{8}{3} + 4 + 6 = \frac{56}{3} - 18 = \frac{2}{3}\ m.\end{aligned} \tag{404}$$

(ii) Find the distance travelled by the particle from $t = 2$ to $t = 4$.
We need to know when $v(t)$ is negative. We'll first find out when $v(t)$ is zero:

$$0 = v(t) = t^2 - 2t - 3 = (t - 3)(t + 1). \tag{405}$$

We have that $v(t) = 0$ at $t = 3$ and $t = -1$. (@ Draw number line.) This reveals *that $v(t) < 0$ for $2 \le t < 3$ and $v(t) > 0$ for $3 < t$. Therefore,*

$$
\begin{aligned}
\int_2^4 |v(t)|\ \mathrm{d}t = \int_2^3 |v(t)|\ \mathrm{d}t + \int_3^4 |v(t)|\ \mathrm{d}t = \int_2^3 -v(t)\ \mathrm{d}t + \int_3^4 v(t)\ \mathrm{d}t \\
= \int_2^3 -\left(t^2 - 2t - 3\right)\ \mathrm{d}t + \int_3^4 t^2 - 2t - 3\ \mathrm{d}t \\
= \left(-\frac{t^3}{3} + t^2 + 3t\Big|_2^3\right) + \left(\frac{t^3}{3} - t^2 - 3t\Big|_3^4\right) \\
= \left(-\frac{27}{3} + 9 + 9\right) - \left(-\frac{8}{3} + 4 + 6\right) + \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{27}{3} - 9 - 9\right) \\
= \frac{27}{3} - \frac{22}{3} - \frac{20}{3} + \frac{27}{3} = \frac{12}{3} = 4\ m. \quad (406)
\end{aligned}
$$

□

Example 18.4 *A bacteria population is 4000 at time $t = 0$ hours and its rate of growth at t hours is $r(t) = (1000)\, 2^t$ bacteria per hour. What is the population after 1 hour?*
The change in the bacteria population is

$$\int_0^1 r(t)\ \mathrm{d}t = \int_0^1 (1000)\, 2^t\ \mathrm{d}t = \frac{1000}{\ln 2} 2^t\Big|_0^1 = \frac{1000}{\ln 2}\left(2^1 - 2^0\right) = \frac{1000}{\ln 2} \approx 1442. \quad (407)$$

Since the original population is 4000, the new population is 5442 bacteria. □

Example 18.5 *The acceleration (in meters per second per second) of a particle moving along a straight line is*

$$a(t) = t^2 - 4t + 6. \quad (408)$$

The particle's position at $t = 0$ is $x(0) = 0$ m and its velocity at $t = 0$ is $v(0) = 1$ m/s.
(i) Find the velocity of the particle.
We know that $a(t) = v'(t)$, so

$$v(t) = \int a(t)\ \mathrm{d}t = \int t^2 - 4t + 6\ \mathrm{d}t = \frac{1}{3}t^3 - 2t^2 + 6t + C_1. \quad (409)$$

Since $v(0) = 1$,

$$1 = \frac{1}{3} - 2 + 6 + C_1, \tag{410}$$

so $C_1 = -\frac{10}{3}$. *Therefore,*

$$v(t) = \frac{1}{3}t^3 - 2t^2 + 6t - \frac{10}{3}. \tag{411}$$

(ii) Find the position of the particle.
We know that $v(t) = x'(t)$, *so*

$$x(t) = \int v(t)\,\mathrm{d}t = \int \frac{1}{3}t^3 - 2t^2 + 6t - \frac{10}{3}\,\mathrm{d}t = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 - \frac{10}{3}t + C_2. \tag{412}$$

We know that $x(0) = 0$, *so*

$$0 = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 - \frac{10}{3}t + C_2, \tag{413}$$

hence $C_2 = 0$. *Therefore,*

$$x(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 - \frac{10}{3}t. \tag{414}$$

□

Example 18.6 *Suppose a particle moving along a straight line has a constant acceleration* a. *In that case, its velocity is*

$$v(t) = \int a\,\mathrm{d}t = at + C_1. \tag{415}$$

Suppose $v(0) = v_0$, *the initial velocity. In that case,*

$$v_0 = v(0) = a(0) + C_1 = C_1, \tag{416}$$

and so

$$v(t) = v_0 + at. \tag{417}$$

Additionally,

$$x(t) = \int v(t)\,\mathrm{d}t = \int at + v_0\,\mathrm{d}t = \frac{1}{2}at^2 + v_0 t + C_2. \tag{418}$$

Suppose $x(0) = x_0$*, the initial position. In that case,*

$$x_0 = x(0) = \frac{1}{2}a(0)^2 + v_0(0) + C_2 = C_2, \tag{419}$$

and so

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2. \tag{420}$$

□

Section 5.5: The substitution rule

Theorem 18.7 *(u-substitution) Let f be a continuous function defined on the real line. If $u(x)$ is a differentiable function, then*

$$\int f\left(u\left(x\right)\right)\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x = \int f\left(u\right)\ \mathrm{d}u \tag{421}$$

Example 18.8 *Evaluate*

$$\int -2xe^{-x^2}\ \mathrm{d}x. \tag{422}$$

We can understand e^{-x^2} as e^u, where $u(x) = -x^2$. In that case, $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x$. Therefore, this is

$$\int e^u\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int e^u\ \mathrm{d}u = e^u + C. \tag{423}$$

Since $u(x) = -x^2$, this is $e^{-x^2} + C$. $\square$

Example 18.9 *Evaluate*

$$\int \sin^2\theta\cos\theta\ \mathrm{d}\theta. \tag{424}$$

If we say $u(x) = \sin\theta$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos\theta$. Therefore, this becomes

$$\int u^2\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int u^2\ \mathrm{d}u = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C. \tag{425}$$

$\square$

Example 18.10 *Evaluate*

$$\int \sqrt{2t+1}\ \mathrm{d}t. \tag{426}$$

Define $u = 2t+1$. In that case, $\frac{\mathrm{d}u}{\mathrm{d}t} = 2$. Notation:

$$\begin{aligned} u &= 2t+1 \\ \mathrm{d}u &= 2\ \mathrm{d}t \end{aligned}. \tag{427}$$

Therefore, $\frac{1}{2}\,\mathrm{d}u = \mathrm{d}t$*, and so*

$$\int \sqrt{u}\frac{1}{2}\,\mathrm{d}u = \frac{1}{2}\int u^{\frac{1}{2}}\,\mathrm{d}u = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2t+1)^{\frac{3}{2}} + C. \tag{428}$$

□

Example 18.11 *Evaluate*

$$\int \sec^2(2\theta)\ \mathrm{d}\theta. \tag{429}$$

We write

$$\begin{aligned} u &= 2\theta \\ \mathrm{d}u &= 2\,\mathrm{d}\theta \end{aligned}. \tag{430}$$

This becomes

$$\int \frac{1}{2}\sec^2 u\,\mathrm{d}u = \frac{1}{2}\tan u + C = \frac{1}{2}\tan(2\theta) + C. \tag{431}$$

□

Example 18.12 *Evaluate*

$$\int e^{\cos t}\sin t\,\mathrm{d}t. \tag{432}$$

Select

$$\begin{aligned} u &= \cos t \\ \mathrm{d}u &= -\sin t\,\mathrm{d}t \end{aligned}. \tag{433}$$

This becomes

$$-\int e^u\,\mathrm{d}u = -e^u + C = -e^{\cos t} + C. \tag{434}$$

□

Example 18.13 *Evaluate*

$$\int \frac{\cos(\ln t)}{t}\,\mathrm{d}t \tag{435}$$

Select

$$\begin{aligned} u &= \ln t \\ \mathrm{d}u &= \tfrac{1}{t}\,\mathrm{d}t \end{aligned}. \tag{436}$$

This becomes

$$\int \cos u \,\mathrm{d}u = \sin u + C = \sin(\ln t) + C. \tag{437}$$

□

Example 18.14 *Evaluate*

$$\int \frac{\mathrm{d}t}{\cos^2 t\sqrt{1+\tan t}}. \tag{438}$$

We can write this as

$$\int \sec^2 t \frac{1}{\sqrt{1+\tan t}} \,\mathrm{d}t. \tag{439}$$

If we select

$$\begin{aligned} u &= 1+\tan t \\ \mathrm{d}u &= \sec^2 t \,\mathrm{d}t \end{aligned}. \tag{440}$$

This becomes

$$\int \frac{1}{\sqrt{u}} \,\mathrm{d}u = \int u^{-\frac{1}{2}} \,\mathrm{d}u = 2\sqrt{u} + C = 2\sqrt{1+\tan t} + C. \tag{441}$$

□

Example 18.15 *Evaluate*

$$\int \frac{x}{1+x^4} \,\mathrm{d}x. \tag{442}$$

Select

$$\begin{aligned} u &= x^2 \\ \mathrm{d}u &= 2x \,\mathrm{d}x \end{aligned}. \tag{443}$$

This becomes

$$\frac{1}{2}\int \frac{1}{1+u^2} \,\mathrm{d}u = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}\left(x^2\right) + C. \tag{444}$$

□

For definite integrals, change the bounds when using u-substitution.

Example 18.16 *Evaluate*

$$\int_0^1 (3t-1)^{50} \,\mathrm{d}t. \tag{445}$$

Select

$$\begin{aligned} u &= 3t - 1 \\ \mathrm{d}u &= 3\ \mathrm{d}t \end{aligned}. \tag{446}$$

When $t = 0$, $u = -1$, and when $t = 1$, $u = 2$. Therefore, this becomes

$$\int_{-1}^{2} u^{50}\ \mathrm{d}u = \frac{1}{51} u^{51}\bigg|_{-1}^{2} = \frac{1}{51}\left((2)^{51} - (-1)^{51}\right) = \frac{1}{51}\left(2^{51} + 1\right). \tag{447}$$

□

Example 18.17 *Evaluate*

$$\int_{0}^{3} \frac{\mathrm{d}x}{5x+1}. \tag{448}$$

Select

$$\begin{aligned} u &= 5x + 1 \\ \mathrm{d}u &= 5\ \mathrm{d}x \end{aligned}. \tag{449}$$

When $x = 0$, $u = 1$ and when $x = 3$, $u = 16$. This becomes

$$\frac{1}{5}\int_{1}^{16} \frac{1}{u}\ \mathrm{d}u = \frac{1}{5}\ln|u|\bigg|_{1}^{16} = \frac{1}{5}\ln(16) = \frac{4}{5}\ln 2. \tag{450}$$

□

Example 18.18 *Evaluate*

$$\int_{0}^{4} \frac{x}{\sqrt{1+2x}}\ \mathrm{d}x. \tag{451}$$

Select

$$\begin{aligned} u &= 1 + 2x \\ \mathrm{d}u &= 2\ \mathrm{d}x \end{aligned}. \tag{452}$$

When $x = 0$, $u = 1$ and when $x = 4$, $u = 9$. This becomes

$$\begin{aligned} \frac{1}{2}\int_{1}^{9} \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\ \mathrm{d}u = \frac{1}{4}\int_{1}^{9} \frac{u-1}{u^{\frac{1}{2}}}\ \mathrm{d}u = \frac{1}{4}\int_{1}^{9} u^{\frac{1}{2}} - u^{-\frac{1}{2}}\ \mathrm{d}u \\ = \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\bigg|_{1}^{9}\right) = \frac{1}{4}\left(14 - \frac{2}{3}\right) = \frac{10}{3}. \end{aligned} \tag{453}$$