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HF = HM, V
Seiberg–Witten Floer homology and handle additions

ÇAĞATAY KUTLUHAN
YI-JEN LEE
CLIFFORD HENRY TAUBES

This is the last of five papers that construct an isomorphism between the Seiberg–Witten Floer homology and the Heegaard Floer homology of a given compact, oriented 3–manifold. See Theorem 1.4 for a precise statement. As outlined in paper I (Geom. Topol. 24 (2020) 2829–2854), this isomorphism is given as a composition of three isomorphisms. In this article, we establish the third isomorphism, which relates the Seiberg–Witten Floer homology on the auxiliary manifold with the appropriate version of Seiberg–Witten Floer homology on the original manifold. This constitutes Theorem 4.1 in paper I, restated in a more refined form as Theorem 1.1 below. The tool used in the proof is a filtered variant of the connected sum formula for Seiberg–Witten Floer homology, in special cases where one of the summand manifolds is $S^1 \times S^2$ (referred to as "handle-addition" in all five articles in this series). Nevertheless, the arguments leading to the aforementioned connected sum formula are general enough to establish a connected sum formula in the wider context of Seiberg–Witten Floer homology with nonbalanced perturbations. This is stated as Proposition 6.7 here. Although what is asserted in this proposition has been known to experts for some time, a detailed proof has not appeared in the literature, and therefore of some independent interest.

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1 Introduction

To summarize what was done in the predecessors to this article [19; 20; 21; 22]: the first article in this series outlined a program for a proof of Theorem 1.4, based on a concatenation of three isomorphisms. The first isomorphism [19, Theorem 2.3] relates a version of embedded contact homology on an auxiliary manifold to the Heegaard Floer homology on the original, and was accomplished in [20; 21]. The second isomorphism [19, Theorem 3.4] relates the relevant version of the embedded contact homology on the auxiliary manifold and a version of the Seiberg–Witten Floer homology on this same manifold. This was established in [22]. This last installment of the HM $=$ HF series contains the proof of the third isomorphism, stated as Theorem 4.1 in [19]. Part of the content of this paper is drawn from unpublished details of the proof of the second author's Corollary 8.4 in [23], which describes the behavior of certain Seiberg–Witten Floer homology under handle addition.

1.1 The main theorem and an outline of proof

Let M be a closed, connected and oriented 3–manifold. Given a Spin^c structure $\mathfrak{s}$ on M, P B Kronheimer and T S Mrowka defined in [17] three flavors of Seiberg–Witten Floer homology, $\widehat{\mathrm{HM}}_*$, $\overline{\mathrm{HM}}_*$ and $\widecheck{\mathrm{HM}}_*$, modeling on three different versions of S^1–equivariant homologies. These homology groups have the structure of modules over the graded ring

$$A_\dagger(M) := \mathbb{Z}[U] \otimes \textstyle\bigwedge^*(H_1(M;\mathbb{Z})/\mathrm{Tors}),$$

where U has degree -2 and elements in $H_1(M;\mathbb{Z})/\mathrm{Tors}$ have degree -1. These modules are graded by an affine space over $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$, where $c_{\mathfrak{s}} \in 2\mathbb{Z}^{\geq 0}$ is the divisibility of $c_1(\mathfrak{s})$, the first Chern class of the Spin^c structure $\mathfrak{s}$. Moreover, as $A_\dagger(M)$–modules, these three flavors of Seiberg–Witten Floer homologies fit into a long exact sequence modeling on the fundamental exact sequence of S^1–equivariant Floer homologies (see equation (3.4) in [17])

$$\cdots \to \widehat{\mathrm{HM}}_* \to \overline{\mathrm{HM}}_* \to \widecheck{\mathrm{HM}}_* \to \cdots . \tag{1-1}$$

This is called the *first fundamental exact sequence of* HM in this article. In [23], the second author defined a fourth flavor of Seiberg–Witten Floer homology, $\widetilde{\mathrm{HM}}_*$, with the same module structure and relative grading. (It was originally denoted by $\mathrm{HM}^{\mathrm{tot}}$ in [23], given here as Definition 5.6.) The definition models on the ordinary homology of an S^1–space. As such, it fits into a second long exact sequence together with $U\widehat{\mathrm{HM}}_*$ and $\widehat{\mathrm{HM}}_*$. This is referred to as the *second fundamental exact sequence of* HM; see Lemma 5.7 below.

In this article, we regard these four flavors of HM as a system, in the order of $\widehat{\mathrm{HM}}_*$, $\overline{\mathrm{HM}}_*$, and $\widecheck{\mathrm{HM}}_*$, $\widetilde{\mathrm{HM}}_*$. They are denoted collectively by $\mathring{\mathrm{HM}}_*$.

As will be detailed in the upcoming Section 2, the Seiberg–Witten Floer homology (also referred to as the monopole Floer homology in this article) $\mathring{\mathrm{HM}}_*$ depends on the cohomology class of the perturbation form ϖ in addition to the Spin^c–structure $\mathfrak{s}$. One may also define a monopole Floer homology with local coefficients Γ compatible with $\mathfrak{s}$ and $[\varpi]$. Of particular interest to us is the case when the perturbation is "balanced", in this case Γ may be taken to be $\mathbb{Z}$. These are denoted by $\mathring{\mathrm{HM}}_*(M,\mathfrak{s},c_b)$; and this is the variant of monopole Floer homology to be equated with the Heegaard Floer homology HF°_*, in Theorem 1.4 below. This is, in a sense, the strongest possible statement of equivalence between HM and HF, as the monopole Floer homology $\overline{\mathrm{HM}} \neq 0$ and $\widehat{\mathrm{HM}} \neq \widecheck{\mathrm{HM}}$ only in the balanced case. The equivalence between other versions of HM and HF may be deduced from this case through the use of local coefficients. It is also worth mentioning that a coarser version of Seiberg–Witten Floer homology, $\mathrm{HM}_\bullet$, defined by taking a completion of the Floer complex with respect to grading,[1] frequently appears in [17] and other literature. In this article we work exclusively with the original version, $\mathring{\mathrm{HM}}_*$.

The upcoming Theorem 1.1 relates $\mathring{\mathrm{HM}}_*(M,\mathfrak{s},c_b)$ with two filtered variants of monopole Floer homology. The first was introduced in [23], originally denoted by HMT° therein. Here, the label $\circ$ stands, in specific order, for $-,\infty,+,\wedge$. The fact that they appear in the superscript (instead of the top) of the notation, and the order in which they appear, reflects the nature of their definition. The latter is done following the algebraic framework of Ozsváth and Szabó in [29]. The second of these two variants was introduced in [22] (see also Section 4 of [19] for a brief summary). They are denoted by $\mathrm{H}^\circ_*(Y)$ in [19], and by $\mathrm{H}^\circ_{\mathrm{SW}}$ in [22]. The construction of both these filtered monopole Floer homologies is based on the same general framework, which we describe in Section 3 below. This framework always produces four flavors of Floer homologies, labeled by $\circ = -,\infty,+,\wedge$; and they are related by two fundamental long exact sequences parallel to those appearing in the Heegaard Floer theory; see (1-6) below. (To be more precise, only the first three flavors appeared in [19; 22], but it shall

[1] After completion $\overline{\mathrm{HM}}$ often becomes trivial even in the balanced case; see eg [17, page 685].

become clear in Section 3 that the aforementioned general construction actually gives rise to a fourth flavor). The basic ingredient of this construction consists of a triple of data: a certain Spin^c 3–manifold Y_Z, a closed 2–form w on Y_Z used to define a monotone perturbation to the Seiberg–Witten equations, and a special 1–cycle γ embedded in Y_Z useful for defining a filtration on the associated monopole Floer complex. Further constraints on the choice of this triple are given in Section 3.2.

The triple that enters the definition of HMT° is what was denoted by $(\underline{M}, *d\underline{f}, \underline{\gamma})$ in [23]. Here, $\underline{M}$ is constructed from M by adding a 1–handle[2] along the extrema of f, the latter being a Morse function giving rise to the Heegaard diagram used to define HF°. Denote this 1–handle by $\mathcal{H}_0$. What was denoted by $\underline{f}$ is an S^1–valued harmonic Morse function obtained by a natural extension of f. The 1–circle $\underline{\gamma}$ is related to the path $\gamma_z \subset M$ used by Ozsváth and Szabó to define a filtration on the Heegaard Floer complex. The triple used for the definition of $\mathrm{H}^\circ_{\mathrm{SW}}$ in [22] was denoted by $(Y, w, \gamma^{(z_0)})$ in [20; 22]. The 3–manifold Y is obtained from $\underline{M}$ by attaching additional 1–handles along pairs of index 1 and index 2 critical points of f. The 2–form w on Y is constructed from a natural extension of $*df$. The 1–cycle $\underline{\gamma}$ in $\underline{M}$ becomes the 1–cycle $\gamma^{(z_0)}$ in Y after the handle-attachment. The precise definitions of HMT° and $\mathrm{H}^\circ_{\mathrm{SW}}$ may be found in Section 3.8. By construction, HMT° and $\mathrm{H}^\circ_{\mathrm{SW}}$ are, respectively, $\boldsymbol{A}_\dagger(\underline{M})$– and $\boldsymbol{A}_\dagger(Y)$–modules, and each is equipped with a pair of fundamental exact sequences parallel to (1-6).

Let G denote the number of 1–handles added to $\underline{M}$ in order to obtain Y and denote these handles by $\mathcal{H}_\mathfrak{p}$ for $\mathfrak{p} \in \Lambda$, where the label set Λ is an ordered set consisting of G elements. Recall that $\gamma_z \subset M$ is defined so that $\partial\gamma_z$ is the attaching 0–cycle of $\mathcal{H}_0$. As described in [23], the path γ_z determines a decomposition of $\underline{M}$ as a connected sum $\underline{M} \simeq M \# (S^1 \times S^2)$ (see [23, equation (15)]), and hence a splitting

$$H_1(\underline{M}; \mathbb{Z}) \simeq H_1(M; \mathbb{Z}) \oplus H_1(S^1 \times S^2; \mathbb{Z}), \tag{1-2}$$

with the second summand generated by $[\underline{\gamma}] \in H_1(\underline{M}; \mathbb{Z})$. Correspondingly, this determines a factorization of the algebra

$$\boldsymbol{A}_\dagger(\underline{M}) \simeq \boldsymbol{A}_\dagger(M) \otimes_{\mathbb{Z}[U]} \boldsymbol{A}_\dagger(S^1 \times S^2) = \boldsymbol{A}_\dagger(M) \otimes \textstyle\bigwedge^* H_1(S^1 \times S^2; \mathbb{Z}). \tag{1-3}$$

The last factor above, $\bigwedge^* H_1(S^1) = \bigwedge^* H_1(S^1 \times S^2; \mathbb{Z})$, has a natural action on its dual algebra $\bigwedge^*(H^1(S^1))$. The latter is regarded as a graded $\mathbb{Z}$–algebra generated by two elements, one of degree 0 and the other of degree 1. This was denoted by $\widehat{V}$

[2] See item (7) in Section 1.3.

in [19] and by $H_*(S^1)$ in the rest of this article (see item (6) of Section 1.3 below). For this reason we shall use the shorthand $H_{-*}(S^1)$ for the factor $\bigwedge^* H_1(S^1\times S^2;\mathbb{Z})$ in (1-3), and the aforementioned dual action is implied whenever we refer to "the $H_{-*}(S^1)$–action on $H_*(S^1)$" below.

The auxiliary manifold Y may be decomposed as a connected sum of $\underline{M}$ and G copies of $S^1\times S^2$, one for each of the 1–handles $\mathcal{H}_{\mathfrak{p}}$, in a similar manner: For each $\mathfrak{p}\in\Lambda$, we fix an arc $\lambda_{\mathfrak{p}}$ in M connecting the attaching 0–cycle of $\mathcal{H}_{\mathfrak{p}}$. Let $S_{\mathfrak{p}}$ denote the boundary sphere of a small tubular neighborhood of $\lambda_{\mathfrak{p}}$, and use the same notation for the corresponding sphere in Y. The precise description of $\lambda_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ is given in Part 1 of Section 9.5 below. Now split Y along these spheres $S_{\mathfrak{p}}$ to get the aforementioned connected sum, and use this to define a splitting

$$H_1(Y;\mathbb{Z})\simeq H_1(\underline{M};\mathbb{Z})\oplus\bigoplus_{\mathfrak{p}\in\Lambda}H_1((S^1\times S^2)_{\mathfrak{p}};\mathbb{Z}),\tag{1-4}$$

where $(S^1\times S^2)_{\mathfrak{p}}$ denotes the copy of $S^1\times S^2$ coming from $\mathcal{H}_{\mathfrak{p}}$. This in turn determines a factorization

$$\boldsymbol{A}_\dagger(Y)\simeq\boldsymbol{A}_\dagger(\underline{M})\otimes_{\mathbb{Z}[U]}\bigotimes_{\mathfrak{p}\in\Lambda}\boldsymbol{A}_\dagger((S^1\times S^2)_{\mathfrak{p}})=\boldsymbol{A}_\dagger(\underline{M})\otimes H_{-*}(S^1)^{\mathrm{G}}\tag{1-5}$$

like (1-3).

The main theorem of this article relates the three versions of monopole Floer homologies: $\mathring{\mathrm{HM}}(M,\mathfrak{s},c_b)$, HMT° and $\mathrm{H}^\circ_{\mathrm{SW}}=\mathrm{H}^\circ(Y)$.

Theorem 1.1 (1) *Use* $\mathrm{HMT}^\circ\boxtimes H_*(S^1)^{\boxtimes\mathrm{G}}$ *to denote the external tensor product*[3] *of the* $\boldsymbol{A}_\dagger(\underline{M})$*–module* HMT° *and* G *copies of the* $H_{-*}(S^1)$*–module* $H_*(S^1)$*. With respect to the factorization* (1-5)*, there exists a system of isomorphisms of* $\boldsymbol{A}_\dagger(Y)=\boldsymbol{A}_\dagger(\underline{M})\otimes H_{-*}(S^1)^{\otimes\mathrm{G}}$*–modules*

$$\mathrm{H}^\circ(Y)\xrightarrow{\simeq}\mathrm{HMT}^\circ\boxtimes H_{-*}(S^1)^{\boxtimes\mathrm{G}},\quad\circ=-,\infty,+,\wedge,$$

which preserves the relative gradings and is natural with respect to the fundamental long exact sequences on both sides.

[3] When the coefficient is left unspecified, the tensor product notation $\otimes$ implicitly refers to $\otimes_{\mathbb{Z}}$. Given two ($\mathbb{Z}$–)algebras A and B, an A–module M and a B–module N, $M\boxtimes N$ denotes $M\otimes N$ viewed as an $A\otimes B$–module.

(2) *The* $H_{-*}(S^1)$ *factor of the factorization* $\boldsymbol{A}_\dagger(\underline{M}) = \boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)$ *in* (1-3) *acts trivially on* HMT. *Regarding* HMT *as an* $\boldsymbol{A}_\dagger(M)$*–module in this manner, there exists a system of isomorphisms of* $\boldsymbol{A}_\dagger(M)$*–modules from*

$$\mathrm{HMT}^\circ,\ \circ = -, \infty, +, \wedge, \quad \textit{to} \quad \overset{\circ}{\mathrm{HM}}(M, \mathfrak{s}, c_b),\ \circ = \wedge, -, \vee, \sim,$$

respectively, that preserves the relative gradings and is natural with respect to the fundamental long exact sequences on both sides.

The proof of this theorem is given in Section 6.3. The remainder of this section gives a brief outline of this proof.

Given how Y is constructed from $\underline{M}$, and $\underline{M}$ in turn from M, it is little surprise that the preceding theorem is a consequence of a certain filtered variant of the connected sum formula for Seiberg–Witten Floer homologies. See Proposition 6.11 in Section 6.3. The first steps of the proof of this formula, via understanding the chain maps on Seiberg–Witten Floer complexes induced by cobordisms associated to the connected sum, lead to a connected sum formula for Seiberg–Witten Floer homologies *sans filtration*. This is stated as Proposition 6.7 below.

The more essential part of the proof, which also constitutes the major technical component of this article, consists of an extension of the framework defining HMT° and $\mathrm{H}^\circ(Y)$ to the context of cobordisms and their associated chain maps. The analytical foundation of such an extension is provided in Sections 7–9 of this article.

The proof of part (2) of Theorem 1.1 also involves some homological algebra computation that turns out to be a manifestation of so-called "Koszul duality". An elementary account of the relevant part of this story is given in Section 4. This algebraic machinery expresses all four flavors of the balanced monopole Floer homology, $\overset{\circ}{\mathrm{HM}}(M, \mathfrak{s}, c_b)$ in terms of a balanced monopole Floer complex of the first flavor, $\widehat{\mathrm{CM}}_*(M, \mathfrak{s}, c_b)$. Meanwhile, the filtered connected sum formula previously mentioned expresses all four flavors of HMT° in terms of a monopole Floer complex with "negative monotone" perturbation, $\mathrm{CM}_*(M, \mathfrak{s}, c_-)$. See Proposition 5.9 below. These two monopole Floer complexes are linked via a chain-level variant of the following result of Kronheimer and Mrowka:

Theorem 1.2 [17, Theorem 31.5.1] *Suppose* $c_1(\mathfrak{s})$ *is not torsion. Then*

$$\widehat{\mathrm{HM}}_*(M, \mathfrak{s}, c_b) \simeq \mathrm{HM}_*(M, \mathfrak{s}, c_-).$$

The right-hand side of the preceding isomorphism refers to the monopole Floer homology for negative monotone perturbations. A brief account of this variant of monopole Floer homology can be found in Section 2.3. The construction of both $\mathrm{H}^{\circ}_{\mathrm{SW}}$ and HMT° are based on negative monotone monopole Floer complexes.

More on the motivation for various constructions in the article may be found in [23].

Remark 1.3 With the hindsight gained from Juhasz's [14] and Kronheimer and Mrowka's [18] definitions of sutured Floer homologies, we feel that HMT° are best interpreted as variants of sutured Floer homology. In particular, $\mathrm{HM}(M(1), \mathfrak{s}(1)) = \widehat{\mathrm{HMT}}(\underline{M}, \underline{\mathfrak{s}})$ in terms of the notation in [14; 18; 23]. From this point of view, Theorem 1.1(2) may be viewed as a reinterpretation of monopole Floer homology of closed 3–manifolds as (generalized) sutured Floer homology. In particular, the $\circ = \wedge$ variant of this statement is a Seiberg–Witten analog of Proposition 2.2 in [14], where the hat version of the Heegaard Floer homology is reinterpreted as a sutured Floer homology. See also Theorem 1.6 announced in [5] for an ECH analog (of the $\circ = \wedge$ variant). We hope to discuss this in more detail elsewhere. (See also the end of Part 4 in Section 9.1.)

1.2 Relating Heegaard and Seiberg–Witten Floer homologies

With all said and done, the main result here combines with those in [19; 20; 21; 22] to reach our ultimate goal:

Theorem 1.4 *Let M be a closed, oriented 3–manifold, and $\mathfrak{s}$ be a Spin^c–structure on M. Then there exists a system of isomorphisms from $\mathrm{HF}^{\circ}_{*}(M, \mathfrak{s})$ for $\circ = -, \infty, +, \wedge$ to $\overset{\circ}{\mathrm{HM}}_{*}(M, \mathfrak{s}, c_b)$ for $\circ = \wedge, -, \vee, \sim$, respectively, as $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$–graded $A_{\dagger}(M)$–modules, which is natural with respect to the fundamental exact sequences of the Heegaard and monopole Floer homologies.*

The result summaries the relation between the Heegaard and monopole Floer homologies, which has been conjectured since the inception of Heegaard Floer theory. See for example Conjecture 1.1 in [28], Section I.3.12 in [17], Conjecture 1 in [16] and Conjecture 1.1 in [23].

As the Heegaard Floer homology HF° makes no other appearances for the rest of this article, the reader is referred to [29; 28] for its definition and properties. In particular, the fundamental exact sequences relating its four flavors take the form

$$\cdots \to \mathrm{HF}^{-} \to \mathrm{HF}^{\infty} \to \mathrm{HF}^{+} \to \cdots, \quad \cdots \to \mathrm{HF}^{-} \xrightarrow{U} \mathrm{HF}^{-} \to \widehat{\mathrm{HF}} \to \cdots. \tag{1-6}$$

Proof of Theorem 1.4 An outline of the proof is already given in [19]. To summarize, by combining the two parts of Theorem 1.1, one has (see Theorem 4.1 in [19])

$$\mathrm{H}^\circ(Y) \simeq \mathring{\mathrm{HM}}(M, \mathfrak{s}, c_b) \boxtimes H_{-*}(S^1)^{\boxtimes \mathrm{G}}, \tag{1-7}$$

as modules over the algebra $\mathbb{A}_\dagger(M) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$. Here, the $\mathbb{A}_\dagger(M) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$–structure on $\mathrm{H}^\circ(Y)$ comes from the latter's $\mathbb{A}_\dagger(Y)$–module structure via the inclusion

$$\mathbb{A}_\dagger(M) \otimes 1 \otimes H_{-*}(S^1)^{\otimes \mathrm{G}} \hookrightarrow \mathbb{A}_\dagger(M) \otimes H_{-*}(S^1) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}} \simeq_{i_{\mathrm{sum}}} \mathbb{A}_\dagger(Y) \tag{1-8}$$

with respect to the factorization combining (1-3) and (1-5).

It is asserted in Theorem 3.4 of [19] and proven in Theorem 1.5 of [22] that the left-hand side of (1-7), $\mathrm{H}^\circ(Y)$, is isomorphic to what was called "ech°" as $\mathbb{A}_\dagger(Y)$–modules. The ech° chain complex, as well as a (particular choice of) $\mathbb{A}_\dagger(Y)$–action on it, is explicitly described in [20; 21]. A computation based on this explicit description yields:

Proposition 1.5 (see also Theorem 2.4 of [19]) *There is a system of isomorphisms*

$$\mathrm{ech}^\circ \simeq \mathrm{HF}^\circ(M, \mathfrak{s}) \boxtimes H_*(S^1)^{\boxtimes \mathrm{G}}$$

as modules over $\mathbb{A}_\dagger(M) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$, which preserves relative gradings and is natural with respect to the fundamental exact sequences on both sides. Here, the $\mathbb{A}_\dagger(M) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$–structure on ech° also refers to the one induced from the latter's $\mathbb{A}_\dagger(Y)$–module structure via the same inclusion (1-8).

The proof of this proposition involves some details of [21]'s description of the $\mathbb{A}_\dagger(Y)$–actions on ech°, as well as some particular choice of the arcs $\lambda_{\mathfrak{p}}$ used to define the factorization (1-5), and will be postponed to Section 9.6.

The assertion of the theorem is a direct consequence of the composition of the three isomorphisms from the preceding proposition, (1-7) and Theorem 1.5 of [22] (which is Theorem 3.4 of [19]). □

1.3 Some notation and conventions

Throughout the remainder of this paper, section numbers, equation numbers, and other references from [19; 20; 21; 22] are distinguished from those in this paper by the use of the appropriate Roman numeral as a prefix. For example, "Section II.1" refers to Section 1 in [20]. In addition, the following conventions are used:

(1) As in [19; 20; 21; 22], we use c_0 to denote a constant in $(1, \infty)$ whose value is independent of all relevant parameters. The value of c_0 can increase between subsequent appearances.

(2) As in [19; 20; 21; 22], we denote by χ a fixed, nonincreasing function on $\mathbb{R}$ that equals 1 on a neighborhood of $(-\infty, 0]$ and equals 0 on a neighborhood of $[1, \infty)$.

(3) When left unspecified, the modules, chain complexes and homologies in this article are over the coefficient ring $\mathbb{K}$, which can be taken to be $\mathbb{Z}$, as was done in [19; 20; 21; 22]. Using a separate notation serves to distinguish different roles the abelian group $\mathbb{Z}$ plays in this article, eg as the group of deck transformations versus the coefficient ring of the chain complexes.

(4) The term "module" in this article refers to either a left module or a right module. Thus, both the monopole Floer homology and monopole Floer cohomology are said to have a module structure over the ring $H^*(BS^1)$. Note in contrast that in [17], a "module" refers specifically to a left module. Moreover, what appears as $U_\dagger$ in [17] is denoted by U in this article for simplicity, since we focus on Floer homology as opposed to cohomology.

(5) The definition of Floer complexes in this article often depends on several parameters, yet there are chain homotopies relating the Floer complexes with the values of some of the parameters changed. In the interest of simplicity, these parameters are usually left unspecified in our notation for the Floer complexes unless necessary.

(6) Due to geometric motivations (see [10]), we view $H_*(S^1)$ and $H^*(BS^1)$ both as free commutative differential graded algebras with zero differential and a single generator, where the odd generator y for $H_1(S^1)$ has degree 1, while the even generator u for $H^*(BS^1)$ has degree -2. In this paper commutativity and the commutator $[\cdot,\cdot]$ are meant in the *graded* sense. In particular, what is called an "antichain map" in [17] is in our terminology an odd chain map. If necessary, we use notation $[\cdot,\cdot]_{\mathrm{odd}}$ or $[\cdot,\cdot]_{\mathrm{even}}$ to emphasize the parity of the commutator. When $H_*(S^1)$ is written as a polynomial algebra in y, $\mathbb{Z}[y]$, $H_{-*}(S^1)$ is often written as $\mathbb{Z}[\partial_y]$, to reflect the action of $H_{-*}(S^1)$ on $H_*(S^1)$.

(7) In this article as well as its prequels, a "1–handle" frequently refers to $[0,1]\times S^2$, and "attaching a 1–handle to a 3–manifold" refers to a 0–dimensional surgery on the 3–manifold.

(8) In the context of fiber bundles over a fixed base manifold, $\underline{\mathbb{F}}$ typical stands for a trivial bundle with fibers $\mathbb{F}$.

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2 Elements of Seiberg–Witten Floer theory

This subsection reviews some background on Seiberg–Witten Floer theory, with the book [17] as the definitive reference. By way of this, we introduce some notation and terminology used in the rest of this article, some of which differ from those in [17]. We focus mostly on the special cases involved in the proof of Theorem 1.1, leaving the general details for the reader to consult [17]. Many notions here have analogs in eg [22; 25], which work with similar settings.

2.1 Seiberg–Witten equations on 3–manifolds

Let M be a closed, oriented, Riemannian 3–manifold. Fix a Spin^c–structure $\mathfrak{s}$ on M and let $\mathbb{S}$ denote its associated spinor bundle. We call a pair, $(\mathbb{A}, \Psi)$, consisting of a Hermitian connection on $\det \mathbb{S}$ and a section of $\mathbb{S}$ a (Seiberg–Witten) *configuration*. The gauge group $C^\infty(M; U(1))$ acts on the space of configurations in the following fashion: Let $\hat{u} : M \to U(1)$. Then $\hat{u}$ sends a configuration, $(\mathbb{A}, \Psi)$, to $(\mathbb{A} - 2\hat{u}^{-1} d\hat{u}, \hat{u}\Psi)$. Two solutions obtained one from the other in this manner are said to be *gauge-equivalent*. Note that this $C^\infty(M; U(1))$–action is free except at pairs of the form $(\mathbb{A}, \Psi = 0)$; these are called *reducible* configurations. Configurations which are not reducible are *irreducible*.

In the most general form, the 3–dimensional Seiberg–Witten equations ask that a configuration $(\mathbb{A}, \Psi)$ obey

$$B_{\mathbb{A}} - \Psi^\dagger \tau \Psi + i\varpi - \mathfrak{T} = 0 \quad \text{and} \quad D^{\mathbb{A}}\Psi - \mathfrak{S} = 0, \tag{2-1}$$

where $B_{\mathbb{A}}$ denotes the Hodge dual of the curvature form of $\mathbb{A}$, $D^{\mathbb{A}}$ denotes the Dirac operator and the quadratic term $\Psi^\dagger\tau\Psi$ is as in Section 1.2 of [25]; ϖ is a closed 2–form, and the pair $(\mathfrak{T},\mathfrak{S})$ is a small perturbation arising as the formal gradient of a gauge-invariant function of $(\mathbb{A},\Psi)$. This is called a tame perturbation in [17], and is in general needed to guarantee the transversality properties necessary for the definition of Seiberg–Witten Floer homology. See Chapters 10 and 11 in [17]. In the simplest case, $(\mathfrak{T},\mathfrak{S})$ may be taken to be of the form

$$(\mathfrak{T},\mathfrak{S}) = (2i *d\mu, 0) \tag{2-2}$$

for a smooth 1–form μ taken from a Banach space called Ω in [22]. This may be assumed to be a subspace of the Banach space of tame perturbations in Chapter 11.6 in [17], and hence inherits the so-called "$\mathcal{P}$–norm" from [17]. This norm bounds the norms of the derivatives of μ to any given order.

Irreducible solutions to (2-1) may exist only when the cohomology class is $[\varpi] = 2\pi c_1(\det\mathbb{S})$. In this case the Seiberg–Witten equations (2-1) is said to have *balanced perturbation*, while it is said to have *exact perturbation* when $[\varpi]=0$. The cases when $[\varpi]=2\mathrm{r}c_1(\det\mathbb{S})$ is said to be *monotone*: when $\mathrm{r}>\pi$ it is said to be *negative monotone*, and when $\mathrm{r}<\pi$ it is said to be *positive monotone*. Note that when $c_1(\det\mathbb{S})$ is torsion, the notions of balanced, exact, and positive or negative monotone perturbations are equivalent. We work in the negative monotone case with nontorsion $c_1(\det\mathbb{S})$ for most of this paper where all Seiberg–Witten solutions are irreducible. Note in contrast that in the closely related series of articles [35; 36; 37; 38; 39], ϖ is taken to be da for a contact 1–form a, which is an exact perturbation.

This said, unless otherwise specified, from now on we set

$$\varpi = 2\mathrm{r}w \tag{2-3}$$

for a closed 2–form w in the cohomology class of $c_1(\det\mathbb{S})$ and a real number $\mathrm{r}>\pi$. When $c_1(\det\mathbb{S})$ is torsion, we always set $w\equiv 0$. Otherwise, the particulars of w for the proof of our main result, Theorem 1.1, are described in Section 3.2.

To make contact with the notation in [22], write

$$\det\mathbb{S} = E^2\otimes K^{-1} \tag{2-4}$$

with $K\to M$ being a fixed complex line bundle. Fix a smooth connection, A_K, on K^{-1}. Where w is nowhere-vanishing (such as over the stable Hamiltonian manifold Y in [22]), K^{-1} is typically given by $\mathrm{Ker}(*w)\subset TM$ and E the $i|w|$–eigenbundle of the Clifford action by w. More constraints on the choice of K and A_K will be specified along the way through the rest of this article.

With A_K chosen, let A denote the connection on the E–summand corresponding to $\mathbb{A}$, and write $\Psi = \sqrt{2\mathrm{r}}\,\psi$. In this case, perturbations of the form (2-2) suffice for our purpose. Since the Riemannian metric and a connection on E determine a Spin^c–connection on $\mathbb{S}$, we often consider the equivalent equations for (A,ψ) of the form

$$\begin{cases} B_A - \mathrm{r}(\psi^\dagger\tau\psi - i *w) + \frac{1}{2}B_{A_K} - i *d\mu = 0, \\ D_A\psi = 0, \end{cases} \tag{2-5}$$

where $D_A = D^{\mathbb{A}}$, B_A is the Hodge star of the curvature 2–form of A and B_{A_K} denotes the Hodge star of the curvature 2–form for the connection, A_K.

Given a Hermitian line bundle $V \to M$, we use $\mathrm{Conn}(V)$ to denote the space of Hermitian connections on V. The equations in (2-1) are the variational equations of the functional $\mathfrak{a}$ of $(A,\psi) \in \mathrm{Conn}(E) \times C^\infty(M;\mathbb{S})$, given by

$$\mathfrak{a} = \tfrac{1}{2}\mathfrak{cs} - \mathrm{r}\mathrm{W} + \mathfrak{e}_\mu + \mathrm{r}\int_M \psi^\dagger D_A\psi, \tag{2-6}$$

where the notation is as follows: The functions $\mathfrak{cs}$ and W are defined using a chosen reference connection on E. Let A_E denote the latter. With A written as

$$A = A_E + \hat{a}_A,$$

then W and $\mathfrak{cs}$ are given by

$$\mathrm{W} = i\int_M \hat{a}_A \wedge w \quad \text{and} \quad \mathfrak{cs} = -\int_M \hat{a}_A \wedge d\hat{a}_A - 2\int_{Y_Z} \hat{a}_A \wedge \big(F_{A_E} + \tfrac{1}{2}F_{A_K}\big). \tag{2-7}$$

What is denoted by $\mathfrak{e}_\mu$ is the integral over M of $i\mu\wedge F_A$. The functionals $\mathfrak{a}$, W and $\mathfrak{cs}$ in general are not invariant under the $C^\infty(M;U(1))$–action on $\mathrm{Conn}(\det\mathbb{S})\times C^\infty(M;\mathbb{S})$, however their differentials descend to the orbit space. These differentials are henceforth denoted by $d\mathfrak{a}$, $d(\mathfrak{cs})$, …, etc.

To define the Seiberg–Witten Floer homology in general, Kronheimer and Mrowka [17] take a real blowup of the space $\mathrm{Conn}(\det\mathbb{S}) \times C^\infty(M;\mathbb{S}) =: \mathcal{C}(M)$ along the set of reducibles (see Chapter 6 of [17]). This blown-up space is denoted by $\mathcal{C}^\sigma(M,\mathfrak{s})$ therein and has a free $C^\infty(M,U(1))$–action (see [17, page 115]). The vector field dual to $d\mathfrak{a}$ extends to $\mathcal{C}^\sigma$, which is then used to define the Seiberg–Witten equations (see [17, Section 6.2]).

A solution $\mathfrak{c}$ to the Seiberg–Witten equations or its corresponding gauge-equivalence class $[\mathfrak{c}] \in \mathcal{C}^\sigma(M,\mathfrak{s})/C^\infty(M,U(1))$ is said to be *nondegenerate* when a certain differential operator $\mathfrak{L}_\mathfrak{c}$ has trivial kernel. The explicit form of this operator is given for irreducible solutions of (2-1) in (7-36) below. In general, this notion of nondegeneracy arises from the interpretation of $[\mathfrak{c}]$ as a zero of the 1–form $d\mathfrak{a}$ on $\mathcal{C}^\sigma(M,\mathfrak{s})/C^\infty(M,U(1)) =: \mathcal{B}^\sigma(M)$. With the metric and ϖ fixed, a choice of $(\mathfrak{T},\mathfrak{S})$ (or, in the case of (2-5), of μ) such that all solutions to (2-1) or (2-5) are nondegenerate is said in what follows to be *suitable.* In the negative monotone case with nontorsion $c_1(\det\mathbb{S})$, a suitable choice for μ can be found with $\mathcal{P}$–norm bounded by any given positive number (see eg (1.18) in [22]). Otherwise, especially when reducible solutions exist, a suitable pair $(\mathfrak{T},\mathfrak{S})$ is typically of more general form than that of (2-2). Nondegenerate gauge-equivalence classes of reducible Seiberg–Witten solutions are further classified into the "stable" and "unstable" types in [17].

2.2 Seiberg–Witten equations on 4–dimensional cobordisms

Let Y_- and Y_+ be closed oriented 3–manifolds. In this paper X will denote a simple cobordism from Y_- to Y_+ of the following sort: X is an oriented complete 4–manifold equipped with the extra structure listed below:

(2-8)
- There is a proper function $s\colon X\to\mathbb{R}$ with nondegenerate critical points with at most one single critical value, 0.
- There exists an orientation-preserving diffeomorphism between the $s<0$ part of X and $(-\infty,0)\times Y_-$ that identifies s with the Euclidean coordinate on the $(-\infty,0)$ factor.
- There exists an orientation-preserving diffeomorphism between the $s>0$ part of X and $(0,\infty)\times Y_+$ that identifies s with the Euclidean coordinate on the $(0,\infty)$ factor.
- There is an even class in $H^2(X;\mathbb{Z})$ that restricts to the $s<0$ and $s>0$ parts of X as the respective Y_- and Y_+ versions of $c_1(\det\mathbb{S})$.

The diffeomorphism in the second bullet of (2-8) is used, often implicitly, to identify the $s<0$ part of X with $(-\infty,0)\times Y_-$; and the diffeomorphism in the third bullet of (2-8) is likewise used to identify the $s>0$ part with $(0,\infty)\times Y_+$. Fix a class satisfying the last bullet of (2-8) and denote it also by $c_1(\det\mathbb{S})$.

Assume that the Riemannian metric on X satisfies the following:

(2-9) • There exists $L \geq 100$ such that the metric on the $s \leq -L$ and $s \geq L$ parts of X are identified by the embeddings in the second and third bullets of (2-8) with the respective product metrics on $(-\infty, -L] \times Y_-$ and $[L, \infty) \times Y_+$.

• The metric pulls back from the $|s| \in [L-8, L]$ part of X via the embeddings from the second and third bullets of (2-8) as the quadratic form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ being an s–dependent metric on either Y_- or Y_+ as the case may be.

The chosen metric on X is used to write $\bigwedge^2 T^*X$ as $\Lambda^+ \oplus \Lambda^-$ with Λ^+ denoting the bundle of self-dual 2–forms and with Λ^- denoting the corresponding bundle of anti-self-dual 2–forms. A given 2–form $\mathfrak{w}$ is written with respect to this splitting as $\mathfrak{w} = \mathfrak{w}^+ + \mathfrak{w}^-$.

Use the metric to define the notion of a Spin^c–structure on X. It follows from the last bullet in (2-8) that there is a Spin^c structure that restricts to the $s \leq -2$ and $s \geq 2$ parts of X as the given Spin^c structures from Y_- and Y_+, and has its first Chern class equal to $c_1(\det \mathbb{S})$. Fix such a Spin^c structure and use $\mathbb{S}^+$ and $\mathbb{S}^-$ to denote the respective bundles of self-dual and anti-self-dual spinors.

The Seiberg–Witten equations on X are equations for a pair $(\mathbb{A}, \Psi)$ with $\mathbb{A}$ being a Hermitian connection on the line bundle $\det \mathbb{S}^+$ and with Ψ being a section of $\mathbb{S}^+$. It takes the general form

$$F_{\mathbb{A}}^+ - (\Psi^\dagger \tau \Psi - i\varpi_X) - \mathfrak{T}^+ = 0 \quad \text{and} \quad \mathcal{D}_{\mathbb{A}}^+ \Psi - \mathfrak{S}^+ = 0, \tag{2-10}$$

where the notation uses $F_{\mathbb{A}}$ to denote the curvature 2–form of $\mathbb{A}$, and it uses $\Psi^\dagger \tau \Psi$ to denote the bilinear map from $\mathbb{S}^+$ to $i\Lambda^+$ that is defined using the Clifford multiplication. Meanwhile, $\mathcal{D}_{\mathbb{A}}^+ \colon \Gamma(\mathbb{S}^+) \to \Gamma(\mathbb{S}^-)$ and $\mathcal{D}_{\mathbb{A}}^- \colon \Gamma(\mathbb{S}^-) \to \Gamma(\mathbb{S}^+)$ are the 4–dimensional Dirac operators on X defined by the metric and the chosen connection $\mathbb{A}$. What is denoted by ϖ_X is a self-dual 2–form satisfying the following list for some $L' \geq L$:

(2-11) • The pullback of ϖ_X from the $s < -L'$ part of X via the embedding from the second bullet of (2-8) is twice the self-dual part of a closed 2–form ϖ_- on Y_-.

• The pullback of ϖ_X from the $s > L'$ part of X via the embedding from the third bullet of (2-8) is twice the self-dual part of a closed 2–form ϖ_+ on Y_+.

The pair $(\mathfrak{T}^+, \mathfrak{S}^+)$ is the 4–dimensional analog of $(\mathfrak{T}, \mathfrak{S})$ in (2-1); see (24.2) in [17].

We denote $X_c := s^{-1}([-L'-1, L'+1]) \subset X$ and call it the "compact piece" of X. Each connected component of $X - X_c$ is called an *end* of X. The diffeomorphisms in

(2-8) identify each end with a product $(-\infty, -L')\times M$ or $M\times(L',\infty)$ for a connected oriented manifold M; in the first case it is said to be a *negative end*, and in the second case a *positive end*. In either case we call this end the M*–end* of X. "The negative end of X" refers to $s^{-1}(-\infty,-L'-1)\simeq(-\infty,-L'-1)\times Y_-$, and "the positive end of X" refers to $s^{-1}(L'+1,\infty)\simeq(L'+1,\infty)\times Y_+$.

Caveat What is denoted by X_c in this article was denoted by X in [17]. Correspondingly, the noncompact manifold X in this article was denoted by X^* in [17].

An important special case is when (2-10) is defined on a *product cobordism*. By this we mean that $X=\mathbb{R}\times M$ for a closed oriented Spin^c 3–manifold M, with the function s as the Euclidean coordinate of the $\mathbb{R}$ factor; the Riemannian metric on X is the product of the affine metric on $\mathbb{R}$ and the Riemannian metric on M, and both ϖ_X and $(\mathfrak{T}^+,\mathfrak{S}^+)$ are invariant under the natural $\mathbb{R}$–action on $\mathbb{R}\times M$. Thus, the conditions in the first bullet of (2-9) and in (2-11) may be paraphrased as saying that the $s^{-1}[L',\infty)$ and $s^{-1}(-\infty,-L']$ part of the Seiberg–Witten equations on X are those of product cobordisms. As explained in [17], Clifford action by ds over product cobordisms may be used to identify $\mathbb{S}^+\simeq\mathbb{S}^-$. Meanwhile, both are the pullback of a spinor bundle $\mathbb{S}$ over M. In this way, (2-10) may be rewritten as a gradient flow equation of the action functional $\mathfrak{a}$; see (IV.1-20). The gradient vector field here is -1 times the left-hand side of (2-1), with (2-10)'s $\varpi_X=2\varpi^+$, and $\mathfrak{T}^+$ and $\mathfrak{S}^+$ induced respectively from the $\mathfrak{T}$ and $\mathfrak{S}$ in (2-1).

A solution $\mathfrak{d}=(\mathbb{A},\Psi)$ to (2-10) is said to be an *instanton* if the constant $s\le -L$ pullbacks converge as $s\to-\infty$ to a pair that can be written as $(\mathbb{A}_-,\Psi_-)$, with $(\mathbb{A}_-,\Psi_-)$ being a solution to (2-1) on Y_-, and if the constant $s\ge L$ pullbacks converge as $s\to\infty$ to a pair $(\mathbb{A}_+,\Psi_+)$, with $(\mathbb{A}_+,\Psi_+)$ being a solution to (2-1) on Y_+. If $\mathfrak{d}$ is an instanton then the convention in what follows will be to say that the respective $s\to-\infty$ and $s\to\infty$ limits of $\mathfrak{d}$ are $(\mathbb{A}_-,\Psi_-)$ and $(\mathbb{A}_+,\Psi_+)$. As in the 3–dimensional case, in [17] they define a real "blowup" of the space $\mathcal{C}_{\mathrm{loc}}(X):=\mathrm{Conn}(\det\mathbb{S}^+)\times C^\infty(X,\mathbb{S}^+)$, this denoted by $\mathcal{C}^\sigma_{\mathrm{loc}}(X)$ below. To describe $\mathcal{C}^\sigma_{\mathrm{loc}}(X)$ in more detail, consider the tautological bundle $C^\infty(X,\mathbb{S}^+)-\{0\}$ over the sphere $\mathbb{U}(C^\infty(X,\mathbb{S}^+)):=(C^\infty(X,\mathbb{S}^+)-\{0\})/\mathbb{R}^+$, and let $\Gamma^\sigma(X;\mathbb{S}^+)$ denote the $\mathbb{R}^{\ge 0}$–bundle associated to this principal $\mathbb{R}^+$–bundle. Then $\mathcal{C}^\sigma_{\mathrm{loc}}(X):=\mathrm{Conn}(\det\mathbb{S}^+)\times\Gamma^\sigma(X,\mathbb{S}^+)$. Alternatively,

$$\mathcal{C}^\sigma_{\mathrm{loc}}(X)=\bigcap_{l\in\mathbb{Z}^+}\mathcal{C}^\sigma_{l,\mathrm{loc}}(X),$$

where $\mathcal{C}^{\sigma}_{l,\mathrm{loc}}(X)$ is the $L^2_{l,\mathrm{loc}}$ variant of $\mathcal{C}^{\sigma}_{\mathrm{loc}}(X)$ defined in [17, page 464]. We write an element in $\Gamma^{\sigma}(X,\mathbb{S}^+)$ in the form of $\Psi^{\sigma}=(\boldsymbol{\Psi},\Psi)$, where $\boldsymbol{\Psi}\in\mathbb{U}(C^{\infty}(X,\mathbb{S}^+))$, and Ψ is in the fiber of the bundle $\Gamma^{\sigma}(X;\mathbb{S}^+)$ over $\boldsymbol{\Psi}$.

The 4–dimensional Seiberg–Witten equations (2-10) may be generalized to elements in $\mathcal{C}^{\sigma}_{\mathrm{loc}}(X)$, and hence also the notion of an instanton (see [17, equation (6.5)]). A (generalized) instanton has its $s\to-\infty$ and $s\to\infty$ limits in $\mathcal{C}^{\sigma}(Y_-)$ and $\mathcal{C}^{\sigma}(Y_+)$, respectively, in the sense explained in [17, page 219]. The 4–dimensional Seiberg–Witten equation is invariant under the actions of the gauge group $C^{\infty}(X;U(1))$. An instanton $(\mathbb{A},\Psi^{\sigma})$, or a gauge-equivalence class of instantons, is said to be *reducible* when $\Psi\equiv 0$; otherwise it is *irreducible*.

The perturbation $(\mathfrak{T}^+,\mathfrak{S}^+)$ is introduced in (2-10) so that a certain operator that is associated to any given instanton solutions to (2-10) is Fredholm with trivial cokernel. See Chapter 24.3 of [17] in general and equation (1-21) in [22] for a special case closely related to this article. Instanton solutions with this property are said to be nondegenerate. We call perturbation term *suitable* when all instanton solutions to the corresponding version of (2-10) are nondegenerate. A suitable perturbation can be found for (2-10) with norm bounded by any given positive number. The relevant norm is also called the $\mathcal{P}$–norm. As in the case with elements in Ω, the $\mathcal{P}$–norm of a perturbation term bounds the norms of its derivatives to all orders.

Just as in the 3–dimensional case, the 4–dimensional cobordisms relevant to Theorem 1.1 are equipped with ϖ_X and pairs $(\mathfrak{T}^+,\mathfrak{S}^+)$ of the form

$$\varpi_X = 2r w_X \quad\text{and}\quad (\mathfrak{T}^+,\mathfrak{S}^+)=(i\mathfrak{w}_{\mu}^+,0).$$

Here, $\mathfrak{w}_{\mu}$ is a 2–form of the form $d(\chi(L+s)\mu_-+\chi(L-s)\mu_+)$ for some 1–forms μ_- and μ_+ on Y_- and Y_+, respectively. However, in the case of a product cobordism $X=\mathbb{R}\times M$, we take $\mathfrak{w}_{\mu}=d\mu_-=d\mu_+$. Meanwhile, w_X is a self-dual 2–form constrained by the properties listed in (2-12) below, among others. These constraints involve another constant, denoted by L_{tor} below. The latter is no smaller than $L+4$. The constraints use X_{tor} to denote the union of the components of the $|s|>0$ part of X where $c_1(\det\mathbb{S})$ is torsion.

(2-12)
- The pullback of w_X to each constant s slice of X is a closed 2–form whose de Rham cohomology class is that of $c_1(\det\mathbb{S})$.
- The embedding from the first bullet of (2-11) pulls back w_X from the $s<-L$ part of $X-X_{\mathrm{tor}}$ as twice the self-dual part of the Y_- version of

the 2–form w. The embedding from the second bullet of (2-11) pulls back w_X from the $s > L$ part of $X - X_{\mathrm{tor}}$ as twice the self-dual part of the Y_- version of the 2–form w. The 2–form w_X is identically zero on any component of the $|s| > L_{\mathrm{tor}}$ part of X_{tor}.

Similarly to the 3–dimensional case, the 4–dimensional Seiberg–Witten equations may be rewritten in terms of the pair $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(\mathbb{S}^+)$ that is obtained from the pair $(\mathbb{A}, \Psi) \in \mathrm{Conn}(\det \mathbb{S}^+) \times C^\infty(\mathbb{S}^+)$ via the same formulas as those in the previous subsection. This requires an extension of K and A_K from the ends $s^{-1}[L', \infty) \cup s^{-1}(-\infty, -L']$. Constraints on such choices will be introduced in subsequent sections as needs arise; typically, where ϖ_X is nowhere-vanishing, E is chosen to be the $i|\varpi_X|$–eigenbundle under the Clifford action of ϖ_X on $\mathbb{S}^+$.

2.3 The monopole Floer chain complex

Fix a closed, oriented, connected Riemannian 3–manifold M and a Spin^c–structure $\mathfrak{s}$ on it. We first give in Part 1 below a precise definition of the monopole Floer complexes involved in the proof of the Theorem 1.4, the main objective of this series of articles. Sketches of how they generalize to other cases are provided in Parts 2 and 3 of this subsection.

Part 1: nontorsion $c_1(\mathfrak{s})$, positive/negative monotone ϖ Suppose for now that $\mathfrak{s}$ has nontorsion first Chern class, and ϖ and $(\mathfrak{T}, \mathfrak{S})$ are as in (2-3) and (2-2), respectively, with $\mathrm{r} > \pi$. Fix also a complex Hermitian line bundle $K \to M$ as specified in Section 2.1 above. The *spectral flow* function on $\mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$ is defined initially on the complement of a certain codimension 1 subvariety just as in Section 1.5 in [22] using a chosen Hermitian connection on E and a suitably generic section of $\mathbb{S}$. As such, it is locally constant and integer-valued. The definition can be extended to the whole of $\mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$ as explained in Sections 7.6 and 7.8 below. This spectral flow function is denoted by $\mathfrak{f}_s$. It suffices for now to know only that this extended function $\mathfrak{f}_s$ has integer values and that the functions

$$\mathfrak{cs}^{\mathfrak{f}} := \mathfrak{cs} - 4\pi^2 \mathfrak{f}_s \quad \text{and} \quad \mathfrak{a}^{\mathfrak{f}} := \mathfrak{a} + 2\pi(\mathrm{r} - \pi)\mathfrak{f}_s$$

are invariant under the action of $C^\infty(M; U(1))$ on $\mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$ that has a $\hat{u} \in C^\infty(M; U(1))$ sending (A, ψ) to $(A - \hat{u}^{-1} d\hat{u}, \hat{u}\psi)$. By way of comparison, $\mathfrak{a}$, $\mathfrak{f}_s$ and $\mathfrak{cs}$ are not invariant under this action. (The notions $\mathfrak{a}^{\mathfrak{f}}$ and $\mathfrak{cs}^{\mathfrak{f}}$ can be generalized to be defined over the blown-up configuration space; see eg [17, equation (16.4)]. The arguments in the proof of Lemma 16.4.4 therein show that this generalization is also invariant under gauge actions.)

Denote by $\mathcal{Z}_{w,\mathrm{r}}$ the set of gauge-equivalence classes of solutions to the corresponding system (2-5). (This was denoted by a slightly different notation, $\mathcal{Z}_{\mathrm{SW},\mathrm{r}}$, in [22].) It is well known that in this case, for a generic choice of r and μ, this set $\mathcal{Z}_{w,\mathrm{r}}$ consists of finitely many, nondegenerate irreducible elements. (See eg (IV.1-18), ignoring the "holonomy nondegenerate" condition there for the moment.) Assume this to be the case. Consider next the 4–dimensional Seiberg–Witten equations on the product cobordism $\mathbb{R}\times M$, with $w_X = 2w^+$, and $\mu_- = \mu_+ = \mu$. Here, w is used to denote the pullback of the 2–form w on M under the projection of $\mathbb{R}\times M$ to its second factor. Given an instanton $\mathfrak{d}$ on this product cobordism with $s\to-\infty$ and $s\to\infty$ limits given respectively by representatives of $\mathfrak{c}_-$ and $\mathfrak{c}_+$ in $\mathcal{Z}_{w,\mathrm{r}}$. The differential operator in (IV.1-21) has a Fredholm extension, whose index we denote by $\iota_{\mathfrak{d}}$. By [1], in this case,

$$\iota_{\mathfrak{d}} = \mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-). \tag{2-13}$$

Let $\mathcal{M}_k(\mathfrak{c}_-,\mathfrak{c}_+)$ denote the space of gauge-equivalence classes of such instantons with $\iota_{\mathfrak{d}} = k$. These spaces are k–dimensional manifolds with a free $\mathbb{R}$–action when the perturbation term in the Seiberg–Witten equations is suitable and $k>0$. In particular, the monotonicity assumption guarantees that $\mathcal{M}_1(\mathfrak{c}_-,\mathfrak{c}_+)/\mathbb{R}$ consists of finitely many elements. With a coherent orientation chosen (this amounts to choices of preferred elements of $\Lambda(\mathfrak{c})$ for all $\mathfrak{c}\in\mathcal{Z}_{w,\mathrm{r}}$ in the language of [17]), each element in $\mathcal{M}_1(\mathfrak{c}_-,\mathfrak{c}_+)/\mathbb{R}$ is assigned a sign.

Fix a ring $\mathbb{K}$, which can be taken to be $\mathbb{Z}$ for the rest of this article. The chain module for the monopole (or, alternatively, Seiberg–Witten) Floer chain group is the free $\mathbb{K}$–module generated by $\mathcal{Z}_{w,\mathrm{r}}$, denoted by $\mathbb{K}(\mathcal{Z}_{w,\mathrm{r}})$ below. The spectral flow function $\mathfrak{f}_s$ descends to define a relative $\mathbb{Z}/c_{\mathfrak{s}}\mathbb{Z}$–grading on this module, where $c_{\mathfrak{s}}\in 2\mathbb{Z}$ is the divisibility of the first Chern class of the Spin^c–structure $\mathfrak{s}$. The differential $\partial_{w,\mathrm{r}}$ of this monopole Floer complex in this situation is the endomorphism of $\mathbb{K}(\mathcal{Z}_{w,\mathrm{r}})$ given by the rule

$$\mathfrak{c}_1 \mapsto \sum_{\mathfrak{c}_2\in\mathcal{Z}_{w,\mathrm{r}}} \mathrm{w}(\mathfrak{c}_1,\mathfrak{c}_2)\mathfrak{c}_2, \tag{2-14}$$

where

$$\mathrm{w}(\mathfrak{c}_1,\mathfrak{c}_2) = \sum_{\mathfrak{d}\in\mathcal{M}_1(\mathfrak{c}_1,\mathfrak{c}_2)/\mathbb{R}} \mathrm{sign}(\mathfrak{d}) = \chi(\mathcal{M}_1(\mathfrak{c}_1,\mathfrak{c}_2)/\mathbb{R}).$$

The aforementioned properties of $\mathcal{Z}_{w,\mathrm{r}}$ and $\mathcal{M}_1(\mathfrak{c}_1,\mathfrak{c}_2)/\mathbb{R}$ for suitable monotone perturbations guarantee that this homomorphism is well defined, and it is of degree 1 according to (2-13). A typical gluing argument shows that $\partial^2_{w,\mathrm{r}} = 0$. (See eg [17, Sections 19 and 22].) The homology of the above monopole Floer complex is the monopole Floer homology, or, alternatively, the Seiberg–Witten Floer homology of the negative monotone perturbation (2-3). This is denoted by $\mathrm{HM}_*(M,\mathfrak{s},c_-)$ below. The monopole Floer homology for positive monotone perturbation forms, still assuming that $c_1(\mathfrak{s})$ is nontorsion, is defined in the same way.

Part 2: local coefficients One may also associate monopole Floer homologies for more general Seiberg–Witten equations (2-1). The construction of monopole Floer complexes in Part 1 may fail to work due mainly to two reasons:

(1) With balanced perturbations, the generating set of the chain group, $\mathcal{Z}$, namely the set of gauge-equivalence classes of solutions to (2-1), may contain reducible elements. (Recall that $\mathcal{Z} = \mathcal{Z}_{w,\mathrm{r}}$ in the previous part, which consists of finitely many irreducible elements.)

(2) The space $\mathcal{M}_1(\mathfrak{c}_1,\mathfrak{c}_2)/\mathbb{R}$ might contain infinitely many elements, making the coefficients appearing in (2-14)'s formula for the Seiberg–Witten differential, $\mathrm{w}(\mathfrak{c}_1,\mathfrak{c}_2)$, undefined.

The second issue above can be dealt with by working with monopole Floer complexes with more general coefficients (as opposed to Part 1's $\mathbb{Z}$–coefficient monopole Floer complex). See [17, Section 22.6].

Assume for simplicity that the perturbation ϖ in (2-1) is nonbalanced, so that the issue (1) above can be ignored: namely, with a generic perturbation, $\mathcal{Z}$ will still consist of finitely many, nondegenerate, irreducible elements. Fix a local system Γ in the sense described in [17]. This assigns to every $\mathfrak{c} \in \mathcal{Z}$ a group $\Gamma(\mathfrak{c})$ and for each relative homotopy class z of paths between $\mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{Z} \subset \mathcal{B}(M)$, a homomorphism $\Gamma(z)\colon \Gamma(\mathfrak{c}_1) \to \Gamma(\mathfrak{c}_2)$. The monopole Floer chain complex with local coefficient Γ, (C,∂), has $C := \bigoplus_{\mathfrak{c}\in\mathcal{Z}} \Gamma(\mathfrak{c})$ as its chain module. As for its differential ∂, regard each $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_1,\mathfrak{c}_2)$ as path in $\mathcal{B}(M)$ and let $\mathcal{M}_z(\mathfrak{c}_1,\mathfrak{c}_2) \subset \mathcal{M}(\mathfrak{c}_1,\mathfrak{c}_2)$ be the subspace consisting of elements of relative homotopy class z. Refining (2-14), the following formula defines the associated differential $\partial \in \mathrm{End}(C)$:

$$\partial = \sum_{\mathfrak{c}_1,\mathfrak{c}_2\in\mathcal{Z}} \sum_{z\in\pi_1\mathcal{B}(M;\mathfrak{c}_1,\mathfrak{c}_2)} \mathrm{w}(\mathfrak{c}_1,\mathfrak{c}_2;z)\Gamma(z), \tag{2-15}$$

where

(2-16) $$\mathrm{w}(\mathfrak{c}_1, \mathfrak{c}_2; z) = \sum_{\mathfrak{d} \in \mathcal{M}_{1,z}(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}} \mathrm{sign}(\mathfrak{d}) = \chi(\mathcal{M}_{1,z}(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}),$$

and $\pi_1 \mathcal{B}(M; \mathfrak{c}_1, \mathfrak{c}_2)$ denotes the space of relative homotopy classes of instantons with $\mathfrak{c}_1$ and $\mathfrak{c}_2$ as its $s \to -\infty$ and $s \to \infty$ limits, respectively. Typical compactness results can be used to ensure that each coefficient $\mathrm{w}(\mathfrak{c}_1, \mathfrak{c}_2; z)$ is finite. (See eg [17, Theorem 8.1.1 and Proposition 16.1.4].) Though (2-15) may have infinitely many nonvanishing terms, the sum may be well defined when Γ is chosen to satisfy certain completeness conditions depending on the choice of $\mathfrak{s}$ and $[\varpi]$. See Definition 30.2.2 in [17]. We call a local system Γ satisfying this completeness condition $(\mathfrak{s}, [\varpi])$–*complete* (as opposed to "c–complete" in [17]). There is also a more stringent notion of completeness which depends only on the cohomology class $[d\mathfrak{a}] \in H^1(\mathcal{B}(M); \mathbb{Z})$ due originally to Novikov. This sort of local system is said to be "strongly c–complete" in [17]; see Definition 30.2.4 therein. We call such Γ *strongly* $(\mathfrak{s}, [\varpi])$*–complete* instead. We shall not encounter local systems other than $\mathbb{Z}$ except in Proposition 6.7(b) below, which is not directly relevant to the proof of Theorem 1.1. The interested reader is therefore referred to [17] for more details on the definition of monopole Floer homology with local coefficients. A brief summary in alternative language may also be found in the last section of [25]. In the monotone case discussed in Part 1 or the balanced case in the upcoming Part 3, the (strong) $(\mathfrak{s}, [\varpi])$–completeness condition is met for all coefficients, and the sum (2-16) has finitely many nonvanishing terms.

Part 3: balanced perturbations We now briefly describe how issue (1) in Part 2 is dealt with in the balanced case. For details, see Chapters VI and VIII in [17]. As already mentioned in Section 2.1, [17] considered the extension of (2-1) to $\mathcal{C}^\sigma$. The set of gauge-equivalence classes of solutions to this extended Seiberg–Witten equation is denoted by $\mathfrak{C}$. Suppose that the perturbation to the Seiberg–Witten equation is suitable. The subsets of irreducible, unstable reducible, and stable reducible elements are respectively denoted by $\mathfrak{C}^o$, $\mathfrak{C}^u$ and $\mathfrak{C}^s$. (In the nonbalanced situation previously considered, $\mathfrak{C} = \mathfrak{C}^o = \mathcal{Z}_{\mathrm{r},w}$.) The first three flavors of monopole Floer homology as defined in [17] use different combinations of $\mathfrak{C}^o$, $\mathfrak{C}^u$ and $\mathfrak{C}^s$ to generate the chain groups: set

$$C^o = \mathbb{K}(\mathfrak{C}^o), \quad C^u = \mathbb{K}(\mathfrak{C}^u), \quad C^s = \mathbb{K}(\mathfrak{C}^s),$$

and let

$$\widehat{C} = C^o \oplus C^u, \quad \overline{C} = C^s \oplus C^u, \quad \check{C} = C^o \oplus C^s.$$

Meanwhile, the operator in (IV.I-21) has a Fredholm generalization for paths $\mathfrak{d}(s)$ in $\mathrm{Conn}(E)\times C^\infty(M,\mathbb{S})$ with $s\to\infty$ or $s\to-\infty$ limits that are nondegenerate elements in $\mathcal{C}^\sigma$. (See Sections 14.4 and 22.3 in [17].) The index of this operator is also denoted by $\iota_{\mathfrak{d}}$ below, and it may be used to generalize the spectral flow function $\mathfrak{f}_s$ to the set of nondegenerate elements in $\mathcal{C}^\sigma$. This in turn defines a relative $\mathbb{Z}/c_{\mathfrak{s}}$–grading, gr, on the modules C^o, C^u and C^s. The chain modules $\widehat{C}$, $\overline{C}$ and $\check{C}$ are also $\mathbb{Z}/c_{\mathfrak{s}}$–graded according to the rule

$$\widehat{C}=\bigoplus_j \widehat{C}_j,\quad \widehat{C}=\bigoplus_j \widehat{C}_j,\quad \widehat{C}=\bigoplus_j \widehat{C}_j,$$

where

$$\widehat{C}_j=C^o_j\oplus C^u_j,\quad \overline{C}_j=C^s_j\oplus C^u_{j+1},\quad \check{C}_j=C^o_j\oplus C^s_j.$$

Note that the $\overline{C}$ chain module above is graded by a modified grading $\overline{\mathrm{gr}}$, related to gr via equation (22.15) in [17]. To define the differentials, define homomorphisms $\partial^\sharp_\natural\colon C^\sharp\to C^\natural$ via rules similar to (2-14) or (2-15) by counting irreducible instantons with $\iota_{\mathfrak{d}}=1$ whose $s\to-\infty$ and $s\to\infty$ limits are in $\mathfrak{C}^\sharp$ and $\mathfrak{C}^\natural$, respectively; see [17, equation (22.8)] for the precise formulas. Here, $\sharp$ and $\natural$ may stand for one of the labels u, o and s; however, due to the way $\mathfrak{C}^u$, $\mathfrak{C}^s$ and $\mathfrak{C}^o$ are defined, only the homomorphisms ∂^o_o, ∂^o_s, ∂^u_o and ∂^u_s are nontrivial. Meanwhile, there are homomorphisms $\overline{\partial}^\sharp_\natural\colon C^\sharp\to C^\natural$, and with $\sharp$ and $\natural$ denoting either the label u or s, by counting reducible instantons whose $s\to-\infty$ and $s\to\infty$ limits are in $\mathfrak{C}^\sharp$ and $\mathfrak{C}^\sharp$, respectively, with $\overline{\mathrm{gr}}$ differing by -1. If the Spin^c–structure and $[\varpi]$ satisfy monotonicity condition, then the differentials for the complexes, $\widehat{\partial}\colon\widehat{C}\to\widehat{C}$, $\overline{\partial}\colon\overline{C}\to\overline{C}$ and $\check{\partial}\colon\check{C}\to\check{C}$, are defined in terms of these homomorphisms via equation (22.7) and Definition 22.1.3 in [17]. To give some examples, $\check{\partial}\colon C^o\oplus C^s\to C^o\oplus C^s$ and $\widehat{\partial}\colon C^o\oplus C^u\to C^o\oplus C^u$ are respectively written in block form as

$$\begin{bmatrix}\partial^o_o & -\partial^u_o\overline{\partial}^s_u\\ \partial^o_s & \overline{\partial}^s_s-\partial^u_s\overline{\partial}^s_u\end{bmatrix},\quad \begin{bmatrix}\partial^o_o & -\partial^u_o\\ -\overline{\partial}^s_u\partial^o_s & -\overline{\partial}^u_u-\overline{\partial}^s_u\partial^u_s\end{bmatrix}. \tag{2-17}$$

The gluing theorems in [17] show that $\widehat{\partial}^2$, $\overline{\partial}^2$ and $\check{\partial}^2$ are indeed all 0. When the perturbation is balanced, such as in the statement of Theorem 1.1, the homology of these chain complexes $(\overset{\circ}{C}_*,\overset{\circ}{\partial}_*)$, namely the corresponding monopole Floer homology, is denoted by $\overset{\circ}{\mathrm{HM}}_*(M,\mathfrak{s},c_b)$ for $\circ=\wedge,-,\vee$.

The aforementioned homomorphisms $\partial^\sharp_\natural$ and $\overline{\partial}^\sharp_\natural$ are also used to define chain maps (denoted by $i\colon\overline{C}\to\check{C}$, $j\colon\check{C}\to\widehat{C}$ and $p\colon\widehat{C}\to\overline{C}$ in [17]) that do not define a short exact sequence, but their induced maps on homologies do, this being the first of the fundamental exact sequences referred to in Theorem 1.1. See [17, Proposition 22.2.1].

Part 4: notation and other remarks When specificality is desired, the notation

$$\overset{\circ}{C}_*(M,\mathfrak{s},[\varpi];\Gamma)=\overset{\circ}{C}_*(M,\mathfrak{s},\varpi;\Gamma),\quad \circ=\wedge,-,\vee,$$

is used to denote the monopole Floer complex corresponding to the cylindrical version of (2-10) with an $(\mathfrak{s},[\varpi])$–complete local coefficients Γ, and $\overset{\circ}{\mathrm{HM}}_*(M,\mathfrak{s},[\varpi];\Gamma)$ is used to denote the corresponding monopole Floer homology. (The Floer chain complex $\overset{\circ}{C}_*(M,\mathfrak{s},\varpi;\Gamma)$ does depend on the choice of ϖ, not just its cohomology class, though its associated Floer homology only depends on the cohomology class $[\varpi]$. The notation $\overset{\circ}{C}_*(M,\mathfrak{s},[\varpi];\Gamma)$ is adopted when the specific representative ϖ of $[\varpi]$ is irrelevant.) In particular, when $[\varpi]=2\pi c_1(\det\mathbb{S})$, $\overset{\circ}{C}_*(M,\mathfrak{s},[\varpi];\Gamma)$ and $\overset{\circ}{\mathrm{HM}}_*(M,\mathfrak{s},[\varpi];\Gamma)$ are also respectively denoted by $\overset{\circ}{C}_*(M,\mathfrak{s},c_b;\Gamma)$ and $\overset{\circ}{\mathrm{HM}}_*(M,\mathfrak{s},c_b;\Gamma)$. The coefficient Γ is dropped from the notation when it is $\mathbb{Z}$, or not important. The following (admittedly sloppy) convention will be adopted for the rest of this article: Since the Floer complexes $(\check{C},\check{\partial})=(\widehat{C},\widehat{\partial})=(C^o,\partial^o_o)$ when the perturbation is nonbalanced, we use CM or (CM,∂) to denote the one complex in this case. When we wish to emphasize the Spin^c–manifold and/or cohomology class of perturbation, etc, used to define the monopole Floer complex, these data are added to the above expression in parentheses, such as $\mathrm{CM}_*(M,\mathfrak{s},[\varpi])$ or $(\mathrm{CM}_*(M,\mathfrak{s}),\partial_*(M,\mathfrak{s}))$.

As final remarks to this subsection, note that in [17] there is an equivalent, geometric version of grading for the monopole Floer complexes in terms of homotopy classes of oriented 2–plane fields. This is briefly described in Part 1 of Section 6.1 below, and denoted by $\mathbb{J}(M)$ therein. A very brief description of this in the special cases relevant to this article will appear in Part 1 of Section 6.1. Meanwhile, the signs sign$(\mathfrak{d})$ assigned according to the rules in [17] depend on a choice of *homology orientation* of M. See Definition 22.5.2 in [17].

2.4 Cobordism-induced maps between monopole Floer complexes

Instantons on cobordisms X described in Section 2.2 are used to define maps between the monopole Floer complexes. Details of the construction of these maps are given in [17, Chapter VII] for cobordisms X between *connected* 3–manifolds Y_- and Y_+, even though properties of moduli spaces of Seiberg–Witten instantons on more general X, where Y_- may be disconnected, are also established therein. In particular, taking X to be a product cobordism $\mathbb{R}\times M$, this construction is used to define chain maps from $\overset{\circ}{C}$ back to itself for $\circ = \wedge, -, \vee$, which induce the $\boldsymbol{A}_\dagger$–module structure on the corresponding Floer homology. Another application of these cobordism-induced chain maps is to define chain homotopies between monopole Floer complexes $\overset{\circ}{C}$ associated to different metrics and $(\mathfrak{T}, \mathfrak{S})$. See eg the proof for Corollary 23.1.6 in [17] and its variants. According to the conventions set forth in Section 1.3, this justifies our notation for the monopole Floer complex, $\overset{\circ}{C}(M, \mathfrak{s}, [\varpi]; \Gamma)$. In fact, this type of arguments show that $\overset{\circ}{C}$ with positively proportional "period class" [17, page 591] are chain homotopic to each other. (See Theorem 31.4.1 in [17].) This in turn justifies using the notation $\mathrm{CM}(M, \mathfrak{s}, c_-)$ for any negatively monotone, nonbalanced perturbation, according to our convention.

The rest of this subsection is divided into four parts. In the first three parts we review some basic elements in the construction to the aforementioned cobordism-induced maps. The last part contains a generalization of [17]'s construction to certain simple cobordisms between possibly disconnected manifolds, in order to accommodate our needs in Section 6.

Part 1: moduli spaces and their compactifications Fix a Spin^c–structure $\mathfrak{s}_X$ on X which restricts to the $s \le -2$ and the $s \ge 2$ part of X respectively as Spin^c–structures $\mathfrak{s}_-$ on Y_- and $\mathfrak{s}_+$ on Y_+. Fix also a self-dual 2–form ϖ_X on X satisfying (2-11) and a suitable pair $(\mathfrak{T}^+, \mathfrak{S}^+)$. Let $c_{\mathfrak{s}_X}$ denote the divisibility of $c_1(\mathfrak{s}_X)$. This number divides both $c_{\mathfrak{s}_-}$ and $c_{\mathfrak{s}_+}$. Assume that $Y_\pm$ are both connected in this part.

Consider instantons $\mathfrak{d}$ defined from (2-10) with representatives of $\mathfrak{c}_-$ and $\mathfrak{c}_+$ respectively as its $s \to -\infty$ and $s \to \infty$ limits. The index of the Fredholm operator that entered the definition of nondegeneracy for instantons is denoted by $\iota_\mathfrak{d}$. This generalizes the notion of index in the case of product cobordisms described in the previous subsection, and it depends only on the relative homotopy class of $\mathfrak{d}$. See again Chapter 24 of [17]. Let $\mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ denote the space of gauge-equivalence classes of such instantons with $\iota_\mathfrak{d} = k$. When $\mathfrak{c}_- \in \mathfrak{C}^\sharp(Y_-)$ and $\mathfrak{c}_+ \in \mathfrak{C}^\flat(Y_+)$ are both reducible, let $\mathcal{M}_k^{\mathrm{red}}(X; \mathfrak{c}_-, \mathfrak{c}_+) \subset \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ be the subspace consisting of reducible instantons. Note that $\mathcal{M}_k^{\mathrm{red}}(X; \mathfrak{c}_-, \mathfrak{c}_+) = \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ in the cases when the pair $(\sharp, \flat)$ is (u, u), (s, s) or (s, u). When $(\mathfrak{T}^+, \mathfrak{S}^+)$ is suitable, $\mathcal{M}_k^{\mathrm{red}}(X; \mathfrak{c}_-, \mathfrak{c}_+)$ is a smooth manifold with dimension respectively k, k, $k+1$ or $k-1$ in the cases when the pair $(\sharp, \flat)$ is (u, u), (s, s), (s, u) or (u, s). The moduli space $\mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)$ is a k–dimensional manifold consisting purely of irreducible instantons in the case when at least one of $\mathfrak{c}_-$ or $\mathfrak{c}_+$ is irreducible, while it is a k–manifold with boundary $\partial \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+) = \mathcal{M}_k^{\mathrm{red}}(X; \mathfrak{c}_-, \mathfrak{c}_+)$ in the case when $(\sharp, \flat) = (u, s)$.

All the spaces $\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}_k^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ are given orientations according to the rules specified in [17]. This depends on a choice of what is called a "homological orientation" of X *as a cobordism* in [17]. (See Definition 3.4.1 in [17].) Let $\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}_{k,z}^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ respectively be subspaces of $\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}_k^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ consisting of instantons with relative homotopy class z. (Given $\mathfrak{c}_-$, $\mathfrak{c}_+$ and z, the spaces $\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ (resp. $\mathcal{M}_{k,z}^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$) are empty for all $k\in\mathbb{Z}$ except one. This is denoted by $\mathcal{M}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)$ (resp. $\mathcal{M}_z^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$) below.) All the moduli spaces introduced above lie in the orbit space of $\mathcal{C}_{\mathrm{loc}}^{\sigma}(X)$ under the gauge action by $C^\infty(X,U(1)) =: \mathcal{G}_{\mathrm{loc}}(X)$. This orbit space is denoted by $\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$. Let $\mathcal{M}_k(X)\subset\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$ denote the union of all spaces $\mathcal{M}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}_z^{\mathrm{red}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ with dimension less than or equal to k for all $\mathfrak{c}_-\in\mathfrak{C}(Y_-)$, $\mathfrak{c}_+\in\mathfrak{C}(Y_+)$ and $z\in\pi_0(\mathcal{B}_{\mathrm{loc}}^{\sigma}(X))$.

It follows from [17, Section 13.6] that the embeddings $\mathcal{M}(X)=\bigcup_k\mathcal{M}_k(X)\hookrightarrow\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$ and $\mathcal{M}(X)\hookrightarrow\mathcal{B}_{l,\mathrm{loc}}^{\sigma}(X)$ factor respectively through subspaces $\mathcal{B}^{\sigma}(X)\subset\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$ and $\mathcal{B}_l^{\sigma}(X)\subset\mathcal{B}_{l,\mathrm{loc}}^{\sigma}(X)$, described below. These subspaces are homotopy equivalent to $\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$ but are sometimes more convenient to work with. In particular, $\mathcal{B}_l^{\sigma}(X)$ has the virtue of carrying a Banach manifold structure. Let

$$\mathcal{B}^{\sigma}(X) := \bigcup_{\mathfrak{c}_-\in\mathcal{B}^{\sigma}(Y_-)}\ \bigcup_{\mathfrak{c}_+\in\mathcal{B}^{\sigma}(Y_+)}\mathcal{B}^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+),$$
$$\mathcal{B}_l^{\sigma}(X) := \bigcup_{\mathfrak{c}_-\in\mathcal{B}_l^{\sigma}(Y_-)}\ \bigcup_{\mathfrak{c}_+\in\mathcal{B}_l^{\sigma}(Y_+)}\mathcal{B}_l^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+),$$

where $\mathcal{B}^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+)=\bigcap_l\mathcal{B}_l^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{B}_{\mathrm{loc}}^{\sigma}(X)$, and $\mathcal{B}_l^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{B}_{l,\mathrm{loc}}^{\sigma}(X)$ is defined as follows. Let $c_\pm=(\mathbb{A}_\pm,(\mathbf{\Psi}_\pm,\Psi_\pm))\in\mathcal{C}_l^{\sigma}(Y_\pm)$ be respectively representatives of $\mathfrak{c}_\pm\in\mathcal{B}_l^{\sigma}(Y_\pm)$, and use the same notation $(\mathbb{A}_\pm,(\mathbf{\Psi}_\pm,\Psi_\pm))$ to denote the corresponding $\mathbb{R}$–invariant element in $\mathcal{C}_{l,\mathrm{loc}}^{\sigma}(\mathbb{R}\times Y_\pm)$. Using the diffeomorphisms in (2-8) to identify connected components of $X-X_c$ with subdomains of $\mathbb{R}\times Y_+$ or $\mathbb{R}\times Y_+$, let $\mathcal{C}_l^{\sigma}(X;c_-,c_+)\subset\mathcal{C}_{l,\mathrm{loc}}^{\sigma}(X)$ be the subspace consisting of $(\mathbb{A},(\mathbf{\Psi},\Psi))\in\mathcal{C}_{l,\mathrm{loc}}^{\sigma}(X)$ such that $\mathbb{A}-\mathbb{A}_+$ and $\Psi-\Psi_+$ are both L^2_l on the positive end of X, and $\mathbb{A}-\mathbb{A}_-$ and $\Psi-\Psi_-$ are both L^2_l on the negative end of X. Let $\mathcal{B}_l^{\sigma}(X;\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{B}_{l,\mathrm{loc}}^{\sigma}(X)$ be the subspace consisting of elements represented by elements in $\mathcal{C}_l^{\sigma}(X;c_-,c_+)\subset\mathcal{C}_{l,\mathrm{loc}}^{\sigma}(X)$. By construction, $\mathcal{B}^{\sigma}(X)$ and $\mathcal{B}_l^{\sigma}(X)$ come equipped with maps

$$\Pi^{\partial}=\Pi^{-\infty}\times\Pi^{\infty}\colon \mathcal{B}^{\sigma}(X)\to\mathcal{B}^{\sigma}(Y)\times\mathcal{B}^{\sigma}(Y),$$
$$\Pi^{\partial}=\Pi^{-\infty}\times\Pi^{\infty}\colon \mathcal{B}_l^{\sigma}(X)\to\mathcal{B}_l^{\sigma}(Y)\times\mathcal{B}_l^{\sigma}(Y),$$

sending $(\mathbb{A},(\mathbf{\Psi},\Psi))$ to $(\mathbb{A}_-,(\mathbf{\Psi}_-,\Psi_-))\times(\mathbb{A}_+,(\mathbf{\Psi}_+,\Psi_+))$.

Let $\mathcal{M}^+_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}^+_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ be respectively the compactification of $\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ by adding (parametrized) "broken trajectories" as described in [17, Definition 24.6.1 and Theorem 24.6.2]. In Definition 24.6.9 of [17], a surjective map $\mathfrak{r}$ from $\mathcal{M}^+_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ to a smaller compactification, $\overline{\mathcal{M}}_{k,z}(X,\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{B}^\sigma_{\mathrm{loc}}(X)$, was introduced. Both compactifications $\mathcal{M}^+_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\overline{\mathcal{M}}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ are "spaces stratified by manifolds" in the sense of [17, Definition 16.5.1]. (See [17, Propositions 24.6.8 and 24.6.10].) For brevity, we refer to such spaces simply as "stratified manifolds" in this article. By definition, $\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ is the top-dimensional stratum of both $\mathcal{M}^+_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\overline{\mathcal{M}}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$, and each $\overline{\mathcal{M}}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ embeds in $\mathcal{B}^\sigma(X)\subset\mathcal{B}^\sigma_{\mathrm{loc}}(X)$ through the stratified manifold

$$\mathcal{M}(X)=\bigcup_k\mathcal{M}_k(X)\subset\mathcal{B}^\sigma(X),\quad \varnothing\subset\cdots\subset\mathcal{M}_{k-1}(X)\subset\mathcal{M}_k(X)\subset\cdots\subset\mathcal{M}(X).$$

Meanwhile, the map $\mathfrak{r}$ sends strata of $\mathcal{M}^+_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ to strata of $\overline{\mathcal{M}}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ (not necessarily of the same dimension), and restricts to an isomorphism on the top stratum. The moduli spaces of reducible instantons $\mathcal{M}^{\mathrm{red}}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ are compactified similarly.

Part 2: integrating cochains on stratified manifolds Generalizing the formula for the differential of monopole Floer complex, (2-16), the purported maps between monopole Floer complexes have coefficients given in terms of "integrals" of the form $\langle u,\mathcal{M}\rangle$, where $u\in\mathrm{C}(\mathcal{B}^\sigma(X);\mathbb{K})$, $(\mathrm{C}(\mathcal{B}^\sigma(X);\mathbb{K}),\delta)$ being a suitable version of cochain complex for $\mathcal{B}^\sigma(X)$, $H(\mathrm{C}(\mathcal{B}^\sigma(X);\mathbb{K}))=H^*(\mathcal{B}^\sigma(X);\mathbb{K})$, and $\mathcal{M}\subset\mathcal{B}^\sigma(X)$ is a compactified moduli space of the types described in Part 1. Explicit formulas for these maps are given below; see (2-19) and thereabouts. Before proceeding to explain the possible choices of $(\mathrm{C}(\mathcal{B}^\sigma(X);\mathbb{K}),\delta)$ and the definition of the integrals $\langle u,\mathcal{M}\rangle$ associated to them, we make a few motivational remarks.

Ideally, the stratification structure of the relevant $\mathcal{M}=\mathcal{M}_k$ is sufficiently simple, eg it is a manifold with corners such that

$$\partial\mathcal{M}_k=\mathcal{M}_{k-1},\quad \partial\mathcal{M}_{k-1}=0. \tag{2-18}$$

(See eg [17, Remark, page 291] for an example of pathological stratified manifolds.) Defined from broken trajectories, the lower-dimensional strata of $\mathcal{M}$ typically have an explicit description in the manner of [17, Propositions 24.6.8 and 24.6.10]. Thus, when (2-18) holds, Stokes' theorem of the form

$$\langle\delta v,\mathcal{M}\rangle=\langle v,\partial\mathcal{M}\rangle=\langle v,\mathcal{M}_{k-1}\rangle$$

can be invoked to derive various essential identities for the associated cobordism maps between monopole Floer complexes. For example, this type of arguments are used to show that when u is closed, the associated map $\mathring{m}[u]$ is a chain map, and thus it induces maps between the corresponding monopole Floer homology groups. Moreover, the induced maps on Floer homologies depend only on the cohomology class of u, $[u] \in H^k(\mathcal{B}^\sigma(X); \mathbb{K})$, rendering the specific choice of $(\mathrm{C}(\mathcal{B}^\sigma(X); \mathbb{K}), \delta)$ irrelevant on the homological level.

With suitable $(\mathfrak{T}^+, \mathfrak{S}^+)$, (2-18) indeed holds in the nonbalanced case, when all the relevant Seiberg–Witten solutions are irreducible. Though the moduli spaces one encounters may in general have more complicated stratification, it was shown in [17] (eg Theorem 24.7.2 therein) that in most settings of interest, the stratification is still simple enough that (2-18) holds in a formal sense (see [17, Lemma 21.3.1] for a precise statement). Thus, via a suitable variant of Stokes' theorem (see [17, equation (21.4)]), the arguments sketched above still apply, leading to the desired identities.

Returning to the issue of choosing $\mathrm{C}(\mathcal{B}^\sigma(X); \mathbb{K})$, a simplest option is the de Rham complex: taking u to be a differential k–form on $\mathcal{B}^\sigma(X)$, its restriction to $\mathcal{M} \subset \mathcal{B}^\sigma(X)$ or any stratum of $\mathcal{M}$ is well defined, and the "integral"

$$\langle u, \mathcal{M}\rangle = \int_{\mathcal{M}} u = \int_{\mathcal{M}_k \setminus \mathcal{M}_{k-1}} u$$

is literally the integral of u over $\mathcal{M}$. This however only works for $\mathbb{K} = \mathbb{R}$. To be able to work with more general $\mathbb{K}$, in particular $\mathbb{K} = \mathbb{Z}$, Kronheimer and Mrowka [17] choose to work with particular types of Čech cochain complexes $(C^*(\mathcal{U}; \mathbb{K}), \delta)$, where $\mathcal{U}$ is an open cover of $\mathcal{B}^\sigma(X)$ satisfying certain transversality conditions relative to $\mathcal{M} \subset \mathcal{B}^\sigma(X)$. It was shown that such a covering $\mathcal{U}$ exists and any two of them have a common refinement. See Chapter 21 of [17]. The exposition in [17] focuses on maps between monopole Floer *homology groups* instead of their underlying chain maps between monopole Floer *complexes*. As mentioned previously, the former depends only on the cohomology class $[u] \in H(C^*(\mathcal{U}; \mathbb{K})) = H^*(\mathcal{B}^\sigma(X); \mathbb{K})$; thus, in [17] the specific choice of the covering $\mathcal{U}$ and the cochain u representing $[u] \in H^*(\mathcal{B}^\sigma(X); \mathbb{K})$ is typically left unspecified. In this article, however, specific maps between monopole Floer *complexes* do play a role, and the cochains u used to define these maps need to be specified. This shall be done without reference to the covering $\mathcal{U}$, as there is no natural choice for the latter. Instead, in the upcoming remark we introduce a notion of equivalence (depending on $\mathcal{M}(X) \subset \mathcal{B}^\sigma(X)$) among cochains possibly from different choices of underlying chain complexes $\mathrm{C}^*(\mathcal{B}^\sigma(X);\mathbb{K})$ for $H^*(\mathcal{B}^\sigma(X);\mathbb{K})$. The maps $\mathring{m}[u]$ between monopole Floer complexes depend only on the equivalence class of u. Abusing terminology, what is called a "k–cochain u on $\mathcal{B}^\sigma(X)$" in this article typically refers to *any representative* $u\in\mathrm{C}^*(\mathcal{B}^\sigma(X);\mathbb{K})$ of a given equivalence class (relative to a fixed $\mathcal{M}(X)$).

Remark 2.1 Let $\mathcal{M}$ be a finite-dimensional compact oriented stratified manifold embedded in a metric space $\mathcal{B}$. Suppose $\mathcal{U}$ is an open covering of $\mathcal{B}$ transverse to $\mathcal{M}$ in the sense defined in [17, Chapter 21]. As explained in [17], the transversality condition on $\mathcal{U}$ makes it possible to associate to each Čech cochain $u\in C^k(\mathcal{U};\mathbb{K})$ a well-defined cohomology class on the k–dimensional stratum of $\mathcal{M}$,

$$[u]\in\check{H}^k(\mathcal{M}_k,\mathcal{M}_{k-1};\mathbb{K})\simeq H^k_c(\mathcal{M}_k\setminus\mathcal{M}_{k-1};\mathbb{K}),$$

and the value of $\langle u,\mathfrak{M}\rangle$ for each stratum $\mathfrak{M}$ of $\mathcal{M}$ is given in terms of this cohomology class. See page 408 of [17]. To rephrase the constructions in [17], we introduce a cochain complex $(C^*_{\mathcal{M}},\delta_{\mathcal{M}})$ defined as follows: let $C^k_{\mathcal{M}}=C^{k;\mathbb{K}}_{\mathcal{M}}:=H^k(\mathcal{M}_k,\mathcal{M}_{k-1};\mathbb{K})$, and let $\delta_{\mathcal{M}}\colon H^k(\mathcal{M}_k,\mathcal{M}_{k-1};\mathbb{K})\to H^{k+1}(\mathcal{M}_{k+1},\mathcal{M}_k;\mathbb{K})$ be the connecting map in the long exact sequence for the triple $(\mathcal{M}_{k+1},\mathcal{M}_k,\mathcal{M}_{k-1})$. (The fact that $\delta^2_{\mathcal{M}}=0$ is inessential in this article and we leave its verification to the reader.) Use $[u]_{\mathcal{M}}\in C^k_{\mathcal{M}}$ to denote the cohomology class of u in $H^k(\mathcal{M}_k,\mathcal{M}_{k-1};\mathbb{K})$ in the preceding expression. Then, by construction,

$$[\delta u]_{\mathcal{M}}=\delta_{\mathcal{M}}[u]_{\mathcal{M}}.$$

Let $(C^{\mathcal{M}}_*,\partial_{\mathcal{M}})$ denote the dual chain complex of $(C^*_{\mathcal{M}},\delta_{\mathcal{M}})$. There is a canonical basis $\{\mu^k_\alpha\}_\alpha$ for $C^k_{\mathcal{M}}$, with α indexing all the connected k–dimensional strata $\mathfrak{M}_\alpha$ of $\mathcal{M}$, and μ^k_α generating $H^k(\mathfrak{M}_\alpha,\mathcal{M}_{k-1};\mathbb{K})=\mathbb{K}\subset H^k(\mathcal{M}_k,\mathcal{M}_{k-1};\mathbb{K})$. The duals of μ^k_α, denoted by $[\mathfrak{M}_\alpha]$ below, then form a corresponding basis for $C^{\mathcal{M}}_k$. This is used to define a notion of "fundamental class" for stratified manifolds: Given a k–dimensional stratum $\mathfrak{M}$ of $\mathcal{M}$, let

$$[\mathfrak{M}]:=\sum_\beta[\mathfrak{M}_\beta]\in C^{\mathcal{M}}_k,$$

where $\mathfrak{M}_\beta$ are the connected components of $\mathfrak{M}=\bigcup_\beta\mathfrak{M}_\beta$. We say that $\mathcal{M}'\subset\mathcal{M}$ is a k–dimensional *stratified submanifold* of $\mathcal{M}$ if $\mathcal{M}'$ is a k–dimensional stratified manifold whose strata are strata of $\mathcal{M}$. Given such $\mathcal{M}'$, let

$$[\mathcal{M}']:=[\mathcal{M}'\setminus\mathcal{M}_{k-1}]\in C^{\mathcal{M}}_k.$$

Then $\langle u, \mathfrak{M}\rangle$ (resp. $\langle u, \mathcal{M}'\rangle$) simply denotes the pairing of $[u]_{\mathcal{M}} \in C^*_{\mathcal{M}}$ and $[\mathfrak{M}] \in C_*^{\mathcal{M}}$ (resp. $[\mathcal{M}'] \in C_*^{\mathcal{M}}$), and [17]'s version of "Stokes' theorem" states

$$\langle \delta v, \mathcal{M}'\rangle = \langle \delta_{\mathcal{M}}[v]_{\mathcal{M}}, [\mathcal{M}']\rangle = \langle [v]_{\mathcal{M}}, \partial_{\mathcal{M}}[\mathcal{M}']\rangle.$$

(See [17, equations (21.3) and (21.4)].) In the case when $\mathcal{M}'$ is a manifold with corners, $\partial_{\mathcal{M}}[\mathcal{M}'] = [\partial\mathcal{M}']$ and the right-hand side of the preceding formula equals $\langle v, \partial\mathcal{M}'\rangle$, reducing the formula to the usual Stokes' theorem. As noted in [17], the compactness of $\mathcal{M}$ ensures the finiteness of the integrals $\langle u, \mathcal{M}'\rangle$, even though $C_*^{\mathcal{M}}$ may have infinite rank.

Now suppose u is a differential k–form on $\mathcal{B}$. Since u restricts to a closed form on any k–dimensional submanifold, it also determines an element $[u]_{\mathcal{M}} \in C^{k;\mathbb{R}}_{\mathcal{M}} = H^k(\mathcal{M}_k, \mathcal{M}_{k-1}; \mathbb{R})$. With $[u]_{\mathcal{M}}$ for differential forms so defined, one has

$$\delta_{\mathcal{M}}[u]_{\mathcal{M}} = [d\,u]_{\mathcal{M}} \in C^{k+1;\mathbb{R}}_{\mathcal{M}} \quad \text{and} \quad \langle [u]_{\mathcal{M}}, [\mathfrak{M}]\rangle = \int_{\mathfrak{M}} u$$

for any k–dimensional stratum $\mathfrak{M}$ of $\mathcal{M}$.

Fix $\mathcal{M} \subset \mathcal{B}$ and $\mathbb{K}$. Let u be a k–cochains in one of the models for $\mathrm{C}^*(\mathcal{B}; \mathbb{K})$ described above, namely, it is a Čech cochain $u \in C^k(\mathcal{U}; \mathbb{K})$ for an arbitrary open cover $\mathcal{U}$ transverse to $\mathcal{M}$, or when $\mathbb{K} = \mathbb{R}$, it can be a differential k–form on $\mathcal{B}$. Let u' be another k–cochain in a possibly different model of $\mathrm{C}^*(\mathcal{B}; \mathbb{K})$. We say that the two "$k$–cochains on $\mathcal{B}$", u and u', are *equivalent on* $\mathcal{M}$ (or simply "equivalent" if the $\mathcal{M}$ being referred to is clear) if $[u]_{\mathcal{M}} = [u']_{\mathcal{M}} \in C^{k;\mathbb{K}}_{\mathcal{M}}$. (In other words, u and u' evaluate identically on all k–dimensional strata of $\mathcal{M}$.) To keep notation simple, we usually omit the subscript $\mathcal{M}$ from $\delta_{\mathcal{M}}$ or $\partial_{\mathcal{M}}$ below.

Now let $\mathcal{B}^\sigma(X)$ be as in Part 1, namely the orbit space of $\mathcal{C}^\sigma(X)$ under gauge group actions. Let u be a k–cochain on $\mathcal{B}^\sigma(X)$ in the sense just explained. For each fixed Spin^c–structure, introduce homomorphisms

$$m^{\sharp}_{\natural}[u](X, \mathfrak{s}_X)\colon C^{\sharp}(Y_-, \mathfrak{s}_-) \to C^{\natural}(Y_+, \mathfrak{s}_+) \quad \text{for } \sharp = o, u,\ \natural = o, s,$$
$$\overline{m}^{\sharp}_{\natural}[u](X, \mathfrak{s}_X)\colon C^{\sharp}(Y_-, \mathfrak{s}_-) \to C^{\natural}(Y_+, \mathfrak{s}_+) \quad \text{for } \sharp = u, s,\ \natural = u, s,$$

respectively, by the rules

$$\begin{aligned} &\mathfrak{C}^{\sharp} \ni \mathfrak{c}_- \mapsto \sum_{\mathfrak{c}_+ \in \mathfrak{C}^{\natural}} \langle u, \mathcal{M}_k(X; \mathfrak{c}_-, \mathfrak{c}_+)\rangle \mathfrak{c}_+, \\ &\mathfrak{C}^{\sharp} \ni \mathfrak{c}_- \mapsto \sum_{\mathfrak{c}_+ \in \mathfrak{C}^{\natural}} \langle u, \mathcal{M}^{\mathrm{red},k}(X; \mathfrak{c}_-, \mathfrak{c}_+)\rangle \mathfrak{c}_+, \end{aligned} \tag{2-19}$$

where $\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+) := \mathcal{M}^{\mathrm{red}}_{\underline{k}}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ with $\underline{k} := k, k, k-1, k+1$ respectively in the cases when the pair $(\natural,\flat)$ is (u,u), (s,s), (s,u) or (u,s). (In other words, $\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ stands for the moduli space of reducible instantons of dimension k). Note that (the interior of) all $\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$, $\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+)$, $\mathcal{M}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}^{\mathrm{red}}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)$ are strata or stratified submanifolds of $\mathcal{M}(X) \subset \mathcal{B}^\sigma(X)$. By the preceding remark, the maps $m^\sharp_\natural[u](X,\mathfrak{s}_X)$ and $\overline{m}^\sharp_\natural[u](X,\mathfrak{s}_X)$ depend only on the class $[u]_{\mathcal{M}} \in C^*_{\mathcal{M}}$.

Once in place, the homomorphisms $m^\sharp_\natural[u](X,\mathfrak{s}_X)$, $\overline{m}^\sharp_\natural[u](X,\mathfrak{s}_X)$, $\partial^\sharp_\natural(Y_\pm,\mathfrak{s}_\pm)$ and $\overline{\partial}^\sharp_\natural(Y_\pm,\mathfrak{s}_\pm)$ can be assembled according to the formulas in (25.5) and Definition 25.3.3 of [17] into homomorphisms

$$\overset{\circ}{m}[u](X,\mathfrak{s}_X)\colon \overset{\circ}{C}_*(Y_-,\mathfrak{s}_-) \to \overset{\circ}{C}_*(Y_+,\mathfrak{s}_+)$$

for $\circ = \vee, -, \wedge$. For example, for $u \in C^k(\mathcal{U};\mathbb{K})$, $\widehat{m}[u]\colon C^o(Y_-) \oplus C^u(Y_-) \to C^o(Y_+) \oplus C^u(Y_+)$ is given in block form as

$$\begin{bmatrix} m^o_o[u] & m^u_o[u] \\ (-1)^k \overline{m}^s_u[u]\partial^o_s - \overline{\partial}^s_u m^o_s[u] & (-1)^k \overline{m}^u_u[u] + (-1)^k \overline{m}^s_u[u]\partial^u_s - \overline{\partial}^s_u m^u_s[u] \end{bmatrix}. \tag{2-20}$$

The gluing theorems in Section 24.7 of [17] show that when u is closed, these are chain maps, with both $\overset{\circ}{C}_*(Y_-,\mathfrak{s}_-)$ and $\overset{\circ}{C}_*(Y_+,\mathfrak{s}_+)$ regarded as chain complexes with relative $\mathbb{Z}/c_{\mathfrak{s}_X}$–grading. As remarked in Section 2.3, gradings on $\overset{\circ}{C}_*(Y_-,\mathfrak{s}_-)$ and $\overset{\circ}{C}_*(Y_+,\mathfrak{s}_+)$ are alternatively described in [17] by $\mathbb{J}(Y_-)$ and $\mathbb{J}(Y_+)$, the geometrically defined grading sets $\mathbb{J}(Y_-)$ and $\mathbb{J}(Y_+)$. A cobordism X determines a relation $\sim_X$ between the grading sets $\mathbb{J}(Y_-)$ and $\mathbb{J}(Y_+)$ mentioned in Section 2.3.

Remark 2.2 In subsequent discussions, we make use of cobordism maps $\overset{\circ}{m}[u]$ associated to more general cochains than those described above. (See in particular Part 3 of Section 2.5 below.) Note that the formula (2-20) defining $\overset{\circ}{m}[u]$ assembles $m^\#_\flat[u]$, $\overline{m}^\#_\flat[u]$, $\partial^\#_\flat$ and $\overline{\partial}^\#_\flat$ in the particular manner specified in [17], so that desirable properties for $\overset{\circ}{m}[u]$ may be obtained by applying the Stokes' theorem for integrands of the form $\mathfrak{r}^*u$ on stratified submanifolds of $\mathcal{M}^+(X)$, with $u \in \mathrm{C}(\mathcal{B}^\sigma_{\mathrm{loc}}(X);\mathbb{K})$. In other words, the integrals defining $\overset{\circ}{m}[u](X)$ factors through integrals over the small compactified moduli space $\overline{\mathcal{M}}(X)$. The more general maps $\overset{\circ}{m}[u](X)$ that we shall encounter are constructed by mapping $\mathcal{M}^+(X)$ to a larger space (typically a bundle over $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$), and considering integrals of pullbacks of cochains on the latter larger space over $\mathcal{M}^+(X)$. To correctly assemble these integrals so as to make the Stokes' theorem useful, the formula defining such $\overset{\circ}{m}[u](X)$ generalizes that given in [17] (as exemplified in (2-20)) by replacing terms of the form $\bar{\partial}^s_u m^\#_s[u]$ or $m^u_\flat[u]\bar{\partial}^s_u$ in [17]'s formulas with a sum, in which it appears as the first terms. In the notation of Part 2 of Section 2.5, the terms in this sum take the general form of $\bar{n}^s_u[\mathrm{u}_+]m^\#_s[u']$ or $m^u_\flat[u']\bar{n}^s_u[\mathrm{u}_+]$, where $\mathrm{u}_\pm$ are cochains on $\mathcal{B}^\sigma(Y_\pm)$, and $\deg(u')+\deg(\mathrm{u}_\pm)=\deg(u)-1$. In particular, $\bar{\partial}^s_u=\bar{n}^s_u[1]$ in this notation.

In order for the maps between Floer homologies induced by these chain maps to behave well when composing cobordisms (exemplified by Proposition 23.2.2 in [17]), one works with the assembled maps

$$\mathring{m}[u](X)=\sum_{\mathfrak{s}_X}\mathring{m}[u](X,\mathfrak{s}_X)\colon \bigoplus_{\mathfrak{s}_-}\mathring{C}_*(Y_-,\mathfrak{s}_-)\to\bigoplus_{\mathfrak{s}_+}\mathring{C}_*(Y_+,\mathfrak{s}_+),\quad \circ=\vee,-,\wedge,$$

where the direct sum $\bigoplus_{\mathfrak{s}_\pm}$ is over the set of all Spin^c–structures on $Y_\pm$, and $\mathfrak{s}_X$ runs through all Spin^c–structures on X. As explained in Remark 24.6.6 in [17], there can be infinitely many $\mathfrak{s}_X$ to sum over for a fixed pair of $\mathfrak{s}_-$ and $\mathfrak{s}_+$. This necessitates the replacement of the chain complexes $\mathring{C}_*(Y_-,\mathfrak{s}_-)$ and $\mathring{C}_*(Y_+,\mathfrak{s}_+)$ in the preceding expression by their "grading-completed" variants, $\mathring{C}_\bullet(Y_-,\mathfrak{s}_-)$ and $\mathring{C}_\bullet(Y_+,\mathfrak{s}_+)$ (see Definition 3.1.3 and paragraphs around (30.1) in [17]). The cobordisms relevant to our proof of Theorem 1.1 however have $H^2(X,Y_-)=0$, and this is why we may use the precompletion Floer complexes $\mathring{C}_*$ as the domain and target of $\mathring{m}[u]$.

Part 3: local coefficients The values $\langle u,\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle$ and $\langle u,\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle$ in (2-19) are finite only if the moduli spaces $\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ have certain compactness properties. The standard compactness arguments can be adapted to work with nonvanishing ϖ_X, when the perturbation form ϖ_X can be written as

$$\varpi_X=2\omega^+ \tag{2-21}$$

for some closed 2–form ω on X. We assume that ϖ_X satisfies (2-21) throughout this article. As with the monopole Floer complex in Section 2.3, the coefficients in (2-19) are finite only when the cohomology classes $c_1[\mathfrak{s}_X]$ and $[\omega]$ are related by certain constraints. A generalization of [17, Lemma 25.3.1] (making use of the modified energy bounds from Section 29.1 therein) guarantees that these constraints are met when

(2-22) $\varpi_X=2\mathrm{r}w_X$ for $\mathrm{r}\neq 0$ and a w_X satisfying (2-12), and when $X_{\mathrm{tor}}\neq s^{-1}(\mathbb{R}-\{0\})$.

For more general pairs of $c_1[\mathfrak{s}_X]$ and $[\omega]$, cobordism maps $\mathring{m}[u]$ may still be well defined for suitable local coefficients. Let Γ_X be an "X–morphism" between local systems Γ_- on $\mathcal{B}^\sigma(Y_-)$ and Γ_+ on $\mathcal{B}^\sigma(Y_+)$ in the sense of [17, Definition 23.3.1]. To each relative homotopy class $z\in\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))$, Γ_X assigns an isomorphism $\Gamma_X(z)\colon \Gamma_-(\mathfrak{c}_-)\to\Gamma_+(\mathfrak{c}_+)$. One then generalizes the homomorphisms (of $\mathbb{Z}$–modules) $m^\sharp_\natural[u](X,\mathfrak{s}_X)$ and $\overline{m}^\sharp_\natural[u](X,\mathfrak{s}_X)$ given by (2-19) to

$$m^\sharp_\natural[u](X,\mathfrak{s}_X;\Gamma_X)\colon C^\sharp(Y_-,\mathfrak{s}_-;\Gamma_-)\to C^\natural(Y_+,\mathfrak{s}_+;\Gamma_+),\quad \sharp=o,u,\ \natural=o,s;$$
$$\overline{m}^\sharp_\natural[u](X,\mathfrak{s}_X;\Gamma_X)\colon C^\sharp(Y_-,\mathfrak{s}_-;\Gamma_-)\to C^\natural(Y_+,\mathfrak{s}_+;\Gamma_+),\quad \sharp=u,s,\ \natural=u,s;$$

these are defined respectively by the formulas

$$\begin{aligned} m^\sharp_\natural[u](X,\mathfrak{s}_X;\Gamma_X)&=\sum_{\mathfrak{c}_-\in\mathfrak{C}^\sharp}\sum_{\mathfrak{c}_+\in\mathfrak{C}^\natural}\sum_{z\in\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))}\langle u,\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle\Gamma_X(z),\\ \overline{m}^\sharp_\natural[u](X,\mathfrak{s}_X;\Gamma_X)&=\sum_{\mathfrak{c}_-\in\mathfrak{C}^\sharp}\sum_{\mathfrak{c}_+\in\mathfrak{C}^\natural}\sum_{z\in\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))}\langle u,\mathcal{M}^{\mathrm{red},k}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle\Gamma_X(z),\end{aligned}\tag{2-23}$$

where $\mathcal{M}_{k,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{M}_k(X;\mathfrak{c}_-,\mathfrak{c}_+)$ and $\mathcal{M}^{\mathrm{red},k}_z(X;\mathfrak{c}_-,\mathfrak{c}_+)\subset\mathcal{M}^{\mathrm{red},k}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ are the subspaces consisting of elements with relative homotopy class z. These $m^\sharp_\natural$ and $\overline{m}^\sharp_\natural$ are assembled in the same manner (eg (2-20) for $\widehat{m}$) into the cobordism maps

$$\mathring{m}[u](X,\mathfrak{s}_X;\Gamma_X)\colon \mathring{C}(Y_-,\mathfrak{s}_-;\Gamma_-)\to\mathring{C}(Y_+,\mathfrak{s}_+;\Gamma_+),\quad \circ=\vee,-,\wedge.$$

Again, for the sums in (2-23) to be well defined, Γ_X and $\Gamma_\pm$ need to satisfy certain completeness conditions depending on $\mathfrak{s}_X$ and ϖ_X. Here we limit ourselves to some general remarks; more details will be provided on a case-by-case basis as occasions arise. See also Section 25.3 in [17], which contains some discussion on the case with $\varpi_X=0$.

Remark 2.3 In the more formal language of [25, Section 6.1], where a "local system" in Floer theory is described as a functor, an "X–morphism" from Γ_- to Γ_+ is a natural transformation that intertwines the fundamental-groupoid structure on both sides. That is to say, it satisfies the composition law in [17, equation (23.7)]. (In [17], $\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))$ is denoted by $\pi(\mathfrak{c}_-,X,\mathfrak{c}_+)$ and an element in $\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+)$ is called an "X–path".) For each pair $\mathfrak{c}_-$ and $\mathfrak{c}_+$, the fundamental groups $\pi_1\mathcal{B}^\sigma(Y_-)\simeq H^1(Y_-;\mathbb{Z})$ and $\pi_1\mathcal{B}^\sigma(Y_+)\simeq H^1(Y_+;\mathbb{Z})$ act respectively from the right and from the left on $\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))$ through "concatenation of paths". Meanwhile,

$$\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))\simeq(j^*)^{-1}(c_1(\mathfrak{s}_X))\subset H^2(X,\partial X;\mathbb{Z})\tag{2-24}$$

in the relative long exact sequence

$$\cdots \to H^1(\partial X;\mathbb{Z}) \xrightarrow{\delta} H^2(X,\partial X;\mathbb{Z}) \xrightarrow{j^*} H^2(X;\mathbb{Z}) \xrightarrow{i^*} H^2(\partial X;\mathbb{Z}) \to \cdots .$$

Note that $(j^*)^{-1}(c_1(\mathfrak{s}_X))$ is an affine space under the abelian group $\mathrm{Im}(\delta) = \mathrm{Ker}(j^*)$. Under the identification (2-24), the $\pi_1\mathcal{B}^\sigma(Y_\pm) \simeq H^1(Y_\pm;\mathbb{Z})$–actions on the group $\pi_0(\mathcal{B}^\sigma(X;\mathfrak{c}_-,\mathfrak{c}_+))$ respectively factor through the aforementioned $\delta H^1(\partial X;\mathbb{Z})$–action on $(j^*)^{-1}(c_1(\mathfrak{s}_X))$ under $\delta_\pm := \delta\circ i_\pm$, where $i_\pm\colon H^1(Y_\pm;\mathbb{Z}) \hookrightarrow H^1(\partial X;\mathbb{Z})$ denotes the inclusion. The following simple consequences of the above observations will be useful in this article:

- When $\delta_\pm\colon H^1(Y_\pm;\mathbb{Z}) \to \mathrm{Im}\,\delta$ are both isomorphisms, any local system Γ_- on $\mathcal{B}^\sigma(Y_-)$ determines a local system Γ_+ on $\mathcal{B}^\sigma(Y_+)$ and a unique (modulo automorphisms of Γ_- and Γ_+) X–morphism Γ_X from Γ_- to Γ_+. Conversely, any local system Γ_+ on $\mathcal{B}^\sigma(Y_+)$ also determines a local system Γ_- on $\mathcal{B}^\sigma(Y_-)$ and a unique X–morphism Γ_X from Γ_- to Γ_+. In this case $\pi_0(\mathcal{B}^\sigma(X,\mathfrak{c}_-,\mathfrak{c}_+))$ is an affine space under both the actions of $\pi_1(\mathcal{B}^\sigma(Y_-))$ and $\pi_1(\mathcal{B}^\sigma(Y_+))$ and a choice of an element $z_0 \in \pi_0(\mathcal{B}^\sigma(X,\mathfrak{c}_-,\mathfrak{c}_+))$ induces isomorphisms $\iota^\pm_{z_0}\colon \pi_1(\mathcal{B}^\sigma(Y_\pm)) \to \pi_0(\mathcal{B}^\sigma(X,\mathfrak{c}_-,\mathfrak{c}_+))$ as $\pi_1(\mathcal{B}^\sigma(Y_\pm))$–spaces.
- It was explained in [25] that the "$(\mathfrak{s},[\varpi])$–completeness" condition for a local system Γ in $\mathring{C}(M,\mathfrak{s},[\varpi];\Gamma)$ is determined by the class $[\varpi]|_{\mathrm{Ker}\,c_1(\mathfrak{s})}$; in particular, when $[\varpi]|_{\mathrm{Ker}\,c_1(\mathfrak{s})} = 0$, any Γ (including $\mathbb{Z}$) is $(\mathfrak{s},[\varpi])$–complete. In the more general setting of cobordisms, the cobordism map $\mathring{m}[u](X;\Gamma_X)$ is well defined via (2-20) when $\Gamma_\pm$ are respectively $(\mathfrak{s}_\pm,[\varpi_\pm])$–complete, *and* an additional completeness condition depending on the class $[\omega]|_{\mathrm{Ker}\,c_1(\mathfrak{s}_X)}$ is satisfied. (Here, $c_1(\mathfrak{s}_X)$ and $[\omega]$ are both viewed as homomorphisms from $H^2(X,\partial X)$ to $\mathbb{Z}$ via the Poincaré–Lefschetz duality.) In particular, this additional completeness condition is vacuous when $[\omega]|_{\mathrm{Ker}\,c_1(\mathfrak{s}_X)} = 0$. Thus, the cobordism map $\mathring{m}[u](X)$ is well defined with coefficient $\mathbb{Z}$ via (2-19) when $[\omega] = 2\mathrm{r}c_1(\mathfrak{s}_X)$ for $\mathrm{r} \in \mathbb{R}$, the setting relevant to the proof of Theorem 1.1.

Part 4: disconnected Y_- or Y_+ Suppose X_i for $i = 1,\dots,k$ are respectively cobordisms from Y_i^+ to Y_i^+, where all $Y_i^\pm$ are connected. Then $X := \coprod_i X_i$ may be viewed as a cobordism from $Y_- := \coprod_i Y_i^-$ to $Y_+ := \coprod_i Y_i^+$. The cobordism map $\widehat{m}[u]$ introduced in Parts 2 and 3 above has a straightforward generalization in this setting: Let $\widehat{C}(Y_\pm) := \bigotimes_{i=1}^k \widehat{C}(Y_i^\pm)$. Observe that in this case $\mathcal{B}^\sigma(X) = \prod_{i=1}^k \mathcal{B}^\sigma(X_i)$, and so given cochains $u_i \in \mathrm{C}^*(\mathcal{B}^\sigma(X_i))$ (in the sense explained in Part 2) and X_i–morphisms

Γ_{X_i} from Γ_i^- to Γ_i^+ for each i, one has a cochain $u := \prod_i u_i \in \prod_i \mathrm{C}^*(\mathcal{B}^\sigma(X_i)) = \mathrm{C}^*(\mathcal{B}^\sigma(X))$ and an X–morphism Γ_X from $\Gamma_- := \prod_i \Gamma_i^-$ to $\Gamma_+ := \prod_i \Gamma_i^+$. Meanwhile, a set of local systems Γ_i^- for each Y_i^+ Define $\widehat{m}[u](X;\Gamma_X)\colon \widehat{C}(Y_-;\Gamma_+) \to \widehat{C}(Y_+;\Gamma_-)$ as

$$\widehat{m}[u](X;\Gamma_-) := \bigotimes_{i=1}^k \widehat{m}[u_i](X_i;\Gamma_{X_i})\colon \bigotimes_{i=1}^k \widehat{C}(Y_i^-;\Gamma_i^-) \to \bigotimes_{i=1}^k \widehat{C}(Y_i^+;\Gamma_i^+). \tag{2-25}$$

The proof of Theorem 1.1 also requires maps associated to more general cobordisms. For this purpose, it suffices to consider the $\widehat{m}$ variant of the chain map for cobordisms X satisfying the following constraint:

(2-26) At most one of Y_- or Y_+ is disconnected, in which case it consists of two components. Moreover, at most one end of X is associated with a balanced perturbation.

Assume that one of Y_- or Y_+ is of the form $Y_\sqcup = Y_1 \sqcup Y_2$ for connected Y_1 and Y_2, while the other is connected. Take $Y_- = Y_\sqcup$ for example, since the case where $Y_+ = Y_\sqcup$ is entirely parallel. Given the self-dual 2–form ϖ_X described in (2-11), we shall always take Y_2 to be the only end of X possibly associated with a balanced perturbation. Thus, $\mathfrak{C}(Y_\sqcup) = \mathfrak{C}(Y_1) \times \mathfrak{C}(Y_2) = \mathfrak{C}^{oo} \sqcup \mathfrak{C}^{ou} \sqcup \mathfrak{C}^{os}$, with $\mathfrak{C}^{oo}$, $\mathfrak{C}^{ou}$ and $\mathfrak{C}^{os}$ denoting $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^o(Y_2)$, $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^u(Y_2)$ and $\mathfrak{C}^o(Y_1) \times \mathfrak{C}^s(Y_2)$, respectively. Let $C^{oo}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{oo}) = \mathrm{CM}(Y_1) \otimes C^o(Y_2)$, $C^{ou}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{ou}) = \mathrm{CM}(Y_1) \otimes C^u(Y_2)$ and $C^{os}(Y_\sqcup) = \mathbb{K}(\mathfrak{C}^{os}) = \mathrm{CM}(Y_1) \otimes C^s(Y_2)$.

In these cases we have the analogs of $m_\natural^\sharp$ in [17], these being the homomorphisms $m_\natural^{o\sharp}\colon \mathrm{CM}(Y_1) \otimes C^\sharp(Y_2) \to C^\natural(Y_+)$ (or $m_{o\natural}^\sharp\colon C^\sharp(Y_-) \to \mathrm{CM}(Y_1) \otimes C^\natural(Y_2)$ in the case where $Y_\sqcup = Y_+$), with $\sharp$ standing for o or u; and with the label $\natural$ standing for o or s. Meanwhile, the analogs of $\overline{m}_\natural^\sharp$ are all trivial, since by (2-26) there are no reducible instantons on X.

As the condition (2-26) implies that $\widehat{C}(Y_\#) = \mathrm{CM}(Y_\#)$ and $\widehat{C}(Y_\sqcup) = C^{oo} \oplus C^{ou}$, the maps $\overline{m} = 0$, $\widehat{m}[u]\colon \mathrm{CM}(Y_\#) \to C^{oo} \oplus C^{ou}$ and $\widehat{m}[u]\colon C^{oo} \oplus C^{ou} \to \mathrm{CM}(Y_\#)$, respectively, take the simple form

$$\begin{bmatrix} m_o^{oo} & m_o^{ou} \end{bmatrix}, \quad \begin{bmatrix} m_{oo}^o \\ -(1 \otimes \overline{\partial}_u^s(Y_2)) \circ m_{os}^o \end{bmatrix}. \tag{2-27}$$

Further properties of the Floer complex $\widehat{C}(Y_\sqcup)$ and the maps $\widehat{m}$ associated to cobordisms X satisfying (2-26) will be discussed in Section 6.1.

Caveat This part assumes implicitly that the X–morphisms and local coefficients involved satisfy appropriate completeness conditions in the sense of Remark 2.3. While we forgo general discussions of this issue, it will be addressed for the special cases in Section 6.

2.5 $A_\dagger$–module actions and geometric cochains

In this subsection we introduce some useful cochains u on $\mathcal{B}^\sigma(X)$. They are described in terms of differential forms on $\mathcal{B}^\sigma(X)$ or $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$. (Note that a differential form on the latter induces a corresponding differential form on the former via pulling back the embedding $\mathcal{B}^\sigma(X) \hookrightarrow \mathcal{B}^\sigma_{\mathrm{loc}}(X)$; since $\mathcal{M}(X) \subset \mathcal{B}^\sigma(X) \hookrightarrow \mathcal{B}^\sigma_{\mathrm{loc}}(X)$, they are equivalent on $\mathcal{M}(X)$ in the sense of Remark 2.1.) To a connected d–dimensional submanifold of X, we associate an element of $\Omega^{2-d}(\mathcal{B}^\sigma(X))$. There are many possible choices of this differential form, but its equivalence class in $C^{*;\mathbb{R}}_{\mathcal{M}}$ will be fixed. To work with more general $\mathbb{K}$, this class is often replaced by a cohomologous element from $C^{*;\mathbb{Z}}_{\mathcal{M}} \subset C^{*;\mathbb{R}}_{\mathcal{M}}$. We then describe $A_\dagger$–module actions on monopole Floer complexes and related chain homotopy maps as maps $\mathring{m}[u]$ associated to product cobordisms $X = \mathbb{R} \times M$ and cochains u of this type.

The significance of such geometrically constructed cochains is that the Seiberg–Witten cobordism maps $\widehat{m}[u]$ have natural counterparts in invariants (some yet to be rigorously defined) constructed from counting pseudoholomorphic curves; in the latter case, the cobordism maps are constructed from submanifolds in X.

Let X be a Spin^c 4–manifold described by (2-8) and (2-9), and let $\mathcal{E} = \{M_i\}_i$ be the set of connected oriented Spin^c–manifolds indexing the ends of X.

Fix a self-dual 2–form ϖ_X on X satisfying (2-11) and a suitable pair of $(\mathfrak{T}^+, \mathfrak{S}^+)$. Let $\mathcal{M}(X)$ be the stratified manifold of instanton solutions to (2-10) introduced in Part 1 of the last subsection, with stratification $\varnothing \subset \mathcal{M}_0(X) \subset \cdots \subset \mathcal{M}_k(X) \subset \mathcal{M}_{k+1}(X) \subset \mathcal{M}(X)$ as before.

Fix a Hermitian line bundle K on X and a smooth connection A_K on K^{-1}, and write

$$\det \mathbb{S}^+ = E^2 \otimes K^{-1}; \tag{2-28}$$

namely, a 4–dimensional version of (2-4). Let $A \in \mathrm{Conn}(E)$ denote the unitary connection induced from $\mathbb{A} \in \mathrm{Conn}(\det \mathbb{S}^+)$. As mentioned previously in the end of Section 2.2, both $(\mathbb{A}, \Psi) \in \mathrm{Conn}(\det \mathbb{S}^+) \times C^\infty(\mathbb{S}^+)$ and its corresponding $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(\mathbb{S}^+)$ are used to denote an element in $\mathcal{C}(X)$. At this point K is not assumed to be related to ϖ_X. In the case when the factorization (2-4) or (2-28) arises from a splitting $\mathbb{S}$ or $\mathbb{S}^+ = E \oplus E \otimes K^{-1}$, we write $\psi = (\alpha, \beta)$, where α and β are respectively the E and the $E \otimes K^{-1}$ component of ψ under the decomposition.

Part 1: cocycles on $\mathcal{B}^\sigma(X)$ from closed d–submanifolds in X The cocycles in this part are constructed from differential forms on $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$. As mentioned previously, they induce differential forms on $\mathcal{B}^\sigma(X)$ and we shall use the same notation for forms on $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$ and their corresponding forms on $\mathcal{B}^\sigma(X)$. Alternatively, one may define the forms on $\mathcal{B}^\sigma(X)$ by carrying out parallel arguments using $\mathcal{C}^\sigma(X)$ in place of $\mathcal{C}^\sigma_{\mathrm{loc}}(X)$.

(a) (when $d = 0$) To a point $x \in X$ we associate an integral 2–cocycle $[e]_{\mathcal{M}(X)} \in C^{2;\mathbb{Z}}_{\mathcal{M}(X)}$ as follows. Consider the subgroup

$$\mathcal{G}_{x,\mathrm{loc}} \subset C^\infty(X, U(1)) := \mathcal{G}_{\mathrm{loc}}(X)$$

consisting of maps $u\colon X \to U(1)$ with $u(x) = 1$. Then

$$\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) := \mathcal{C}^\sigma_{\mathrm{loc}}(X)/\mathcal{G}_{x,\mathrm{loc}}$$

admits a free $U(1) = \mathcal{G}_{\mathrm{loc}}(X)/\mathcal{G}_{x,\mathrm{loc}}$–action, and $\mathcal{B}^\sigma_{x,\mathrm{loc}}(X)$ is the orbit space of this action. Let

$$\pi_x\colon \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) \to \mathcal{B}^\sigma_{\mathrm{loc}}(X)$$

denote the quotient map of this action. We use $\vartheta \in \Omega^1(\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X))$ to denote a Thom form of the $U(1)$–fibration $\pi_x\colon \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) \to \mathcal{B}^\sigma_{\mathrm{loc}}(X)$, so that

$$d\vartheta = \pi_x^* e,$$

$e \in \Omega^2(\mathcal{B}^\sigma_{\mathrm{loc}}(X))$ being an Euler form. Choose ϑ so that it defines a principal $U(1)$–connection on $\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)$, now regarded as a principal $U(1)$–bundle. In this setting $(\pi_x)_* := (\pi_x^*)^{-1}$ is well defined at $d\vartheta$, and we formally write $e = (\pi_x)_*(d\vartheta)$. Let $\mathcal{E}_x$ be the Hermitian line bundle associated to the principal $U(1)$–bundle $\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)$. The latter is identified with the $(U(1))$ fiber product

$$\mathcal{E}_x(X) := \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) \times_{U(1)} E_x = (\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) \times E_x)/\text{diagonal } U(1)\text{–action},$$

where $E_x \simeq \mathbb{C}$ is the fiber of the bundle E over $x \in X$, equipped with the $U(1) = \mathcal{G}_{\mathrm{loc}}(X)/\mathcal{G}_{x,\mathrm{loc}}(X)$–action. Then e has an alternative interpretation as $\frac{i}{2\pi}$ times the curvature form of the unitary connection associated to ϑ on $\mathcal{E}_x$.

The following alternative interpretation of $\mathcal{E}_x(X)$ will come in handy later: let the map $\boldsymbol{\pi}\colon \boldsymbol{\mathcal{E}}(X) \to X \times \mathcal{B}^\sigma_{\mathrm{loc}}(X)$ be the "universal family" (described below) for the bundle

$\pi_E \colon E \to X$; then

$$\mathcal{E}_x(X) = \boldsymbol{\mathcal{E}}(X)|_{\{x\}\times\mathcal{B}^\sigma_{\mathrm{loc}}(X)}.$$

The bundle $\boldsymbol{\mathcal{E}}(X)$ is constructed in the following manner. Consider the Hermitian line bundle $\pi_E\times\mathrm{Id}\colon E\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)\to X\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)$. This bundle is equipped with a tautological unitary connection $\widetilde{A}$ characterized by the following property: $\widetilde{A}|_{X\times\{(\mathbb{A},(\boldsymbol{\Psi},\Psi))\}} = A$ for all $(\mathbb{A},(\boldsymbol{\Psi},\Psi))\in\mathcal{C}^\sigma_{\mathrm{loc}}(X)$, and $\widetilde{A}|_{\{x\}\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)}$ is trivial for each $x\in X$. In the case when $E\subset\mathbb{S}^+$ is a summand of a splitting of $\mathbb{S}^+$, the bundle $\pi_E\times\mathrm{Id}\colon E\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)\to X\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)$ also carries a tautological section $\widetilde{\alpha}$, characterized by the property that $\widetilde{\alpha}|_{X\times\{(\mathbb{A},(\boldsymbol{\Psi},\Psi=\sqrt{2r}(\alpha,\beta)))\}}=\alpha$. Let $\boldsymbol{\mathcal{E}}(X)$ be the quotient of $E\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)$ by the diagonal $\mathcal{G}_{\mathrm{loc}}(X)$–action. The map $\pi_E\times\mathrm{Id}\colon E\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)\to X\times\mathcal{C}^\sigma_{\mathrm{loc}}(X)$ then descends to define a Hermitian line bundle

$$\boldsymbol{\pi}\colon \boldsymbol{\mathcal{E}}(X)\to X\times\mathcal{B}^\sigma_{\mathrm{loc}}(X),$$

and $\widetilde{\alpha}$ (when defined) and $\widetilde{A}$ descend respectively to define a tautological section and a tautological unitary connection on $\boldsymbol{\mathcal{E}}(X)$, also denoted by $\widetilde{\alpha}$ and $\widetilde{A}$ below. Let $\overline{X}\supset X$ denote the compactification of X over which the diffeomorphisms in (2-8) extend to define a diffeomorphism between $([-\infty,-L')\times Y_-)\sqcup((L',\infty]\times Y_-)$ and $\overline{X}-X_c$. When restricted to $X\times\mathcal{B}^\sigma(X)\subset X\times\mathcal{B}^\sigma_{\mathrm{loc}}(X)$, the bundle $\boldsymbol{\mathcal{E}}(X)|_{X\times\mathcal{B}^\sigma(X)}$ extends to define a bundle over $\overline{X}\times\mathcal{B}^\sigma(X)$, denoted by

$$\boldsymbol{\pi}\colon \boldsymbol{\mathcal{E}}(\overline{X})\to\overline{X}\times\mathcal{B}^\sigma(X)$$

below. The tautological section and connection, $\widetilde{\alpha}$ (when defined) and $\widetilde{A}$, extend over $\boldsymbol{\mathcal{E}}(\overline{X})$ and will be denoted by the same notation.

Restricting the tautological connection $\widetilde{A}$ to $\boldsymbol{\mathcal{E}}(X)|_{\{x\}\times\mathcal{B}^\sigma_{\mathrm{loc}}(X)}=\mathcal{E}_x(X)$, one has a unitary connection on $\mathcal{E}_x(X)$. Let $\widetilde{\vartheta}$ denote the corresponding principal $U(1)$–connection on $\widetilde{\mathcal{B}}_{x,\mathrm{loc}}(X)$, ie the principal $U(1)$–bundle associated to $\mathcal{E}_x(X)$, and let

$$\theta := (\pi_x)_*(\vartheta-\widetilde{\vartheta})\in\Omega^1(\mathcal{B}^\sigma_{\mathrm{loc}}(X)). \tag{2-29}$$

The form ϑ (and consequently its associated $\mathfrak{e}$) is far from unique. However, as mentioned in Remark 2.1, we are only interested in $\mathfrak{e}$'s equivalence class rel $\mathcal{M}(X)$ or ϑ's equivalence class rel $\pi_x^{-1}\mathcal{M}(X)$, where $\pi_x^{-1}\mathcal{M}(X)\subset\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)$ is viewed a stratified manifold with stratification $\varnothing\subset\cdots\subset\pi_x^{-1}\mathcal{M}_{k+1}(X)\subset\pi_x^{-1}\mathcal{M}_k(X)\subset\cdots\subset\pi_x^{-1}\mathcal{M}(X)$. For this purpose it suffices to describe $\vartheta|_{\pi_x^{-1}\mathcal{M}_1(X)}$.

We say that the connection ϑ is *integral* over $\mathcal{M}_1(X)$ if it is induced from a trivialization $\rho_\vartheta\colon\underline{\mathbb{C}}\xrightarrow{\sim}\mathcal{E}_x|_{\mathcal{M}_1(X)}$, where $\underline{\mathbb{C}}$ denotes the trivial $\mathbb{C}$–bundle $\mathcal{M}_1(X)\times\mathbb{C}$. Note that conversely ρ_ϑ is uniquely determined by ϑ modulo constant $U(1)$–actions. We also use ρ_ϑ to denote the associated trivialization $\underline{U(1)} \xrightarrow{\sim} \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)|_{\mathcal{M}_1(X)}$. We require that

$$\vartheta \text{ be } \textit{integral} \text{ over } \mathcal{M}_1(X). \tag{2-30}$$

Such ϑ exists since there is no obstruction to trivializing $U(1)$–bundles over 1–complexes. Since the boundary of each 2–dimensional stratum $\mathfrak{M}$ of $\mathcal{M}(X)$ lies in $\mathcal{M}_1(X)$, a choice of such ϑ determines a well-defined relative Euler class for the $U(1)$–bundle $\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)|_{\mathcal{M}_2(X)\setminus\mathcal{M}_1(X)}$ (or equivalently, for $\mathcal{E}_x|_{\mathcal{M}_2(X)\setminus\mathcal{M}_1(X)}$). This class in $H^2(\mathcal{M}_2(X), \mathcal{M}_1(X); \mathbb{Z})$ is by definition the equivalence class $[e]_{\mathcal{M}(X)} \in C^{2;\mathbb{Z}}_{\mathcal{M}(X)}$.

Two connections ϑ_1 and ϑ_2 that are both integral over $\mathcal{M}_1(X)$ differ by $\vartheta_2 - \vartheta_1 = \pi_x^* df$ on $\mathcal{M}_1(X)$, where f is a map $f\colon \mathcal{M}_1(X) \to U(1) = \mathbb{R}/\mathbb{Z}$ (such f is unique modulo constant maps). Thus, $[(\pi_x)_*(\vartheta_2 - \vartheta_1)]_{\mathcal{M}(X)} \in C^{0,\mathbb{R}}_{\mathcal{M}(X)}$ is a closed element. We say that ϑ_1 and ϑ_2 are *δ–cohomologous* if $[df] = 0 \in H^1(\mathcal{M}_1(X); \mathbb{Z})$. In this case f factors through a map $\tilde{f}\colon \mathcal{M}_1(X) \to \mathbb{R}$, and the restriction $\tilde{f}|_{\mathcal{M}_0(X)}$ defines a class $[\tilde{f}]_{\mathcal{M}} \in C^{0;\mathbb{R}}_{\mathcal{M}(X)}$. We have $[(\pi_x)_*(\vartheta_2 - \vartheta_1)]_{\mathcal{M}(X)} = \delta[\tilde{f}]_{\mathcal{M}(X)}$ for δ–cohomologous ϑ_1 and ϑ_2, and hence $[e]_{\mathcal{M}(X)}$ depends only on the δ–cohomology class of ϑ.

Convention When we wish to emphasize the choice of x, we add a subscript x to the forms ϑ, $\tilde{\vartheta}$, θ and e introduced above. For example, ϑ_x denotes the ϑ associated to x.

(b) (when $d = 1$) Let $\gamma \subset X$ be an embedded oriented circle in the interior of X. To such a γ, we associate a real 1–cocycle $[\theta_\gamma]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$. Modifying $[\theta_\gamma]_{\mathcal{M}(X)}$, γ is also associated an integral 1–cocycle $[u_\gamma]_{\mathcal{M}(X)}$ cohomologous to $[\theta_\gamma]_{\mathcal{M}(X)}$. Let

$$\mathrm{hol}_\gamma\colon \mathcal{B}^\sigma_{\mathrm{loc}}(X) \to U(1) = \mathbb{R}/\mathbb{Z}$$

be the map sending an element $\mathfrak{d} \in \mathcal{C}^\sigma_{\mathrm{loc}}(X)$ to the holonomy of $A \in \mathrm{Conn}(E)$ associated to $\mathfrak{d}$. Let

$$\theta_\gamma := d\,\mathrm{hol}_\gamma \in \Omega^1(\mathcal{B}^\sigma_{\mathrm{loc}}(X)).$$

This is an integral closed 1–form on $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$ and defines a class $[\theta_\gamma]_{\mathcal{M}(X)} \in C^{1,\mathbb{R}}_{\mathcal{M}(X)}$. For the purpose of defining cobordism maps, it is often desirable to replace $[\theta_\gamma]_{\mathcal{M}(X)} \in C^{1,\mathbb{R}}_{\mathcal{M}(X)}$ with a cohomologous element

$$u_\gamma = [\theta_\gamma]_{\mathcal{M}(X)} - \delta\varepsilon_\gamma,$$

$\varepsilon_\gamma \in C^{0;\mathbb{R}}_{\mathcal{M}(X)}$, so that $u_\gamma \in C^{1,\mathbb{Z}}_{\mathcal{M}(X)} \subset C^{1,\mathbb{R}}_{\mathcal{M}(X)}$. We call u_γ an "integral correction" of $[\theta_\gamma]_{\mathcal{M}(X)}$. A choice of integral correction u_γ is equivalent to a choice of lifting,

$$h_\gamma\colon \mathcal{M}_0(X) \to \mathbb{R}$$

for $\mathrm{hol}_\gamma|_{\mathcal{M}_0(X)}\colon \mathcal{M}_0(X) \to \mathbb{R}/\mathbb{Z}$, as $\delta\varepsilon_\gamma = \delta[h_\gamma]_{\mathcal{M}(X)}$. Different choices of h_γ differ by elements in $C^{0;\mathbb{Z}}_{\mathcal{M}(X)}$.

Part 2: product cobordisms and $A_\dagger(M)$–actions In this part we apply the construction in Part 1 to the case of product cobordisms. Let $X = \mathbb{R} \times M$, M being a closed connected Spin^c 3–manifold. The cocycles e and θ_γ on $\mathcal{B}^\sigma(X)$ described below, loosely speaking, will take the form of pullbacks from corresponding cocycles on $\mathcal{B}^\sigma(M)$. The latter cocycles are chosen to represent generators of the cohomological algebra

$$\begin{aligned} H^*(\mathcal{B}^\sigma(M);\mathbb{Z}) &= H^*(\mathcal{B}^\sigma_{\mathrm{loc}}(M);\mathbb{Z}) \\ &\simeq H^*(\mathbb{C}P^\infty;\mathbb{Z}) \otimes H^*(H^1(M;\mathbb{R})/H^1(M;\mathbb{Z});\mathbb{Z}) \\ &= A_\dagger(M). \end{aligned} \tag{2-31}$$

(See eg Proposition 9.7.1 of [17].) Let $U \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ be the generator of the polynomial algebra $H^*(\mathbb{C}P^\infty;\mathbb{Z})$, and let $\{\mathfrak{t}_i\}_i$ be a basis of $H_1(M;\mathbb{Z})/\mathrm{Tors} \simeq H^1(H^1(M;\mathbb{R})/H^1(M;\mathbb{Z});\mathbb{Z})$. We use the same notation U and $\mathfrak{t}_i$ to denote the corresponding generating elements of the algebra (2-31). We shall introduce 2–cocycles μ_U representing U and 1–cocycles $\mu_{\mathfrak{t}_i}$ representing $\mathfrak{t}_i$, and the cobordism maps associated to pullbacks of these cocycles are referred to respectively as U–actions or $\mathfrak{t}_i$–actions on the monopole Floer complex $\mathring{C}(M)$. Together they generate the $A_\dagger(M)$–actions on $\mathring{C}(M)$. The choice of μ_U depends on a choice of a point $p \in M$, while the choice of $\mu_{\mathfrak{t}_i}$ depends on the choice of an embedded circle $\gamma_i \subset M$ representing $\mathfrak{t}_i$.

Before proceeding, we make some preparatory remarks on $\mathcal{B}^\sigma_{\mathrm{loc}}(X)$ and its variants in the case $X = \mathbb{R} \times M$. As explained in [17], by a unique continuation theorem $\mathcal{M}(\mathbb{R} \times M)$ falls in a smaller blown-up configuration space

$$\mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R} \times M) = \mathrm{Conn}(\det \mathbb{S}^+) \times \Gamma^\tau(\mathbb{R} \times M, \mathbb{S}^+) \subset \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R} \times M),$$

which is often more convenient to work with. (See Section 6.3 of [17] for more details on the "τ–model" $\mathcal{B}^\tau$.) Here, $\Gamma^\tau(\mathbb{R} \times M, \mathbb{S}^+) \subset \Gamma^\sigma(\mathbb{R} \times M, \mathbb{S}^+)$ consists of elements $(\mathbb{A}, (\boldsymbol{\Psi}, \Psi))$ such that $\Psi|_{\{s\}\times M} \neq 0$ for all $s \in \mathbb{R}$. By construction, there exists for each $s \in \mathbb{R}$ a map

$$\Pi^s\colon \mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R} \times M) \to \mathcal{B}^\sigma, (M)$$

which is defined by restricting $\mathbb{A}$ and Ψ to $\{s\}\times M\subset X$. When restricted to $\mathcal{B}^\tau(\mathbb{R}\times M):=\mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R}\times M)\cap\mathcal{B}^\sigma(X)$, Π^s has well-defined limits as $s\to\pm\infty$,

$$\Pi^{\pm\infty}\colon \mathcal{B}^\tau(\mathbb{R}\times M)\to\mathcal{B}^\sigma(M).$$

An element in $\mathfrak{d}\in\mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R}\times M)$ defines a path $\mathfrak{d}(s)$ in $\mathcal{B}^\sigma(M)$: $s\in\mathbb{R}\mapsto\Pi^s\mathfrak{d}\in\mathcal{B}^\sigma(M)$. Conversely, a path $\mathfrak{d}(\cdot)\colon\mathbb{R}\to\mathcal{B}^\sigma(M)$ *together* with a ∇_s, the latter being the $\frac{\partial}{\partial s}$ component of an $A\in\mathrm{Conn}(E)$, determines a $\mathfrak{d}\in\mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R}\times M)$. Denote the ∇_s associated to $\mathfrak{d}$ by $\nabla_s^{\mathfrak{d}}$. This corresponds to the second term in [17, (4.10)], and is a lift of the vector field $\frac{\partial}{\partial s}$ on the base $\mathbb{R}\times M$ to the total space of the bundle E.

As $\mathcal{M}(\mathbb{R}\times M)\subset\mathcal{B}^\tau(\mathbb{R}\times M)$, the cocycles introduced in Part 1 may be defined using $\mathcal{B}^\tau(\mathbb{R}\times M)$ in place of $\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$.

Translations on $\mathbb{R}\times M$ induce an $\mathbb{R}$–action on $\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ or $\mathcal{B}^\tau(\mathbb{R}\times M)$ in the following manner: For each $a\in\mathbb{R}$, let

$$\tau_a\colon \mathbb{R}\times M\to\mathbb{R}\times M$$

denote the map sending $(s,p)\in\mathbb{R}\times M$ to $(s+a,p)$. For each $\mathfrak{d}\in\mathcal{B}^\tau_{\mathrm{loc}}(\mathbb{R}\times M)$, its associated $\nabla_s^{\mathfrak{d}}$ defines a lift of τ_a to a bundle automorphism of E (or equivalently of $\mathbb{S}^+$), denoted by $\tau_a^{\mathfrak{d}}$ below. Let

$$\tau_a^{\mathcal{B}}\colon \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)\to\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$$

send $\mathfrak{d}$ to the pullback of $\mathfrak{d}$ (as a gauge-equivalence class of $\mathrm{Conn}(\det\mathbb{S}^+)\times\Gamma^\sigma(\mathbb{S}^+)$) via $\tau_{-a}^{\mathfrak{d}}$. Use the same notation, $\tau_a^{\mathcal{B}}$, to denote the similarly defined map from $\mathcal{B}^\tau(\mathbb{R}\times M)$ to itself. In particular, $\tau_a^{\mathcal{B}}$ sends $\mathfrak{d}(s)$ to $\mathfrak{d}(s+a)$. Let

$$\bar{\imath}\colon \mathcal{B}^\sigma(M)\to\mathcal{B}^\tau(\mathbb{R}\times M)$$

be the embedding that sends a $\mathfrak{c}\in\mathcal{B}^\sigma(M)$ to $\mathbb{R}$–invariant element $\mathfrak{d}_{\mathfrak{c}}$ with $\mathfrak{d}_{\mathfrak{c}}(s)=\mathfrak{c}$ for all $s\in\mathbb{R}$. The fixed-point set of the $\mathbb{R}$–action on $\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ is the image of $\bar{\imath}$, and the action is free on the rest of $\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$.

The $\mathbb{R}$–actions $\tau_a^{\mathcal{B}}$ preserve the subspace $\mathcal{M}(\mathbb{R}\times M)\subset\mathcal{B}^\tau(\mathbb{R}\times M)$, together with all of its strata. The fixed-point set of the aforementioned $\mathbb{R}$–action on $\mathcal{M}(\mathbb{R}\times M)$ is

$$\mathcal{M}_0(\mathbb{R}\times M)\simeq\mathfrak{C}(M)\subset\mathcal{B}^\sigma(M),$$

and the action is free on all higher-dimensional strata of $\mathcal{M}(\mathbb{R}\times M)$. Thus, the orbit space $\mathcal{N}_k(M):=(\mathcal{M}_{k+1}(\mathbb{R}\times M)\setminus\mathcal{M}_k(\mathbb{R}\times M))/\mathbb{R}$ is a k–dimensional manifold. As explained in Section 16.1 of [17], the spaces $\mathcal{N}_k(M)$ are compactified into a stratified manifold $\mathcal{N}_k^+(M)$ by adding "(unparametrized) broken trajectories", and the quotient map

$$q_{\mathbb{R}}\colon \mathcal{M}_{k+1}(\mathbb{R}\times M)\setminus\mathcal{M}_k(\mathbb{R}\times M)\xrightarrow{/\mathbb{R}}\mathcal{N}_k(M)$$

extends to a map between the stratified manifolds $(\mathcal{M}_{k+1}(\mathbb{R}\times M)/\mathcal{M}_k(\mathbb{R}\times M))^+$ and $\mathcal{N}_k^+(M)$, also denoted by $q_{\mathbb{R}}$. Recalling that each $\mathcal{M}_{k+1}(\mathbb{R}\times M)/\mathcal{M}_k(\mathbb{R}\times M)$ is a disjoint union of moduli spaces of the form $\mathcal{M}_z(\mathfrak{c}_-,\mathfrak{c}_+)$ or $\mathcal{M}_z^{\mathrm{red}}(\mathfrak{c}_-,\mathfrak{c}_+)$, the space $(\mathcal{M}_{k+1}(\mathbb{R}\times M)/\mathcal{M}_k(\mathbb{R}\times M))^+$ above denotes the disjoint union of their respective compactifications, $\mathcal{M}_z^+(\mathfrak{c}_-,\mathfrak{c}_+)$ or $\mathcal{M}_z^{\mathrm{red}+}(\mathfrak{c}_-,\mathfrak{c}_+)$. Correspondingly, $\mathcal{N}_k^+(M)$ is a disjoint union of compactified spaces of the form $\mathcal{N}_z^+(\mathfrak{c}_-,\mathfrak{c}_+):=(\mathcal{M}_z(\mathfrak{c}_-,\mathfrak{c}_+)/\mathbb{R})^+$ or $\mathcal{N}_z^{\mathrm{red}+}(\mathfrak{c}_-,\mathfrak{c}_+):=(\mathcal{M}_z^{\mathrm{red}}(\mathfrak{c}_-,\mathfrak{c}_+)/\mathbb{R})^+$.

(a) (the U–map) Fix $p\in M$ and let $x=(0,p)\in\mathbb{R}\times M=X$. Let

$$\pi_x\colon \widetilde{\mathcal{B}}_x^\tau(\mathbb{R}\times M):=\pi_x^{-1}\mathcal{B}^\tau(\mathbb{R}\times M)\to\mathcal{B}^\tau(\mathbb{R}\times M)$$

be the principal $U(1)$–bundle obtained by pulling back

$$\pi_x\colon \widetilde{\mathcal{B}}_{x,\mathrm{loc}}^\sigma(\mathbb{R}\times M)\to\mathcal{B}_{\mathrm{loc}}^\sigma(\mathbb{R}\times M)$$

via the embedding $\mathcal{B}^\tau(\mathbb{R}\times M)\hookrightarrow\mathcal{B}_{\mathrm{loc}}^\sigma(\mathbb{R}\times M)$. Define $\pi_x\colon\widetilde{\mathcal{B}}_x^\tau(\mathbb{R}\times M)\to\mathcal{B}^\tau(\mathbb{R}\times M)$ similarly. Let $\widetilde{\mathcal{B}}_p^\sigma(M)$ be the 3–dimensional analog of $\widetilde{\mathcal{B}}_x^\sigma(X)$; namely, $\widetilde{\mathcal{B}}_p^\sigma(M):=\mathcal{C}^\sigma(M)/\mathcal{G}_p(M)$, with $\mathcal{G}_p(M)\subset C^\infty(M,U(1))$ being the subgroup that consists of maps with value 1 at $p\in M$. Then, by construction, the map Π^0 lifts to a map $\widetilde{\Pi}^0$, that fits into the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{B}}_x^\tau(\mathbb{R}\times M) & \xrightarrow{\widetilde{\Pi}^0} & \widetilde{\mathcal{B}}_p^\sigma(M)\\
{\scriptstyle\pi_x}\big\downarrow & & \big\downarrow{\scriptstyle\pi_p}\\
\mathcal{B}^\tau(\mathbb{R}\times M) & \xrightarrow{\Pi^0} & \mathcal{B}^\sigma(M)
\end{array}$$

Regard $\pi_p\colon\widetilde{\mathcal{B}}_p^\sigma(M)\to\mathcal{B}^\sigma(M)$ as a principal $U(1)$–bundle and let $\vartheta_p'\in\Omega^1(\widetilde{\mathcal{B}}_p^\sigma(M))$ denote a principal $U(1)$–connection on this bundle. We will choose the principal $U(1)$–connection ϑ from Part 1(a) to be of the form

$$\vartheta=(\widetilde{\Pi}^0)^*\vartheta_p'.$$

By the unique continuation theorem (see [17, Proposition 7.2.1]), $\Pi^0|_{\mathcal{M}(\mathbb{R}\times M)}$ is an isomorphism, and we choose ϑ_p' to be integral over $\Pi^0\mathcal{M}_1(\mathbb{R}\times M)$, so that ϑ meets the integrability requirement (2-30). Given $p\in M$, the 2–cocycle μ_U on $\mathcal{B}^\sigma(M)$ used to define the U–action is the Euler form $e'_p = (\pi_p)_* d\vartheta'_p$ of the bundle $\widetilde{\mathcal{B}}^\sigma_p(M)$. It is straightforward to verify that indeed $[e'_p] = U \in H^*(\mathcal{B}^\sigma(M); \mathbb{Z})$. Let

$$\mathring{U}_p := \mathring{m}[e](\mathbb{R}\times M)\colon \mathring{C}(M) \to \mathring{C}(M),$$

where $e = (\pi_x)_*(d\vartheta) = \Pi_0^* e'_p$ as before. We call this degree -2 cobordism chain map *the U–map associated to p* on the monopole Floer complex $\mathring{C}(M)$.

It is desirable to express U_p in terms of integrals over the unparametrized moduli spaces, $\mathcal{N}_k(M)$, in a way similar to the formulas (2-15)–(2-16) for the differential ∂ of the monopole Floer complex. For this purpose we digress to make some preparatory observations.

Let $(a,b) \in \mathbb{R}\times\mathbb{R} \mapsto \tau_a \times \tau^{\mathcal{B}}_b$ be the product $\mathbb{R}\times\mathbb{R}$–action on $(\mathbb{R}\times M)\times\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$, ie the base space of the bundle $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)$, and use the same notation to denote the lift via $\widetilde{A}$ of this $\mathbb{R}\times\mathbb{R}$–action to the total space, $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)$. By construction, the tautological $\widetilde{A}$ and $\widetilde{\alpha}$ (when defined) on $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)$ are invariant under pullback of the antidiagonal $\mathbb{R}$–action; namely,

$$(\mathrm{Id}\times\tau^{\mathcal{B}}_a)^*\widetilde{\alpha} = (\tau_a\times\mathrm{Id})^*\widetilde{\alpha}, \quad \text{and similarly for } \widetilde{A}.$$

Let $\mathbb{R}\mathfrak{d} \subset \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ denote the $\mathbb{R}$–orbit through a $\mathfrak{d} \in \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ and let $\hat{p} := \mathbb{R}\times\{p\} \subset \mathbb{R}\times M$ denote the $\mathbb{R}$–orbit through $x = (0,p)$. Then the aforementioned antidiagonal $\mathbb{R}$–action on $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)$ defines a bundle isomorphism ι_Δ between $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)|_{\hat{p}\times\{\mathfrak{d}\}} \simeq E|_{\hat{p}}$ and $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)|_{\{x\}\times\mathbb{R}\mathfrak{d}} \simeq \mathcal{E}_x(X)|_{\mathbb{R}\mathfrak{d}}$, and parallel transports via $\widetilde{A}$ along the two paths $\hat{p}\times\{\mathfrak{d}\}$ and $\{x\}\times\mathbb{R}\mathfrak{d}$ in $(\mathbb{R}\times M)\times\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ are identified under ι_Δ. Note that the connection $\widetilde{A}|_{\hat{p}\times\{\mathfrak{d}\}}$ on $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)|_{\hat{p}\times\{\mathfrak{d}\}} \simeq E|_{\hat{p}}$ is precisely the restriction of $\nabla^{\mathfrak{d}}_s$ to $E|_{\hat{p}}$. This is identified via ι_Δ with the connection $\widetilde{A}|_{\{x\}\times\mathbb{R}\mathfrak{d}}$ on $\boldsymbol{\mathcal{E}}(\mathbb{R}\times M)|_{\{x\}\times\mathbb{R}\mathfrak{d}} \simeq \mathcal{E}_x(X)|_{\mathbb{R}\mathfrak{d}}$, which corresponds to $\widetilde{\vartheta}|_{\mathbb{R}\mathfrak{d}}$ on $\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(\mathbb{R}\times M)|_{\mathbb{R}\mathfrak{d}}$. (Recall the definition of $\widetilde{\vartheta}$ from Part 1(a).) Observe, by the way, that $\mathcal{E}_x(X) \simeq \boldsymbol{\mathcal{E}}(\mathbb{R}\times M)|_{\{x\}\times\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)}$ admits an $\mathbb{R}$–action $a \in \mathbb{R} \mapsto \mathrm{Id}\times\tau^{\mathcal{B}}_a$, and the associated $\mathbb{R}$–action on $\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(\mathbb{R}\times M)$ is precisely the lift of the $\mathbb{R}$–action on $\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)$ via $\widetilde{\vartheta}$. Namely, denoting the lift of $\tau^{\mathcal{B}}_a$ by the same notation, we have the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(\mathbb{R}\times M) & \xrightarrow{\tau^{\mathcal{B}}_a} & \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(\mathbb{R}\times M) \\
{\scriptstyle\pi_x}\big\downarrow & & \big\downarrow{\scriptstyle\pi_x} \\
\mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M) & \xrightarrow{\tau^{\mathcal{B}}_a} & \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R}\times M)
\end{array}$$

Let $\overline{\mathbb{R}} := [-\infty, \infty] \supset \mathbb{R}$, and so in the present setting $\overline{X} = \overline{\mathbb{R}} \times M$. Suppose $\mathfrak{d} \in \mathcal{B}^\tau(\mathbb{R} \times M)$. Then, by definition, $\tau_a^{\mathcal{B}}(\mathfrak{d})$ converges as $a \to \pm\infty$ (in the subspace topology of $\mathcal{B}^\tau(\mathbb{R} \times M) \subset \mathcal{B}^\sigma_{\mathrm{loc}}(\mathbb{R} \times M)$) respectively to

$$\tau^{\mathcal{B}}_{\pm\infty}(\mathfrak{d}) = \bar{\iota}(\Pi^{\pm\infty}\mathfrak{d}).$$

Let $\overline{\mathbb{R}}\mathfrak{d} \subset \mathcal{B}^\tau(\mathbb{R} \times M)$ denote $\{\tau_s^{\mathcal{B}}(\mathfrak{d})\}_{s \in \overline{\mathbb{R}}}$. Thus, the paths $\hat{p} \times \{\mathfrak{d}\}$ and $\{x\} \times \mathbb{R}\mathfrak{d}$ in $(\overline{R} \times M) \times \mathcal{B}^\tau(\mathbb{R} \times M)$ extend respectively to arcs $(\overline{\mathbb{R}} \times \{p\}) \times \{\mathfrak{d}\}$ and $\{x\} \times \overline{\mathbb{R}}\mathfrak{d}$. The previously introduced bundle isomorphism ι_Δ extends to define a bundle isomorphism

$$\iota_\Delta \colon \boldsymbol{\mathcal{E}}(\overline{\mathbb{R}} \times M)|_{(\overline{\mathbb{R}} \times \{p\}) \times \{\mathfrak{d}\}} \simeq E|_{\overline{\mathbb{R}} \times \{p\}} \xrightarrow{\sim} \boldsymbol{\mathcal{E}}(\overline{\mathbb{R}} \times M)|_{\{x\} \times \overline{\mathbb{R}}\mathfrak{d}} \simeq \mathcal{E}_x(\overline{\mathbb{R}} \times M)|_{\overline{\mathbb{R}}\mathfrak{d}}.$$

The assumption that $\mathfrak{d} \in \mathcal{B}^\tau(\mathbb{R} \times M)$ also ensures that parallel transport via $\nabla_s^{\mathfrak{d}}$ along $\overline{\mathbb{R}} \times \{p\}$ gives a well-defined unitary holonomy map

$$\mathrm{hol}^E_{\hat{p}}(\mathfrak{d}) \in \mathrm{Hom}(E|_{(-\infty,p)}, E|_{(\infty,p)}) \simeq \mathrm{Hom}(\boldsymbol{\mathcal{E}}|_{(-\infty,p)\times\{\mathfrak{d}\}}, \boldsymbol{\mathcal{E}}|_{(\infty,p)\times\{\mathfrak{d}\}}).$$

As ι_Δ^* preserves $\widetilde{A}$, the holonomy of $\widetilde{A}$ along $\{x\} \times \overline{\mathbb{R}}\mathfrak{d}$ also gives a well-defined unitary element agreeing with $\iota_\Delta \circ \mathrm{hol}^E_{\hat{p}}(\mathfrak{d}) \circ \iota_\Delta^{-1}$ in

$$\mathrm{Hom}(\mathcal{E}_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_-)\}}, \mathcal{E}_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_+)\}}),$$

where $\mathfrak{c}_\pm := \Pi^{\pm\infty}(\mathfrak{d})$. The space of unitary elements in

$$\mathrm{Hom}(\mathcal{E}_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_-)\}}, \mathcal{E}_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_+)\}})$$

is precisely

$$\widetilde{\mathcal{B}}^\tau_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_-)\}} \times_{U(1)} \widetilde{\mathcal{B}}^\tau_x(\mathbb{R} \times M)|_{\{\bar{\iota}(\mathfrak{c}_+)\}} \simeq \widetilde{\mathcal{B}}^\sigma_p(M)|_{\{\mathfrak{c}_-\}} \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_p(M)|_{\{\mathfrak{c}_+\}}.$$

This is the fiber over $(\mathfrak{c}_-, \mathfrak{c}_+)$ of the $U(1)$–bundle

$$\widetilde{\mathcal{B}}^\sigma_p(M) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_p(M) = (\widetilde{\mathcal{B}}^\sigma_p(M) \times \widetilde{\mathcal{B}}^\sigma_p(M))/\text{diagonal } U(1)\text{–actions} \xrightarrow{\pi_{p-p}} \mathcal{B}^\sigma(M) \times \mathcal{B}^\sigma(M),$$

where π_{p-p} is the quotient map by the residual $U(1)$–action. Let

$$\mathrm{hol}_{\hat{p}} \colon \mathcal{B}^\tau(\mathbb{R} \times M) \to \widetilde{\mathcal{B}}^\sigma_p(M) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_p(M)$$

be the map that sends $\mathfrak{d}$ to the element in $\widetilde{\mathcal{B}}^\sigma_p(M) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_p(M)$ corresponding to $\iota_\Delta \circ \mathrm{hol}^E_{\hat{p}}(\mathfrak{d}) \circ \iota_\Delta^{-1}$. This map is a lift of the map $\Pi^\partial = \Pi^{-\infty} \times \Pi^\infty \colon \mathcal{B}^\tau(\mathbb{R} \times M) \to B^\sigma(M) \times B^\sigma(M)$ in the sense that $\pi_{p-p} \circ \mathrm{hol}_{\hat{p}} = \Pi^\partial$. Meanwhile, letting $\widetilde{\Pi}^a = \widetilde{\Pi}^0 \circ \tau_a$, the map $\widetilde{\Pi}^\partial := \widetilde{\Pi}^{-\infty} \times \widetilde{\Pi}^\infty \colon \widetilde{\mathcal{B}}^\tau_x(\mathbb{R} \times M) \to \widetilde{\mathcal{B}}^\sigma_p(M) \times \widetilde{\mathcal{B}}^\sigma_p(M)$ is in turn a lift of $\mathrm{hol}_{\hat{p}}$ under the quotient map

$$\pi_\Delta\colon \widetilde{\mathcal{B}}^\sigma_p(M)\times\widetilde{\mathcal{B}}^\sigma_p(M)\to\widetilde{\mathcal{B}}^\sigma_p(M)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_p(M).$$

A choice of ϑ'_p determines a principal $U(1)$–connection on the bundle

$$\pi_{p-p}\colon \widetilde{\mathcal{B}}^\sigma_p(M)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_p(M)\to\mathcal{B}^\sigma(M)\times\mathcal{B}^\sigma(M),$$

which we denote by ϑ'_{p-p}. Since ϑ'_p is integral over $\Pi_0\mathcal{M}_1(\mathbb{R}\times M)$, it also determines a trivialization $\rho_{p-p}\colon \underline{U(1)}\xrightarrow{\sim}\widetilde{\mathcal{B}}^\sigma_p(M)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_p(M)|_{\Pi_0\mathcal{M}_1(\mathbb{R}\times M)\times\Pi_0\mathcal{M}_1(\mathbb{R}\times M)}$ of the bundle π_{p-p} over $\Pi_0\mathcal{M}_1(\mathbb{R}\times M)\times\Pi_0\mathcal{M}_1(\mathbb{R}\times M)$. As $\pi_{p-p}(\mathcal{M}(\mathbb{R}\times M))\subset\mathfrak{C}(M)\times\mathfrak{C}(M)\subset\Pi_0\mathcal{M}_1(\mathbb{R}\times M)\times\Pi_0\mathcal{M}_1(\mathbb{R}\times M)$, combining the trivialization ρ_{p-p} with $\widetilde{\Pi}^\partial$, we get a map

$$\mathrm{h}_{\hat{p}}\colon \mathcal{M}(\mathbb{R}\times M)\to U(1)=\mathbb{R}/\mathbb{Z}.$$

Observe that the maps $\widetilde{\Pi}^\partial$, Π^∂, Π^∞, $\widetilde{\Pi}^\infty$, $\Pi^{-\infty}$, $\widetilde{\Pi}^{-\infty}$ and $\mathrm{h}_{\hat{p}}$ are all invariant under the respective $\mathbb{R}$–actions on their domains, and therefore descend to define maps from the orbit spaces under the $\mathbb{R}$–actions. Our convention is to denote the corresponding maps from $\mathcal{B}^\tau(\mathbb{R}\times M)/\mathbb{R}$, $\widetilde{\mathcal{B}}^\tau(\mathbb{R}\times M)/\mathbb{R}$ or $\mathcal{M}(\mathbb{R}\times M)/\mathbb{R}$ by adding underlines to the notation. For example, $\mathrm{h}_{\hat{p}}=\underline{\mathrm{h}}_{\hat{p}}\circ q_{\mathbb{R}}$. By construction, we have

$$(\mathrm{hol}_{\hat{p}})^*\vartheta'_{p-p}=-d\mathrm{h}_{\hat{p}}\quad\text{and}\quad(\underline{\mathrm{hol}}_{\hat{p}})^*\vartheta'_{p-p}=-d\underline{\mathrm{h}}_{\hat{p}}\quad\text{over } \mathcal{N}_k(M).$$

Let $\rho_{\vartheta'_p}\colon \underline{\mathbb{C}}\to\widetilde{\mathcal{B}}_p(M)|_{\Pi_0\mathcal{M}_1(\mathbb{R}\times M)}$ be a trivialization inducing ϑ'_p, and use the notation to denote the associated trivialization of $\mathcal{E}_p(M)|_{\Pi_0\mathcal{M}_1(\mathbb{R}\times M)}$, $\mathcal{E}_p(M)$ being the Hermitian line bundle associated to $\widetilde{\mathcal{B}}_p(M)$. Using ι_Δ to identify $E|_{(\pm\infty,p)}$ respectively with $\mathcal{E}_p(M)|_{\{\mathfrak{c}_\pm\}}$, we have

$$e^{2\pi i\,\mathrm{h}_{\hat{p}}(\mathfrak{d})}=(\rho_{\vartheta'_p})^{-1}\circ\mathrm{hol}^E_{\hat{p}}(\mathfrak{d})\circ\rho_{\vartheta'_p}\in\mathbb{C}^*.$$

Meanwhile, given $\mathfrak{d}\in\mathcal{B}^\tau(\mathbb{R}\times M)$ and an arbitrary $\tilde{\mathfrak{d}}\in\pi_x^{-1}(\mathfrak{d})$,

$$\int_{\mathbb{R}\tilde{\mathfrak{d}}}\vartheta=\int_{\mathbb{R}\mathfrak{d}}\theta=:-h_{\hat{p}}(\mathfrak{d})\in\mathbb{R},$$

where θ is as in Part 1(a)'s (2-29), with x set to be $\dot{p}:=(0,p)\in\mathbb{R}\times M$. In particular, when $\mathfrak{d}\in\mathcal{M}_1(\mathbb{R}\times M)$,

$$\mathrm{h}_{\hat{p}}(\mathfrak{d})=h_{\hat{p}}(\mathfrak{d})\mod\mathbb{Z}.$$

Like $\mathrm{h}_{\hat{p}}$, the function $h_{\hat{p}}(\mathfrak{d})\colon \mathcal{M}(\mathbb{R}\times M)\to\mathbb{R}$ is invariant under the $\mathbb{R}$–action on $\mathcal{M}$, and hence induces a function $\underline{h}_{\hat{p}}\colon \mathcal{N}(M):=\bigcup_k\mathcal{N}_k(M)\to\mathbb{R}$, with

$$\underline{\mathrm{h}}_{\hat{p}}=\underline{h}_{\hat{p}}\mod\mathbb{Z}\quad\text{over } \mathcal{N}_0(M).$$

This function can be used to write

$$[\theta_{\hat{p}}]_{\mathcal{M}(\mathbb{R}\times M)} = -\sum_{\underline{\mathfrak{d}}\in\mathcal{N}_0(M)} \underline{h}_{\hat{p}}(\underline{\mathfrak{d}})\mu^1_{\underline{\mathfrak{d}}}, \tag{2-32}$$

where $\{\mu^1_{\underline{\mathfrak{d}}}\}_{\underline{\mathfrak{d}}\in\mathcal{N}_0(M)}$ is the canonical basis for $C^1_{\mathcal{M}(\mathbb{R}\times M)}$. One may extend by continuity both $\underline{\mathrm{h}}_{\hat{p}}$ and $\underline{h}_{\hat{p}}$ to the spaces of broken trajectories $\mathcal{N}_k^+(M)$ as respectively $\mathbb{R}/\mathbb{Z}$– and $\mathbb{R}$–valued functions. With this extension we have $\underline{h}_{\hat{p}}(\mathfrak{d}) = \sum_{\underline{\mathfrak{d}}_i} \underline{h}_{\hat{p}}(\underline{\mathfrak{d}}_i)$ for $\mathfrak{d} = \{\underline{\mathfrak{d}}_i\}_i \in \mathcal{N}_k^+(M)$, and $\underline{\mathrm{h}}_{\hat{p}} = \underline{h}_{\hat{p}}$ mod $\mathbb{Z}$ over the 0–dimensional strata $(\mathcal{N}_k^+(M))_0$. Consequently,

$$\mathrm{u}_p := [d\underline{\mathrm{h}}_{\hat{p}}]_{\mathcal{N}_1^+(M)} - \delta[\underline{h}_{\hat{p}}]_{\mathcal{N}_1^+(M)} \in C^{1;\mathbb{Z}}_{\mathcal{N}_1^+(M)} \subset C^{1;\mathbb{R}}_{\mathcal{N}_1^+(M)};$$

namely, $[\mathrm{u}_p]_{\mathcal{N}_1^+(M)}$ is an integral correction of $[-(\underline{\mathrm{hol}}_{\hat{p}})^*\vartheta'_{p-p}]_{\mathcal{N}_1^+(M)} = [d\underline{\mathrm{h}}_{\hat{p}}]_{\mathcal{N}_1^+(M)}$.

We next express $\mathring{U}_p$ in terms of integrals of u_p over $\mathcal{N}_1^+(M)$. According to (2-19) and (2-23), the coefficients of $\mathring{U}_p$ take the form of

$$\langle e, \mathcal{M}_{2,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle = \langle e, \overline{\mathcal{M}}_{2,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle$$

or

$$\langle e, \mathcal{M}_z^{\mathrm{red},2}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle = \langle e, \overline{\mathcal{M}}_z^{\mathrm{red},2}(X;\mathfrak{c}_-,\mathfrak{c}_+)\rangle,$$

where $X = \mathbb{R}\times M$. Let $\overline{\mathcal{M}}$ be one of the compactified moduli spaces $\overline{\mathcal{M}}_{2,z}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ or $\overline{\mathcal{M}}_z^{\mathrm{red},2}(X;\mathfrak{c}_-,\mathfrak{c}_+)$ named above. This is a 2–dimensional stratified submanifold of $\mathcal{M}(X)$. Let $\mathfrak{M} = \overline{\mathcal{M}} \setminus \mathcal{M}_1(X)$ denote the top-dimensional stratum of $\overline{\mathcal{M}}$, and let $\mathcal{M}^+$ be the larger compactification of $\mathfrak{M}$ by adding (parametrized) broken trajectories. The latter carries a stratification of the form

$$\varnothing \subset (\mathcal{M}^+)_0 \subset (\mathcal{M}^+)_1 \subset (\mathcal{M}^+)_2 = \mathcal{M}^+.$$

Meanwhile, $\mathfrak{M}$ consists of $\mathbb{R}$–orbits; let $\mathfrak{N} := \mathfrak{M}/\mathbb{R} \subset \mathcal{N}_1(M)$, and use $\mathcal{N}^+ \subset \mathcal{N}_1^+(M)$ to denote the compactification of $\mathfrak{N}$ by adding unparametrized broken trajectories. It is stratified as $\varnothing \subset (\mathcal{N}^+)_0 \subset (\mathcal{N}^+)_1 = \mathcal{N}^+$. The strata of $\mathcal{M}_z^+(\mathfrak{c}_-,\mathfrak{c}_+)$, $\mathcal{M}_z^{\mathrm{red}+}(\mathfrak{c}_-,\mathfrak{c}_+)$, $\overline{\mathcal{M}}_z(\mathfrak{c}_-,\mathfrak{c}_+)$, $\overline{\mathcal{M}}_z^{\mathrm{red}}(\mathfrak{c}_-,\mathfrak{c}_+)$, $\mathcal{N}_z^+(\mathfrak{c}_-,\mathfrak{c}_+)$ or $\mathcal{N}_z^{\mathrm{red}+}(\mathfrak{c}_-,\mathfrak{c}_+)$ are described in [17, (24.27)–(24.28) and Proposition 24.6.10]. Applied to the case under discussion, this entails:

(2-33)
- $(\overline{\mathcal{M}})_0 = \bar{\imath}(\{\mathfrak{c}_-,\mathfrak{c}_+\}) \subset \bar{\imath}(\mathfrak{C}(M))$.
- $(\mathcal{N}^+)_0$ consists of finitely many once- or twice-broken trajectories. We denote such a broken trajectory in the form $\mathfrak{d} = (\mathfrak{d}_i)_i$, where each $\mathfrak{d}_i \in \mathcal{N}_0(M)$ and $i \in \{1,2\}$ or $\{1,2,3\}$.

- The 1–dimensional strata of $\overline{\mathcal{M}}$ consists of $\mathbb{R}$–orbits in $\mathcal{B}^\sigma(X)$. More precisely, $(\overline{\mathcal{M}})_1 \setminus (\overline{\mathcal{M}})_0 = \bigcup_{\mathfrak{d}=(\mathfrak{d}_i)_i\in(\mathcal{N}^+)_0} \bigcup_i \mathbb{R}\mathfrak{d}_i$.
- $(\mathcal{M}^+)_1$ is a union of two parts,

$$(\mathcal{M}^+)_1 = \mathfrak{r}^{-1}(\overline{\mathcal{M}})_0 \cup \mathfrak{r}^{-1}((\overline{\mathcal{M}})_1 \setminus (\overline{\mathcal{M}})_0).$$

  In the first part, $\mathfrak{r}^{-1}\overline{\mathfrak{c}} \simeq (\mathcal{N}^+)_1$ for each $\overline{\mathfrak{c}} \in (\overline{\mathcal{M}})_0 \subset \mathfrak{C}(M)$, while on the second part, $\mathfrak{r}$ restricts to an isomorphism from $\mathfrak{r}\colon \mathfrak{r}^{-1}((\overline{\mathcal{M}})_1 \setminus (\overline{\mathcal{M}})_0)$ to $(\overline{\mathcal{M}})_1 \setminus (\overline{\mathcal{M}})_0$.

Our strategy to compute $\langle e, \overline{\mathcal{M}}\rangle$ is to introduce a map $\tilde{\varsigma}\colon \mathcal{M}^+ \to \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)$ so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M}^+ \arrow[r, "\tilde{\varsigma}"] \arrow[d, "\mathfrak{r}"'] & \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X) \arrow[d, "\pi_x"] \\ \overline{\mathcal{M}} \arrow[r, "\varsigma"] & \mathcal{B}^\sigma_{\mathrm{loc}}(X) \end{tikzcd} \tag{2-34}$$

where $\varsigma\colon \mathcal{M}_2(X) \to \mathcal{B}^\sigma_{\mathrm{loc}}(X)$ denotes the embedding. This means that $\tilde{\varsigma}|_{\mathfrak{M}}$ is then a lift of the embedding $\varsigma|_{\mathfrak{M}}$ under π_x. We choose this lifting so that $\tilde{\varsigma}(\mathfrak{M}) \subset \widetilde{\mathcal{B}}^\tau(\mathbb{R}\times M) \subset \widetilde{\mathcal{B}}^\sigma_{x,\mathrm{loc}}(X)$ is tangent to the $\mathbb{R}$–action. Such a choice is specified in turn by a lift $\tilde{\varsigma}_{\mathcal{N}}$ of $\mathfrak{N} \subset \mathcal{N}_1(M)$ to $\widetilde{\mathcal{N}}_1(M)$. As an extension of $\tilde{\varsigma}|_{\mathfrak{M}}$, $\tilde{\varsigma}$'s image is also tangent to the $\mathbb{R}$–action on $\widetilde{\mathcal{B}}^\tau(\mathbb{R}\times M)$. With $\tilde{\varsigma}$ chosen, we then write

$$\begin{aligned} \langle e, \overline{\mathcal{M}}\rangle &= \langle e, \mathfrak{M}\rangle = \langle \tilde{\varsigma}^* e, \mathfrak{M}\rangle = \langle \tilde{\varsigma}^* e, \mathcal{M}^+\rangle \\ &= \langle \tilde{\varsigma}^*\vartheta, \partial[\mathcal{M}^+]\rangle = \langle \tilde{\varsigma}^*\vartheta, [(\mathcal{M}^+)_1]\rangle, \end{aligned} \tag{2-35}$$

using [17, Theorem 24.7.2 and Lemma 21.3.1]. By (2-33), the last term above is written as a sum

$$\begin{aligned} \langle \tilde{\varsigma}^*\vartheta, [(\mathcal{M}^+)_1]\rangle &= \langle \tilde{\varsigma}^*\vartheta, \mathfrak{r}^{-1}(\overline{\mathcal{M}})_0\rangle + \langle \tilde{\varsigma}^*\vartheta, \mathfrak{r}^{-1}(\overline{\mathcal{M}}_1 \setminus \overline{\mathcal{M}}_0)\rangle \\ &= \langle \tilde{\varsigma}^*_{\mathcal{N}}(\underline{\widetilde{\Pi}}^-)^*\vartheta'_p, \mathcal{N}^+\rangle - \langle \tilde{\varsigma}^*_{\mathcal{N}}(\underline{\widetilde{\Pi}}^+)^*\vartheta'_p, \mathcal{N}^+\rangle + \langle \theta_{\dot{p}}, \overline{\mathcal{M}}_1 \setminus \overline{\mathcal{M}}_0\rangle \\ &= -\langle (\underline{\mathrm{hol}}_{\hat{p}})^*\vartheta'_{p-p}, \mathcal{N}^+\rangle - \sum_{\mathfrak{d}\in(\mathcal{N}^+)_0} \mathrm{sign}(\mathfrak{d})\underline{h}_{\hat{p}}(\mathfrak{d}). \end{aligned}$$

Note that the first term in the last line above is independent of the choice of $\tilde{\varsigma}_{\mathcal{N}}$; it is also independent of the choice of ϑ'_p, since, by (2-34), $\Pi_{\mathcal{N}}\mathfrak{N}$ lies in a fiber of $\widetilde{\mathcal{B}}^\sigma(M) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma(M)$, over which ϑ'_p is the standard $U(1)$–invariant volume form generating $H^1(U(1);\mathbb{Z})$. To summarize, we have

$$\langle e, \overline{\mathcal{M}}\rangle = \langle \mathrm{u}_p, \mathcal{N}^+\rangle, \quad \langle d\underline{\mathrm{h}}_{\hat{p}}, \mathcal{N}^+\rangle = \langle e - \delta[\theta_{\dot{p}}]_{\mathcal{M}(\mathbb{R}\times M)}, \overline{\mathcal{M}}\rangle. \tag{2-36}$$

(b) (the $\mathfrak{t}_i$–maps) Given an embedded oriented circle $\underline{\gamma} \subset M$, one may define a 3–dimensional counterpart of the 1–form θ_γ in Part 1(b) above: let

$$\mathrm{h}'_{\underline{\gamma}} \colon \mathcal{B}^\sigma(M) \to U(1) = \mathbb{R}/\mathbb{Z}$$

be the map sending a $\mathfrak{d} \in \mathcal{B}^\sigma(M)$ to the holonomy along $\underline{\gamma}$ of $A \in \mathrm{Conn}(E)$ associated to $\mathfrak{d}$, and set

$$\theta'_{\underline{\gamma}} = d\mathrm{h}'_{\underline{\gamma}} \in \Omega^1(\mathcal{B}^\sigma(M)).$$

By construction $\theta'_{\underline{\gamma}}$ is closed and its cohomology class $[\theta'_{\underline{\gamma}}] \in H^1(\mathcal{B}^\sigma(M);\mathbb{Z})$ equals $[\underline{\gamma}] \in H_1(M;\mathbb{Z})/\mathrm{Tors}$ under the isomorphism

$$H_1(M;\mathbb{Z})/\mathrm{Tors} \simeq H^1(H^1(M;\mathbb{R})/H^1(M;\mathbb{Z});\mathbb{Z}) \simeq H^1(\mathcal{B}^\sigma(M);\mathbb{Z}).$$

Let $\dot\gamma := \{0\} \times \underline{\gamma} \subset \mathbb{R} \times M$. Then $\theta_{\dot\gamma} = \Pi_0^* \theta'_{\underline{\gamma}}$ and $[\theta_{\dot\gamma}]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$. Let

$$u_{\dot\gamma} = [\theta_{\dot\gamma}]_{\mathcal{M}(X)} - \delta[h_{\dot\gamma}]_{\mathcal{M}(X)} \in C^{1;\mathbb{Z}}_{\mathcal{M}(X)}$$

be an integral correction of $[\theta_{\dot\gamma}]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$, as described in Part 1(b). In the present case, $\mathcal{M}_0(X) \simeq_{\Pi_0} \mathfrak{C}(M) \subset \mathcal{B}^\sigma(M)$, and the function $h_{\dot\gamma} \colon \mathcal{M}_0(X) \to \mathbb{R}$ in Part 1(b) takes the form of $\Pi_0^* h'_{\underline{\gamma}}$, where

$$h'_{\underline{\gamma}} \colon \mathfrak{C}(M) \to \mathbb{R}$$

is a lift of $\mathrm{h}'_{\underline{\gamma}} |_{\mathfrak{C}(M)} \colon \mathfrak{C}(M) \to \mathbb{R}/\mathbb{Z}$. Noting that the strata of $\mathcal{M}(X)$ are $\mathbb{R}$–spaces in this product cobordism case, we have

$$\begin{aligned} h_{\dot\gamma} &= \Pi_0^* h'_{\underline{\gamma}} = \Pi_s^* h'_{\underline{\gamma}} = h_{\{s\}\times\underline{\gamma}}, \\ [\theta_{\dot\gamma}]_{\mathcal{M}(X)} &= [\Pi_0^* \theta'_{\underline{\gamma}}]_{\mathcal{M}(X)} = [\Pi_s^* \theta'_{\underline{\gamma}}]_{\mathcal{M}(X)} = [\theta_{\{s\}\times\underline{\gamma}}]_{\mathcal{M}(X)} \end{aligned}$$

for all $s \in [-\infty, \infty]$. Thus, for our purpose, $\dot\gamma$ may be taken to be $\{s\} \times \underline{\gamma} \subset \mathbb{R} \times M$ for arbitrary s.

Let $\mathfrak{t} := [\underline{\gamma}] \in H_1(M;\mathbb{Z})/\mathrm{Tors}$ and let

$$\mathrm{p}_{\mathfrak{t}} \colon \widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M) \to \mathcal{B}^\sigma(M)$$

be the $\mathbb{Z}$–covering of $\mathcal{B}^\sigma(M)$ with $\pi_1(\widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}) \subset \pi_1(\mathcal{B}^\sigma(M))$ being the kernel of the map $\mathfrak{t} \colon \pi_1(\mathcal{B}^\sigma(M)) \simeq H^1(M;\mathbb{Z}) \to \mathbb{Z}$. The function $\mathrm{h}'_{\underline{\gamma}} \colon \mathcal{B}^\sigma(M) \to U(1) = \mathbb{R}/\mathbb{Z}$ lifts to an $\mathbb{R}$–valued function

$$\widehat{h}_{\underline{\gamma}} \colon \widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M) \to \mathbb{R}.$$

This lift $\widehat{h}_{\underline{\gamma}}$ is unique modulo addition by constant, $\mathbb{Z}$–valued functions, and it can be fixed by choosing a basepoint $\hat{\mathfrak{c}}_0 \in \widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M)$ with $\mathrm{h}'_{\underline{\gamma}}(\mathfrak{c}_0) = 0 \bmod \mathbb{Z}$, where $\mathfrak{c}_0 := \mathrm{P}_{\mathfrak{t}}(\hat{\mathfrak{c}}_0)$. Namely, let $\widehat{h}_{\underline{\gamma}}$ be such that

$$\widehat{h}_{\underline{\gamma}}(\hat{\mathfrak{c}}_0) = 0 \in \mathbb{R}.$$

Given an arc $\mathfrak{b}$: $[0,1] \to \mathcal{B}^\sigma(M)$, the difference $\widehat{h}_{\underline{\gamma}}(\hat{\mathfrak{b}}(1)) - \widehat{h}_{\underline{\gamma}}(\hat{\mathfrak{b}}(0))$ takes the same value for any lift $\hat{\mathfrak{b}}$: $[0,1] \to \widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M)$ of $\mathfrak{b}$; we denote this value by $\Delta_{\mathfrak{b}}\widehat{h}_{\underline{\gamma}} \in \mathbb{R}$. It depends only on $\mathrm{h}'_{\underline{\gamma}}$ and not on the choice of the lift $\widehat{h}_{\underline{\gamma}}$. In particular, any $\underline{\mathfrak{d}} \in \mathcal{N}(M)$ defines an arc $\mathfrak{b}_{\underline{\mathfrak{d}}}$ in $\mathcal{B}^\sigma(M)$, and we adopt the shorthand $\Delta_{\underline{\mathfrak{d}}}\widehat{h}_{\underline{\gamma}} := \Delta_{\mathfrak{b}_{\underline{\mathfrak{d}}}}\widehat{h}_{\underline{\gamma}}$. This value only depends on the relative homotopy class of $\underline{\mathfrak{d}}$. Observing that $\int_{\mathbb{R}\underline{\mathfrak{d}}} \theta_{\dot{\gamma}} = \Delta_{\underline{\mathfrak{d}}}\widehat{h}_{\underline{\gamma}}$, we have

$$[\theta_{\dot{\gamma}}]_{\mathcal{M}(\mathbb{R}\times M)} = \sum_{\underline{\mathfrak{d}}\in\mathcal{N}_0(M)} (\Delta_{\underline{\mathfrak{d}}}\widehat{h}_{\underline{\gamma}})\mu^1_{\underline{\mathfrak{d}}} \in C^{1;\mathbb{R}}_{\mathcal{M}(\mathbb{R}\times M)}.$$

An integral correction $u_{\dot{\gamma}}$ of $[\theta_{\dot{\gamma}}]_{\mathcal{M}(\mathbb{R}\times M)}$ can be written in a similar fashion by replacing the function $\widehat{h}_{\underline{\gamma}}$: $\widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M) \to \mathbb{R}$ in the preceding discussion by a modified function

$$\mathrm{x}_{\underline{\gamma}}\colon \widehat{\mathcal{B}}^\sigma_{\mathfrak{t}}(M) \to \mathbb{R},$$

where $\mathrm{x}_{\underline{\gamma}} = \widehat{h}_{\underline{\gamma}} - \mathrm{P}^*_{\mathfrak{t}}\varepsilon'_{\underline{\gamma}}$ for a function $\varepsilon'_{\underline{\gamma}}$: $\mathcal{B}^\sigma(M) \to \mathbb{R}$ satisfying $\varepsilon'_{\underline{\gamma}}|_{\mathfrak{C}(M)} = h'_{\underline{\gamma}}$.

Returning to the subject of $\boldsymbol{A}_\dagger$–actions, take $\underline{\gamma} = \gamma_i \subset M$ to be one that represents $\mathfrak{t}_i \in H_1(M;\mathbb{Z})/\mathrm{Tors}$. The $\mathfrak{t}_i$*–map associated to* γ_i is defined to be

$$\mathring{\mathfrak{m}}_{\mathfrak{t}_i} = \mathring{\mathfrak{m}}_{\gamma_i} := \mathring{m}[u_{\dot{\gamma}_i}](\mathbb{R}\times M)\colon \mathring{C}(M) \to \mathring{C}(M).$$

This corresponds to the 1–cocycle $\mu_{\mathfrak{t}_i} = (\mathrm{P}_{\mathfrak{t}_i})_*(d\mathrm{x}_{\underline{\gamma}})$ on $\mathcal{B}^\sigma(M)$.

It will also be handy to introduce an analog of Part 2(a)'s u_p (see (2-36)): Given $\underline{\mathfrak{d}} \in \mathcal{N}(M)$, let $\Delta_{\underline{\mathfrak{d}}}\,\mathrm{x}_{\gamma_i}$ be defined in the same way as $\Delta_{\underline{\mathfrak{d}}}\widehat{h}_{\gamma_i}$ above. Let u_{γ_i} denote the function on $\mathcal{N}(M)$ that sends each $\underline{\mathfrak{d}} \in \mathcal{N}(M)$ to $\Delta_{\underline{\mathfrak{d}}}\,\mathrm{x}_{\gamma_i}$. Note that u_{γ_i} is $\mathbb{Z}$–valued, and hence defines a class in $C^{0;\mathbb{Z}}_{\mathcal{N}(M)}$, denoted by the same notation. The coefficients appearing in the formula for $\mathfrak{m}_{\mathfrak{t}_i}$ then may be reexpressed as integrals of u_{γ_i} over $\mathcal{N}(M)$:

$$\langle u_{\dot{\gamma}_i}, \mathcal{M}\rangle = \langle \mathrm{u}_{\gamma_i}, \mathcal{N}\rangle,$$

where $\mathcal{M}$ is a 1–dimensional stratum in $\mathcal{M}(\mathbb{R}\times M)$ and $\mathcal{N} = \mathcal{M}/\mathbb{R}$ is the corresponding stratum in $\mathcal{N}(M)$. In general, we use the notation $\mathring{n}[\mathrm{u}] := \mathring{m}[u]$ when $X = \mathbb{R}\times M$ is a product cobordism and the coefficients in the formula for $\mathring{m}[u]$ may be expressed as integrals of $\mathrm{u} \in C^k_{\mathcal{N}(M)}$ over $\mathcal{N}(M)$ in the way described above. For example, we write

$$\mathring{\mathfrak{m}}_{\gamma_i} = \mathring{n}[\mathrm{u}_{\gamma_i}], \quad \mathring{U}_p = \mathring{n}[\mathrm{u}_p]. \tag{2-37}$$

Part 3: cochains on $\mathcal{B}^\sigma(X)$ from noncompact d–submanifolds of X In this part we consider d–submanifolds in X that are "asymptotically cylindrical" in the sense described below, and use them to define cochains on $\mathcal{B}^\sigma(X)$ (or, more generally, on various bundles over $\mathcal{B}^\sigma(X)$) in a manner similar to Part 1. These cochains are often useful for defining chain homotopy equivalences between Floer complexes, as will be demonstrated by examples.

(a) (when $d=1$) Let $M_1, M_2 \in \mathcal{E}$ label two ends of X, allowing $M_1 = M_2$. We say that an oriented connected 1–submanifold $\lambda \subset X$ is a *path from* $p_1 \in M_1$ *to* $p_2 \in M_2$ if $\lambda \cap (X - X_c)$ consists of two connected components of the following form: the first component is $(-\infty, L) \times \{p_1\} \subset (-\infty, L) \times M_1$ or $(L, \infty) \times \{-p_1\} \subset (L, \infty) \times M_1$ under the diffeomorphisms in (2-8), depending on whether M_1 is a negative end or a positive end, and the second component is $(-\infty, L) \times \{-p_2\} \subset (-\infty, L) \times M_2$ or $(L, \infty) \times \{p_2\} \subset (L, \infty) \times M_2$ under the diffeomorphisms in (2-8). We shall define a 1–cochain $[\theta_\lambda]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$ and its integral correction $[\kappa_\lambda]_{\mathcal{M}(X)} \in C^{1;\mathbb{Z}}_{\mathcal{M}(X)}$, beginning by introducing generalizations of notions such as $\mathrm{hol}_{\hat{p}}$, ϑ_{p-p}, π_{p-p}, etc, previously encountered in Part 2(a).

Fix choices of $\vartheta'_{p_1} \in \Omega^1(\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1))$ and $\vartheta'_{p_2} \in \Omega^1(\widetilde{\mathcal{B}}^\sigma_{p_1}(M_2))$ as described in Part 2(a) and note that $\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)$ is a principal $U(1)$–bundle over $\mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2)$ and ϑ'_{p_1} and ϑ'_{p_2} together define a principal $U(1)$–connection on this bundle, which we denote by $\vartheta'_{p_2-p_1}$: Consider the commutative diagram

$$\begin{tikzcd}
 & \widetilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times \widetilde{\mathcal{B}}^\sigma_{p_2}(M_2) \arrow[dl, "\widetilde{\mathrm{pr}}_i"'] \arrow[d, "\pi_\Delta"] \arrow[dd, bend left=60, "\pi_{p_1} \times \pi_{p_2}"] \\
\widetilde{\mathcal{B}}^\sigma_{p_i}(M_i) \arrow[d, "\pi_{p_i}"] & \widetilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_{p_2}(M_2) \arrow[l, "\mathrm{pr}'_i"] \arrow[dl, "\underline{\mathrm{pr}}_i"] \arrow[d, "\pi_{p_2-p_1}"] \\
\mathcal{B}^\sigma(M_i) & \mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2) \arrow[l, "\mathrm{pr}_i"]
\end{tikzcd} \tag{2-38}$$

for $i = 1$ or 2, where pr_i denotes projecting to the i^{th} factor and π_Δ denotes quotienting by the diagonal $U(1)$–action. Then

$$\vartheta'_{p_2-p_1} = (\pi_\Delta)_!(\widetilde{\mathrm{pr}}_1^* \vartheta'_{p_1} \wedge \widetilde{\mathrm{pr}}_2^* \vartheta'_{p_2}) = (\mathrm{pr}'_2)^* \vartheta'_{p_2} = -(\mathrm{pr}'_1)^* \vartheta'_{p_1}, \tag{2-39}$$

where $(\pi_\Delta)_!\colon \Omega^2(\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)\to\Omega^1(\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2))$ is integration over the fibers of π_Δ. Let $\widetilde{\mathcal{B}}^\sigma_\lambda(X)$ and $\widetilde{\widetilde{\mathcal{B}}}^\sigma_\lambda(X)$ be pullback bundles defined by the commutative diagram

$$\begin{tikzcd}
\widetilde{\widetilde{\mathcal{B}}}^\sigma_\lambda(X) \arrow[r, "\widetilde{\widetilde{\Pi}}^\partial_\lambda"] \arrow[d, "\widetilde{\pi}"] \arrow[dd, bend right=60, "\widetilde{\pi}_\lambda"'] & \widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2) \arrow[d, "\pi_\Delta"] \arrow[dd, bend left=60, "\pi_{p_1}\times\pi_{p_2}"] \\
\widetilde{\mathcal{B}}^\sigma_\lambda(X) \arrow[r, "\widetilde{\Pi}^\partial_\lambda"] \arrow[d, "\pi_\lambda"] & \widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2) \arrow[d, "\pi_{p_2-p_1}"] \\
\mathcal{B}^\sigma(X) \arrow[r, "\Pi^\partial_\lambda"] & \mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(M_2) \\
\mathcal{M}(X) \arrow[r, "\Pi^\partial_\lambda"] \arrow[u, hook] & \mathfrak{C}(M_1)\times\mathfrak{C}(M_2) \arrow[u, hook]
\end{tikzcd} \tag{2-40}$$

where $\Pi^\partial_\lambda := \Pi^{M_1}\times\Pi^{M_2}$, and $\Pi^{M_i} := \Pi^{\pm\infty}|_{M_i\subset Y_\pm}$. For $i=1,2$, let

$$\rho_{\vartheta'_{p_i}}\colon \underline{U(1)}\to\widetilde{\mathcal{B}}^\sigma(M_i)|_{\Pi_0\mathcal{M}_1(\mathbb{R}\times M_i)}$$

be a trivialization of the $U(1)$–bundle $\widetilde{\mathcal{B}}^\sigma(M_i)$ over $\Pi_0\mathcal{M}_1(\mathbb{R}\times M_i)\subset\mathcal{B}^\sigma(M_i)$. Over $\mathfrak{C}(M_1)\times\mathfrak{C}(M_2)\subset\Pi_0\mathcal{M}_1(\mathbb{R}\times M_1)\times\Pi_0\mathcal{M}_1\subset(\mathbb{R}\times M_2)\subset\mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(M_2)$, the $U(1)\times U(1)$–bundle $\pi_{p_1}\times\pi_{p_2}\colon\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)\to\mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(M_2)$ is equipped with a trivialization $\rho_{\vartheta'_{p_1}}\times\rho_{\vartheta'_{p_1}}$. This trivialization factors through a trivialization, $\rho_{p_2-p_1}$, of the $U(1)$–bundle, $\pi_{p_2-p_1}\colon\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)\to\mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(M_2)$ over $\mathfrak{C}(M_1)\times\mathfrak{C}(M_2)\subset\mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(M_2)$ and a trivialization, ρ_Δ, of the $U(1)$–bundle $\pi_\Delta\colon\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)\to\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)$ over

$$\widetilde{\mathfrak{c}} := \pi^{-1}_{p_2-p_1}(\mathfrak{C}(M_1)\times\mathfrak{C}(M_2))\subset\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2).$$

The trivializations $\rho_{p_2-p_1}$ and ρ_Δ above are compatible respectively with $\vartheta'_{p_2-p_1}$ and $\widetilde{\mathrm{pr}}_1^*\vartheta'_{p_1}+\widetilde{\mathrm{pr}}_2^*\vartheta'_{p_2}$, which are in turn integral respectively over $\mathfrak{C}(M_1)\times\mathfrak{C}(M_2)$ and $\widetilde{\mathfrak{c}}$ as the ϑ'_{p_i} satisfy (2-30). All the trivializations above are determined by ϑ'_{p_1} and ϑ'_{p_2} modulo constant $U(1)$–maps.

Identify the Hermitian line bundle associated to the principal $U(1)$–bundle

$$\pi_\Delta\colon\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)\to\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)$$

with the bundle $\mathrm{Hom}(\underline{\mathrm{pr}}_1^*\mathcal{E}_{p_1}(M_1),\underline{\mathrm{pr}}_2^*\mathcal{E}_{p_2}(M_2))$, and use E_{p_i} to denote the fiber of the bundle $E\to M_i$ at $p_i\in M_i$. Given $\underline{\mathfrak{d}}\in\mathcal{C}^\sigma(X)$, let $\mathrm{hol}^E_\lambda(\underline{\mathfrak{d}})\in\mathrm{Hom}(E_{p_1},E_{p_2})$ denote the holonomy along λ of the $A \in \mathrm{Conn}(E)$ associated to $\underline{\mathfrak{d}}$. Observing that given a $\tilde{\mathfrak{d}} \in \tilde{\mathcal{B}}^\sigma_\lambda(X)$, the value $\mathrm{hol}^E_\lambda(\underline{\mathfrak{d}})$ is identical for all representatives $\underline{\mathfrak{d}} \in \mathcal{C}^\sigma(X)$ of $\tilde{\mathfrak{d}}$, this then defines a map, also denoted by hol_λ, from $\tilde{\mathcal{B}}^\sigma_\lambda(X)$ to $\tilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times \tilde{\mathcal{B}}^\sigma_{p_2}(M_2)$ that fits into the commutative diagram

$$\begin{tikzcd} \tilde{\tilde{\mathcal{B}}}^\sigma_\lambda(X) \arrow[r, "\mathrm{hol}_\lambda"] \arrow[d, "\tilde{\pi}"] & \tilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times \tilde{\mathcal{B}}^\sigma_{p_2}(M_2) \arrow[d, "\pi_\Delta"] \\ \tilde{\mathcal{B}}^\sigma_\lambda(X) \arrow[ur, "\mathrm{hol}_\lambda"] \arrow[r, "\tilde{\Pi}^\partial_\lambda"] & \tilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times_{U(1)} \tilde{\mathcal{B}}^\sigma_{p_2}(M_2) \end{tikzcd} \tag{2-41}$$

Since $\Pi^\partial_\lambda(\mathcal{M}(X)) \subset \mathfrak{C}(M_1) \times \mathfrak{C}(M_2) \subset \mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2)$ and $\tilde{\Pi}^\partial_\lambda(\tilde{\mathcal{M}}(X)) \subset \tilde{\mathfrak{c}} \subset \tilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times_{U(1)} \tilde{\mathcal{B}}^\sigma_{p_2}(M_2)$, we may compose $\mathrm{hol}_\lambda |_{\tilde{\mathcal{M}}(X)}$ with the trivialization ρ_Δ to get a map $\mathrm{h}_\lambda\colon \tilde{\mathcal{M}}(X) \subset \tilde{\mathcal{B}}^\sigma_\lambda(X) \to U(1) = \mathbb{R}/\mathbb{Z}$. Let

$$\vartheta_\lambda := d\mathrm{h}_\lambda,$$

a closed 1–form on $\tilde{\mathcal{M}}(X)$. Note that ϑ_λ depends only on ϑ'_{p_1} and ϑ'_{p_2}, not the choices of $\rho_{\vartheta'_{p_1}}$ and $\rho_{\vartheta'_{p_2}}$. Let $\tilde{\Pi}^{M_i}_\lambda := \mathrm{pr}'_i \circ \tilde{\Pi}^\partial_\lambda$ and observe that both ϑ_λ and

$$(\tilde{\Pi}^\partial_\lambda)^* \vartheta'_{p_2 - p_1} = (\tilde{\Pi}^{M_2}_\lambda)^* \vartheta'_{p_2} = -(\tilde{\Pi}^{M_1}_\lambda)^* \vartheta'_{p_1} \tag{2-42}$$

define principal $U(1)$–connections on the bundle $\pi_\lambda\colon \tilde{\mathcal{M}}(X) \to \mathcal{M}(X)$. Thus,

$$\vartheta_\lambda - (\tilde{\Pi}^\partial_\lambda)^* \vartheta'_{p_2 - p_1} = \pi^*_\lambda \theta_\lambda \tag{2-43}$$

for a 1–form θ_λ on $\mathcal{M}(X)$, and, correspondingly, a $[\theta_\lambda]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$. Note that θ_λ does not depend on the choice of either ϑ'_{p_1} or ϑ'_{p_2}, since varying the choice of either changes ϑ_λ and $(\tilde{\Pi}^\partial_\lambda)^* \vartheta'_{p_2 - p_1}$ by the same amount.

As observed in Remark 2.2, with θ_λ constructed from forms on the bundle $\tilde{\mathcal{B}}^\sigma_\lambda(X)$, the cobordism map $\mathring{m}[\theta_\lambda]$ is defined by a generalization of the formula in [17]. Let $m^\#_\flat[\theta_\lambda]$ and $\overline{m}^\#_\flat[\theta_\lambda]$ be defined as a sum of integrals in the usual way, ie by (2-23), the explicit formula for $\hat{m}[\theta_\lambda](X)$, generalizing (2-20), is

$$\begin{bmatrix} m^o_o[\theta_\lambda] & m^u_o[\theta_\lambda] \\ \hat{m}^o_u[\theta_\lambda] & \hat{m}^u_u[\theta_\lambda] \end{bmatrix}, \tag{2-44}$$

where

$$\begin{aligned} \hat{m}^o_u[\theta_\lambda] &:= -\overline{m}^s_u[\theta_\lambda]\partial^o_s - \overline{\partial}^s_u m^o_s[\theta_\lambda] + \overline{n}^s_u[d\underline{\mathrm{h}}_{\hat{p}_2}] m^o_s[1], \\ \hat{m}^u_u[\theta_\lambda] &:= -\overline{m}^u_u[\theta_\lambda] - \overline{m}^s_u[\theta_\lambda]\partial^u_s - \overline{\partial}^s_u m^u_s[\theta_\lambda] + \overline{n}^s_u[d\underline{\mathrm{h}}_{\hat{p}_2}] m^u_s[1] \end{aligned} \tag{2-45}$$

when $p_1 \in Y_-$ and $p_2 \in Y_+$; when p_1 and p_2 are both in Y_+, then the $\bar{n}^s_u[d\underline{\mathrm{h}}_{\hat{p}_2}]$'s in the formulas above are replaced by $\bar{n}^s_u[d\underline{\mathrm{h}}_{\hat{p}_2}]-\bar{n}^s_u[d\underline{\mathrm{h}}_{\hat{p}_1}]$. When p_1 and p_2 both belong to Y_-, $\widehat{m}[\theta_\lambda]$ is given by (2-20). (In this case it is $\check{m}[\theta_\lambda]$ that gains additional terms.)

Example Let M be connected, and take $X = \mathbb{R}\times M$ to be a product cobordism. Let $\lambda \subset \mathbb{R}\times M$ be the graph of a path $\bar{\lambda}(\cdot)\colon \mathbb{R}\to M$ that sends the $(-\infty, -L') \subset \mathbb{R}$ to $p_1 \in M$ and $(L', \infty) \subset \mathbb{R}$ to $p_2 \in M$. Let $\overline{\mathcal{M}}$ be as in Part 2(a). We choose a lifting $\tilde{\varsigma}_\lambda$ of the embedding $\varsigma\colon \overline{\mathcal{M}} \to \mathcal{B}^\sigma(X)$ in a way parallel to (2-34), namely such that the following diagram commutes:

$$\tag{2-46}
\begin{tikzcd}
\mathcal{M}^+ \arrow[r, "\tilde{\varsigma}_\lambda"] \arrow[d, "\mathfrak{r}"'] & \widetilde{\mathcal{B}}^\sigma_\lambda(X) \arrow[r, "\widetilde{\Pi}^\partial_\lambda"] \arrow[d, "\pi_\lambda"'] & \widetilde{\mathcal{B}}^\sigma_{p_1}(M)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M) \arrow[d, "\pi_{p_2-p_1}"'] \\
\overline{\mathcal{M}} \arrow[r, "\varsigma"] & \mathcal{B}^\sigma(X) \arrow[r, "\Pi^\partial_\lambda"] & \mathcal{B}^\sigma(M)\times\mathcal{B}^\sigma(M)
\end{tikzcd}$$

As observed previously, over $(\varsigma\circ\Pi^\partial_\lambda)\overline{\mathcal{M}} \subset \mathfrak{C}(M_1)\times\mathfrak{C}(M_2) \subset \mathcal{B}^\sigma(M)\times\mathcal{B}^\sigma(M)$, the bundle $\pi_{p_2-p_1}\colon \widetilde{\mathcal{B}}^\sigma_{p_1}(M)\times_{U(1)}\widetilde{\mathcal{B}}^\sigma_{p_2}(M)$ is trivialized by $\rho_{p_2-p_1}$. This induces a trivialization of its pullback bundle $\pi_\lambda\colon \widetilde{\mathcal{B}}^\sigma_\lambda(X)\to\mathcal{B}^\sigma(X)$ (via $(\widetilde{\Pi}^\partial_\lambda)^*$) over $\overline{\mathcal{M}} \xhookrightarrow{\varsigma} \mathcal{B}^\sigma(X)$. Choose $\tilde{\varsigma}_\lambda$ to be constant with respect to this trivialization. Then $(\widetilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}$ vanishes over $\tilde{\varsigma}_\lambda((\mathcal{M}^+)_1\setminus(\mathcal{M}^+)_0) = \pi_\lambda^{-1}(\overline{\mathcal{M}}_1\setminus\overline{\mathcal{M}}_0)$. As $\vartheta_\lambda \in \Omega^1(\mathcal{B}^\sigma(X))$ is closed by construction, arguing as in (2-35) and the subsequent discussions, again using [17, Theorem 24.7.2 and Lemma 21.3.1] and (2-33), we have

$$\tag{2-47}
\begin{aligned}
0 &= \langle\tilde{\varsigma}_\lambda^*(d\vartheta_\lambda), \mathcal{M}^+\rangle \\
&= \langle\tilde{\varsigma}_\lambda^*\vartheta_\lambda, \partial[\mathcal{M}^+]\rangle = \langle\tilde{\varsigma}_\lambda^*\vartheta_\lambda, [(\mathcal{M}^+)_1]\rangle \\
&= \langle\tilde{\varsigma}_\lambda^*\vartheta_\lambda, \mathfrak{r}^{-1}(\overline{\mathcal{M}})_0\rangle + \langle\tilde{\varsigma}_\lambda^*\vartheta_\lambda, \mathfrak{r}^{-1}(\overline{\mathcal{M}}_1\setminus\overline{\mathcal{M}}_0)\rangle \\
&= \langle\tilde{\varsigma}_\lambda^*(\widetilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}, \{\bar{\imath}\mathfrak{c}_-, \bar{\imath}\mathfrak{c}_+\}\times\mathcal{N}^+\rangle + \langle\theta_\lambda, \overline{\mathcal{M}}_1\setminus\overline{\mathcal{M}}_0\rangle \\
&= \langle\tilde{\varsigma}_\lambda^*(\widetilde{\Pi}^{M_1}_\lambda)^*\vartheta'_{p_1}, \{\bar{\imath}\mathfrak{c}_-\}\times\mathcal{N}^+\rangle - \langle\tilde{\varsigma}_\lambda^*(\widetilde{\Pi}^{M_2}_\lambda)^*\vartheta'_{p_2}, \{\bar{\imath}\mathfrak{c}_+\}\times\mathcal{N}^+\rangle \\
&\qquad + \langle[\theta_\lambda]_{\mathcal{M}(X)}, \partial[\overline{\mathcal{M}}]\rangle \\
&= \langle e_{\dot{p}_1}, \overline{\mathcal{M}}\rangle - \langle e_{\dot{p}_2}, \overline{\mathcal{M}}\rangle \\
&\qquad + \langle[\theta_\lambda]_{\mathcal{M}(\mathbb{R}\times M)} - [\theta_{\dot{p}_1}]_{\mathcal{M}(\mathbb{R}\times M)} + [\theta_{\dot{p}_2}]_{\mathcal{M}(\mathbb{R}\times M)}, \partial[\overline{\mathcal{M}}]\rangle.
\end{aligned}$$

(To see the last two lines in the preceding expression, recall (2-36) and (2-42).) Summarizing, we have

$$\tag{2-48} e_{\dot{p}_2} - e_{\dot{p}_1} = \delta u_\lambda \in C^{2;\mathbb{Z}}_{\mathcal{M}(\mathbb{R}\times M)},$$

where

$$u_\lambda := [\theta_\lambda]_{\mathcal{M}(\mathbb{R}\times M)} - [\theta_{\dot p_1}]_{\mathcal{M}(\mathbb{R}\times M)} + [\theta_{\dot p_2}]_{\mathcal{M}(\mathbb{R}\times M)} \in C^{1;\mathbb{R}}_{\mathcal{M}(\mathbb{R}\times M)}.$$

Note that u_λ in fact has integral coefficients, ie $u_\lambda \in C^{1;\mathbb{Z}}_{\mathcal{M}(\mathbb{R}\times M)} \subset C^{1;\mathbb{R}}_{\mathcal{M}(\mathbb{R}\times M)}$. To see this, recall (2-32) and write

$$u_\lambda = \sum_{\underline{\mathfrak{d}}\in\mathcal{N}_0(M)} \left(\int_{\mathbb{R}\underline{\mathfrak{d}}} \theta_\lambda + \underline{h}_{\hat p_1}(\underline{\mathfrak{d}}) - \underline{h}_{\hat p_2}(\underline{\mathfrak{d}})\right)\mu^1_{\underline{\mathfrak{d}}}.$$

Meanwhile, for a $\underline{\mathfrak{d}}$ in $\mathcal{M}(\mathfrak{c}_-,\mathfrak{c}_+)/\mathbb{R}$,

$$\int_{\mathbb{R}\underline{\mathfrak{d}}} \theta_\lambda \bmod \mathbb{Z} = \int_{\mathbb{R}\underline{\mathfrak{d}}} \tilde\varsigma_\lambda^*(d\mathrm{h}_\lambda) \bmod \mathbb{Z} = \mathrm{h}_\lambda\big(\tilde\varsigma_\lambda(\bar\iota(\mathfrak{c}_+))\big) - \mathrm{h}_\lambda\big(\tilde\varsigma_\lambda(\bar\iota(\mathfrak{c}_-))\big) \in \mathbb{R}/\mathbb{Z}.$$

Let $\mathfrak{d}\in\mathcal{C}^\tau(\mathbb{R}\times M)$ represent an element in $\tilde\pi_{\underline{\lambda}}^{-1} q_{\mathbb{R}}^{-1}(\underline{\mathfrak{d}})$ and let $\gamma\subset\overline{\mathbb{R}}\times M$ be the loop formed by the union of four arcs $(\overline{\mathbb{R}}\times\{p_1\})\cup(\overline{\mathbb{R}}\times\{-p_2\})\cup(\{\infty\}\times\underline{\lambda})\cup(\{-\infty\}\times(-\underline{\lambda}))$ in $\overline{\mathbb{R}}\times M$, where $\underline{\lambda}\subset M$ is the closure of the image of the path $\lambda(\cdot)\colon \mathbb{R}\to M$. Then

$$\mathrm{h}_\lambda\big(\tilde\varsigma_\lambda(\bar\iota(\mathfrak{c}_+))\big) - \mathrm{h}_\lambda\big(\tilde\varsigma_\lambda(\bar\iota(\mathfrak{c}_-))\big) + \underline{\mathrm{h}}_{\hat p_1}(\underline{\mathfrak{d}}) - \underline{\mathrm{h}}_{\hat p_2}(\underline{\mathfrak{d}}) = -\frac{i}{2\pi}\ln(\mathrm{hol}^E_\gamma(\mathfrak{d})) = 0 \in \mathbb{R}/\mathbb{Z},$$

and hence the coefficients in u_λ are $\int_{\mathbb{R}\underline{\mathfrak{d}}}\theta_\lambda - \underline{h}_{\hat p_1}(\underline{\mathfrak{d}}) + \underline{h}_{\hat p_2}(\underline{\mathfrak{d}}) \in \mathbb{Z}$.

Let $\mathring{\mathrm{K}}_\lambda = \mathring{m}[u_\lambda]\colon \mathring{C}(M)\to\mathring{C}(M)$, a degree -1 map defined in the same manner as $\mathring{m}[\theta_\lambda]$, namely as in (2-44)–(2-45) with $\bar n^s_u[d\underline{\mathrm{h}}_{\hat p_i}]$ there replaced by $\bar n^s_u[\mathrm{u}_{p_i}] = (\overline{U}_{p_i})^s_u$. It follows from (2-48) and [17, Proposition 25.3.4] that

$$\mathring{U}_{p_2} - \mathring{U}_{p_1} = [\mathring{\mathrm{K}}_\lambda, \mathring{\partial}]. \tag{2-49}$$

Namely, $\mathring{\mathrm{K}}_\lambda$ defines a chain homotopy equivalence between the two U–maps $\mathring{U}_{p_2}$ and $\mathring{U}_{p_1}$.

The arguments in the preceding example generalizes readily to cobordisms X of the types considered in Section 2.4. Note that the diagram (2-46) and the first three lines of (2-47) hold in general. When X is not a product cobordism, the fourth line of (2-47) has a simple modification by replacing its first term by the more general $\langle\tilde\varsigma_\lambda^*(\tilde\Pi^\partial_\lambda)^*\vartheta'_{p_2-p_1}, \mathfrak{r}^{-1}(\overline{\mathcal{M}})_0\rangle$, where $\mathfrak{r}^{-1}(\overline{\mathcal{M}})_0$ fibers over $(\overline{\mathcal{M}})_0$, with fibers consisting of 1–dimensional strata of $\mathcal{N}^+(M_1)$ or $\mathcal{N}^+(M_2)$.

The map $\hat{\mathrm{K}}_\lambda$ in the example has an analog in this setting, which we denote by the same notation,

$$\mathring{\mathrm{K}}_\lambda(X) := \mathring{m}[\theta_\lambda](X) + \mathring{\Theta}_{p_2} * \mathring{m}[1](X) - \mathring{\Theta}_{p_1} * \mathring{m}[1](X). \tag{2-50}$$

In the above,

$$\mathring{\Theta}_{p_i} := \mathring{m}[\theta_{\dot{p}_i}](\mathbb{R}\times M_i) = -\mathring{n}[\underline{h}_{\hat{p}}](M_i)$$

is used to denote both an endomorphism on $\mathring{C}(M_i)$ and its associated endomorphism, $\mathring{\Theta}_{p_i}\otimes 1$, on $\mathring{C}(Y_\pm)$. Meanwhile, $\mathring{\Theta}_{p_i} * \mathring{m}[1](X)$ denotes either the composition $\mathring{\Theta}_{p_i}\mathring{m}[1](X)$ or $\mathring{m}[1](X)\mathring{\Theta}_{p_i}$, depending on whether $p_i\in Y_-$ or $p_i\in Y_+$. Note that while $\mathring{m}[\theta_\lambda](X)$ is defined for coefficient ring $\mathbb{K}=\mathbb{R}$, arguments similar to those in the preceding example show that $\mathring{\mathrm{K}}_\lambda$ is in fact defined for coefficient ring $\mathbb{K}=\mathbb{Z}$. The arguments also give rise to an analog of the identity (2-49),

$$\mathring{U}_{p_2} * \mathring{m}[1](X) - \mathring{U}_{p_1} * \mathring{m}[1](X) = [\mathring{\mathrm{K}}_\lambda, \mathring{\partial}]. \tag{2-51}$$

Remark 2.4 Instead of the formula given in (2-50), it is possible to express $\mathring{\mathrm{K}}_\lambda$ as

$$\mathring{\mathrm{K}}_\lambda = \mathring{m}[u_\lambda](X),$$

with $u_\lambda\in C^{1;\mathbb{Z}}_{\mathcal{M}(X)}$, in a way parallel to (2-48). This often yields cleaner formulas in later discussions but is less practical, being not as concrete as (2-50). In what follows we alternate between these two equivalent description of $\mathring{\mathrm{K}}_\lambda$, depending on which is more convenient in the context.

(b) (when $d=2$) For each $M_i\in\mathfrak{E}$, let $\gamma_i\subset M_i$ be an embedded (oriented) circle or the empty set. Let $\Sigma\subset X$ be an embedded oriented surface *asymptotic to* $\{\gamma_i\}_{i\in\mathfrak{E}}$ in the following sense: $\Sigma\cap(X-X_c)$ is the union of connected components of the following form: under the diffeomorphisms in (2-8), for each M_i there is a component $(-\infty,L')\times(-\gamma_i)\subset(-\infty,L')\times M_i$ if M_i is a negative end, and it is $(L',\infty)\times\gamma_i\subset(L',\infty)\times M_i$ if M_i is a positive end. Let $F_\Sigma\colon\mathcal{B}^\sigma(X)\to\mathbb{R}$ be the function sending a $\mathfrak{d}\in\mathcal{B}^\sigma(X)$ to

$$F_\Sigma(\mathfrak{d}) := \int_\Sigma \frac{iF_A}{2\pi},$$

where $A\in\mathrm{Conn}(E)$ is the connection associated to an arbitrary representative of $\mathfrak{d}$. The function F_Σ depends only on the relative homology class of Σ: for another embedded surface Σ' asymptotic to the end $\{\gamma_i\}_{i\in\mathfrak{E}}$,

$$F_{\Sigma'}-F_\Sigma = \tfrac{1}{2}\langle c_1(\mathfrak{s})-c_1(K^{-1}),[\Sigma'-\Sigma]\rangle.$$

Let $\theta'_{\gamma_i}\in\Omega^1(\mathcal{B}^\sigma(M_i))$ and $\mathrm{h}'_{\gamma_i}\colon\mathcal{B}^\sigma(M_i)\to\mathbb{R}/\mathbb{Z}$ be as defined in Part 2(b) if $\gamma_i\neq\varnothing$, and let $\theta'_{\gamma_i}:=0$ and $\mathrm{h}'_{\gamma_i}=0$ if $\gamma_i=\varnothing$. Then

$$dF_\Sigma = \sum_{i\in\mathfrak{E}}(\Pi^{M_i})^*\theta'_{\gamma_i}. \tag{2-52}$$

Thus, an integral correction of $[dF_\Sigma]_{\mathcal{M}(X)} = \delta[F_\Sigma]_{\mathcal{M}(X)} \in C^{1;\mathbb{R}}_{\mathcal{M}(X)}$ takes the form of $\delta\mathrm{F}_\Sigma$, where $\mathrm{F}_\Sigma \in C^{0;\mathbb{Z}}_{\mathcal{M}(X)} \subset C^{0;\mathbb{R}}_{\mathcal{M}(X)}$ is given by

$$\mathrm{F}_\Sigma := [F_\Sigma]_{\mathcal{M}(X)} - \sum_{i\in\mathcal{E}} [(\Pi^{M_i})^* h'_{\gamma_i}]_{\mathcal{M}(X)}, \tag{2-53}$$

where $h'_{\gamma_i}\colon \mathfrak{C}(M_i) \to \mathbb{R}$ is as in Part 2(b).

Example Take $X = \mathbb{R}\times M$ again to be the product cobordism, and let Σ be such that $s \mapsto \Sigma\cap(\{s\}\times M)$ forms a homotopy between the circles $\gamma_-, \gamma_+ \subset M$, both representing the element $\mathfrak{t}_i \in H_1(M;\mathbb{Z})/\mathrm{Tors}$. Applying equations (2-53) and (2-52) to this setting, and recalling from Part 2(b) the definition and properties of $u_{\dot\gamma}$, we have

$$\delta\mathrm{F}_\Sigma = u_{\dot\gamma_+} - u_{\dot\gamma_-}. \tag{2-54}$$

By Proposition 25.3.4 of [17], this implies that

$$\mathring{\mathfrak{m}}_{\gamma_-} - \mathring{\mathfrak{m}}_{\gamma_+} = [\mathring{\partial}, \mathring{m}[\mathrm{F}_\Sigma](\mathbb{R}\times M)]. \tag{2-55}$$

Namely, $\mathring{m}[\mathrm{F}_\Sigma]$ defines a chain homotopy equivalence between the two $\mathfrak{t}_i$–maps $\mathring{\mathfrak{m}}_{\gamma_-} := \mathring{m}[u_{\dot\gamma_-}]$ and $\mathring{\mathfrak{m}}_{\gamma_+} := \mathring{m}[u_{\dot\gamma_+}]$.

The preceding example also generalizes readily. When X is not a product cobordism, the identities (2-54) and (2-55) have respectively the analogs

$$\delta\mathrm{F}_\Sigma = \sum_{i\in\mathcal{E}} [(\Pi^{M_i})^*\mu_{\gamma_i}], \quad -\sum_i \mathring{m}[1](X) * \mathring{\mathfrak{m}}_{\gamma_i} = [\mathring{\partial}, \mathring{m}[\mathrm{F}_\Sigma](X)], \tag{2-56}$$

where $\mathring{m}[1](X) * \mathring{\mathfrak{m}}_{\gamma_i}$ denotes the composition map $\mathring{m}[1](X)\mathring{\mathfrak{m}}_{\gamma_i}$ when $\gamma_i \subset Y_+$, and it denotes $-\mathring{\mathfrak{m}}_{-\gamma_i}\mathring{m}[1](X)$ when $\gamma_i \subset Y_-$.

Remark 2.5 In view of (2-25), the actions $\mathring{U}_p$ and $\mathring{\mathfrak{m}}_{\mathfrak{t}_i}$ defined in Part 2 above extend to the case when M is not necessarily connected, and together they define a $A_\dagger(M) := \mathbb{K}[U]\otimes \bigwedge^* H_1(M;\mathbb{Z})/\mathrm{Tors}$–action associated to each choice of p and $\{\mathfrak{t}_i\}_i$ for possibly disconnected M. These more general $\mathring{U}_p$ and $\mathring{\mathfrak{m}}_{\mathfrak{t}_i}$ are chain maps as in the connected case; in fact it follows as a straightforward consequence of the case for connected 3–manifolds, already verified in [17] in the process of defining the $A_\dagger$–actions on the monopole Floer homology $\widehat{\mathrm{HM}}$. The arguments in Part 3(a) show that in this more general setting, $\mathring{U}_{p_1}$ and $\mathring{U}_{p_2}$ are chain homotopy equivalent when p_1 and p_2 belong to the same connected component of M, but *not* if p_1 and p_2 lie on different components of M. In fact, the cohomology classes $[\mu_{p_1}], [\mu_{p_2}] \in H^*(\mathcal{B}^\sigma(M);\mathbb{Z}) = H^*(\mathcal{B}^\sigma(M_1);\mathbb{Z}) \otimes H^*(\mathcal{B}^\sigma(M_2);\mathbb{Z})$ are independent. Generalizing the definition of $\mathring{U}_p$, given a 0–cycle $\boldsymbol{p}$ consisting of finitely many signed points p_i in M, let

$$\mathring{U}_{\boldsymbol{p}} := \sum_i \operatorname{sign}(p_i)\mathring{U}_{p_i}.$$

Suppose $M = M_\sqcup := M_1 \sqcup M_2$ consists of two connected components M_1 and M_2, and so $\mathcal{B}^\sigma(M_\sqcup) = \mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2)$. Suppose $p_i \in M_i$ for $i = 1, 2$. Then the spaces $\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times \widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)$ and $\widetilde{\mathcal{B}}^\sigma_{p_1}(M_1) \times_{U(1)} \widetilde{\mathcal{B}}^\sigma_{p_2}(M_2)$ in the second column of the diagram (2-38) are respectively $U(1)\times U(1)$– and $U(1)$–bundles over $\mathcal{B}^\sigma(M_\sqcup)$, and we abbreviate them respectively as $\widetilde{\widetilde{\mathcal{B}}}^\sigma_{p_1,p_2}(M_\sqcup)$ and $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$. It is worth noting that $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ is of the same homotopy type as $\mathcal{B}^\sigma(M_1 \# M_2)$ and $M_1 \# M_2$ being the connected sum of M_1 of M_2 along p_1 and p_2. While the Floer complex $\widehat{C}(M_\sqcup)$ in Part 2 above (heuristically) reflects the topology of $\mathcal{B}^\sigma(M_\sqcup)$, the connected sum theorem in Section 6 relates the Floer complex $\widehat{C}(M_\#)$ (associated with $\mathcal{B}^\sigma(M_\#)$) not directly to $\widehat{C}(M_\sqcup)$, but to a "Floer complex associated with $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$". Using the description of $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ as an S^1–bundle over $\mathcal{B}^\sigma(M_\sqcup)$, the latter complex is constructed using what was called the "algebraic S^1–bundle" operation in [23], described in more detail in Section 4 below. The ingredients of this construction consist of a chain-complex for the orbit space of the S^1–action, endowed with a "U–map" associated to its Euler class. The Euler class of the bundle $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ is $\mathrm{pr}_2^*\, e_{\dot{p}_2} - \mathrm{pr}_1^*\, e_{\dot{p}_1}$; so, in the setting under discussion, these are $\widehat{C}(M_\sqcup)$, endowed with U–map

$$\widehat{U}_\sqcup := 1 \otimes \widehat{U}_{p_2} - \widehat{U}_{p_1} \otimes 1 = \widehat{U}_{p_2-p_1}. \tag{2-57}$$

The precise definition of (the hat flavor of) "the Floer complex for $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$" is then what is called $S_{\widehat{U}_\sqcup}\widehat{C}_*(M_\sqcup)$ in Part 3 of Section 6.1. There, for any given $p \in M_\sqcup = M$ we also introduce an associated U–map on this Floer complex. Two such U–maps associated to different points $p, p' \in M$ are chain homotopy equivalent even if p and p' belong to different connected components of M. (See Lemma 6.4 below.)

Part 4: $A_\dagger$–actions under large r perturbations Let M be connected and let Q denote one of the generating elements of U or $\mathfrak{t}_i$ of $A_\dagger(M)$, U and $\mathfrak{t}_i$ being as defined in the beginning of Part 2. In the nonbalanced setting discussed in [22], a particular choice of p and the γ_i was made for the case when M is the auxiliary manifold Y in Theorem 1.1 (see Part 7 of Section IV.1.3), and the associated U–maps and $\mathfrak{t}_i$–maps were defined concretely. In this part we relate the description therein with the more general and abstract construction given in Part 2 above. The same arguments can be used to reinterpret the type of cobordism maps in Parts 1 and 3 under large r perturbations in a manner similar to [22]. Details will be provided for some particular 3–manifolds and cobordisms (including Y and the product cobordism $\mathbb{R}\times Y$ from [22] as special cases) in Section 3 below.

In the context of [22] as well those to be discussed in Section 3, the spinor bundle $\mathbb{S}$ on M splits as $E \oplus E \otimes K^{-1}$, and hence also a splitting of $\mathbb{S}^+$ on $\mathbb{R}\times M$, which we denote by the same notation. As pointed out in Part 2(a), in this case the tautological section $\tilde{\alpha}$ on $\mathcal{E}(\mathbb{R}\times M)$ or $\mathcal{E}(\overline{\mathbb{R}}\times M)$ is well defined.

The nonbalanced assumption implies that there are no reducible Seiberg–Witten solutions, leading to significant simplifications. To name a few: this allows one to replace the blowup space $\mathcal{B}^\sigma$ occurring in last part by the space $\mathcal{B}$ before blowing up. It also implies that $\overline{U}$– and $\overline{\mathfrak{t}}_i$–maps are trivial, and $\hat{U} = \check{U} =: U$ and $\hat{\mathfrak{m}}_{\mathfrak{t}_i} = \check{\mathfrak{m}}_{\mathfrak{t}_i} := \mathfrak{m}_{\mathfrak{t}_i}$. Moreover, the relevant moduli spaces are manifolds with corners in this setting; namely (2-18) holds and $[\partial\mathcal{M}] = \partial[\mathcal{M}]$.

The generating set of the relevant Floer complex, $\mathfrak{C}(M)$, in [22] is denoted by $\mathcal{Z} = \mathcal{Z}_{\mathrm{SW,r}}$. For large r, this is a finite set, and its elements are all represented by elements of the form $(A,(\alpha,\beta)) \in \mathrm{Conn}(E)\times\Gamma(E\oplus E\otimes K^{-1})$ with $\alpha^{-1}(0)$ consisting of finitely many points in M. This makes it possible to choose the point $p\in M$ used to define the U–map and the embedded circles γ_i used to define the $\mathfrak{t}_i$–maps to be mutually disjoint and to all lie in the complement of $\alpha^{-1}(0)\subset M$. Write the map, $\mathfrak{m}_Q$, associated to each Q in a form similar to (2-14) and (2-15) (with $\mathfrak{m}_U := U$):

$$\mathfrak{m}_Q = \sum_{\mathfrak{c}_1,\mathfrak{c}_2\in\mathcal{Z}} \sum_{z\in\pi_1\mathcal{B}(M;\mathfrak{c}_1,\mathfrak{c}_2)} \mathrm{w}_Q(\mathfrak{c}_1,\mathfrak{c}_2;z)\Gamma(z);$$

and, in the monotone case, let $\mathrm{w}_Q(\mathfrak{c}_1,\mathfrak{c}_2) = \sum_z \mathrm{w}_Q(\mathfrak{c}_1,\mathfrak{c}_2;z)$. The discussion in the rest of this part works for both $\mathrm{w}_Q(\mathfrak{c}_1,\mathfrak{c}_2;z)$ and $\mathrm{w}_Q(\mathfrak{c}_1,\mathfrak{c}_2)$, but for simplicity only the latter will be mentioned.

(a) (the U–map associated to $p\in M$) In the formulation of Part 2(a), the coefficients of the U_p–map are given by

$$\mathrm{w}_U(\mathfrak{c}_1,\mathfrak{c}_2) = \langle e, \overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2)\rangle.$$

This is the Euler number of the bundle $\mathcal{E}(\mathbb{R}\times M)|_{\overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2)}$ relative to the trivialization $\rho_\vartheta|_{\partial\overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2)\subset\mathcal{M}_1(\mathbb{R}\times M)}$. In comparison, Section IV.1.3's $\mathrm{w}_U(\mathfrak{c}_1,\mathfrak{c}_2)$ is taken to be the signed count of elements in $\mathcal{M}_{2,p}(\mathfrak{c}_1,\mathfrak{c}_2)$, where $\mathcal{M}_{k,p}(\mathfrak{c}_1,\mathfrak{c}_2)\subset$

$\mathcal{M}_k(\mathfrak{c}_1,\mathfrak{c}_2)$ consists of elements $\mathfrak{d} \in \mathcal{M}_k(\mathfrak{c}_1,\mathfrak{c}_2)$ represented by some $(A,(\alpha,\beta)) \in \operatorname{Conn}(E) \times \Gamma(\mathbb{S}^+)$ with α vanishing at $x = (0,p) \in X = \mathbb{R} \times M$. Suitable genericity assumptions on $(\mathfrak{T},\mathfrak{S})$ and p were imposed so that for all $\mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{Z}$, $\mathcal{M}_{k,p}(\mathfrak{c}_1,\mathfrak{c}_2) = \varnothing$ for $k < 2$ and $\mathcal{M}_{2,p}(\mathfrak{c}_1,\mathfrak{c}_2)$ consists of finitely many regular points. Let $\tilde{\alpha}_x \in \Gamma(\mathcal{E}_x(\overline{\mathbb{R}} \times M))$ be the section obtained by restricting the tautological section $\tilde{\alpha}$ to $\mathcal{E}|_{\{x\}\times\mathcal{B}(\overline{\mathbb{R}}\times M)\subset X\times\mathcal{B}(\mathbb{R}\times M)} = \mathcal{E}_x(\mathbb{R} \times M)$. Then the space $\mathcal{M}_{k,p}(\mathfrak{c}_1,\mathfrak{c}_2)$ is precisely the zero locus of the section $\tilde{\alpha}_x$ on

$$\mathcal{M}_k(\mathfrak{c}_1,\mathfrak{c}_2) \subset \mathcal{B}_{\mathrm{loc}}(\mathbb{R} \times M).$$

The fact that $\mathcal{M}_{k,p}(\mathfrak{c}_1,\mathfrak{c}_2) = \varnothing$ for all $\mathfrak{c}_1$ and $\mathfrak{c}_2$ for $k < 2$ implies that $\tilde{\alpha}_x$ is nowhere-vanishing on $\mathcal{M}_1(\mathbb{R} \times M)$, and hence $\tilde{\alpha}_x/|\tilde{\alpha}_x|$ defines a trivialization of $\mathcal{E}_x(\overline{\mathbb{R}} \times M)|_{\partial\overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2)\subset\mathcal{M}_1(\mathbb{R}\times M)}$, and the Euler number of the complex line bundle $\mathcal{E}_x(\overline{\mathbb{R}} \times M)|_{\overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2)}$ relative to this trivialization is precisely the Euler characteristic of $\mathcal{M}_{2,p}(\mathfrak{c}_1,\mathfrak{c}_2) = \overline{\mathcal{M}}_2(\mathfrak{c}_1,\mathfrak{c}_2) \cap \tilde{\alpha}_x^{-1}(0)$, namely the value of $\mathrm{w}_U(\mathfrak{c}_1,\mathfrak{c}_2)$ defined in [22]. This agrees with the expression from Part 2(a) if ϑ therein is *chosen so that* $\tilde{\alpha}_x/|\tilde{\alpha}_x|$ is constant with respect to the trivialization ρ_ϑ on $\mathcal{M}_1(\mathbb{R} \times M)$. As observed in Part 1(a), the cocycle $e \in C^{2;\mathbb{Z}}_{\mathcal{M}(\mathbb{R}\times M)}$ depends on the δ–cohomology class of ϑ, which in turn depends on the class $[\theta_p]_{\mathcal{M}(\mathbb{R}\times M)} \in C^{2;\mathbb{Z}}_{\mathcal{M}(\mathbb{R}\times M)}$. The aforementioned choice in the large-perturbation setting is natural in the sense that under proper setup, one expects

$$[\theta_p]_{\mathcal{M}(\mathbb{R}\times M)} \to 0 \quad\text{and}\quad [\underline{h}_{\hat{p}}]_{\mathcal{N}_1^+(M)} \to 0 \quad \text{as } \mathrm{r} \to \infty, \tag{2-58}$$

which in turn is based on the expectation that, roughly speaking,

$$|\nabla_A \alpha| \to 0 \quad \text{pointwise away from } \alpha^{-1}(0) \text{ as } \mathrm{r} \to \infty; \tag{2-59}$$

or, put in another way, a variant of [30, Proposition 4.1] holds. A weak version of the latter in the setting of Section 3 is provided in Lemma 7.6.

To see how (2-58) would follow from (2-59), recall (2-32) and note that, as

$$\overline{\mathcal{M}}_1(\mathbb{R} \times M) \cap \tilde{\alpha}^{-1}(0) = \varnothing,$$

$|\alpha|\big|_{\hat{p}}$ is nowhere-vanishing for all $\mathfrak{d} \in \mathcal{M}_1(\mathbb{R} \times M)$. Let $(A,(\alpha,\beta)) \in \mathcal{C}(\mathbb{R} \times M)$ be a representative of the aforementioned $\mathfrak{d}$, and use $\hat{A}_\alpha$ to denote the connection defined on $(\mathbb{R} \times M) \setminus \alpha^{-1}(0)$ satisfying $\nabla_{\hat{A}_\alpha}(\alpha/|\alpha|) = 0$. Thus, for $\mathfrak{d} \in \mathcal{M}_1(\mathbb{R} \times M)$,

$$h_{\hat{p}}(\mathfrak{d}) = -\int_{\mathbb{R}\mathfrak{d}} \theta_p = -\int_{\hat{p}} (\iota_\Delta)^* \theta_{\dot{p}} = \frac{i}{2\pi} \int_{\hat{p}} (A - \hat{A}_\alpha) \to 0 \quad \text{as } \mathrm{r} \to 0$$

if (2-59) holds, and (2-58) follows as a consequence.

(b) (the $\mathfrak{t}_i$–map associated to $\gamma_i \subset M$) According to Part 2(b),

$$\begin{aligned}\text{(2-60)}\qquad \mathrm{w}_{\mathfrak{t}_i}(\mathfrak{c}_1,\mathfrak{c}_2) &= \langle u_{\dot\gamma_i}, \overline{\mathcal{M}}_1(\mathfrak{c}_1,\mathfrak{c}_2)\rangle \\ &= \sum_{\underline{\mathfrak{d}}\in\mathcal{N}_0(\mathfrak{c}_1,\mathfrak{c}_2)} \operatorname{sign}(\underline{\mathfrak{d}})(\Delta_{\underline{\mathfrak{d}}}\,\mathrm{x}_{\gamma_i}) \\ &= \langle \theta_{\dot\gamma_i}, \overline{\mathcal{M}}_1(\mathfrak{c}_1,\mathfrak{c}_2)\rangle - \langle \Pi_0^* h'_{\gamma_i}, \partial\overline{\mathcal{M}}_1(\mathfrak{c}_1,\mathfrak{c}_2)\rangle.\end{aligned}$$

If γ_i lies on the complement of $\alpha^{-1}(0)$ for all $(A,(\alpha,\beta))$ representing elements $\mathfrak{c}$ in $\mathfrak{C}(M)$, there is a natural choice of $h'_{\gamma_i}\colon \mathfrak{C}(M)\to\mathbb{R}$ among the $\mathbb{Z}^{\mathfrak{C}(M)}$–many possible lifts of $\mathrm{h}'_{\gamma_i}|_{\mathfrak{C}(M)}$, leading to a natural choice of u_{γ_i}. Namely, one sets

$$\text{(2-61)}\qquad h'_{\gamma_i}(\mathfrak{c}) = \frac{i}{2\pi}\int_{\gamma_i}(A-\hat{A}_\alpha)$$

in this case. With this choice of h'_{γ_i}, the corresponding x_{γ_i} satisfies

$$\mathrm{x}_{\gamma_i}(\mathfrak{c}) = \mathrm{hol}_{\gamma_i}(\hat{A}_\alpha) \bmod \mathbb{Z} \quad \text{for all } \mathfrak{c}\in\mathfrak{C}(M),$$

where $\mathrm{hol}_{\gamma_i}(\hat{A}_\alpha)\in U(1)=\mathbb{R}/\mathbb{Z}$ denotes the holonomy of $\hat{A}_\alpha$ along γ_i. Such γ_i can be found in large r–perturbation settings when a suitable variant of (2-59) holds; in fact, with such γ_i,

$$\text{(2-62)}\qquad h'_{\gamma_i}\to 0 \quad \text{as } \mathrm{r}\to\infty.$$

This may indeed be arranged in the setting of this series of articles. In various parts of [22] as well as in latter parts of this article (eg (3-10)), choices of x_{γ_i} were made via explicit formulas, and (2-62) in this context is given a precise reformulation in terms of x_{γ_i} in Lemma 3.2.

In Part 7 of Section IV.1.3, the integer $\Delta_{\underline{\mathfrak{d}}}\,\mathrm{x}_{\gamma_i}$ in (2-60) is given an alternative description as the algebraic intersection number between $\alpha^{-1}(0)$ and the cylinder $\mathbb{R}\times\gamma_i\subset\mathbb{R}\times M$. To relate this with the definition in Part 2(b), note that by the choice of γ_i, the section $\alpha|_{\mathbb{R}\times\gamma_i\subset\mathbb{R}\times M}$ of the bundle $E|_{\mathbb{R}\times\gamma_i}$ is nowhere-vanishing over the boundary of the cylinder $\partial(\overline{\mathbb{R}}\times\gamma_i)\subset\{-\infty,\infty\}\times M$, and the aforementioned intersection number agrees with the relative Chern number of $E|_{\overline{\mathbb{R}}\times\gamma_i}$ relative to the trivialization over $\partial(\overline{\mathbb{R}}\times\gamma_i)$ defined by $\alpha/|\alpha|$. This relative Chern number in turn can be expressed as

$$\int_{\overline{\mathbb{R}}\times\gamma_i}\frac{i}{2\pi}F_{\hat{A}_\alpha} = \Delta_{\underline{\mathfrak{d}}}\,\mathrm{x}_{\gamma_i}.$$

3 Filtered monopole Floer homologies

The algebraic recipe for Ozsváth and Szabó's definition of the four flavors of Heegaard Floer homologies, labeled by the superscripts $-$, ∞, $+$ and $\wedge$, was summarized abstractly in Section 4 of [23]. In this section, we explain how the same recipe may be applied in the Seiberg–Witten context to define analogs of Ozsváth and Szabó's Floer homologies. These intermediate Floer homologies play a pivotal role in the proof of Theorem 1.1.

3.1 Motivation and sketches of construction

The aforementioned recipe hinges on the existence of certain filtration on a Floer chain complex with local coefficients in the group ring $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$, with U corresponding to the generator $1 \in \mathbb{Z}$. This Floer complex with local coefficients constitutes the ∞ flavor of the Ozsváth–Szabó construction, while the "filtration" refers to the filtration of the coefficient ring $\mathbb{K}[U, U^{-1}]$ by submodules

$$\cdots \subset U\mathbb{K}[U] \subset \mathbb{K}[U] \subset U^{-1}\mathbb{K}[U] \subset \cdots \subset \mathbb{K}[U, U^{-1}].$$

If the differential of the ∞ flavor of the Floer complex preserves this filtration, then it induces a filtration on the ∞ flavor Floer complex by $\mathbb{K}[U]$–subcomplexes, which are all isomorphic via multiplication by powers of U. This defines the $-$ flavor Floer complex. With these two basic flavors in place, the $+$ and the $\wedge$ flavors are defined so that they fit into short exact sequences (see (3-18) below) inducing what are called the *fundamental exact sequences* of corresponding Floer homologies. In [23], the existence of such a filtration is attributed to the existence of what was termed a "semipositive 1–cocycle". The 1–cocycle used here refers to the cocycle that defines the local system on the ∞ flavor Floer complex. The "semipositivity" condition serves to guarantee that the differential is filtration-preserving. Note that the ∞ flavor of Floer homology depends only on the cohomology class of this cocycle. The other three flavors of Ozsváth and Szabó's construction depend on the choice the cocycle that defines the semipositivity condition.

Section 4.2 of [23] provides some examples where this recipe may be applied. Section 6 of the same article sketched how such semipositive 1–cocycles might arise in certain versions of Seiberg–Witten Floer theory associated to equations of the form of (2-5). In particular, choosing the metric and 2–form w in (2-5) to reflect the data that go into the definition of Heegaard Floer homology provides a bridge to relate the Heegaard and Seiberg–Witten Floer homologies.

To elaborate, the local system underlying the Seiberg–Witten analog of the Ozsváth–Szabó construction is closely related to what was denoted by Γ_η in [17] (see the example in the end of their Section 22.6), where η is a singular 1–cycle in a certain 3–manifold $\underline{M}$. Use $[(\mathbb{A}, \Psi)] \in \mathcal{B}^\sigma$ to denote the gauge-equivalence of $(\mathbb{A}, \Psi)$. In [17], this local system associates to each point on $\mathcal{B}^\sigma$ the "fiber" $\mathbb{R}$, and, to each path $\{[(\mathbb{A}(\tau), \Psi(\tau))]\}_\tau$ from $[(\mathbb{A}_-, \Psi_-)]$ to $[(\mathbb{A}_+, \Psi_+)]$, an isomorphism $\mathbb{R}^\times \subset \mathrm{End}(\mathbb{R})$ between the fibers over the endpoints. The latter isomorphism is given by multiplication by the real number

$$e^{(i/2\pi)\int_\tau \int_\eta \frac{d}{d\tau}\mathbb{A}(\tau)}. \tag{3-1}$$

Note that the exponent is the difference of the holonomy of $\mathbb{A}_-$ along the cycle $\eta \subset M$ from that of $\mathbb{A}_+$, and it defines a real 1–cocycle in $\mathcal{B}^\sigma$. Meanwhile, as only points in $\mathfrak{C} \subset \mathcal{B}^\sigma$ and paths constituting the sets $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ with $\mathfrak{c}_-, \mathfrak{c}_+ \in \mathfrak{C}$ enter the definition of a monopole Floer complex, it suffices to consider the holonomy difference of paths corresponding to elements in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$. The observation leading to [23]'s construction of filtered monopole Floer homologies (in the sense of Ozsváth and Szabó) is the following:

(3-2) For monopole Floer complexes associated to certain ϖ in the form of (2-3) with large r and a certain choice of η, the value of the aforementioned holonomy difference is very close to a nonnegative integer.

(See also (2-62).)

Associating to each element in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ its corresponding integer, one has a (partially defined) integer 1–cocycle on $\mathcal{B}^\sigma$ with which one may define a Floer complex with more refined local coefficients than Γ_η. We denote the latter local system by Λ_η. It replaces the fibers $\mathbb{R}$ over $\mathfrak{C}$ of Γ_η by the group ring $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$; and it replaces the isomorphism induced by an element $\mathfrak{d}$ in $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ between these fibers, namely (3-1), by U^n where n denotes the aforementioned nonnegative integer associated to $\mathfrak{d}$. The fact that $n \geq 0$ in all cases has the following consequence: Use the corresponding monopole Floer complex with local coefficients, Λ_η, as the ∞ *flavor Floer complex*. There is filtration on this chain complex, $\mathrm{CM}_*(\underline{M}; \Lambda_\eta)$, by subcomplexes of $\mathbb{K}[U]$–modules. This can be used to define the other three flavors of Floer complexes.

The program described in [23] assumes various plausible conjectures and assertions that come from an extension of the geometric picture in the last author's work relating the Seiberg–Witten and Gromov invariant for closed 4–manifolds (see [31; 30]). A proof of these conjectures constitute a major part of the technical hurdle for implementing the program in [23]. The difficulties arise because the 2–form ϖ in [23] must have zeros.

In this series of articles [19; 20; 21; 22], the roadblock to the approach in [23] is circumvented by a modification of [23]'s outline. Very roughly, the manifold $\underline{M}$ in [23] is replaced by the manifold denoted by Y in [20]. This is obtained from $\underline{M}$ by adding further 1–handles along the zeros of w on $\underline{M}$. The 2–form w extends into Y as a nowhere-vanishing closed 2–form, which we also denote by w. Over the middle of the added 1–handle, this w approximates da for a certain contact form a, and as the special 1–cycle η (denoted by $\underline{\gamma}$ therein) lies away from the zeros of w on $\underline{M}$, this 1–cycle also embeds in Y. This was denoted by $\gamma^{(z_0)}$ in [19; 20; 21; 22]. The technical challenge in this new approach involves, among other things, the analog of (3-2) for the monopole Floer complex associated to Y, w and $\eta = \gamma^{(z_0)}$. Some of these technical issues are dealt with in [22]. Those that remain are dealt with in Sections 7–9 of this article.

In Section 3.2 below, we specify the class of 3–manifolds, denoted by Y_Z therein, together with the 2–form w on it and the 1–cycle η for which positively results of the kind (3-2) hold. Section 3.3 describes the sort of cobordisms X for which the companion statements hold. See Propositions 3.4, 3.5, 3.10, 3.12 and 3.15. The remaining subsections give precise statements of the desired positivity results. The formulation here involves a "cut-off" version of the connection A (called $\widehat{A}$), so that in place of (3-2), its associated holonomy difference is integer-valued. (See Lemma 3.2.) The conditions on Y_Z and X are introduced more for technical convenience rather than essential reasons, and the statements in Sections 3.4–3.7 may conceivably hold for more general 3–manifolds and 4–dimensional cobordisms.

3.2 The 3–manifold Y_Z

Let Z denote a given connected, oriented closed 3–manifold; and let Y_Z denote the manifold that is obtained from Z by attaching a 1–handle at a chosen pair of points, denoted by (p_0, p_3) below. In the proof of the main Theorem 1.1, Z is taken to be either S^3, the manifold M in the statement of Theorem 1.1, or a manifold that is obtained from M by attaching some number of 1–handles. Although Y_Z is diffeomorphic to the connected sum of Z and $S^1 \times S^2$, it is viewed for the most part as

$Z_\delta \cup \mathcal{H}_0$ with $\mathcal{H}_0$ the attached 1–handle and with Z_δ being the complement of a pair of coordinate balls about the chosen points p_0 and p_3 in Z. The manifold Y_Z has a distinguished embedded loop that crosses the handle $\mathcal{H}_0$ once. This loop is denoted by γ. The three parts of this subsection say more about the geometry of Y_Z near $\mathcal{H}_0$, near γ, and in general.

Part 1 The geometry of Y_Z near $\mathcal{H}_0$ is just like that given in Section II.1A. By way of a reminder, the description of the geometry requires the a priori specification of constants $\delta_* \in (0,1)$ and $R > -100\ln\delta_*$. Also needed are coordinate charts centered on p_0 and p_3. The latter are used to identify respective neighborhoods of these points with balls of radius $10\delta_*$ in $\mathbb{R}^3$. The pullback of the standard spherical coordinates on $\mathbb{R}^3$ gives spherical coordinate functions on the neighborhood of p_0, these denoted by $(\mathrm{r}_+, (\theta_+, \phi_+))$. There are corresponding coordinate functions for the neighborhood of p_3; these are denoted in what follows by $(\mathrm{r}_-, (\theta_-, \phi_-))$.

The handle $\mathcal{H}_0$ is diffeomorphic to the product of an interval with S^2. The interval factor is written as $[-R-7\ln\delta_*, R+7\ln\delta_*]$ and u is used to denote the Euclidean coordinate for this interval. The spherical coordinates for the S^2 factor are written as (θ,ϕ). The handle $\mathcal{H}_0$ is attached to the coordinate balls centered on p_0 and p_3 as follows: Delete the $\mathrm{r}_+ < e^{-2R}(7\delta_*)^{-1}$ part of the coordinate ball centered on p_0 and the corresponding part of the coordinate ball centered on p_3. Having done so, identify $\mathcal{H}_0$ with the respective $\mathrm{r}_+ \in [e^{-2R}(7\delta_*)^{-1}, 7\delta_*]$ and $\mathrm{r}_- \in [e^{-2R}(7\delta_*)^{-1}, 7\delta_*]$ parts of these coordinate balls with $\mathcal{H}_0$ by writing

$$(\mathrm{r}_+ = e^{-R+u}, (\theta_+ = \theta, \phi_+ = \phi)) \quad \text{and} \quad (\mathrm{r}_- = e^{-R-u}, (\theta_- = \pi - \theta, \phi_- = \phi)). \tag{3-3}$$

The handle $\mathcal{H}_0$ has a distinguished closed 2–form, this being $\frac{1}{2}\sin\theta\, d\theta\, d\phi$. This 2–form is nowhere zero on the constant u cross-sectional spheres and thus orients these spheres. Granted this orientation, then $\frac{1}{2}\sin\theta\, d\theta\, d\phi$ has integral 2 over constant u spheres.

Part 2 The loop γ intersects $\mathcal{H}_0$ as the $\theta = 0$ arc. Thus it has geometric intersection number 1 with each $u =$ constant sphere. This loop is oriented so that the corresponding algebraic intersection number is $+1$. A tubular neighborhood of γ is specified with a diffeomorphism to the product of S^1 and a disk about the origin in $\mathbb{C}$. The latter is denoted by D_γ and its complex coordinate is denoted by z. The diffeomorphism identifies the $z = 0$ circle in $S^1 \times D$ with γ. The circle S^1 is written in what follows as $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ with $\ell_\gamma > 0$ being a chosen constant. The affine coordinate for $\mathbb{R}/(\ell_\gamma\mathbb{Z})$

is denoted by t. The product structure on such a neighborhood is constrained where it intersects $\mathcal{H}_0$ by the requirement that the $\mathcal{H}_0$ coordinate u on the intersection depend only on t. A neighborhood with these coordinates is fixed once and for all; it is denoted by U_γ.

Part 3 Use the Mayer–Vietoris principle to write the second homology of Y_Z as

$$H_2(Y_Z;\mathbb{Z}) = H_2(Z;\mathbb{Z}) \oplus H_2(\mathcal{H}_0;\mathbb{Z}). \tag{3-4}$$

The convention in what follows is to take the generator of $H_2(\mathcal{H}_0;\mathbb{Z})$ to be the class of any cross-sectional sphere with the orientation given by the 2–form $\sin\theta\, d\theta\, d\phi$. Fix a class in $H^2(Y_Z;\mathbb{Z})$ which has even pairing with the classes in $H_2(Y_Z;\mathbb{Z})$ and pairing 2 with the generator of the $H_2(\mathcal{H}_0;\mathbb{Z})$ summand in (3-4). This class is denoted in what follows by $c_1(\det\mathbb{S})$, and it is necessarily nontorsion by the above assumption.

There is a corresponding closed 2–form on Y_Z whose de Rham cohomology class is that of $c_1(\det\mathbb{S})$. In particular, there are forms w of this sort satisfying the following additional constraints:

(3-5)
- The form restricts to $\mathcal{H}_0$ as $\frac{1}{2\pi}\sin\theta\, d\theta\, d\phi$.
- The form restricts to U_γ as $\frac{i}{2\pi} g(|z|)\, dz \wedge d\bar{z}$ with g denoting a strictly positive function.
- There is a closed 1–form on Y_Z, typically denoted by υ below, with the following properties:
  - (a) It has nonnegative wedge product with w.
  - (b) It restricts to U_γ as dt, and restricts to $\mathcal{H}_0$ as $\mathrm{H}(u)\, du$ with $\mathrm{H}(u) > 0$ for all u.

Fix such a 2–form as the perturbation form w in (2-5).

The metric on Y_Z is chosen to satisfy the following constraints:

(3-6)
- The metric appears on $\mathcal{H}_0$ as the product metric of an S^2–independent metric on the interval $[-R - \ln(7\delta_*), R + \ln(7\delta_*)]$ and the round metric $d\theta^2 + \sin^2\theta\, d\phi^2$ on the S^2 factor. Meanwhile, the curvature 2–form of A_K on $\mathcal{H}_0$ is $\frac{i}{2\pi}\sin\theta\, d\theta\, d\phi$.
- The metric appears on U_γ as $dt^2 + g(|z|)\, dz \otimes d\bar{z}$ with g being the function in the second bullet of (3-5). Meanwhile, A_K has holonomy 1 on γ and its curvature 2–form on U_γ is iw.

Many of the lemmas and propositions in the rest of this section depend implicitly on the radius of D_{r} and on the injectivity radius of the Riemannian metric. They also depend implicitly on the norms of w, the curvature of A_K, the Riemannian curvature, and the norms of their derivatives up to some order less than 10.

There are suitable choices for μ with positive but small as desired $\mathcal{P}$–norm that vanish on $\mathcal{H}_0 \cup U_\gamma$. This last property is not a direct consequence of an explicit assertion in [17] but it follows nonetheless from their constructions.

The reference connection A_E is chosen constrained only to the extent that it is flat on $\mathcal{H}_0$ and is flat with holonomy 1 on U_γ.

The function on $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ of central concern in what follows is the analog here of the function that is defined in (IV.1-16). This function is denoted by x. The definition requires the a priori choice of a smooth function $\wp\colon [0,\infty) \to [0,\infty)$ which is nondecreasing, obeys $\wp(x) = 0$ for $x < \frac{7}{16}$ and $\wp(x) = 1$ for $x \geq \frac{9}{16}$. As in [22], it proves convenient to choose $\wp$ so that its derivative, $\wp'$, is bounded by $2^{10}(1-\wp)^{3/4}$. The definition of x uses the fact that w is nowhere zero on U_γ. In particular, Clifford multiplication by $*w$ on U_γ splits $\mathbb{S}$ over U_γ as the direct sum of eigenbundles. This splitting is

$$\mathbb{S} = E \oplus E \otimes K^{-1} \tag{3-7}$$

with the convention being that $*w$ acts as $+i|w|$ on E. A given section ψ of $\mathbb{S}$ is written with respect to this splitting over U_γ as a pair denoted by $|w|^{1/2}(\alpha, \beta)$.

Granted this notation, use $\wp$ with a given pair $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ to define the connection

$$\hat{A} = A - \tfrac{1}{2}\wp(|\alpha|^2)|\alpha|^{-2}(\bar{\alpha}\nabla_A\alpha - \alpha\nabla_A\bar{\alpha}) \tag{3-8}$$

on $E|_{U_\gamma}$. The salient point is that the connection $\hat{A}$ is flat on the part of U_γ where $|\alpha|^2 > \frac{9}{16}$ (this is where $\wp = 1$) and the A–derivative of $\alpha/|\alpha|$ is zero on this same part of U_γ. This can be seen from the formulas

(3-9)
- $F_{\hat{A}} = (1-\wp(|\alpha|^2))F_A + \wp'(|\alpha|^2)\nabla_A\alpha \wedge \nabla_A\bar{\alpha}$;
- $\nabla_{\hat{A}}\alpha = (1-\wp(|\alpha|^2))\nabla_A\alpha + \wp(|\alpha|^2)\, d(\ln|\alpha|)\alpha$.

Meanwhile, the connections $\hat{A}$ and A are equal where $|\alpha|^2 \leq \frac{7}{16}$ (this is where $\wp = 0$).

With $\widehat{A}$ understood, then the value of the function $\mathrm{x} = \mathrm{x}_\gamma$ on the given configuration $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ is defined by rule whereby

$$\mathrm{x}(\mathfrak{c}) = \frac{i}{2\pi} \int_\gamma (\widehat{A} - A_E). \tag{3-10}$$

Remark 3.1 To relate with the general discussion in Part 2(b) of Section 2.5, note that $\widehat{A}$ from (3-8) agrees with the connection $\widehat{A}_\alpha$ over γ; and so setting $\varepsilon'_\gamma(\mathfrak{c}) = \int_\gamma (\widehat{A} - A)$ for $\mathfrak{c} \in \mathcal{B}^\sigma(M)$ would meet the requirement that $\varepsilon'_\gamma|_{\mathfrak{C}(M)} = h'_\gamma$ when h'_γ is given by (2-61). Meanwhile, the reference connection A_E plays the role of the basepoint $\hat{\mathfrak{c}}_0 \in \widehat{\mathcal{B}}^\sigma(M)$ in Section 2.5 in the following sense: Let $(A_0, (\alpha_0, \beta_0))$ be an arbitrary representative of $\hat{\mathfrak{c}}_0$ and for any $\hat{\mathfrak{c}} \in \widehat{\mathcal{B}}^\sigma_{\mathfrak{k}}(M)$, let $(A, (\alpha, \beta))$ be an arbitrary representative of $\hat{\mathfrak{c}}$. Then $\mathrm{x}_\gamma(\hat{\mathfrak{c}})$, as defined in Section 2.5's Part 2(a), equals

$$\mathrm{x}_\gamma(\hat{\mathfrak{c}}) = \hat{h}_\gamma(\hat{\mathfrak{c}}) - \Pi_0^* \varepsilon'_\gamma(\hat{\mathfrak{c}}) = \frac{i}{2\pi}\left(\int_\gamma (A - A_0) - \int_\gamma (A - \widehat{A})\right) = \frac{i}{2\pi} \int_\gamma (\widehat{A} - A_0).$$

The last term above equals (3-8) when $A_0 = A_E$. Note that $\hat{\mathfrak{c}}_0$ and A_E are required to satisfy certain constraints, namely both $\mathrm{hol}_\gamma(A_0) = 0 \bmod \mathbb{Z}$ and $\mathrm{hol}_\gamma(A_E) = 0 \bmod \mathbb{Z}$.

The following lemma supplies a fundamental observation about x:

Lemma 3.2 *If the conditions in* (3-5)–(3-6) *hold, then there exists* $\kappa > \pi$ *with the following significance: Fix* $\mathrm{r} > \kappa$ *and a* 1*–form* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1*. The function* x *has only integer values on the solutions to the corresponding* (r, μ) *version of* (2-5).

This lemma is proved in Section 7.3.

3.3 4–dimensional cobordisms

This subsection describes in general terms the sorts of cobordisms that are considered.

To start, let Z_- and Z_+ denote two versions of the manifold Z and let Y_- and Y_+ denote the respective $Z = Z_-$ and $Z = Z_+$ versions of Y_Z. There is no need to assume that either Y_- or Y_+ is connected, but if not, then the handle $\mathcal{H}_0$ is attached to the same connected component. Use γ_- to denote the Y_- version of the curve γ and use γ_+ to denote the Y_+ version. The corresponding versions of U_γ are denoted in what follows by U_- and U_+.

Of interest here is a smooth, oriented, 4–dimensional manifold X with the properties listed below, in addition to those in (2-8):

(3-11)
- There exists an embedding of $\mathbb{R}\times[-R-\ln(7\delta_*), R+\ln(7\delta_*)]\times S^2$ into X that pulls back s as the Euclidean coordinate on the $\mathbb{R}$ factor. Moreover, the composition of this embedding with the diffeomorphism in the second bullet identifies the $s<0$ part with $(-\infty,0)\times\mathcal{H}_0$ in $(-\infty,0)\times Y_-$; and the composition with the diffeomorphism from the third bullet identifies the $s>0$ part with $(0,\infty)\times\mathcal{H}_0$ in $(0,\infty)\times Y_+$.
- There exists an embedding of $\mathbb{R}\times S^1$ into X that pulls back s as the Euclidean coordinate on the $\mathbb{R}$ factor. Moreover, the composition of this embedding with the diffeomorphism in the second bullet identifies the $s<0$ part of $\mathbb{R}\times S^1$ with $(-\infty,0)\times\gamma_-$; and the composition with the diffeomorphism from the third bullet identifies the $s>0$ part of $\mathbb{R}\times S^1$ with $(0,\infty)\times\gamma_+$.

The image in X of the embedding of $\mathbb{R}\times[-R-\ln(7\delta_*), R+\ln(7\delta_*)]\times S^2$ from the first bullet above is denoted by U_0.

The notation used in the next constraint has C denoting the image in X of $\mathbb{R}\times S^1$ as described by the second bullet of (3-11). This constraint requires that the γ_- and γ_+ versions of ℓ_γ are equal:

(3-12)
- There exists $\ell_\gamma>0$ and a diffeomorphism of a neighborhood of C to the product of $\mathbb{R}\times\mathbb{R}/(\ell_\gamma\mathbb{Z})$ with a disk about the origin in $\mathbb{C}$. This disk is denoted by D.
- The diffeomorphism identifies the Euclidean coordinate on $\mathbb{R}\times\mathbb{R}/(\ell_\gamma\mathbb{Z})\times D$ with s.
- The $s<0$ and $s>0$ parts of the neighborhood are in $(-\infty,0)\times U_-$ and in $(0,\infty)\times U_+$, respectively. Moreover, the diffeomorphism on these parts of the neighborhood respects the respective splittings of U_- and U_+ as $(-\infty,0)\times\mathbb{R}/(\ell_\gamma\mathbb{Z})\times D$ and $(0,\infty)\times\mathbb{R}/(\ell_\gamma\mathbb{Z})\times D$.

By way of an explanation, a diffeomorphism of this sort exists if the conormal bundle to C in X has a nowhere-zero section that restricts to the $s<0$ part of X as the real part of the $\mathbb{C}$–valued 1–form dz along γ_- and restricts to the $s>0$ part of X as the real part of the $\mathbb{C}$–valued 1–form dz along γ_+. The tubular neighborhood in (3-12) is denoted in what follows by U_C. The diffeomorphism in (3-12) is used, often implicitly, to identify U_C with $\mathbb{R}\times\mathbb{R}/(\ell_\gamma\mathbb{Z})\times D$.

In addition to those listed in (2-12), the 2–form w_X to use in the Seiberg–Witten equations is required to satisfy the following additional constraint:

(3-13) The pullback of w_X to U_0 via the embedding from the fourth bullet of (3-11) is twice the self-dual part of $\frac{1}{2}\sin\theta\, d\theta\, d\phi$ and its pullback to U_C via the embedding in (3-12) is twice the self-dual part of $\frac{i}{2\pi} g(|z|)\, dz \wedge d\bar{z}$.

Meanwhile, the metric on X is required to satisfy the following constraints in addition to those in (2-9):

(3-14)
- The metric pulls back from U_0 via the embedding of the first bullet of (3-11) as the product metric defined by the Euclidean metric on the $\mathbb{R}$ factor and an $\mathbb{R}$–independent product metric on the $[-R-\ln(7\delta_*), R+\ln(7\delta_*)] \times S^2$ factor.
- The metric pulls back from U_C via the embedding in (3-12) as the product metric given by the quadratic form $ds^2 + dt^2 + g(|z|)\, dz \otimes d\bar{z}$.

Extensions to U_C and U_0 of the Y_- and Y_+ versions of the line bundles K and E and their connections A_K and A_E are needed for what follows. There is no obstruction to making these extensions. Even so, it is necessary to constrain A_K and A_E on Y_- and Y_+ so that extended versions of A_K and A_E on $U_C \cup U_0$ exist with the curvature of the extended version of A_K pulling back via the embeddings from the first bullet of (3-11) and (3-12) as $\sin\theta\, d\theta\, d\phi$ and $g(|z|)\, dz \wedge d\bar{z}$. Meanwhile, the pullbacks of the curvature of A_E via these embeddings is zero. Extensions with this property are assumed implicitly.

The definitions in [17] are sufficiently flexible so as to allow for the following: for any given $\mathrm{r} > \pi$, there are suitable perturbation terms for (2-10) with positive but as-small-as desired $\mathcal{P}$–norm that vanish on U_C and on the image of $\mathbb{R} \times \mathcal{H}_0$ via the embedding map from the first bullet of (3-11).

With regards to notation and conventions, the propositions and lemmas that follow refer only to (2-10). Even so, all assertions still hold for the versions with an extra perturbation term if the perturbation term has $\mathcal{P}$–norm bounded by $e^{-\mathrm{r}^2}$ or has small, r–independent $\mathcal{P}$–norm and vanishes on U_C and on the image of $\mathbb{R} \times \mathcal{H}_0$ via the embedding from the first bullet of (3-11). Proofs of the propositions and lemmas will likewise refer only to (2-10). The modifications that are needed to deal with the extra perturbation terms are straightforward and so left to the reader.

The second set of constraints require the choice of constants $c \geq 1$ and $r \geq 1$. By way of notation, one of the upcoming constraints uses the embeddings from the second and third bullets of (2-8) to write w_X on the $|s| \in [L-4, L]$ part of X as $w_X = ds \wedge *w_* + w_*$ with w_* denoting a closed, s–dependent 2–form on Y_- or Y_+, and with $*$ here denoting the Hodge star for the metric $\mathfrak{g}$ in the second bullet of (2-9).

(3-15) (1) The constant L in (2-9) is less than c. The constant L_{tor} in (2-12) is equal to $c \ln r$.

(2) The norm of the Riemannian curvature tensor and those of its covariant derivatives up to order 10 are less than $r^{1/c}$ on the $s \in [-L, L]$ part of X.

(a) The injectivity radius is larger than $r^{-1/c}$ on the $s \in [-L, L]$ part of X.

(b) The metric volume of the s–inverse image in X of any unit interval is bounded by c.

(3) The metric $\mathfrak{g}$ from (2-9)'s second bullet obeys $|\frac{\partial}{\partial s}\mathfrak{g}| \le r^{1/c}$.

(4) The norm of w_X is bounded by c. The norms of its covariant derivatives to order 10 are bounded by $r^{1/c}$ on the $s \in [-L, L]$ part of X.

(a) The 2–form w_X is closed on the $|s| \le L-4$ part of X.

(b) Use the embeddings from the second and third bullets of (2-8) to write w_X on the $|s| \in [L-4, L]$ parts of $X - X_{\mathrm{tor}}$ as $w_X = ds \wedge *w_* + w_*$. Then $\frac{\partial}{\partial s} w_* = d\mathfrak{b}$, where $\mathfrak{b}$ is a smooth, s–dependent 1–form on the relevant components of Y_- or Y_+ with $\int_{(X-X_{\mathrm{tor}})\cap|s|^{-1}([L-4,L])} |\mathfrak{b}|^2 < r^{-1/c}$.

(c) The 2–form w_X is closed on the components of the $L-4 \le |s| \le L_{\mathrm{tor}}-4$ part of X_{tor}.

(d) Use the embeddings from the second and third bullets of (2-8) to write w_X on the $|s| \in [L_{\mathrm{tor}}-4, L_{\mathrm{tor}}]$ parts of X_{tor} as $w_X = ds \wedge *w_* + w_*$. Then $\frac{\partial}{\partial s} w_* = d\mathfrak{b}$ where $\mathfrak{b}$ is a smooth, s–dependent 1–form on the relevant components of Y_- or Y_+ with $\int_{X_{\mathrm{tor}}\cap|s|^{-1}([L_{\mathrm{tor}}-4,L_{\mathrm{tor}}])} |\mathfrak{b}|^2 < r^{-1/c}$.

(5) There is a smooth, closed 1–form on X, denoted by υ_X below, with norm bounded by c and such that:

(a) The pullback of υ_X to $(-\infty, -L] \times Y_-$ and to $[L, \infty) \times Y_+$ via the embeddings from the second and third bullets of (2-8) is an s–independent 1–form on Y_- and Y_+.

(b) The pullback of υ_X to U_C via the embedding from (3-12) is dt and its pullback to U_0 via the embedding from the first bullet of (3-11) is $\mathrm{H}(u)\, du$ with $\mathrm{H}(\cdot) \ge c^{-1}$.

(c) $*(ds \wedge \upsilon_X \wedge w_X) \ge -r^{-1/c}$ on the $|s| \in [L-4, \infty)$ part of X.

Note that item (4) of the preceding constraints ensures that the condition (2-21) holds.

Definition 3.3 The metric and w_X on X are said to be (c, r)*–compatible* when one of the following conditions are met:

(3-16)
- The space $X = \mathbb{R} \times Y_Z$, the metric has the form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ being an s–independent metric on Y_Z, and the 2–form w_X is the s–independent form $ds \wedge *w + w$. Moreover, there exists a closed 1–form on Y_Z, denoted by υ below, that restricts to U_γ as dt, and restricts to $\mathcal{H}_0$ as $\mathrm{H}(u)\, du$ with $\mathrm{H}(\cdot) > c^{-1}$, and is such that $\upsilon \wedge w \geq -r^{-1/c}$.
- The metric and w_X obey the constraints in (2-9), (2-12), (3-13), (3-14) and (3-15).

By way of a look ahead, the notion of (c, r)–compatibility is invoked below with r given by the constant r in (2-10).

3.4 Positivity on cobordisms

An analog of the connection that is defined in (3-8) plays a role in what follows. This connection is denoted in what follows by $\hat{A}$. To define it, keep in mind that $w_X \neq 0$ on U_C and so Clifford multiplication by w_X^+ on $\mathbb{S}^+$ over U_C or $(-\infty, -2] \times \mathcal{H}_0$ or $[2, \infty) \times \mathcal{H}_0$ splits $\mathbb{S}^+$ as a direct sum of eigenbundles, this written as

$$\mathbb{S}^+ = E \oplus (E \otimes K^{-1}) \tag{3-17}$$

with it understood that w_X acts as multiplication by $i|w_X|$ on the leftmost summand (namely, E). (This splitting is the analog of the splitting in (3-7)). A section, ψ, of $\mathbb{S}$ is written with respect to this splitting over U_C as

$$\psi = |w_X|^{1/2}(\alpha, \beta).$$

Meanwhile, $\mathbb{A}$ is written as $A_K + 2A$ with A being a connection on E. Granted this notation, write $\hat{A}$ using the formula in (3-8) with it understood that the covariant derivatives of α that appear have nonzero pairing with the vector field $\frac{\partial}{\partial s}$. This connection is flat where $|\alpha|^2 > \frac{9}{16}$ and $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant. Meanwhile, $\hat{A}$ is equal to A where $|\alpha|^2 \leq \frac{7}{16}$. The formulas for the curvature of $\hat{A}$ and the $\hat{A}$–covariant derivative of α is given in (3-9) with it understood that F_A and $\nabla_A \alpha$ now have components that have nonzero pairing with $\frac{\partial}{\partial s}$.

With a look ahead at the upcoming propositions, note that the integral of $iF_{\hat{A}}$ over C is proved to be well defined when (A, ψ) is an instanton solution to (2-10). This is proved using integration by parts to express the integral of $iF_{\hat{A}}$ as the difference between integrals of the $i\mathbb{R}$–valued 1–form $\hat{A} - A_E$ over respective $s \gg 1$ and $s \ll -1$ slices of C.

The first proposition below concerns the integral of $iF_{\hat{A}}$ on C when X, its metric, and the 2–forms w_X and $\mathfrak{w}_\mu$ define the product cobordism.

Proposition 3.4 *Assume that X, the metric and w_X can be used to define a product cobordism once μ is chosen. Assume in addition that Y_Z has a closed 1–form, $\upsilon_\diamond$, such that $\upsilon_\diamond \wedge w \geq 0$, whose restriction to U_γ is dt and whose restriction to $\mathcal{H}_0$ is $\mathrm{H}\,du$ with H being a strictly positive function of u. Given $c \geq 1$, there exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with either $\mathcal{P}$–norm bounded by $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$–norm bounded by 1 but vanishing on $\mathbb{R} \times (\mathcal{H}_0 \cup U_\gamma)$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the (r, μ) version of* (2-5) *on Y_Z with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to the corresponding version of* (2-10) *on X with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. Then $i\int_C F_{\hat{A}} \geq 0$.*

Proposition 3.4 is a special case of the next proposition, which concerns the integral of $iF_{\hat{A}}$ on C when the relevant data does not necessarily define the product cobordism.

Proposition 3.5 *Assume that X and w_X obey the conditions in Section 3.3, and that the metric on X obeys* (2-9) *and* (3-14)*. Then there exists $\kappa > \pi$ such that given any $c \geq \kappa$, there exists κ_c with the following property: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$–compatible data. Fix μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with either $\mathcal{P}$–norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$–norm less than 1 but vanishing on the respective Y_- and Y_+ versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the (r, μ_-) version of* (2-5) *on Y_- and (r, μ_+) version of* (2-5) *on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. If $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to* (2-10) *with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$, then $i\int_C F_{\hat{A}} \geq 0$.*

Proposition 3.5 is proved in Section 8.2.

3.5 The bound for $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$ in Proposition 3.5

Proposition 3.5 concerns only those instanton solutions to (2-10) that obey the added constraint $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. The two propositions that are stated in a moment are used to guarantee that this constraint is met in the cases that are relevant to the body of this paper. What follows sets the stage for the first proposition.

Definition 3.6 Fix $c > 1$. The metric on Y_Z and the 2–form w are said to define *c–tight* data when there exists a positive, c–dependent constant with the following significance: Use the metric, the 2–form w, a choice of r greater than this constant and a chosen 1–form from Ω with $\mathcal{P}$–norm less than 1 to define (2-5). If $\mathfrak{c}$ is a solution, then $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c})| < \mathrm{r}^{2-1/c}$.

Proposition 3.7 *Let Y_Z denote a compact, oriented Riemannian 3–manifold with a chosen Riemannian metric and a* Spin^c*–structure with nontorsion first Chern class. Let w denote a harmonic 2–form on Y_Z whose de Rham class is this first Chern class. Assume that w has nondegenerate zeros on any component of Y_Z where it is not identically zero. Then the metric and w define a c–tight data set if c is sufficiently large.*

This proposition is proved in Section 7.8.

This notion of being c–tight is used in the second of the promised propositions. To set the stage for this one, suppose that X is a cobordism of the sort that is described in Section 3.3. Fix a metric on X and the auxiliary data as described in (2-9), (2-12) and (3-13), and let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to a given $\mathrm{r} > \pi$ version of (2-10). Use $\mathfrak{c}_-$ and $\mathfrak{c}_+$ to denote the respective $s \to -\infty$ and $s \to \infty$ limits of $\mathfrak{d}$. Associated to $\mathfrak{d}$ is a certain first-order, elliptic differential operator, this being the operator that is depicted in (IV.1-21) when X is the product cobordism. The operator in the general case is written using slightly different notation in (2.61) of [37]. This operator has a natural Fredholm incarnation when the respective Y_- version of $\mathfrak{f}_s$ is constant on a neighborhood of $\mathfrak{c}_-$ and the Y_+ version is constant on a neighborhood of $\mathfrak{c}_+$. Use $\iota_{\mathfrak{d}}$ to denote the corresponding Fredholm index. By way of a relevant example, $\iota_{\mathfrak{d}}$ is equal to $\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)$ when X and the associated data define the product cobordism. Section 8.7 associates an integer, $\iota_{\mathfrak{d}+}$, to $\mathfrak{d}$, which is defined without preconditions on $\mathfrak{c}_-$ and $\mathfrak{c}_+$. The latter is equal to the maximum of $\iota_{\mathfrak{d}}$ and 0 in the case when $\iota_{\mathfrak{d}}$ can be defined.

Proposition 3.8 *Assume that X obeys the conditions in Sections 2.2 and 3.3, that the metric on X obeys* (2-9) *and* (3-14) *for a given $L > 100$, and that w_X obeys the conditions in* (2-11) *and* (2-12) *for a given $L_* \geq L + 4$. Then there exists $\kappa > \pi$ such that for any given $c \geq \kappa$, there exists κ_c with the following significance: Suppose that the respective pairs of metric and version of w on Y_- and Y_+ define c–tight data. Fix $\mathrm{r} > \kappa_c$ and fix μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with*

$\mathcal{P}$–norm less than 1 *so as to define* (2-10) *on* X*. Let* $\mathfrak{d}$ *denote an instanton solution to these equations with* $\iota_{\mathfrak{d}+} \le c$*. Use* $\mathfrak{c}_-$ *and* $\mathfrak{c}_+$ *to denote the respective* $s \to -\infty$ *and* $s \to \infty$ *limits of* $\mathfrak{d}$*. Then* $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) < \mathrm{r}^{2-1/c}$*.*

This proposition is proved in Section 8.7.

3.6 The cases when Y_Z is from $\{M \sqcup (S^1 \times S^2), Y\}$, $\{Y_k\}_{k=0,\dots,\mathrm{G}}$ or $\{Y_k \sqcup (S^1 \times S^2)\}_{k=0,\dots,\mathrm{G}-1}$

In what follows, the notation Y_Z stands, in addition to the manifold itself, also implicitly for its associated metric and 2–form w from Part 3 of Section 3.2.

The body of this article is concerned with $2\mathrm{G}+3$ specific versions of Y_Z, these being as follows: The first manifold of interest is M and $S^1 \times S^2$ and the second is the manifold Y from Section II.1. The next $\mathrm{G}+1$ manifolds are labeled as $\{Y_k\}_{k=0,\dots,\mathrm{G}}$ with a given $k \in \{0, \dots, \mathrm{G}\}$ version being the manifold that is obtained from M by attaching the handle $\mathcal{H}_0$ as directed in Part 2 of Section II.1A and attaching k of the handles from the set $\{\mathcal{H}_{\mathfrak{p}}\}_{\mathfrak{p} \in \Lambda}$ as directed in Part 1 of Section II.1A. Note in this regard that Y and Y_{G} are the same manifold, endowed with different metric and 2–form w. Their main difference is the behavior of w over the attached handles $\mathcal{H}_{\mathfrak{p}}$: for Y it approximates certain standard contact form (see (9-51) below), while for Y_{G} it is harmonic (see Proposition 3.9). The last G manifolds of interest are the disjoint unions of the various $k \in \{0, \dots, \mathrm{G}-1\}$ versions of Y_k and $S^1 \times S^2$.

Part 1 Let Y_Z denote the disjoint union of M and $S^1 \times S^2$. To see about the constraints in Section 3.2, take Z to be the disjoint union of M and S^3. The handle $\mathcal{H}_0$ is attached to S^3 so as to obtain $S^1 \times S^2$. Write S^1 as $\mathbb{R}/(2\pi\mathbb{Z})$ and let t denote the corresponding affine coordinate. Use the spherical coordinates (θ, ϕ) for S^2. The loop γ is the $\theta = 0$ circle in $S^1 \times S^2$.

To see about w and the metric, consider first their appearance on $S^1 \times S^2$. Take the 2–form w on $S^1 \times S^2$ to be $\sin\theta \, d\theta \, d\phi$ and the metric to be $\mathrm{H}\, dt^2 + d\theta^2 + \sin^2\theta \, d\phi^2$ with H denoting a positive constant. If the first Chern class of $\det(\mathbb{S}|_M)$ is torsion, take $w = 0$ on M and take any smooth metric. If the first Chern class of $\det(\mathbb{S}|_M)$ is not torsion, take a metric on M such that the associated harmonic 2–form with de Rham cohomology class that of $c_1(\det(\mathbb{S}|_M))$ has nondegenerate zeros. Take w in this case to be this same harmonic 2–form. By way of a parenthetical remark, a sufficiently generic metric on M will have this property. See for example [11] for a proof that such is the case.

The data just described obeys the conditions in Section 3.2. Use Proposition 3.7 to see that this data is also c–tight for a suitably large version of c.

Part 2 Let Y_Z denote the manifold Y that is described in Section II.1. Suffice it to say for now that Y is obtained from Y_0 via a surgery that attaches some positive number of 1–handles to the M_δ part of Y_0. This number is denoted by G.

The 2–form w is described in Section II.1E. See also Part 3 of Section IV.1.1. Let b_1 denote the first Betti number of M. Part 2 of Section II.1D describes a set of $b_1 + 1$ closed integral curves of the kernel of w that have geometric intersection number 1 with each u = constant 2–sphere in $\mathcal{H}_0$. One of these curves intersects $\mathcal{H}_0$ as the $\theta = 0$ arc. This is the curve $\gamma^{(z_0)}$ in the notation from Part 2 of Section III.1A. Use the latter for γ. It follows from what is said in (II.1-5) and Part 2 of Section II.1D that the γ has a tubular neighborhood with coordinates as described in Section 3.2 such that the 2–form w has the desired appearance. Section II.1E and (IV.1-5) describe a closed 1–form on Y that can be used to satisfy the requirements in the third bullet in (3-5). This 1–form is denoted by $\upsilon_\diamond$.

A set of Riemannian metrics on Y that have the desired form on $\mathcal{H}_0$ are described in Part 5 of Section IV.1.1. Although not stated explicitly, a metric of the sort that is described in Part 5 of Section IV.1.1 can be chosen so that it has the desired behavior on some small radius tubular neighborhood of γ. Note that the set of metrics under consideration are obtained from the choice of an almost complex structure on the kernel of a 1–form $\hat{a}$ given in (IV.1-6). These almost complex structures are taken from the set $\mathcal{J}_{\mathrm{ech}}$ that is described in Theorem II.A.1 and Section III.1C. None of the conclusions in [20; 21; 22] are compromised if the almost complex structure from $\mathcal{J}_{\mathrm{ech}}$ is chosen near γ so that the metric obeys the constraints in (3-6). To be sure, the chosen almost complex structure must have certain genericity properties to invoke the propositions and theorems in these papers. These genericity results are used to preclude the existence of certain pseudoholomorphic subvarieties in $\mathbb{R} \times Y$. An almost complex structure giving a metric near γ that obeys (3-6) is not generic. Even so, the subvarieties that must be excluded can be excluded using a suitably almost generic almost complex structure from the subset described in $\mathcal{J}_{\mathrm{ech}}$ that give a metric that is described by (3-6) near γ. What follows is the key observation that is used to prove this: the curves to be excluded have image via the projection from $\mathbb{R} \times Y$ that intersects the complement of small radius neighborhoods of γ. A detailed argument for the existence of the desired almost complex structures from $\mathcal{J}_{\mathrm{ech}}$ amounts to a relatively straightforward application of the Sard–Smale theorem along the lines used in the proof of Theorem 4.1 in [12].

It follows from Lemma IV.2.5 and Proposition IV.2.7 that the metric just described together with w define a c–tight data set on Y for a suitably large choice for c.

Part 3 This part of the subsection considers the case when Y_Z is some $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. As noted previously, the manifold Y_k is obtained from M by attaching the handle $\mathcal{H}_0$ in the manner that is described in Part 2 of Section II.1A and attaching k of the handles from the set $\{\mathcal{H}_\mathfrak{p}\}_{\mathfrak{p}\in\Lambda}$ as described in Part 1 of Section II.1A. Part 3 in Section II.1A defines a subset $M_\delta \subset M$ and the constructions of both Y and Y_k identify M_δ as a subset of both. The curve $\gamma^{(z_0)}$ that was introduced above in Part 2 sits in the latter part of Y and so it can be viewed using this identification as a curve in Y_k. Use this Y_k incarnation of $\gamma^{(z_0)}$ for the curve γ.

The proposition that follows says what is needed with regards to the 2–form w and the metric to use on Y_k.

Proposition 3.9 *Fix $k \in \{0, \dots, \mathrm{G}\}$. There exists a nonempty set of Riemannian metrics on Y_k with the following two properties: Let w denote the metric's associated harmonic 2–form with de Rham cohomology class that of $c_1(\det \mathbb{S})$. Then w has nondegenerate zeros. Moreover, the metric and w obey the conditions in Section 3.2.*

This proposition is proved in Section 9.2.

The set of metrics in Proposition 3.9 is denoted by *Met* in what follows. Take the metric on Y_k from this set and take w to be the associated harmonic 2–form with de Rham cohomology class that of $c_1(\det \mathbb{S})$. Proposition 3.7 asserts that the resulting data set is c–tight for a suitably large choice of c.

Part 4 This part of the subsection discusses the case when Y_Z is the disjoint union of some $k \in 0, \dots, \mathrm{G}-1$ version of Y_k and $S^1 \times S^2$. The metric on Y_0 comes from Proposition 3.9's set *Met*, and the 2–form w on Y_k is the corresponding harmonic 2–form with de Rham cohomology class that of $c_1(\det \mathbb{S})$. Any smooth metric can be chosen for a given $S^1 \times S^2$ component. The class $c_1(\det \mathbb{S})$ is taken equal to zero on each $S^1 \times S^2$ component and this understood, the 2–form w is identically zero on each such component.

What is said in Proposition 3.7 implies that the resulting data set is c–tight for a suitably large choice of c.

3.7 Cobordisms with Y_+ and Y_- either Y, $M \sqcup (S^1 \times S^2)$, from $\{Y_k\}_k$, or from $\{Y_k \sqcup (S^1 \times S^2)\}_k$

The first proposition concerns the product cobordisms when Y_Z is one of the manifolds from the set Y, $M \sqcup (S^1 \times S^2)$, $\{Y_k\}_{k=0,\dots,\mathrm{G}}$ or $\{Y_k \sqcup (S^1 \times S^2)\}_{k=0,\dots,\mathrm{G}-1}$. The subsequent propositions concern certain cobordisms of the sort described in Section 3.3 with Y_+ and Y_- as follows:

- One is Y and the other is Y_{G}.
- One is Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$ for some $k \in \{1, \dots, \mathrm{G}\}$.
- One is Y_0 and the other is $M \sqcup (S^1 \times S^2)$.

These propositions assume implicitly that the metric and version of w on these manifolds are those supplied by the relevant part of Section 3.5. In particular, the metric and w on $M \sqcup (S^1 \times S^2)$ is described by Part 1 of Section 3.5, and this data on Y is described in Part 2 of Section 3.5. Meanwhile, the metric on the relevant $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k is from the set *Met* and w is the associated harmonic 2–form with de Rham cohomology class that of $c_1(\det \mathbb{S})$.

Proposition 3.10 *Let Y_Z denote either $M \sqcup (S^1 \times S^2)$ or Y or some $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k with the 2–form w and metric as described in the preceding paragraph. Given $\iota \geq 0$, there exists $\kappa > \pi$ with the following significance: Fix any $\mathrm{r} > \kappa$ and a 1–form $\mu \in \Omega$ with either $\mathcal{P}$–norm less than $e^{-\mathrm{r}^2}$ or $\mathcal{P}$–norm less than 1 but vanishing on $\mathcal{H}_0 \cup U_\gamma$. Use this data with the metric and w to define the product cobordism $X = \mathbb{R} \times Y_Z$ as prescribed in Section 2.1. Suppose that $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are solutions to the (r, μ) version of (2-5) on Y_Z with $|\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)| \leq \iota$, and suppose that $\mathfrak{d}$ is an instanton solution to (2-10) on X with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Then $\mathrm{X}(\mathfrak{c}_+) \geq \mathrm{X}(\mathfrak{c}_-)$.*

Proof This follows directly from Propositions 3.4 and 3.7 given what is said in Section 3.5 about w and the metric. □

The next proposition describes cobordisms between Y_0 and $M \sqcup (S^1 \times S^2)$ of the sort that obey the conditions in Section 3.3.

Proposition 3.11 *Take the metric on $M \sqcup (S^1 \times S^2)$ and harmonic 2–form w to be as described in Part 1 of Section 3.5. The metric on $M \sqcup (S^1 \times S^2)$ determines a corresponding set of metrics in the Y_0 version of Met. Choose a metric from this set* *and take w on Y_0 to be the associated harmonic 2–form with de Rham class $c_1(\det \mathbb{S})$. Denote one of Y_0 or $M \sqcup (S^1 \times S^2)$ by Y_- and the other by Y_+. There exists a cobordism that obeys the conditions in Section 3.3 and the conditions in the list below. This list uses X to denote the cobordism manifold*:

- *The function s on X has exactly one critical point. This critical point has index 3 when $Y_- = Y_0$ and index 1 when $Y_- = M \sqcup (S^1 \times S^2)$.*
- *There is a metric on X with an associated self-dual 2–form that are (c, r)–compatible if L and c are sufficiently large and if $r > \pi$.*

This proposition is proved in Section 9.4.

The next proposition uses C to denote the cylinder in Proposition 3.11's cobordism that is described by the first bullet of (3-11). The proposition also reintroduces the notation in (3-9).

Proposition 3.12 *Take w and the metric on Y_0 and on $M \sqcup (S^1 \times S^2)$ to be as described in Proposition 3.11. Denote one of Y_0 or $M \sqcup (S^1 \times S^2)$ by Y_- and the other by Y_+. Take the cobordism space X, the metric on X, and the associated self-dual 2–form w_X to be as described by Proposition 3.11. Given $k \geq 0$, there exists $\kappa > \pi$ with the following significance*: *Fix $\mathrm{r} > \kappa$ and 1–forms μ_- and μ_+ from the Y_- and Y_+ versions of Ω with either $\mathcal{P}$–norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$–norm less than 1 but vanishing on the Y_- and Y_+ versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the resulting version of* (2-10) *with $\iota_{\mathfrak{d}+} \leq k$. Then $i \int_C F_{\hat{A}} \geq 0$.*

Proof The proposition follows directly from Propositions 3.5, 3.8 and 3.11 given what is said in Section 3.5 about the respective Y_0 and $M \sqcup (S^1 \times S^2)$ metrics and versions of w. □

The next set of propositions are analogs of Propositions 3.11 and 3.12 in the case when one of Y_- and Y_+ is some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or when one is Y and the other is Y_{G}. The propositions that follow assume that $c_1(\det \mathbb{S})$ on each $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k vanishes on the cross-sectional spheres in any $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ and that it has pairing 2 with the cross-sectional spheres in $\mathcal{H}_0$. This class is also assumed to be zero on the $S^1 \times S^2$ component of any $k \in \{0, \dots, \mathrm{G}\}$ version of $Y_{k-1} \sqcup (S^1 \times S^2)$. Meanwhile, its restriction to the $H_2(M; \mathbb{Z})$–summand from the associated Mayer–Vietoris sequence for the various $k \in \{0, \dots, \mathrm{G}\}$ versions of $H_2(Y_k; \mathbb{Z})$ is assumed to be independent of k.

Proposition 3.13 *There exists, for each $k \in \{0, \dots, \mathrm{G}\}$, a subset to be denoted by $\mathrm{Met}(Y_k)$ in the Y_k version of Met with the following significance: Let $\mathrm{Met}(Y_0)$ denote the subset from Proposition 3.11. For each $k \in \{1, \dots, \mathrm{G}\}$, take a metric from an open subset of $\mathrm{Met}(Y_{k-1})$ and a metric on $S^1 \times S^2$ to define a metric on $Y_{k-1} \sqcup (S^1 \times S^2)$. Take w on $Y_{k-1} \sqcup (S^1 \times S^2)$ to be the associated harmonic 2–form with de Rham class $c_1(\det \mathbb{S})$. The chosen metric determines a corresponding subset of metrics $\mathrm{Met}(Y_k) \subset Met$. Take a metric from the latter subset and take w to be the associated harmonic 2–form with de Rham class $c_1(\det \mathbb{S})$. Take Y_- to be one of Y_k and $Y_{k-1} \sqcup (S^1 \times S^2)$, and take Y_+ to be the other. There exists a cobordism that obeys the conditions in Section 3.3 and the conditions listed below. This list uses X to denote the cobordism manifold:*

- *The function s on X has precisely one critical point. This critical point has index 3 when Y_+ has the $S^1 \times S^2$ component and it has index 1 when Y_- has the $S^1 \times S^2$ component.*
- *There is a metric on X with an associated self-dual 2–form that are (c, r)–compatible if L, c and $r > \pi$.*

This proposition is proved in Section 9.5.

The next proposition considers the case when one of Y_- and Y_+ is Y and the other is Y_{G}.

Proposition 3.14 *Take w and the metric on Y to be as described in the opening paragraphs of this subsection. Take the metric on Y_{G} from a certain nonempty subset of $\mathrm{Met}(Y_{\mathrm{G}})$ and take w on Y_{G} to be the associated harmonic 2–form with de Rham class that of $c_1(\det \mathbb{S})$. Take Y_- to be one of Y and Y_{G} and take Y_+ to be the other. There exists a cobordism that obeys the conditions in Section 3.2 and the conditions listed below. This list uses X to denote the cobordism manifold:*

- *The function s on X has no critical points.*
- *There is a metric on X with an associated self-dual 2–form that are (c, r)–compatible if L, c and $r > \pi$.*

The proof of Proposition 3.14 is in Section 9.7.

The upcoming proposition uses C to denote the cylinder in Propositions 3.13 and 3.14's cobordism that is described by the first bullet of (3-11). Notation from (3-9) is also used.

Proposition 3.15 *Let X denote one of the cobordism manifolds that are described in Propositions 3.13 and 3.14 with $c_1(\det \mathbb{S})$ and the 2–form and metrics on Y_-, Y_+ and X as described therein. Given $\iota \geq 0$, there exists $\kappa > \pi$ with the following property: Fix $\mathrm{r} > \kappa$ and 1–forms μ_- and μ_+ from the Y_- and Y_+ versions of Ω with either $\mathcal{P}$–norm less than $e^{-\mathrm{r}^2}$ or with $\mathcal{P}$–norm less than 1 but vanishing on the Y_- and Y_+ versions of $\mathcal{H}_0 \cup U_\gamma$. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the resulting version of* (2-10) *with $\iota_{\mathfrak{d}+} \leq \iota$. Then $i \int_C F_{\hat{A}} \geq 0$.*

Proof The proposition follows directly from Propositions 3.5, 3.8, 3.13 and 3.14 given what is said in Section 3.5. □

3.8 Filtered Floer homologies and filtration-preserving chain maps

This subsection is divided into two parts. In the first part, we associate to each triple (Y_Z, w, γ) described in Section 3.2 a system of filtered monopole Floer homologies $\mathrm{HM}^\circ(Y_Z, \mathrm{r}w; \Lambda_\gamma)$ for $\circ = -, \infty, +, \wedge$ and $\mathrm{r} > \pi$, in the manner described in Section 3.1. Recall the constraint on the cohomology class $[w]$ from Part 3 of Section 3.2. Together with the first bullet of (3-5), this implies that $\mathrm{CM}_*(Y_Z, \mathrm{r}w; \Lambda_\gamma)$ is associated with a negative-monotone, nonbalanced perturbation. For reasons that will become clear in a moment, we use $\mathrm{CM}^\circ(Y_Z, \langle w \rangle; \Lambda_\gamma)$ to denote $\mathrm{CM}_*(Y_Z, \mathrm{r}w; \Lambda_\gamma)$ for $\mathrm{r} \gg \pi$ and similarly for its homology. (The notation $\langle w \rangle$ stands for the ray $\mathbb{R}^+[w] \subset H^2(Y_Z; \mathbb{R})$.) This includes, as special cases, the triple $(\underline{M}, \underline{w}, \underline{\gamma})$ in [23] ($\underline{M}$ is denoted by Y_0 in this article), and the triple $(Y, w, \gamma^{(z_0)})$ in Section II.1A.

In the second part, a filtration-preserving chain map from $\mathrm{CM}^-(Y_-, \langle w_- \rangle; \Lambda_{\gamma_-})$ to $\mathrm{CM}^-(Y_+, \langle w_+ \rangle; \Lambda_{\gamma_+})$ is associated to each triple (X, ϖ_X, C) described in Section 3.3. To explain the notation, X is a cobordism from the 3–manifold Y_- to Y_+, while ϖ_X is a self-dual 2–form on X related to w_- and w_+ as prescribed by (2-11). What is denoted by C signifies an embedded surface in X, with ends $\gamma_- \subset Y_-$ and $\gamma_+ \subset Y_+$; see the second bullet of (3-11).

Part 1 To accomplish this task, begin by introducing the (partially defined) integral 1–cocycle on $\mathcal{B}^\sigma(Y_Z)$ defining Λ_γ. This local system associates each $\mathfrak{c} \in \mathcal{Z}_{w,\mathrm{r}}$ the group algebra $\mathbb{K}[\mathbb{Z}] = \mathbb{K}[U, U^{-1}]$. To each $\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ it associates $U^{n(\mathfrak{d})} \in \mathrm{End}(\mathbb{K}[U, U^{-1}])$, where $n(\mathfrak{d}) = \mathrm{x}(\mathfrak{c}_+) - \mathrm{x}(\mathfrak{c}_-)$. Here, x is the "modified holonomy function" given in (3-10). Lemma 3.2 asserts that $n(\mathfrak{d}) \in \mathbb{Z}$ for $\mathfrak{c}_-, \mathfrak{c}_+ \in \mathcal{Z}_{w,\mathrm{r}}$. Following the recipe in Section 3.1, we then set $(\mathrm{CM}^\infty, \partial^\infty)$ to be the monopole Floer complex with twisted coefficients:

$$\mathrm{CM}^\infty = \mathbb{K}[U, U^{-1}](\mathcal{Z}_{w,\mathrm{r}}),$$
$$\partial^\infty \mathfrak{c}_- = \sum_{\mathfrak{c}_+ \in \mathcal{Z}_{w,\mathrm{r}}} \sum_{\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)/\mathbb{R}} \operatorname{sign}(\mathfrak{d}) U^{n(\mathfrak{d})} \mathfrak{c}_+ \quad \text{for } \mathfrak{c}_- \in \mathcal{Z}_{w,\mathrm{r}}.$$

The monotonicity condition guarantees that the sum here is finite. The $\operatorname{sign}(\mathfrak{d}) \in \{\pm 1\}$ in the preceding expression is assigned according to the orientation convention laid out in [17].

One may regard CM^∞ as a chain complex over $\mathbb{K}$, generated by $\hat{\mathcal{Z}}_{w,\mathrm{r}} = \mathcal{Z}_{w,\mathrm{r}} \times \mathbb{Z}$. The generating set $\hat{\mathcal{Z}}_{w,\mathrm{r}}$ lies in $\tilde{\mathcal{B}}^\sigma = \mathcal{C}^\sigma/\mathcal{G}_\gamma$, a $\mathbb{Z}$–covering of $\mathcal{B}^\sigma$. Here, $\mathcal{G}_\gamma \subset \mathcal{G}$ consists of smooth maps $u\colon Y_Z \to S^1$, with $\deg(u|_\gamma) = 0$. Multiplication by U^n then corresponds to a deck transformation on this $\mathbb{Z}$–covering, and the condition on $c_1(\det \mathbb{S})$ set forth in Part 3 of Section 3.2 then implies that $\deg U = -2$. The grading set of $\hat{\mathcal{Z}}_{w,\mathrm{r}}$ is an affine space over $\mathbb{Z}/c_Z\mathbb{Z}$, where $c_Z \in 2\mathbb{Z}$ is the gcd of the values of $c_1(\det \mathbb{S})$ on $H_2(Z;\mathbb{Z})$ according to the splitting (3-4).

Remark 3.16 (a) Here, we use the same notation U for the map on monopole Floer complexes described in Part 2 of Section 2.4 and deck transformation here. This is because for the kind of Y_Z considered in this article, they turn out identical by the arguments for the last bullet of Proposition IV.7.6.

(b) The way the monopole Floer chain complex with local coefficients is graded follows some definitions in the literature, eg what is called a Floer–Novikov complex [24]. The book [17] does not seem to contain an explicit discussion on the grading of Floer complex with local coefficients.

Suppose that (Y_Z, w) define c–tight data for $c > 1$ (see Definition 3.6). Take X to be the product cobordism $\mathbb{R} \times Y_Z$, $w_X = w + ds \wedge *w$ and $C = \mathbb{R} \times \gamma \subset X$. Let $\mathfrak{d} \in \mathcal{M}(\mathfrak{c}_-, \mathfrak{c}_+)$ be as in Proposition 3.5. In this case, $i \int_C F_{\hat{A}} = 2\pi(\mathrm{x}(\mathfrak{c}_+) - \mathrm{x}(\mathfrak{c}_-))$ and Proposition 3.5 asserts that one has $n(\mathfrak{d}) \geq 0$. Thus,

$$\mathrm{CM}^- = \mathbb{K}(\mathcal{Z} \times \mathbb{Z}^{\geq 0}) \subset \mathrm{CM}^\infty$$

is a subcomplex of $\mathbb{K}[\mathbb{Z}^{\geq 0}] = \mathbb{K}[U]$–modules. One may then introduce

$$\mathrm{CM}^+ = \mathrm{CM}^\infty/\mathrm{CM}^-, \quad \widehat{\mathrm{CM}} = \mathrm{CM}^-/U\mathrm{CM}^-.$$

The resulting short exact sequences

$$0 \to \mathrm{CM}^- \to \mathrm{CM}^\infty \to \mathrm{CM}^+ \to 0 \quad \text{and} \quad 0 \to U\mathrm{CM}^- \to \mathrm{CM}^- \to \widehat{\mathrm{CM}} \to 0 \tag{3-18}$$

induce the *fundamental exact sequences* on the homologies. As the $\bigwedge^* H_1(Y_Z)/\mathrm{Tors}$–action on the monopole Floer complexes commute with U, the exact sequences above preserve the $\boldsymbol{A}_\dagger$–module structure.

In Section 3.5, the assumption that (Y_Z, w) is c–tight is verified for the particular manifolds listed therein. In particular:

(3-19)
- When $Y_Z = Y$ and its associated $\mathfrak{s}$, w, γ and metric are as in Part 2 of Section 3.5,
$$\mathrm{HM}^\circ(Y, \langle w\rangle; \Lambda_\gamma) = \mathrm{H}^\circ(Y) = \mathrm{H}^\circ_{\mathrm{SW}}$$
in the notation of [19; 22].
- When $Y_Z = Y_k$, $k \in \{0, \dots, \mathrm{G}\}$ and its associated $\mathfrak{s}$, w, γ and metric are in Part 3 of Section 3.5, the corresponding $\mathrm{HM}^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$ are instrumental in the proof for Theorem 1.1. Recalling that Y_0 and its associated $\mathfrak{s}$, w, γ and metric are respectively what was denoted by $\underline{M}$, $\underline{\mathfrak{s}}$, $\underline{w}$ and $\underline{\gamma}$ in [23], we observe that
$$\mathrm{HM}^\circ(Y_0, \langle w\rangle; \Lambda_\gamma) = \mathrm{HMT}^\circ,$$
introduced in [23].

Note that $\mathrm{CM}^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$ and $\mathrm{HM}^\circ(Y_Z, \langle w\rangle; \Lambda_\gamma)$ introduced above implicitly depend on r and $(\mathfrak{T}, \mathfrak{S})$. According to the convention set forth in Section 1.3, this is permissible if there are chain homotopies between the monopole Floer complexes associated with different parameters preserving the $\boldsymbol{A}_\dagger$–module structure. This is justified by combining the arguments proving Proposition IV.1.4 with what is said in the upcoming Part 2.

Part 2 We now consider chain maps induced by (nonproduct) cobordisms X described in Section 3.3. To begin, we introduce an X–morphism from Λ_{γ_-} to Λ_{γ_+}. (See Definition 23.3.1 in [17] for "X–morphism".) This is done in a way similar to the definition of Γ_C in equation (23.8) in [17]. In [17], a "cobordism" from Y_- to Y_+ refers to a compact 4–manifold with boundary $Y_+ \sqcup (-Y_-)$. This corresponds to the compact part of our X, denoted by $X_c = s^{-1}([-L_{\mathrm{tor}}, L_{\mathrm{tor}}])$. The surface $C \cap X_c$ plays the role of the singular 2–chain ν in (23.8) of [17]. It has boundary $\gamma_+ - \gamma_-$, with $\gamma_+ \simeq \gamma \simeq \gamma_-$. Given $\mathfrak{c}_- \in \mathcal{Z}_{w_-,\mathrm{r}}(Y_-)$ and $\mathfrak{c}_+ \in \mathcal{Z}_{w_+,\mathrm{r}}(Y_+)$, let $\mathfrak{d}$ denote an element in $\mathcal{B}^\sigma(X)$ with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. Then Γ_C is an isomorphism from $\Gamma_{\gamma_-}(\mathfrak{c}_-) \simeq \mathbb{R}$ to $\Gamma_{\gamma_+}(\mathfrak{c}_+) \simeq \mathbb{R}$ given by multiplication by $e^{(i/2\pi)\int_C F_{\mathbb{A}}}$. The analog of Γ_C in our setting, denoted by Λ_C below, is given by an homomorphism from $\Lambda_{\gamma_-}(\mathfrak{c}_-) \simeq \mathbb{K}[U, U^{-1}]$ to $\Lambda_{\gamma_+}(\mathfrak{c}_+) \simeq \mathbb{K}[U, U^{-1}]$ for each pair $\mathfrak{c}_-$ and $\mathfrak{c}_+$. This is given by multiplication with $U^{n(\mathfrak{d})}$, where

$$n(\mathfrak{d}) = \frac{i}{2\pi} \int_C F_{\hat{A}} = \mathrm{x}_{\gamma_+}(\mathfrak{c}_+) - \mathrm{x}_{\gamma_-}(\mathfrak{c}_-), \tag{3-20}$$

the rightmost equality being a consequence of Stokes' theorem. This is again an integer according to Lemma 3.2. With Λ_C in place, given a k–cochain $u \in \mathrm{C}^k(\mathcal{B}^\sigma(X); \mathbb{K})$ in the notation of Section 2.4, we define the map

$$\begin{aligned} m^\infty[u](X, \langle w_X \rangle; \Lambda_C)\colon \mathrm{CM}^\infty(Y_-) = \mathbb{K}[U, U^{-1}](\mathcal{Z}_{w_-,\mathrm{r}}) \\ \to \mathrm{CM}^\infty(Y_+) = \mathbb{K}[U, U^{-1}](\mathcal{Z}_{w_+,\mathrm{r}}) \end{aligned}$$

by the rule

$$\mathcal{Z}_{w_-,\mathrm{r}}(Y_-) \ni \mathfrak{c}_- \mapsto \sum_{\mathfrak{c}_+ \in \mathcal{Z}_{w_+,\mathrm{r}}(Y_+)} \sum_i \langle u, \mathcal{M}_k(X, \mathfrak{c}_-, \mathfrak{c}_+) \rangle U^{n(\mathfrak{d}_i)} \mathfrak{c}_+,$$

where i runs through each connected component of $\mathcal{M}_k(X, \mathfrak{c}_-, \mathfrak{c}_+)$ and, for every i, $\mathfrak{d}_i$ is an element in the corresponding connected component. In order for the sum on the right-hand side to be well defined, we assume that $H^2(X, Y_-) = 0$ and w_X satisfies (2-22).

To proceed, suppose (Y_-, w_-) and (Y_+, w_+) are c–tight and consider

$$C(X, \langle w_X \rangle; \Lambda_C)|_{\mathrm{CM}^-(Y_-)}.$$

Suppose furthermore that (X, w_X) satisfies the conditions in Propositions 3.5 and 3.8. By these propositions, the integers $n(\mathfrak{d}_i)$ in (3-20) are nonnegative, implying that the image of $C(X, \langle w_X \rangle; \Lambda_C)|_{\mathrm{CM}^-(Y_-)}$ under m^∞ lies in $\mathrm{CM}^-(Y_+)$. Use

$$m^-[u](X, \langle w_X \rangle; \Lambda_C)\colon \mathrm{CM}^-(Y_-) \to \mathrm{CM}^-(Y_+)$$

to denote this map. It is straightforward to see that both m^∞ and m^- are chain maps, given that CM^∞ is a variant of monopole Floer complexes, and the nonnegativity of the integers $n(\mathfrak{d})$ appearing in the formulas for ∂^∞ and m^∞. These then induce homomorphisms between the respective homologies,

$$\mathrm{HM}_*(X, \langle w_X \rangle; \Lambda_C)\colon \mathrm{HM}^\circ(Y_-, \langle w_- \rangle; \Lambda_{\gamma_-}) \to \mathrm{HM}^\circ(Y_+, \langle w_+ \rangle; \Lambda_{\gamma_+})$$

for $\circ = -, \infty$. Like those in Part 1, these maps preserve the $\boldsymbol{A}_\dagger$–module structure.

4 Some homological algebra

As mentioned in Section 1, the purpose of this section is to review the algebraic background for the upcoming Proposition 5.9. The latter is used to relate the formula for monopole Floer homology of a connected sum, given in Proposition 6.11 below, in terms of the monopole Floer homology with balanced perturbation that appears in Theorems 1.1 and 1.4. This computation turns out to be a simplest manifestation of the so-called "Koszul duality", well known in certain circles. For a sampling of literature on this subject, see eg [4; 15; 10]. The variant most relevant to this article is discussed in [10], which relates the ordinary chain complex of an S^1–space, equipped with an $H_*(S^1)$–module structure capturing the S^1–action, with the S^1–equivariant chain complexes of the same space, which are naturally endowed with $H^*(BS^1)$–module structures. We need however only a small portion of the full machinery in [10]. Thus, in this section we give a self-contained though elementary exposition of the relevant part of this story, tailored to our needs.

4.1 Terminology and conventions

By a *modules over* $H^*(BS^1)$ we mean a chain complex with a module structure over $\mathbb{K}[u]$, where u acts as a chain map of degree -2. The prime examples of such modules in this article are the monopole Floer complexes. In parallel, a *module over* $H_*(S^1)$ stands for a chain complex with a module structure over $\mathbb{K}[y]$, where y acts as a degree 1 chain map. An example that appears later is the chain complex to compute the monopole Floer homology of a connected sum; see (6-13) in Proposition 6.7. Meanwhile, a graded homology module H_* will be viewed as a chain complex with zero differentials. We use capital letters U and Y to denote the chain maps corresponding to the action of u and y.

Definition 4.1 A *morphism* from one module over $H^*(BS^1)$ to another is a $\mathbb{K}$–chain map which commutes with U–actions. Morphisms between $H_*(S^1)$–modules are defined similarly, with Y replacing U. We shall also often encounter a weaker notion: a p*–morphism* between two $H^*(BS^1)$–modules is a $\mathbb{K}$–chain map which commutes with U–actions up to $\mathbb{K}$–chain homotopy.

4.2 From $H^*(BS^1)$–modules to $H_*(S^1)$–modules

Given a module (C, ∂_C) over $H^*(BS^1)$, we define the module $S_U(C)$ over $H_*(S^1) = \mathbb{K}[y]$ as follows:

(4-1) $$(S_U(C), S_U(\partial_C)) = (C \otimes \mathbb{K}[y], \partial_C \otimes \jmath + U \otimes y),$$

where the homomorphism $\jmath\colon \mathbb{K}[y] \to \mathbb{K}[y]$ is defined by

$$\jmath(a + by) = a - by \quad \text{for } a, b \in \mathbb{K},$$

and the y–action is simply the multiplication $1 \otimes y$ ($\jmath$ was denoted by σ in [23]; compare equation (5.1) therein).

To see that $S_U(C)$ is indeed a chain complex, note that the condition $S_U(\partial_C)^2 = 0$ is equivalent to the pair of identities $\partial_C^2 = 0$, and $[\partial_C, U] = 0$.

Lemma 4.2 *A p–morphism Φ between two $H^*(BS^1)$–modules $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$ induces an $H_*(S^1)$–module morphism $S_U(\Phi)$ between $S_{U_{(1)}}(C_{(1)})$ and $S_{U_{(2)}}(C_{(2)})$, where $U_{(1)}$ and $U_{(2)}$ denote the u–action on $C_{(1)}$ and $C_{(2)}$, respectively. Furthermore:*

- *$S_U(\Phi)$ is injective if Φ is injective, and it is surjective if Φ is surjective.*
- *Let Φ' be another p–morphism of $H^*(BS^1)$–modules from $(C_{(1)}, \partial_{(1)})$ to $(C_{(2)}, \partial_{(2)})$. Then $\Phi + \Phi'$ is a p–morphism as well, and*
$$S_U(\Phi + \Phi') = S_U(\Phi) + S_U(\Phi').$$
- *Let Ψ be a p–morphism of $H^*(BS^1)$–modules from $(C_{(2)}, \partial_{(2)})$ to $(C_{(3)}, \partial_{(3)})$. Then $\Psi \circ \Phi$ is a p–morphism as well, and*

(4-2) $$S_U(\Psi \circ \Phi) = S_U(\Psi) \circ S_U(\Phi).$$

Proof As a p–morphism, Φ satisfies both

(4-3) $$\Phi\partial_{(1)} - (-1)^{\deg(\Phi)}\partial_{(2)}\Phi = 0, \quad \Phi U_{(1)} - U_{(2)}\Phi = K_\Phi\partial_{(1)} + (-1)^{\deg(\Phi)}\partial_{(2)}K_\Phi$$

for a $\mathbb{K}$–linear homomorphism K_Φ. This is equivalent to the identity

(4-4) $$S_U(\Phi)S_U(\partial_{(1)}) - (-1)^{\deg(\Phi)}S_U(\partial_{(2)})S_U(\Phi) = 0,$$

where $S_U(\Phi)\colon C_{(1)} \otimes \mathbb{K}[y] \to C_{(2)} \otimes \mathbb{K}[y]$ is defined as

(4-5) $$S_U(\Phi) = \Phi \otimes \jmath^{\deg(\Phi)} + K_\Phi \otimes y.$$

This verifies that $S_U(\Phi)$ is a chain map. Moreover, since the y–action on $S_U(C_{(1)})$ and $S_U(C_{(2)})$ is multiplication by $1 \otimes y$, it is immediate that $S_U(\Phi)$ commutes with the y–actions on both sides. The claim that S_U preserves injectivity and surjectivity can be checked directly from the definition of $S_U(\Phi)$.

Since the construction of $S_U(\Phi)$ is linear, the second item in the statement of the lemma is obvious.

To verify the third bullet about the composition of p–morphisms, let

$$S_U(\Psi) = \Psi \otimes \jmath^{\deg(\Psi)} + K_\Psi \otimes y.$$

Then (4-2) is straightforward to verify, given that

$$K_{\Psi\circ\Phi} = K_\Psi \circ \Phi + (-1)^{\deg(\Psi)} \Psi \circ K_\Phi.$$

The fact that $\Psi \circ \Phi$ is a p–morphism follows directly from (4-4) and its analog for Ψ.

The first bullet may be directly verified after writing out the definition of $S_U(\Phi)$ explicitly. More is said in the proof of Lemma 4.7 below. □

Remark 4.3 Given a p–morphism Φ, the $H_*(S^1)$–morphism $S_U(\Phi)$ given in (4-5) apparently depends on the choice of the degree $(\deg \Phi + 1)$ map K_Φ. By (4-3), two different choices of K_Φ, say K_Φ and K'_Φ, differ by a chain map:

$$[\partial_C, K_\Phi - K'_\Phi] := (K_\Phi - K'_\Phi)\partial_{(1)} + (-1)^{\deg(\Phi)} \partial_{(2)}(K_\Phi - K'_\Phi) = 0.$$

We say that K_Φ and K'_Φ are *homotopic* if there exists a degree $\deg \Phi$ linear map $Z_\Phi \colon C_{(1)} \to C_{(2)}$ such that

$$K_\Phi - K'_\Phi = [\partial_C, Z_\Phi] := \partial_{(2)} Z_\Phi - Z_\Phi (-1)^{\deg(\Phi)} \partial_{(1)}.$$

Let $S_U(\Phi)$ and $S_U(\Phi)'$, respectively, denote the versions of $S_U(\Phi)$ defined using K_Φ and K'_Φ. They are chain homotopic when K_Φ and K'_Φ are homotopic:

$$S_U(\Phi) - S_U(\Phi)' = ([\partial_C, Z_\Phi]) \otimes y = [S_U(\partial_C), -Z_\Phi \otimes y]. \tag{4-6}$$

(Keep in mind that in our notation, $[\cdot,\cdot]$ stands for a commutator in a graded sense.) Thus, for a given Φ, the homology $H_*(S_U(\Phi); \mathbb{K})$ depends only on the (relative) homotopy class of K_Φ.

Definition 4.4 Two $H^*(BS^1)$–modules $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$ are said to be *p–homotopic* if there exist p–morphisms $\Phi \colon C_{(1)} \to C_{(2)}$ and $\Psi \colon C_{(2)} \to C_{(1)}$, and $H_1 \colon C_{(1)} \to C_{(1)}$ and $H_2 \colon C_{(2)} \to C_{(2)}$, such that

$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2].$$

They are said to be *homotopic* if Φ, Ψ, H_1 and H_2 are morphisms. The notion of two $H_*(S^1)$–modules being *homotopic* is defined similarly.

Lemma 4.5 *Suppose two $H^*(BS^1)$–modules $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$ are p–homotopic* via *p–morphisms $\Phi\colon C_{(1)} \to C_{(2)}$ and $\Psi\colon C_{(2)} \to C_{(1)}$ as above. Then the $H_*(S^1)$–modules $S_U(C_{(1)})$ and $S_U(C_{(2)})$ are homotopic via the maps $S_U(\Phi)$ and $S_U(\Psi)$.*

Proof By assumption, there exist H_1 and H_2 such that Φ and Ψ satisfy

$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2].$$

We need to verify the identities

$$\begin{aligned} S_U(\Psi) \circ S_U(\Phi) - \mathrm{Id}_{(1)} &= [S_U(\partial_{(1)}), S_U(H_1)], \quad [S_U(\Phi), Y] = 0, \\ S_U(\Phi) \circ S_U(\Psi) - \mathrm{Id}_{(2)} &= [S_U(\partial_{(2)}), S_U(H_2)], \quad [S_U(\Psi), Y] = 0. \end{aligned}$$

It suffices to verify the first and the third identities, since the second and the fourth are entirely parallel.

To verify the first identity, use (4-2) and the fact that $\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1]$ to reduce it to

$$S_U(\mathrm{Id}_{(1)}) = \mathrm{Id}.$$

This holds by taking $\Psi = \mathrm{Id}_{(1)}$ and $K_\Psi = 0$ in (4-5).

To verify the third identity, simply plug in the definition of $S_U(\Phi)$ and $Y = I \otimes y$. □

4.3 From $H_*(S^1)$–modules to $H^*(BS^1)$–modules

First, introduce the $\mathbb{K}[u]$–modules

$$\begin{aligned} &V^- := u\mathbb{K}[u], && V^\infty := \mathbb{K}[u, u^{-1}], \\ &V^+ := \mathbb{K}[u, u^{-1}]/u\mathbb{K}[u], && V^\wedge := \mathbb{K}[u]/u\mathbb{K}[u]. \end{aligned} \tag{4-7}$$

These modules by definition fit into the short exact sequences

$$0 \to V^- \xhookrightarrow{i_V} V^\infty \to V^+ \to 0, \tag{4-8}$$

$$0 \to V^- \xrightarrow{u} V^- \to V^\wedge \to 0. \tag{4-9}$$

We shall frequently view these four modules as a system, and write them collectively as V°. The same convention applies to the various systems of modules we construct out of these four below.

Definition 4.6 [13; 23] Given a module (C, ∂_C) over $H_*(S^1)$, we define the following system of modules over $H^*(BS^1) = \mathbb{K}[u]$:

$$(E_Y^\circ(C), E_Y(\partial_C)) := (C \otimes V^\circ, \partial_C \otimes 1 + Y \otimes u) \quad \text{for } \circ = -, \infty, +, \wedge, \tag{4-10}$$

where the u–action is the multiplication $1 \otimes u$.

The fact that $E_Y(\partial_C)^2 = 0$ again follows directly from the definition of $H_*(S^1)$–modules: $\partial_C^2 = 0$, $[Y, \partial_C] = 0$ and $Y^2 = 0$. By taking tensor product of (4-8)–(4-9) with C, one has the corresponding short exact sequences of $\mathbb{K}[u]$–modules

$$0 \to E_Y^-(C) \xrightarrow{\mathrm{Id} \otimes i_V} E_Y^\infty(C) \to E_Y^+(C) \to 0, \tag{4-11}$$

$$0 \to E_Y^-(C) \xrightarrow{\mathrm{Id} \otimes u} E_Y^-(C) \to E_Y^\wedge(C) \to 0. \tag{4-12}$$

It is also straightforward to verify that the maps in the above exact sequences commute with $E_Y(\partial_C)$, and therefore they induce long exact sequences of $H^*(BS^1)$–modules associated to (C, ∂_C),

$$\cdots \to H_*(E_Y^-(C)) \xrightarrow{i_{V*}} H_*(E_Y^\infty(C)) \to H_*(E_Y^+(C)) \xrightarrow{\delta_{V*}} H_{*-1}(E_Y^-(C)) \to \cdots, \tag{4-13}$$

$$\cdots \to H_*(E_Y^-(C)) \xrightarrow{u} H_*(E_Y^-(C)) \to H_*(E_Y^\wedge(C)) \to H_{*-1}(E_Y^-(C)) \to \cdots. \tag{4-14}$$

We call (4-13)–(4-14) the (first and second) *fundamental exact sequences* for the $H_*(S^1)$–module C. For convenience of later reference, we denote the short exact sequences of $H^*(BS^1)$–modules (4-11) and (4-12) by $\mathbb{E}_Y(C)$ and $\boldsymbol{E}_Y(C)$, respectively. Correspondingly, the long exact sequences (4-13) and (4-14) are denoted by $H(\mathbb{E}_Y(C))$ and $H(\boldsymbol{E}_Y(C))$. It is straightforward to verify the assertion in the following lemma and so we leave it to the reader to check that:

Lemma 4.7 *A morphism ϕ between $H_*(S^1)$–modules $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$ induces a system of $H^*(BS^1)$–module morphisms*

$$E^\circ(\phi)\colon E_{Y_{(1)}}^\circ(C_{(1)}) \to E_{Y_{(2)}}^\circ(C_{(2)}), \quad \phi \mapsto \phi \circ 1,$$

for $\circ = -, \infty, +, \wedge$, where $Y_{(1)}$ and $Y_{(2)}$ denote the y–actions on $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$, respectively. Moreover:

- *$E^\circ(\phi)$ is injective if ϕ is injective; it is surjective if ϕ is surjective.*
- *Let ϕ' be another morphism of $H_*(S^1)$–modules between $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$. Then $\phi + \phi'$ is an $H_*(S^1)$–morphism as well, and*

$$E^\circ(\phi + \phi') = E^\circ(\phi) + E^\circ(\phi').$$

- *Let ψ be another morphism of $H_*(S^1)$–modules between $(C_{(2)}, \partial_{(2)})$ and $(C_{(3)}, \partial_{(3)})$. Then $\psi \circ \phi$ is an $H_*(S^1)$–morphism as well, and*

$$E^\circ(\psi \circ \phi) = E^\circ(\psi) \circ E^\circ(\phi). \tag{4-15}$$

- *The system of morphisms $E^\circ(\phi)$ combine to define morphisms of short exact sequences of $H^*(BS^1)$–modules*

$$\mathbb{E}(\phi)\colon \mathbb{E}_Y(C_{(1)}) \to \mathbb{E}_Y(C_{(2)}) \quad \text{and} \quad \boldsymbol{E}(\phi)\colon \boldsymbol{E}_Y(C_{(1)}) \to \boldsymbol{E}_Y(C_{(2)}).$$

  Correspondingly, their induced maps on homologies $H_*(E_Y(\phi))$ combine to define morphisms of long exact sequences of $H^*(BS^1)$–modules

$$H(\mathbb{E}_Y(\phi))\colon H(\mathbb{E}_Y(C_{(1)})) \to H(\mathbb{E}_Y(C_{(2)})),$$
$$H(\boldsymbol{E}_Y(\phi))\colon H(\boldsymbol{E}_Y(C_{(1)})) \to H(\boldsymbol{E}_Y(C_{(2)})).$$

Proof The proofs are straightforward; thus we shall say no more than making the following remarks: Both E°_Y and S_U preserve injectivity and surjectivity due to the same reason, namely they can be written in polynomial form (in u and y, respectively, which defines a filtration), where their 0^{th} order term takes the form of a tensor product of the original morphism and an automorphism. This in turn implies that both of them takes short exact sequences to short exact sequences. □

Lemma 4.8 *Let $C_{(1)}$ and $C_{(2)}$ denote homotopic $H_*(S^1)$–modules. Then $E^\circ_Y(C_{(1)})$ and $E^\circ_Y(C_{(2)})$ are homotopic $H^*(BS^1)$–modules.*

Proof By assumption, there exist morphisms $\Phi\colon C_{(1)} \to C_{(2)}$ and $\Psi\colon C_{(2)} \to C_{(1)}$, and $H_1\colon C_{(1)} \to C_{(1)}$ and $H_2\colon C_{(2)} \to C_{(2)}$, such that

$$\Psi \circ \Phi - \mathrm{Id}_{(1)} = [\partial_{(1)}, H_1], \quad \Phi \circ \Psi - \mathrm{Id}_{(2)} = [\partial_{(2)}, H_2]. \tag{4-16}$$

Lemma 4.7 claims that $E^\circ(\Phi)\colon E^\circ_Y(C_{(1)}) \to E^\circ_Y(C_{(2)})$ and $E^\circ(\Psi)\colon E^\circ_Y(C_{(2)}) \to E^\circ_Y(C_{(1)})$ are systems of morphisms. Meanwhile, the desired identities are

$$\begin{aligned} E^\circ(\Psi) \circ E^\circ(\Phi) - \mathrm{Id}_{(1)} &= [E_Y(\partial_{(1)}), E^\circ(H_1)], \\ E^\circ(\Phi) \circ E^\circ(\Psi) - \mathrm{Id}_{(2)} &= [E_Y(\partial_{(2)}), E^\circ(H_2)]. \end{aligned} \tag{4-17}$$

We shall only verify the first identity, since the second is similar. For this purpose, apply E° to the first identity in (4-16), then apply Lemma 4.7 and subtract the first line of (4-17) to the resulting identity. This leads to

$$E^\circ(\mathrm{Id}_{(1)}) - \mathrm{Id} = [Y, H_1] \otimes u.$$

This is true because of the definition of E° and the fact that H_1 is a morphism. □

4.4 Koszul duality

The functors S_U and E^- may be viewed as inverses of each other in the following sense:

Proposition 4.9 (a) *Let (C, ∂_C) be an $H^*(BS^1)$–module. Then there is a system of isomorphisms of $H^*(BS^1)$–modules*

$$H_*(E^\circ_Y S_U(C)) \simeq H_*(C \otimes_{\mathbb{K}[u]} V^\circ). \tag{4-18}$$

Moreover, these isomorphisms have the following naturality properties:

(i) *They are natural with respect to p–morphisms of $H^*(BS^1)$–modules.*

(ii) *They combine to define isomorphisms of long exact sequences of $H^*(BS^1)$–modules*

$$H(\mathbb{E}_Y S_U(C)) \simeq H(C \otimes_{\mathbb{K}[u]} \mathbb{V}), \quad H(\boldsymbol{E}_Y S_U(C)) \simeq H(C \otimes_{\mathbb{K}[u]} \boldsymbol{V}).$$

Here, $H(C \otimes_{\mathbb{K}[u]} \mathbb{V})$ and $H(C \otimes_{\mathbb{K}[u]} \boldsymbol{V})$ respectively denote the long exact sequence induced by the short exact sequences of $H^*(BS^1)$–modules,

$$0 \to C \otimes_{\mathbb{K}[u]} V^- \to C \otimes_{\mathbb{K}[u]} V^\infty \to C \otimes_{\mathbb{K}[u]} V^+ \to 0,$$
$$0 \to C \otimes_{\mathbb{K}[u]} V^- \xrightarrow{1\otimes u} C \otimes_{\mathbb{K}[u]} V^- \to C \otimes_{\mathbb{K}[u]} V^\wedge \to 0.$$

(b) *Let (C, ∂_C) be an $H_*(S^1)$–module. Then there is an isomorphism of $H_*(S^1)$–modules*

$$H_*(S_U E^-_Y(C)) \simeq H_*(C).$$

Proof (a) Written out explicitly,

$$E^\circ_Y S_U(C) = C \otimes \mathbb{K}[y] \otimes V^\circ, \quad E_Y S_U(\partial_C) = \partial_C \otimes \jmath \otimes 1 + U \otimes y \otimes 1 + 1 \otimes y \otimes u.$$

View this as a filtered complex by the total degree in the $C \otimes V^\circ$ factor. Then the E_1–term of the associated spectral sequence is simply

$$C \otimes \mathbb{K}\{y\} \otimes V^\circ / \big((U \otimes y \otimes 1 + 1 \otimes y \otimes u)(C \otimes \mathbb{K}\{1\} \otimes V^\circ)\big) \simeq C \otimes_{\mathbb{K}[u]} V^\circ, \tag{4-19}$$

with differential d_1 given by $-\partial_C$. This spectral sequence degenerates at E_2, and we have $H_*(E^\circ_Y S_U(C)) \simeq H_*(C \otimes_{\mathbb{K}[u]} V^\circ)$, as claimed. As the u–action on $E^\circ_Y S_U(C)$ is $1 \otimes 1 \otimes u$ and the u–action on C is U, the quotient in (4-19) shows that the isomorphism preserves the $\mathbb{K}[u]$–module structure. Property (ii) also follows immediately from this computation.

On the other hand, given a p–morphism Φ between $H^*(BS^1)$–modules $(C_{(1)}, \partial_{(1)})$ and $(C_{(2)}, \partial_{(2)})$, by Lemmas 4.2 and 4.7 there is a corresponding system of morphisms of $H^*(BS^1)$–modules $E_Y^{\circ} S_U(\Phi)$. The naturality property (i) follows from the fact that these morphisms preserve the filtration.

(b) Written out explicitly,

$$S_U E_Y^-(C) = C \otimes u\mathbb{K}[u] \otimes \mathbb{K}[y], \quad S_U E_Y^-(\partial_C) = \partial_C \otimes 1 \otimes \sigma + Y \otimes u \otimes \sigma + 1 \otimes u \otimes y.$$

Filtrate by the total degree of the factor $C \otimes u\mathbb{K}[u]$ as in the previous part. Then the E_1–term is

$$C \otimes u\mathbb{K}[u] \otimes R\{y\} / \big((1 \otimes u \otimes y)(C \otimes u\mathbb{K}[u] \otimes R\{1\})\big) \simeq C,$$

on which d_1 acts as $-\partial_C$. The spectral sequence again degenerates at E_2, yielding the claimed isomorphism $H_*(S_U E_Y^-(C)) \simeq H_*(C)$. To see that the module structures agree, note that a cycle in the E_1–term given by an element $-z_1 \in C$ with $\partial_C z_1 = 0$ corresponds to a cycle in $S_U E_Y^-(C)$ of the form $Z_0 \otimes 1 + z_1 \otimes u \otimes y$, where $Z_0 \in C \otimes u\mathbb{K}[u]$ satisfies

$$-((Y \otimes u)Z_0) \otimes 1 + ((1 \otimes u)Z_0) \otimes y - (Yz_1) \otimes u^2 \otimes y = 0.$$

In other words, $Z_0 = -(Yz_1) \otimes u$, and the cycle in $S_U E_Y^-(C)$ has the form

$$-(Yz_1) \otimes u \otimes 1 + z_1 \otimes u \otimes y.$$

The y–action $1 \otimes 1 \otimes y$ takes this element to $-(Yz_1) \otimes u \otimes y$, while the element corresponding to $-Yz_1 \in C$ in the E_1–term is $-(Yz_1) \otimes u \otimes y$ as well, since $Y^2 = 0$. □

Remark 4.10 (a) Spelled out explicitly, (4-18) says that $H_*(E_Y^- S_U(C)) \simeq H_*(C)$, and $H_*(E_Y^{\infty} S_U(C))$ is the localization of $H_*(C)$ as a $\mathbb{K}[u]$–module. On the other hand, note that since $V^\wedge = \mathbb{K}[u]/u\mathbb{K}[u]$, $E^\wedge S_U(\partial_C)$ reduces to $S_U(\partial_C) \otimes 1$, and therefore $H_*(E_Y^{\wedge} S_U(C)) \simeq H_*(S_U(C))$.

(b) The constructions E_Y^-, E_Y^{∞} and E_Y^+ above are directly copied from J Jones's formulation of the "co-Borel", "Tate" and Borel (the usual) versions of equivariant homologies [13]. It is proved in [10] that S_U and E_Y induce isomorphisms of derived categories.

Remark 4.11 As stated, the general $H^*(BS^1)$–module (C, ∂_C) in the present section is assumed to be $\mathbb{Z}$–graded. For our application however, results in this section are typically applied to monopole chain complexes $\mathring{C}$. These are only relatively graded, and the grading group is $\mathbb{Z}$ only when $c_1(\mathfrak{s})$ is torsion, in other cases it is $\mathbb{Z}/c_{\mathfrak{s}}$, where $c_{\mathfrak{s}} \in 2\mathbb{Z}$. These chain complexes $\mathring{C}$ are also equipped with a canonical absolute $\mathbb{Z}/2$–grading. (See [17].) Nevertheless, we observe that such a monopole chain complex $\mathring{C}$ can be alternatively interpreted as a $\mathbb{Z}$–graded chain complex (C, ∂) with periodicity $c_{\mathfrak{s}} \in 2\mathbb{Z}$, namely $(C_k, \partial_k) = (C_{k+c_{\mathfrak{s}}}, \partial_{k+c_{\mathfrak{s}}})$ for all $k \in \mathbb{Z}$. (For prior appearance of such interpretation, see eg [24].)

To do this, let $\mathring{C}$ be a certain monopole chain complex associated to the Spin^c–manifold $(M, \mathfrak{s})$, and let $\mathcal{Z} \subset \mathcal{B} := \mathcal{C}^\sigma(M, \mathfrak{s})/C^\infty(M, U(1))$ denote the generating set of $\mathring{C}$. Denote the coefficient ring of $\mathring{C}$ by $\mathbb{K}$. Recall that $H^1(\mathcal{B};\mathbb{Z}) \simeq H^2(M;\mathbb{Z})$, and therefore the class $c_1(\mathfrak{s}) \in H^2(M;\mathbb{Z})$ defines a $\mathbb{Z}$–covering $\pi\colon \widetilde{\mathcal{B}} \to \mathcal{B}$. Let $\widetilde{\mathcal{Z}} := \pi^{-1}\mathcal{Z}$, and consider the chain complex (C, ∂_C) with

$$C := \mathbb{K}(\widetilde{\mathcal{Z}}) \quad \text{and} \quad \langle \tilde{\mathfrak{c}}_1, \partial_C \tilde{\mathfrak{c}}_2 \rangle := \sum_{\tilde{\mathfrak{d}}} \operatorname{sign}(\tilde{\mathfrak{d}})$$

for any pair $\tilde{\mathfrak{c}}_1, \tilde{\mathfrak{c}}_2 \in \widetilde{\mathcal{Z}}$. Regarding elements in $\mathcal{M}_1(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}$ as paths in $\mathcal{B}$ from $\mathfrak{c}_1$ to $\mathfrak{c}_2$, $\tilde{\mathfrak{d}}$ in the sum above stands for any lift of some $\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}$ to $\widetilde{\mathcal{B}}$ ending in $\tilde{\mathfrak{c}}_1$ and $\tilde{\mathfrak{c}}_2$, where $\mathfrak{c}_* := \pi\tilde{\mathfrak{c}}_*$ and $\operatorname{sign}(\tilde{\mathfrak{d}}) := \operatorname{sign}(\mathfrak{d})$. Since the spectral flow on $\mathcal{B}$ is controlled by $c_1(\mathfrak{s})$, the relative grading of this complex (C, ∂_C), defined by spectral flow along $\tilde{\mathfrak{d}}$, is $\mathbb{Z}$–valued. Fix a $\tilde{\mathfrak{c}}_0 \in \widetilde{\mathcal{Z}}$. Then $\mathfrak{c}_0 = \pi\tilde{\mathfrak{c}}_0$ is either even or odd according to the canonical absolute $\mathbb{Z}/2$–grading. Set $\operatorname{gr}(\tilde{\mathfrak{c}}_0) = 0$ if $\mathfrak{c}_0$ is even, and set $\operatorname{gr}(\tilde{\mathfrak{c}}_0) = 1$ if $\mathfrak{c}_0$ is odd. Together with the relative $\mathbb{Z}$–grading $\operatorname{gr}(\cdot,\cdot)$ on (C, ∂_C), we have an absolute $\mathbb{Z}$–grading by setting

$$\operatorname{gr}(\tilde{\mathfrak{c}}) := \operatorname{gr}(\tilde{\mathfrak{c}}_0) + \operatorname{gr}(\tilde{\mathfrak{c}}_0, \tilde{\mathfrak{c}})$$

for any $\tilde{\mathfrak{c}} \in \widetilde{\mathcal{Z}}$. (If $\widetilde{\mathcal{Z}} = \varnothing$, $\widehat{C}$ is trivial.) With this definition, we then have $C_k = \mathring{C}_{k'}$ for any pair $k \in \mathbb{Z}$ and $k' \in \mathbb{Z}/c_{\mathfrak{s}}$ with $k = k' \bmod c_{\mathfrak{s}}$. Moreover, given $\tilde{\mathfrak{c}}_1 \in \widetilde{\mathcal{Z}}$ with $\operatorname{gr}(\tilde{\mathfrak{c}}_1) = k$ and a $\mathfrak{d} \in \mathcal{M}_1(\mathfrak{c}_1, \mathfrak{c}_2)/\mathbb{R}$, there is a unique lift $\tilde{\mathfrak{d}}$ of $\mathfrak{d}$ starting from $\tilde{\mathfrak{c}}_1$, whose endpoint is a lift $\tilde{\mathfrak{c}}_2 \in \widetilde{\mathcal{Z}}$ of $\mathfrak{c}_2$. We have $\operatorname{gr}(\tilde{\mathfrak{c}}_2) = k - 1$. Thus, $\partial_C|_{C_k}\colon C_k \to C_{k-1}$ is identical to $\mathring{\partial}|_{\widehat{C}_{k'}}\colon \widehat{C}_{k'} \to \widehat{C}_{k'-1}$ for the same pair k and k' as before.

5 Balanced Floer homologies from monotone Floer chain complexes

This section reintroduces the fourth flavor of monopole Floer homology, denoted by $\mathrm{HM}^{\mathrm{tot}}$ in [23], now renamed $\widetilde{\mathrm{HM}}$ in deference to Donaldson's notation. (See page 187 of [7].) This definition is a natural byproduct of a reinterpretation of $\overset{\circ}{\mathrm{HM}}_*(M,\mathfrak{s},c_b)$ in terms *purely of the* $\mathbb{K}[U]$*–module* $\hat{C}_*(M,\mathfrak{s},c_b)$ (Corollary 5.3 in [23], restated as Proposition 5.9 below). This result enables us to appeal to the third author's "SW = Gr" program, which in our context was carried out in part IV of this series [22]. The latter constructed an isomorphism from an appropriate variant of ech to a *negative monotone* version of monopole Floer homology, which is in turn related to the balanced version via the following theorem of Kronheimer and Mrowka:

Theorem 5.1 [17, Theorem 31.5.1] *Suppose $c_1(\mathfrak{s})$ is nontorsion. Let $\hat{C}_*(M,\mathfrak{s},c_b)$ and $C_*(M,\mathfrak{s},c_-)=\hat{C}_*(M,\mathfrak{s},c_-)$ respectively denote the Seiberg–Witten Floer chain complexes with balanced and negative monotone perturbations. Then there is a chain homotopy equivalence from the former to the latter. In particular, $\widehat{\mathrm{HM}}_*(M,\mathfrak{s},c_b)\simeq \mathrm{HM}_*(M,\mathfrak{s},c_-)$.*

To be more precise, the statement of Theorem 31.5.1 in [17] concerns only the Floer homologies. However, the chain homotopy equivalence referred to above was constructed in its proof.

Remark 5.2 The variant of ech relevant in this series of papers is related to the negative monotone version of monopole Floer homology, and therefore to $\widehat{\mathrm{HM}}_*$ by the preceding theorem of [17]. This is because the stable Hamiltonian structure used to define the relevant ech is associated to an *nonexact* closed 2–form. Note in contrast that the ordinary embedded contact homology associated to a contact structure is related to $\widecheck{\mathrm{HM}}_*$ instead, since the relevant 2–form in this case is exact. As such, it belongs to the positive monotone situation, and the companion theorem to the one just cited states that $\widecheck{\mathrm{HM}}_*(M,\mathfrak{s},c_b)\simeq \mathrm{HM}_*(M,\mathfrak{s},c_+)$.

5.1 Some properties of the maps i, j and p

In this section, unless otherwise specified, $\overset{\circ}{C}_*=\overset{\circ}{C}_*(M,c_b)$ denotes the monopole Floer chain complex associated to an oriented Riemannian, Spin^c 3–manifold with a *balanced perturbation*. Similarly, let $\overset{\circ}{\mathrm{HM}}_*=\overset{\circ}{\mathrm{HM}}_*(M,c_b)$.

Recall from Proposition 22.2.1 in [17] that $\widehat{\mathrm{HM}}_*$, $\widecheck{\mathrm{HM}}_*$ and $\overline{\mathrm{HM}}_*$ are related by a long exact sequence

$$\cdots\to\overline{\mathrm{HM}}_*\xrightarrow{i_*}\widecheck{\mathrm{HM}}_*\xrightarrow{j_*}\widehat{\mathrm{HM}}_*\xrightarrow{p_*}\overline{\mathrm{HM}}_{*-1}\xrightarrow{i_*}\cdots,\tag{5-1}$$

which we shall call the *fundamental exact sequence of monopole Floer homologies*. The maps i_*, j_* and p_* in the sequence above are respectively induced by maps

$$i\colon \overline{C} \to \check{C}, \quad j\colon \check{C} \to \widehat{C}, \quad p\colon \widehat{C} \to \overline{C},$$

which, written in block form with respect to the decomposition

$$\overline{C} = C^s \oplus C^u, \quad \check{C} = C^o \oplus C^s, \quad \widehat{C} = C^o \oplus C^u, \tag{5-2}$$

are given by

$$i = \begin{bmatrix} 0 & -\partial^u_o \\ 1 & -\partial^u_s \end{bmatrix}, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -\overline{\partial}^s_u \end{bmatrix}, \quad p = \begin{bmatrix} \partial^o_s & \partial^u_s \\ 0 & 1 \end{bmatrix}. \tag{5-3}$$

It is shown in [17] that they are respectively chain maps of degree 0, degree 0 and degree -1.

Lemma 5.3 *The maps i, j and p are p–morphisms of $H^*(BS^1)$–modules.*

Proof It is verified in [17] (for $\mathbb{K} = \mathbb{Z}$) that $[\mathring{\partial}, \mathring{U}] = 0$ for the to, from and bar versions of monopole Floer chain complexes. A straightforward though tedious computation using (5-3) shows that

$$\begin{aligned} i\overline{U} - \check{U}i + K_i\overline{\partial} + \check{\partial}K_i &= 0, \\ j\check{U} - \widehat{U}j + K_j\check{\partial} + \widehat{\partial}K_j &= 0, \\ p\widehat{U} - \overline{U}p - K_p\widehat{\partial} + \overline{\partial}K_p &= 0, \end{aligned} \tag{5-4}$$

where K_i, K_j and K_p, written in block form with respect to the same decompositions (5-2), are

$$K_i = \begin{bmatrix} 0 & -U^u_o \\ 0 & -U^u_s \end{bmatrix}, \quad K_j = \begin{bmatrix} 0 & 0 \\ 0 & -\overline{U}^s_u \end{bmatrix}, \quad K_p = \begin{bmatrix} U^o_s & U^u_s \\ 0 & 0 \end{bmatrix}. \qquad \square$$

As was explained in the proof of Lemma 4.2, the identities (5-4) can be rewritten as

$$\begin{aligned} S_U(i)S_U(\overline{\partial}) - S_U(\check{\partial})S_U(i) &= 0, \\ S_U(j)S_U(\check{\partial}) - S_U(\widehat{\partial})S_U(j) &= 0, \\ S_U(p)S_U(\widehat{\partial}) + S_U(\overline{\partial})S_U(p) &= 0, \end{aligned}$$

where $S_U(i)$, $S_U(j)$ and $S_U(p)$, when written in block form with respect to the same decomposition (5-2), as matrices with coefficients in $\mathbb{K}[y]$, are given as follows:

$$S_U(i) = \begin{bmatrix} 0 & -n^u_o \\ 1 & -n^u_s \end{bmatrix}, \quad S_U(j) = \begin{bmatrix} 1 & 0 \\ 0 & -\overline{n}^s_u \end{bmatrix}, \quad S_U(p) = \begin{bmatrix} N^o_s & N^u_s \\ 0 & 1 \end{bmatrix},$$

where

$$\overline{n}^s_u = \overline{\partial}^s_u + \overline{U}^s_u y, \quad n^*_* = \partial^*_* + U^*_* y, \quad \overline{N}^s_u = \overline{\partial}^s_u \jmath + \overline{U}^s_u y, \quad N^*_* = \partial^*_* \jmath + U^*_* y,$$

these being homomorphisms of $\mathbb{K}[y]$–modules for any pair of super- and subscripts $*$ among u, s and o.

Lemma 5.4 *The induced maps from i, j and p fit into the long exact sequences*

$$\begin{aligned}\text{(5-5)}\quad \cdots \to H_*(S_U(\overline{C})) &\xrightarrow{S_U(i)_*} H_*(S_U(\check{C})) \xrightarrow{S_U(j)_*} H_*(S_U(\hat{C})) \\ &\xrightarrow{S_U(p)_*} H_{*-1}(S_U(\overline{C})) \xrightarrow{S_U(i)_*} \cdots,\end{aligned}$$

$$\begin{aligned}\text{(5-6)}\quad \cdots \to H_*(E^\circ_Y S_U(\overline{C})) &\xrightarrow{E^\circ_Y S_U(i)_*} H_*(E^\circ_Y S_U(\check{C})) \xrightarrow{E^\circ_Y S_U(j)_*} H_*(E^\circ_Y S_U(\hat{C})) \\ &\xrightarrow{E^\circ_Y S_U(p)_*} H_{*-1}(E^\circ_Y S_U(\overline{C})) \xrightarrow{E^\circ_Y S_U(i)_*} H_{*-1}(E^\circ_Y S_U(\check{C})) \cdots.\end{aligned}$$

The first sequence is a sequence of $H_*(S^1)$–modules, and the second one is a sequence of $H^*(BS^1)$–modules.

Proof (a) The proof is based on a modification of the proof of Proposition 22.2.1 in [17]. Recall from [17] the definition of a "mapping cone of $-p$" $(\check{E}, \check{e})$:

$$\check{E} = \hat{C} \oplus \overline{C}, \quad \check{e} = \begin{bmatrix} \hat{\partial} & 0 \\ p & \overline{\partial} \end{bmatrix}.$$

The short exact sequence associated with $(\check{E}, \check{e})$, $0 \to \overline{C} \to \check{E} \to \hat{C} \to 0$, induces a long exact sequence connecting the triple $\overline{\mathrm{HM}}$, $H(\check{E})$ and $\widehat{\mathrm{HM}}$, with connecting map p_*. Kronheimer and Mrowka [17] show that $\check{E}$ is chain homotopic to $\check{C}$. The following diagram summarizes the construction:

$$\begin{tikzcd}
& & \check{C} \arrow[d, "k"] & & \\
0 \arrow[r] & \overline{C} \arrow[r, "\bar{\imath}"] \arrow[ur, "i"] & \check{E} \arrow[r, "\bar{\jmath}"] \arrow[d, "l"] & \hat{C} \arrow[r] \arrow[ul, "j"'] & 0 \\
& & \check{C} & &
\end{tikzcd}$$

where

$$k = \begin{bmatrix} j \\ \Pi_s \end{bmatrix}, \quad l = [\, \Pi_o \;\; i \,],$$

with

$$\Pi_s\colon \check{C} = C^o \oplus C^s \to \overline{C} = C^s \oplus C^u,$$
$$\Pi_o\colon \hat{C} = C^o \oplus C^u \to \check{C} = C^o \oplus C^s,$$
$$\Pi_u\colon \overline{C} = C^s \oplus C^u \to \hat{C} = C^o \oplus C^u$$

denoting projections to the s, o and u components, respectively.

In terms of these, the proof in [17] reduces to the verification of the identities

$$lk = \mathrm{Id}, \tag{5-7}$$
$$kl = \mathrm{Id} + \check{e}K + K\check{e}, \tag{5-8}$$
$$j = \bar{\jmath}k, \tag{5-9}$$
$$ki - \bar{\imath} = \check{e}(K\bar{\imath}) + (K\bar{\imath})\overline{\partial}, \tag{5-10}$$

where

$$K = \begin{bmatrix} 0 & -\Pi_u \\ 0 & 0 \end{bmatrix}.$$

We now want to apply the preceding constructions and identities to the S_U versions. To do so, first observe that the identities (5-4) imply that $\check{E}$ is an $H^*(BS^1)$–module, with the U–map given by

$$U_{\check{e}} = \begin{bmatrix} \hat{U} & 0 \\ K_p & \overline{U} \end{bmatrix}.$$

With this defined, it is straightforward to check that $\bar{\imath}$ and $\bar{\jmath}$ are $H^*(BS^1)$–morphisms. We can then use what was said in the previous subsection to form the $H_*(S^1)$–modules $(S_U(\check{E}), S_U(\check{e}))$, and morphisms $S_U(\bar{\imath})$ and $S_U(\bar{\jmath})$. Lemma 4.2 ensures that

$$0 \to S_U(\overline{C}) \xrightarrow{S_U(\bar{\imath})} S_U(\check{E}) \xrightarrow{S_U(\bar{\jmath})} S_U(\hat{C}) \to 0$$

is a short exact sequence of $H_*(S^1)$–modules. Meanwhile, the identities (5-4) can be used again to verify that

$$k\check{U} - U_{\check{E}}k + \check{K}_j\check{\partial} + \check{e}\check{K}_j = 0 \quad \text{and} \quad lU_{\check{E}} - \check{U}l + \check{K}_i\check{e} + \check{\partial}\check{K}_i = 0,$$

where

$$\check{K}_j = \begin{bmatrix} -K_j \\ 0 \end{bmatrix} \quad \text{and} \quad \check{K}_i = \begin{bmatrix} 0 & K_i \end{bmatrix}.$$

This means that l and k are both p–morphisms of $H^*(BS^1)$–modules. By Lemma 4.2 we can then form the $H_*(S^1)$–module morphisms $S_U(l)$ and $S_U(k)$. The analogs of

(5-7)–(5-8),

$$S_U(l)S_U(k) = \mathrm{Id}, \quad S_U(j) = S_U(\bar{\jmath}S_U(k), \tag{5-11}$$

now follow readily from the naturality property of S_U described in Lemma 4.2. Meanwhile, the analogs of (5-8) and (5-10),

$$\begin{gathered} S_U(k)S_U(l) = \mathrm{Id} + S_U(\check{e})(K \otimes \jmath) + (K \otimes \jmath)S_U(\check{e}), \\ S_U(k)S_U(i) - S_U(\bar{\imath}) = S_U(\check{e})\mathbb{K} + \mathbb{K}S_U(\bar{\partial}), \quad \mathbb{K} := (K \otimes \jmath)S_U(\bar{\imath}), \end{gathered} \tag{5-12}$$

reduce to the identities

$$K_j\Pi_o - \Pi_u K_p = 0, \quad \Pi_s K_i + K_p \Pi_u = 0, \quad \hat{U}\Pi_u - \Pi_u \bar{U} - K_j i + jK_i = 0,$$

and these can be directly verified. This proves (5-5).

To verify (5-6), we simply apply E_Y° to the S_U version of [17]'s constructions and identities obtained above. Since we have shown that $(S_U(\check{E}), S_U(\check{e}))$, $S_U(\bar{\imath})$, $S_U(\bar{\jmath})$, $S_U(l)$ and $S_U(k)$ are $H_*(S^1)$–morphisms, Lemma 4.7 implies that $(E_Y^\circ S_U(\check{E}), E_Y^\circ S_U(\check{e}))$, $E_Y^\circ S_U(\bar{\imath})$, $E_Y^\circ S_U(\bar{\jmath})$, $E_Y^\circ S_U(l)$ and $E_Y^\circ S_U(k)$ are $H^*(BS^1)$–morphisms, and the analogs of the identities (5-11) and (5-12) follow without much ado by applying E_Y° to them and the naturality properties of E_Y° described in Lemma 4.7. □

5.2 The $\overline{C}_*$ complex and localization

Lemma 5.5 $H_*(S_U(\overline{C})) = 0.$

Proof To compute $H_*(S_U(\overline{C}))$, write

$$S_U(\overline{C}) = \overline{C} \otimes \mathbb{K}[y], \quad S_U(\bar{\partial}) = \bar{\partial} \otimes \jmath + \bar{U} \otimes y.$$

Filtrate this complex by the degree in the factor $\overline{C}$; this is done just as in the proof of Proposition 4.9. The E_1–term is $\overline{\mathrm{HM}}_*$, and d_1 is the u–map on $\overline{\mathrm{HM}}_*$. We claim that this map is invertible, and therefore $H_*(S_U(\overline{C}))$ vanishes.

To see that this is indeed the case, write

$$\overline{C} = C_{\mathbb{T}} \otimes \mathbb{K}[x, x^{-1}], \tag{5-13}$$

where $C_{\mathbb{T}}$ is the Morse complex of a Morse function on the torus of flat connections, which is finitely generated. (See Section 25.6 of [17] for a more thorough discussion of computations of $\overline{\mathrm{HM}}_*$ as well as the relevant moduli spaces.) Recall that a generator $a \otimes x^m$ for $C_{\mathbb{T}} \otimes \mathbb{K}[x, x^{-1}]$ corresponds to the m^{th} eigenvalue of D_a, the Dirac operator with the flat connection, where the eigenvalues are ordered by their value in $\mathbb{R}$, and $1 = x^0$ corresponds to the minimal positive eigenvalue.

The index of the $C_{\mathbb{T}}$ factor defines a finite-length filtration on $\overline{C}$, with respect to which $\overline{U}$ can be written as $\sum_{k=0}^{N} \overline{U}_k$ for some $N \in \mathbb{Z}^{\geq 0}$. However, $\overline{U}_0 = x$ (understood as multiplication), because the only possible contribution to $\overline{U}_0$ comes from the moduli space of instantons from $a \otimes x^m$ to $a \otimes x^{m-1}$, and this consists of the space of gradient flows of the quadratic function $\sum_{m\in\mathbb{Z}} \lambda_m |\xi_m|^2$ on $\mathbb{P}(\mathrm{Span}_{\mathbb{C}}\{\eta_m\}_m)$. Here, η_m denotes a chosen unit-norm eigenvector of λ_m. This moduli space is $\mathbb{C}P^1$. The fact that $\overline{U}_0 = x$ is an invertible operator on $C_{\mathbb{T}} \otimes \mathbb{K}[x, x^{-1}]$ then means that $\overline{U}$ is invertible as well. $\square$

It follows from the preceding lemma and Lemma 5.4 that $S_U(j)$ induces an $H_*(S^1)$–module isomorphism from $H_*(S_U(\check{C}))$ to $H_*(S_U(\widehat{C}))$.

Definition 5.6 (see [23, equation (5.6)]) We call the following group the "total" version of monopole Floer homology:

$$\widetilde{\mathrm{HM}}_* := H_*(S_U(\widehat{C})) \simeq H_*(S_U(\check{C})).$$

The motivation for this definition comes from the theory of S^1–equivariant theory; it is related to the equivariant versions of Floer homologies $\widehat{\mathrm{HM}}$, $\widecheck{\mathrm{HM}}$ and $\overline{\mathrm{HM}}$ by properties expected of the homology of their corresponding S^1–space. (The choice of the accent $\sim$ in the notation reflects the fact that this is supposed to come from the space of framed configurations, in accordance with the notation (5.1.1) in [8].) In particular, the following lemma is a consequence of Proposition 4.9(a)(ii) and Remark 4.10:

Lemma 5.7 *$\widetilde{\mathrm{HM}}_*$ is related to $\widehat{\mathrm{HM}}_*$ by the long exact sequence*

$$\cdots \to \widehat{\mathrm{HM}}_* \xrightarrow{U} \widehat{\mathrm{HM}}_{*-2} \to \widetilde{\mathrm{HM}}_* \to \widehat{\mathrm{HM}}_{*-1} \to \cdots. \tag{5-14}$$

The following lemma is invoked in the next subsection:

Lemma 5.8 (localization) *Let $\widehat{C}$, $\overline{C}$, $\widehat{\mathrm{HM}}$ and $\overline{\mathrm{HM}}$ denote the monopole Floer complexes or homologies for a balanced perturbation. Then*:

(a) *The map i_{V*}: $H_*(E_Y^- S_U(\overline{C})) \to H_*(E_Y^\infty S_U(\overline{C}))$ is an isomorphism.*

(b) *The map p_* induces an isomorphism of $\mathbb{K}[u, u^{-1}]$–modules,*

$$p_*\colon \widehat{\mathrm{HM}}_* \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}] \to \overline{\mathrm{HM}}_* \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}].$$

Proof (a) By Proposition 4.9, it is equivalent to consider the localization map $H_*(\overline{C}) \to H_*(\overline{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}])$. However, we saw in the proof of Lemma 5.5 that the u–action is invertible on $H_*(\overline{C})$.

(b) $\operatorname{Tor}(\mathbb{K}[u], \mathbb{K}[u, u^{-1}]) = 0$, so we can work at the chain level:

$$H_*(\widehat{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}]) = H_*((\widehat{U}^N \widehat{C}) \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}])$$

for any $N \in \mathbb{Z}^{\geq 0}$. There are finitely many irreducible Seiberg–Witten solutions; and, with a balanced perturbation, the Seiberg–Witten action functional is real-valued. We can therefore order these finitely many irreducibles by their values of action functional. A nonconstant Seiberg–Witten instanton always decreases the actions unless it is reducible, so, for sufficiently large N, $\widehat{U}^N \widehat{C} \subset C^u$.

Meanwhile, we saw in (5-13) that $C^u = C_{\mathbb{T}} \otimes (x\mathbb{K}[x])$ and $C^s = C_{\mathbb{T}} \otimes \mathbb{K}[x^{-1}]$. We also saw in the proof of Lemma 5.5 that $\overline{U}_0 = x$. Therefore, C^u generates $\overline{C} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}]$. This understood, the assertion follows because we can restrict our attention to C^u and the $u - u$ component of p is the identity. □

5.3 Monopole Floer homologies from twisted tensor products

The modules $S_U(C)$ and $E_Y(C)$ are "twisted tensor products" (in the sense of eg [40; 26]), on which $H^*(BS^1)$ and $H_*(S^1)$ respectively act by simple multiplications. On the other hand, the duality theorem, Proposition 4.9, tells us the following: On the homological level, we can replace any $H^*(BS^1)$– or $H_*(S^1)$–modules by such twisted tensor products by applying $E_Y^- S_U$ or $S_U E_Y^-$, respectively. We shall reformulate the monopole Floer homologies $\widehat{\mathrm{HM}}_*$, $\widecheck{\mathrm{HM}}_*$ and $\overline{\mathrm{HM}}_*$ defined in [17] accordingly. In addition to these three flavors of monopole Floer homologies, we will introduce a fourth flavor, $\widetilde{\mathrm{HM}}_*$, from this point of view. These four flavors of monopole Floer homologies will be regarded as a system and denoted collectively by $\mathring{\mathrm{HM}}_*$ below. We call $\widehat{\mathrm{HM}}_*$, $\widecheck{\mathrm{HM}}_*$, $\overline{\mathrm{HM}}_*$ and $\widetilde{\mathrm{HM}}_*$ the *from, to, bar and total* versions of monopole Floer homology, respectively. Just as $\widehat{\mathrm{HM}}_*$, $\widecheck{\mathrm{HM}}_*$ and $\overline{\mathrm{HM}}_*$ are to be viewed as versions of equivariant homologies of the equivariant Seiberg–Witten Floer stable homotopy type (represented by a pointed S^1–space) $\mathrm{SWF}(Y, \mathfrak{c})$ that is introduced in [27], what we denoted by $\widetilde{\mathrm{HM}}_*$ can be viewed as the (nonequivariant) homology of $\mathrm{SWF}(Y, \mathfrak{c})$ itself.

We now state the main result of this subsection:

Proposition 5.9 [23, Corollary 5.3] *Let $\widehat{C}$ denote $\widehat{C}(M,\mathfrak{s},c_b)$ and $\overset{\circ}{\mathrm{HM}}$ denote $\overset{\circ}{\mathrm{HM}}(M,\mathfrak{s},c_b)$. There is a system of isomorphisms (as $H^*(BS^1)$–modules) from $H_*(E_Y^\circ S_U(\widehat{C}))$ to $\overset{\circ}{\mathrm{HM}}_*$, taking the fundamental exact sequence of equivariant homologies for $S_U(\widehat{C})$ to the fundamental exact sequence of monopole Floer homologies. In particular, we have the following commutative diagram of $H^*(BS^1)$–modules:*

$$\begin{tikzcd}
\cdots H_*(E_Y^- S_U(\widehat{C})) \arrow[r, "i_{V*}"] \arrow[d] & H_*(E_Y^\infty S_U(\widehat{C})) \arrow[r] \arrow[d] & H_*(E_Y^+ S_U(\widehat{C})) \arrow[r, "\delta_{V*}"] \arrow[d] & H_{*-1}(E_Y^- S_U(\widehat{C}))\cdots \arrow[d] \\
\cdots\widehat{\mathrm{HM}}_* \arrow[r, "p_*"] & \overline{\mathrm{HM}}_{*-1} \arrow[r, "i_*"] & \widetilde{\mathrm{HM}}_{*-1} \arrow[r, "j_*"] & \widehat{\mathrm{HM}}_{*-1}\cdots
\end{tikzcd} \tag{5-15}$$

where the vertical arrows are $H^*(BS^1)$–module-isomorphisms.

Proof of Proposition 5.9 Regard $\overset{\circ}{C}$ as $\mathbb{Z}$–graded complexes as prescribed in Remark 4.11 and consider the following diagram, denoted by $\mathfrak{D}$ below:

$$\begin{tikzcd}
{} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] & {} \arrow[d, "E_YS_U(j)"] \\
\cdots H_{*+2}(E_Y^+ S_U(\widehat{C})) \arrow[r] \arrow[d, "E_YS_U(p)"] & H_{*+1}(E_Y^- S_U(\widehat{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(p)"] & H_{*+1}(E_Y^\infty S_U(\widehat{C})) \arrow[r] \arrow[d, "E_YS_U(p)", "\simeq"'] \arrow[dl, dotted, "\bar{h}"'] & H_{*+1}(E_Y^+ S_U(\widehat{C}))\cdots \arrow[d, "E_YS_U(p)"] \\
\cdots H_{*+1}(E_Y^+ S_U(\overline{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^- S_U(\overline{C})) \arrow[r, "i_{V*}", "\simeq"'] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^\infty S_U(\overline{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^+ S_U(\overline{C}))\cdots \arrow[d, "E_YS_U(i)"] \\
\cdots H_{*+1}(E_Y^+ S_U(\check{C})) \arrow[r, "\delta_{V*}", "\simeq"'] \arrow[d, "E_YS_U(j)", "\simeq"'] & H_*(E_Y^- S_U(\check{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(j)"] & H_*(E_Y^\infty S_U(\check{C})) \arrow[r] \arrow[d, "E_YS_U(j)"] & H_*(E_Y^+ S_U(\check{C}))\cdots \arrow[d, "E_YS_U(j)", "\simeq"'] \\
\cdots H_{*+1}(E_Y^+ S_U(\widehat{C})) \arrow[r] \arrow[d, "E_YS_U(p)"] \arrow[ur, dotted, "\check{h}"'] & H_*(E_Y^- S_U(\widehat{C})) \arrow[r, "i_{V*}"] \arrow[d, "E_YS_U(p)"] & H_*(E_Y^\infty S_U(\widehat{C})) \arrow[r] \arrow[d, "E_YS_U(p)", "\simeq"'] & H_*(E_Y^+ S_U(\widehat{C}))\cdots \arrow[d, "E_YS_U(p)"] \\
\cdots H_{*+1}(E_Y^+ S_U(\overline{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^- S_U(\overline{C})) \arrow[r, "i_{V*}", "\simeq"'] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^\infty S_U(\overline{C})) \arrow[r] \arrow[d, "E_YS_U(i)"] & H_*(E_Y^+ S_U(\overline{C}))\cdots \arrow[d, "E_YS_U(i)"] \\
{} & {} & {} & {}
\end{tikzcd}$$

All rows and columns above are exact sequences of $H^*(BS^1)$–modules: the rows are fundamental exact sequences of equivariant homologies of $S_U(\widehat{C})$, $S_U(\overline{C})$ and $S_U(\check{C})$, and the columns are the exact sequences from (5-6).

By Proposition 4.9, the exact sequence in the second column is isomorphic to the first fundamental exact sequence of the monopole Floer homologies (5-1), namely the second row in (5-15). Therefore we shall henceforth replace the second column by

$$\cdots \xrightarrow{j_*} \widehat{\mathrm{HM}}_* \xrightarrow{p_*} \overline{\mathrm{HM}}_* \xrightarrow{i_*} \widecheck{\mathrm{HM}}_* \xrightarrow{j_*} \widehat{\mathrm{HM}}_{*-1} \xrightarrow{p_*} \cdots .$$

Our goal is therefore to construct an isomorphism from the exact sequence in the first or fourth row to the exact sequence in the second column:

$$\begin{tikzcd}
\cdots H_{*+1}(E_Y^- S_U(\hat{C})) \arrow[r, "i_{V*}"] \arrow[d, "\hat{}"'] & H_{*+1}(E_Y^{\infty} S_U(\hat{C})) \arrow[r] \arrow[d, "\bar{h}"'] & H_{*+1}(E_Y^+ S_U(\hat{C}))\cdots \arrow[d, "\check{h}"'] \\
\cdots H_{*+1}(E_Y^- S_U(\hat{C})) \arrow[r, "E_Y S_U(p)"] & H_*(E_Y^- S_U(\bar{C})) \arrow[r, "E_Y S_U(i)"] & H_*(E_Y^- S_U(\check{C}))\cdots
\end{tikzcd}$$

To see this, note that in the third column of $\mathfrak{D}$, the map

$$E_Y S_u(p)\colon H_*(E_Y^{\infty} S_U(\hat{C})) \to H_*(E_Y^{\infty} S_U(\bar{C}))$$

is an isomorphism by the preceding lemma and Proposition 4.9. Thus, $H_*(E_Y^{\infty} S_U(\check{C}))$ is trivial. This in turn implies that the map

$$\delta_{V*}\colon H_{*+1}(E_Y^+ S_U(\check{C})) \to H_*(E_Y^- S_U(\check{C}))$$

in the third row of $\mathfrak{D}$ is an isomorphism. For the same reasons, the map

$$i_{V_*}\colon H_*(E_Y^- S_U(\bar{C})) \to H_*(E_Y^{\infty} S_U(\bar{C}))$$

on the first and fifth rows of $\mathfrak{D}$ is an isomorphism as well, and thus $H_*(E_Y^+ S_U(\bar{C}))$ is trivial too. This in turn implies that the map

$$E_Y S_U(j)\colon H_*(E_Y^+ S_U(\check{C})) \to H_*(E_Y^+ S_U(\hat{C}))$$

on the first and fourth columns is an isomorphism. We now take

$$\hat{h} = \mathrm{Id}, \quad \bar{h} = i_{V*}^{-1} \circ E_Y S_U(p), \quad \check{h} = \delta_{V_*} \circ (E_Y S_U(j))^{-1},$$

and the proposition follows. □

Remark 5.10 The chain complex $\hat{C}$ in the statement of the preceding proposition may be replaced by $\mathrm{CM}(M, \mathfrak{s}, c_-)$ (yet $\mathring{\mathrm{HM}}$ still stands for $\mathring{\mathrm{HM}}(M, \mathfrak{s}, c_b)$). When $c_(\mathfrak{s})$ is nontorsion, this follows from Lemmas 4.5 and 4.8 and Theorem 5.1. When $c_1(\mathfrak{s})$ is torsion, this is simply because $\mathrm{CM}(M, \mathfrak{s}, c_-) = \mathrm{CM}(M, \mathfrak{s}, c_b)$. As we remarked previously in Section 2.1, in this case, monotone, balanced and exact perturbations are identical notions.

6 Monopole Floer homology under connected sum

We follow the (by now) traditional approach to connected sum formula for Floer homologies that appeared in the instanton Floer homology setting in [9; 7]. To proceed, some setting-up is required.

6.1 Preparations

Let M_1 and M_2 be closed, oriented, connected 3–manifolds, and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be Spin^c–structures on M_1 and M_2, respectively. Denote by $M_\sqcup = M_1 \sqcup M_2$ the disjoint union of M_1 and M_2. Let $\mathfrak{s}_\sqcup = (\mathfrak{s}_1, \mathfrak{s}_2)$ denote the Spin^c–structure on $M_\sqcup$ given by $\mathfrak{s}_1$ and $\mathfrak{s}_2$.

Part 1: Spin^c–structures and gradings Recall from [17] the interpretation of Spin^c–structures and grading via oriented 2–plane fields on the 3–manifold M. Denote by $\mathbb{J}(M)$ the set of homotopy classes of oriented 2–plane fields on the 3–manifold M. According to Proposition 23.1.8 of [17], this may be identified with the set of gradings of the manifold M, as defined in [17, page 424]. There is a $\mathbb{Z}$–action on $\mathbb{J}(M)$, defined by modifying a representing plane field in a ball in M [17, Definition 3.1.2]. Its quotient is the set of Spin^c–structures over M, $\mathrm{Spin}(M) = \mathbb{J}(M)/\mathbb{Z}$. The orbit over $\mathfrak{s} \in \mathrm{Spin}(M)$ is the set of gradings for the Spin^c–structure $\mathfrak{s}$, which we denote by $\mathbb{J}(M, \mathfrak{s})$. Let $c_\mathfrak{s}$ be the divisibility of $c_1(\mathfrak{s})$. The stabilizer of the orbit $\mathbb{J}(M, \mathfrak{s})$ is $c_\mathfrak{s}\mathbb{Z}$; therefore $\mathbb{J}(M, \mathfrak{s})$ is a torsor under $\mathbb{Z}/c_\mathfrak{s}\mathbb{Z}$.

Let $B(p_1)$ and $B(p_2)$ be respectively open balls centered at $p_1 \in M_1$ and $p_2 \in M_2$, and $\varphi\colon B(p_1)\setminus\{p_1\} \to B(p_2)\setminus\{p_2\}$ be an orientation-reversing map such that

$$M_\# := M_1 \# M_2 := (M_1 \setminus \{p_1\} \cup M_2 \setminus \{p_2\})/\sim_\varphi. \tag{6-1}$$

As described in [17], the $\mathbb{Z}$–action on $\mathbb{J}(M_i)$ is induced from the $C^0(Y_i, \mathrm{SO}(3))$–action on the space of plane fields on M_i. Each element in the group $\mathbb{Z}$ is represented by an even-degree element in $C^0(M_i, \mathrm{SO}(3))$ sending $M_i \setminus B(p_i)$ to $1 \in \mathrm{SO}(3)$. Since this map has even degree, we may choose it to send $p_i \in B(p_i)$ to 1 as well. The orientation-reversing map φ then defines an isomorphism

$$\iota_\mathbb{J}\colon (\mathbb{J}(M_1) \times \mathbb{J}(M_2))/\bar{\Delta} \to \mathbb{J}(M_\#),$$

where $\bar{\Delta} \subset \mathbb{Z}\times\mathbb{Z}$ denotes the antidiagonal. This isomorphism is equivariant with respect to the residual $\mathbb{Z}$–action on $\iota_\mathbb{J}\colon (\mathbb{J}(M_1) \times \mathbb{J}(M_2))/\bar{\Delta}$ and the $\mathbb{Z}$–action of $\mathbb{J}(M_\#)$.

Thus, by taking the quotient on both sides above, one has an induced isomorphism

$$\iota_S\colon \mathrm{Spin}(M_1) \times \mathrm{Spin}(M_2) \to \mathrm{Spin}(M_\#).$$

Let $\mathfrak{s}_\# = \iota_S(\mathfrak{s}_1, \mathfrak{s}_2)$.

Note that restrictions of $\iota_{\mathbb{J}}$ to orbits of the $\mathbb{Z}$–actions give rise to isomorphisms (also denoted by $\iota_{\mathbb{J}}$)

$$\iota_{\mathbb{J}}\colon (\mathbb{J}(M_1, \mathfrak{s}_1) \times \mathbb{J}(M_2, \mathfrak{s}_2))/\bar{\Delta} \to \mathbb{J}(M_\#, \mathfrak{s}_\#)$$

as affine spaces under $\mathbb{Z}/c_\#\mathbb{Z}$, where $c_\#$ is the gcd of $c_{\mathfrak{s}_1}$ and $c_{\mathfrak{s}_2}$.

Recall from Part 4 of Section 2.4 the definition of $(\widehat{C}(M_\sqcup), \widehat{\partial}_{M_\sqcup})$ as a product complex of $\widehat{C}(M_1)$ and $\widehat{C}(M_2)$. Through out this section, we adopt the same assumption that the Floer complex $\mathring{C}(M_1)$ comes from a *nonbalanced* perturbation; in particular, $\widehat{C}(M_1) = CM(M_1)$ in our notation. As observed there, this assumption implies that $\mathring{C}(M_\#)$ is also associated with a nonbalanced perturbation, and $\widehat{C}(M_\#) = CM(M_\#)$ as well. Given an $\mathfrak{s}_\sqcup = (\mathfrak{s}_1, \mathfrak{s}_2) \in \mathrm{Spin}(M_\sqcup) \simeq \mathrm{Spin}(M_1) \times \mathrm{Spin}(M_2)$, we use $\mathbb{J}(M_\sqcup, \mathfrak{s}_\sqcup)$ to denote the $\mathbb{Z}_{c_\#}$–grading $(\mathbb{J}(M_1) \times \mathbb{J}(M_2))/\bar{\Delta}$ on $(\widehat{C}(M_\sqcup, \mathfrak{s}_\sqcup), \widehat{\partial}_{M_\sqcup})$ induced from the natural bigrading of the latter as a product complex.

Remark 6.1 The canonical $\mathbb{Z}/2\mathbb{Z}$–grading [17, Section 22.4] of $\iota_{\mathbb{J}}(\xi_1, \xi_2)$ differs from the sum of the canonical $\mathbb{Z}/2\mathbb{Z}$–grading of ξ_1 and ξ_2 by 1.

Part 2: $A_\dagger$–actions on $\widehat{C}(M_\sqcup)$ and CM($M_\#$) Use the connected sum decomposition of $M_\#$, (6-1), to define a splitting

$$H_1(M_\#; \mathbb{Z}) \simeq H_1(M_1; \mathbb{Z}) \oplus H_1(M_2; \mathbb{Z}),$$

and, correspondingly, a factorization of the algebra

$$\begin{aligned} A_\dagger(M_\#) &\simeq A_\dagger(M_1) \otimes_{\mathbb{K}[u]} A_\dagger(M_2) \\ &\simeq \mathbb{K}[U] \otimes \textstyle\bigwedge^*(H_1(M_1; \mathbb{Z})/\mathrm{Tors}) \otimes \textstyle\bigwedge^*(H_1(M_2; \mathbb{Z})/\mathrm{Tors}). \end{aligned} \tag{6-2}$$

This factorization is used to identify $A_\dagger(M_\#)$ with

$$\begin{aligned} A_\dagger(M_\sqcup) &:= \mathbb{K}[U] \otimes \textstyle\bigwedge^*(H_1(M_\sqcup; \mathbb{Z})/\mathrm{Tors}) \\ &= \mathbb{K}[U] \otimes \textstyle\bigwedge^*(H_1(M_1; \mathbb{Z})/\mathrm{Tors}) \otimes \textstyle\bigwedge^*(H_1(M_2; \mathbb{Z})/\mathrm{Tors}). \end{aligned}$$

(Note that $A_\dagger(M_\sqcup) \neq H^*(\mathcal{B}^\sigma(M_\sqcup); \mathbb{Z}) = A_\dagger(M_1) \otimes_{\mathbb{Z}} A_\dagger(M_2)$ with $A_\dagger(M_\sqcup)$ so defined.)

Recall the description of $\boldsymbol{A}_\dagger(M)$–actions on monopole Floer complex from Section 2.5's Part 2 and Remark 2.5. These depend on the choice of a point $p \in M$ for the U–map, and a circle $\gamma \subset M$ representing $\mathfrak{t}$ for each element $\mathfrak{t} \in \{\mathfrak{t}_i\}_i$, the latter being a basis of $H_1(M;\mathbb{Z})/\mathrm{Tors}$. We denote the associated maps on the Floer complex by $\mathring{\mathfrak{m}}_U = \mathring{U}_p = \mathring{U}$ or $\mathring{\mathfrak{m}}_\gamma = \mathring{\mathfrak{m}}_\mathfrak{t}$, depending on emphasis and context. The choices for the manifolds discussed in this section, $M_* = M_1$, M_2, $M_\#$ or $M_\sqcup$, are given as follows.

For the $H_1(M_*;\mathbb{Z})/\mathrm{Tors}$–actions, choose $b_1(M_1)$ mutually disjoint embedded circles $\gamma_i^{[1]}$ in M_1 so that $p_1 \notin \bigcup_i \gamma_i^{[1]}$ and $\{\mathfrak{t}_i^{[1]} = [\gamma_i^{[1]}]\}_i$ forms a basis of $H_1(M_1;\mathbb{Z})/\mathrm{Tors}$. Choose similarly $b_1(M_2)$ mutually disjoint embedded circles $\gamma_j^{[2]}$ in M_2 so that $p_2 \notin \bigcup_j \gamma_j^{[2]}$ and $\{\mathfrak{t}_j^{[2]} = [\gamma_j^{[2]}]\}_j$ forms a basis of $H_1(M_2;\mathbb{Z})/\mathrm{Tors}$. Use $\{\gamma_i^{[1]}\}_i$, $\{\gamma_j^{[2]}\}_j$ and $\{\gamma_i^{[1]}\}_i \cup \{\gamma_j^{[2]}\}_j$ to define the $H_1(M_*;\mathbb{Z})/\mathrm{Tors}$–actions on $\widehat{C}(M_*)$ respectively for $M_* = M_1, M_2, M_\sqcup$. For $M_* = M_\#$, regard all the $\gamma_i^{[1]}$'s and $\gamma_j^{[2]}$'s as embedded circles in $M_\#$ through (6-1), and use them to define the $H_1(M_\#;\mathbb{Z})/\mathrm{Tors}$–action on $CM(M_\#)$.

The $U \in \boldsymbol{A}_\dagger(M_*)$–actions for $M_* = M_1$, $M_\sqcup$ or $M_\#$ are given as follows. Choose a point $p \in M_1$ disjoint from $\{p_1\} \cup \{\gamma_i^{[1]}\}_i$. This p can also be viewed as a point in $M_* = M_1$ or $M_\sqcup$, or a point in $M_* = M_\#$ via (6-1). We use $\widehat{U}_p = \widehat{U}_p(M_*)$ to denote the associated U–action on the monopole chain complex $\widehat{C}(M_*)$ for such M_*. Note that $\widehat{U}_p(M_\sqcup) = U_p(M_1) \otimes 1$ on the product complex $\widehat{C}(M_\sqcup) = CM(M_1) \otimes \widehat{C}(M_2)$.

Remark 6.2 As already noted, $\widehat{U}_p$ and $\widehat{U}_{p'}$ induce different U–actions on $\widehat{\mathrm{HM}}(M_\sqcup)$. This is however irrelevant for our purposes, namely deriving and applying the connected sum formulas, Propositions 6.7 and 6.11. See Lemma 6.4 below. We choose p to be on M_1 because $CM(M_1)$ is assumed to be associated with a nonbalanced perturbation, and for our application this perturbation is of the type discussed in Part 4 of Section 2.5, where U_p has a nice geometric interpretation.

Part 3: $\boldsymbol{A}_\dagger(M_\sqcup)$–actions on $S_{U_\sqcup}\widehat{C}_*(M_\sqcup)$ Let $\widehat{U}_\sqcup = \widehat{U}_{p_2-p_1}$ be as in (2-57). (To simplify notation, we shall frequently drop the hat from $\widehat{U}_\sqcup$; that is, $U_\sqcup := \widehat{U}_\sqcup$ in what follows.) The statement of the upcoming connected sum theorem relates $CM_*(M_\#)$ with $S_{U_\sqcup}(\widehat{C}_*(M_\sqcup))$; the Floer complex is obtained by applying Section 4.2's S_U operation to $\widehat{C}_*(M_\sqcup)$, with the latter regarded as an $H^*(BS^1)$–module generated by the $U = \widehat{U}_\sqcup$–action above.

Abstractly, an $\boldsymbol{A}_\dagger(M_\sqcup)$–action on $\widehat{C}_*(M_\sqcup)$ can be used to define a corresponding $\boldsymbol{A}_\dagger(M_\sqcup)$–action on $S_{U_\sqcup}\widehat{C}_*(M_\sqcup)$, due to the following observations: Given any generator $Q = U$ or $\mathfrak{t}_i$ of $\boldsymbol{A}_\dagger(M_\sqcup)$ and any map $\hat{\mathfrak{m}}_Q$ on $\widehat{C}(M_\sqcup, \mathfrak{s}_\sqcup)$ underlying the

Q–action on the Floer homology, $\hat{\mathfrak{m}}_Q$ is a p–morphism from $\hat{C}(M_\sqcup, \mathfrak{s}_\sqcup)$ to itself in the language of Section 4. This in turn is because $\hat{\mathfrak{m}}_Q$ and $\hat{U}_\sqcup$ induce commutative maps on Floer homology. Thus, by Lemma 4.2, $S_{U_\sqcup}(\hat{\mathfrak{m}}_Q)$ is defined (albeit nonuniquely), and is an $H_*(S^1)$–morphism from $S_{U_\sqcup}(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup))$ to itself.

With the $\boldsymbol{A}_\dagger(M_\sqcup)$–action on $\hat{C}_*(M_\sqcup)$ fixed in the previous part, we define the Q–action on $S_{U_\sqcup}\hat{C}(M_\sqcup, \mathfrak{s}_\sqcup)$ to be the map $S_{U_\sqcup}(\hat{\mathfrak{m}}_Q)$. In the notation of (4-5), these take the form

$$S_{U_\sqcup}(\hat{U}_p) = \hat{U}_p \otimes 1 + \hat{K}_{\hat{U}_p} \otimes y, \quad S_{U_\sqcup}(\hat{\mathfrak{m}}_\gamma) = \hat{\mathfrak{m}}_\gamma \otimes \jmath + \hat{K}_{\mathfrak{m}_\gamma} \otimes y,$$

where $\hat{K}_{\mathfrak{m}_Q}$ are chain homotopy maps satisfying (4-3). In other words, they satisfy

$$\begin{aligned} [\hat{U}_p, \hat{U}_\sqcup] &= \hat{\partial}(M_\sqcup)\hat{K}_{U_p} + \hat{K}_{U_p}\hat{\partial}(M_\sqcup), \\ [\hat{\mathfrak{m}}_\gamma, \hat{U}_\sqcup] &= \hat{\partial}(M_\sqcup)\hat{K}_{\mathfrak{m}_\gamma} - \hat{K}_{\mathfrak{m}_\gamma}\hat{\partial}(M_\sqcup). \end{aligned} \tag{6-3}$$

As previously noted, the choice of the chain homotopy maps $\hat{K}_{\mathfrak{m}_Q}$ is not unique. In fact, it was observed in Remark 4.3 that even the homology $H_*(S_{U_\sqcup}\mathfrak{m}_Q)$ depends on the choice of $\hat{K}_{\mathfrak{m}_Q}$ (modulo homotopy). In this article we adopt a particular choice of these $\hat{K}_{\mathfrak{m}_Q}$ that suits our purposes best and has certain nice properties. In particular, the homotopy class of $\hat{K}_{U_p}$ or $\hat{K}_{\mathfrak{m}_\gamma}$ varies in a consistent manner with p or γ, leading to the desired invariance result, Corollary 6.5 below.

To describe these particular choices of $\hat{K}_{\mathfrak{m}_Q}$, first recall the notions $\mathring{n}[\mathrm{u}]$, $\underline{\mathrm{h}}_{\hat{p}}$, u_γ and $\hat{\Theta}_p$ from Section 2.5. Fixing a set of choices for p and the γ_i from Part 2, we set

$$\hat{K}_{U_p} := \hat{n}[d\underline{\mathrm{h}}_\sqcup \wedge d\underline{\mathrm{h}}_{\hat{p}}] + [\hat{n}[d\underline{\mathrm{h}}_\sqcup], \hat{\Theta}_p] + [\hat{\Theta}_\sqcup, \hat{U}_\sqcup] = \hat{n}[\mathrm{u}_\sqcup \wedge \mathrm{u}_p], \tag{6-4}$$

where $\mathrm{u} := \mathrm{u}_{p_2} - \mathrm{u}_{p_1}$. For each $\gamma \in \{\gamma_i^{[1]}\}_i \cup \{\gamma_i^{[2]}\}_j$, set

$$\hat{K}_{\mathfrak{m}_\gamma} := \hat{n}[\mathrm{u}_\gamma d\underline{\mathrm{h}}_\sqcup] + [\hat{\mathfrak{m}}_\gamma, \hat{\Theta}_\sqcup] = \hat{n}[\mathrm{u}_\gamma \mathrm{u}_\sqcup]. \tag{6-5}$$

We now verify that:

Lemma 6.3 *The maps $\hat{K}_{U_p}$ and $\hat{K}_{\mathfrak{m}_\gamma}$ given in* (6-4) *and* (6-5) *above satisfy the identities* (6-3).

Proof (i) To verify that (6-4) satisfies the first identity in (6-3), let $\mathcal{N}^+ = (\mathcal{N}^+)_3$ be a 3–dimensional stratified submanifold of $\mathcal{N}_3^+(M_\sqcup)$. (Recall the notation $\mathcal{N}_k(M)$ and $\mathcal{N}_k^+(M)$ from Section 2.5. Recall also that $(\mathcal{N}^+)_k$ stands for the k^{th} step in the stratification, $\varnothing \subset \cdots \subset (\mathcal{N}^+)_k \subset \cdots \subset \mathcal{N}^+$, of $\mathcal{N}^+$.) Since both $d\underline{\mathrm{h}}_p$ and $d\underline{\mathrm{h}}_\sqcup$ are closed forms, by [17, Lemma 21.3.1, equation (21.4) and Theorem 19.5.4],

$$0 = \langle d(d\underline{\mathrm{h}}_\sqcup \wedge d\underline{\mathrm{h}}_{\hat{p}}), [\mathcal{N}^+]\rangle = \langle d\underline{\mathrm{h}}_\sqcup \wedge d\underline{\mathrm{h}}_{\hat{p}}, \partial[\mathcal{N}^+]\rangle = \langle d\underline{\mathrm{h}}_\sqcup \wedge d\underline{\mathrm{h}}_{\hat{p}}, (\mathcal{N}^+)_2\rangle.$$

By construction, $(\mathcal{N}^+)_2$ is a union of two types of product spaces, the first being of the form $\mathcal{N}_0^+(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_2^+(\mathfrak{c}, \mathfrak{c}_+)$ or $\mathcal{N}_2^+(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_0^+(\mathfrak{c}, \mathfrak{c}_+)$, and the second of the form $\mathcal{N}_1^+(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_1^+(\mathfrak{c}, \mathfrak{c}_+)$. Integrals of $d\underline{\mathrm{h}}_{\hat{p}} \wedge d\underline{\mathrm{h}}_\sqcup$ over these two subspaces of $(\mathcal{N}^+)_2$ give respectively the first and the second term of the right-hand side of the identity

$$0 = [\hat{\partial}, \hat{n}[d\underline{\mathrm{h}}_\sqcup \wedge d\underline{\mathrm{h}}_{\hat{p}}]] - [\hat{n}[d\underline{\mathrm{h}}_{\hat{p}}], \hat{n}[d\underline{\mathrm{h}}_\sqcup]]. \tag{6-6}$$

By (2-36), $\hat{n}[d\underline{\mathrm{h}}_{\hat{p}}] = \hat{U}_p - [\hat{\partial}, \hat{\Theta}_p]$, and similarly for $\hat{n}[d\underline{\mathrm{h}}_\sqcup]$. Together with the fact that $\hat{U}_p, \hat{U}_\sqcup$ are both chain maps, this implies

$$\begin{aligned}
[\hat{n}[d\underline{\mathrm{h}}_{\hat{p}}], \hat{n}[d\underline{\mathrm{h}}_\sqcup]] &= [\hat{U}_p, \hat{U}_\sqcup] - [[\hat{\partial}, \hat{\Theta}_p], \hat{n}[d\underline{\mathrm{h}}_\sqcup]] + [\hat{U}_p, [\hat{\partial}_\sqcup, \hat{\Theta}_\sqcup]] \\
&= [\hat{U}_p, \hat{U}_\sqcup] - \big[\hat{\partial}, [\hat{\Theta}_p, \hat{n}[d\underline{\mathrm{h}}_\sqcup]]\big] - [\hat{\partial}_\sqcup, [\hat{\Theta}_\sqcup, \hat{U}_p]].
\end{aligned}$$

Inserting this back into (6-6), the first line of (6-3) follows readily.

(ii) The second identity in (6-3) is verified using similar arguments. Take now $\mathcal{N}^+ = (\mathcal{N}^+)_2$ to be a 2–dimensional stratified submanifold of $\mathcal{N}_2^+(M_\sqcup)$. The coefficients in $[\hat{\partial}, \hat{n}[\mathrm{u}_\gamma\, d\underline{\mathrm{h}}_\sqcup]]$ are given by terms of the form

$$\langle \mathrm{u}_\gamma d\underline{\mathrm{h}}_\sqcup, \partial[\mathcal{N}^+]\rangle = \langle \mathrm{u}_\gamma d\underline{\mathrm{h}}_\sqcup, (\mathcal{N}^+)_1\rangle,$$

where $(\mathcal{N}^+)_1$ is a union of product spaces of either the form

$$\mathcal{N}_0^+(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_1^+(\mathfrak{c}, \mathfrak{c}_+) \quad \text{or} \quad \mathcal{N}_1^+(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_0^+(\mathfrak{c}, \mathfrak{c}_+).$$

Since u_γ is locally constant, integrals of $\mathrm{u}_\gamma d\underline{\mathrm{h}}_\sqcup$ over these spaces take the form of products

$$\langle \mathrm{u}_\gamma, \mathcal{N}_0^+(\mathfrak{c}_-, \mathfrak{c})\rangle\langle d\underline{\mathrm{h}}_\sqcup, \mathcal{N}_1^+(\mathfrak{c}, \mathfrak{c}_+)\rangle \quad \text{or} \quad \langle d\underline{\mathrm{h}}_\sqcup, \mathcal{N}_1^+(\mathfrak{c}_-, \mathfrak{c})\rangle\langle \mathrm{u}_\gamma, \mathcal{N}_0^+(\mathfrak{c}, \mathfrak{c}_+)\rangle.$$

By (2-37) and (2-36), this shows that

$$[\hat{\partial}, \hat{n}[\mathrm{u}_\gamma\, d\underline{\mathrm{h}}_\sqcup]] = [\mathfrak{m}_\gamma, \hat{U}_\sqcup - [\hat{\partial}, \hat{\Theta}_\sqcup]] = [\mathfrak{m}_\gamma, \hat{U}_\sqcup] - [\hat{\partial}, [\mathfrak{m}_\gamma, \hat{\Theta}_\sqcup]],$$

leading directly to the second identity of (6-3). □

This understood, we may now justify the claim in Remark 2.5 that the U–action on $H_*(S_{U_\sqcup}(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup))$ is independent of p.

Lemma 6.4 *For any given $p, p' \in M_\sqcup$, $S_{U_\sqcup}(\hat{U}_{p'})$ and $S_{U_\sqcup}(\hat{U}_p)$ are chain homotopic.*

Proof We wish to show that there is a map $Z_*\colon S_{U_\sqcup}(\widehat{C}) \to S_{U_\sqcup}(\widehat{C})$ such that

$$S_{U_\sqcup}(\widehat{U}_{p'}) - S_{U_\sqcup}(\widehat{U}_p) = (\widehat{U}_{p'} - \widehat{U}_p) \otimes 1 + (\widehat{K}_{U_p'} - \widehat{K}_{U_p}) \otimes y = [Z_*, D_\sqcup].$$

We choose Z_* to be of the form $Z_* = Z_0 \otimes \jmath + Z_1 \otimes y$, with Z_0 and Z_1 being maps from $\widehat{C}(M_\sqcup)$ to itself, respectively of degree -1 and -2. The preceding identity now reads

$$\widehat{U}_{p'} - \widehat{U}_p = [Z_0, \widehat{\partial}], \quad \widehat{K}_{U_p'} - \widehat{K}_{U_p} = [Z_1, \widehat{\partial}] + [\widehat{U}_\sqcup, Z_0]. \tag{6-7}$$

(i) In the case when p and p' belong to the same connected component of $M_\sqcup$, there is a path λ in $M_\sqcup$ from p to p' and an associated map $\widehat{\mathrm{K}}_\lambda$, defined in (2-49). Since, by (2-49), $\widehat{U}_{p'} - \widehat{U}_p = [\widehat{\mathrm{K}}_\lambda, \widehat{\partial}]$, setting Z_0 to be

$$Z_0^\lambda := \widehat{\mathrm{K}}_\lambda$$

suffices to validate the first line of (6-7). We claim that with Z_0 so chosen, and with K_{U_p} given by (6-4), the second line of (6-7) also holds if Z_1 is set to be

$$Z_1^\lambda := \widehat{m}[u_\sqcup u_\lambda](\mathbb{R} \times M_\sqcup),$$

where $u_\sqcup = u_{\widehat{p}_2} - u_{\widehat{p}_1}$, and $\widehat{p}_i$ denotes $\mathbb{R} \times \{p_i\} \subset \mathbb{R} \times M_\sqcup$. In other words,

$$\widehat{n}[\mathrm{u}_{p'}\mathrm{u}_\sqcup] - \widehat{n}[\mathrm{u}_p\mathrm{u}_\sqcup] = [\widehat{m}[u_\sqcup u_\lambda](\mathbb{R} \times M_\sqcup), \widehat{\partial}_\sqcup] + [\widehat{n}[\mathrm{u}_\sqcup], \widehat{m}[u_\lambda](\mathbb{R} \times M_\sqcup)].$$

This identity is essentially a higher-degree variant of (2-49), and is proved by arguments similar to (2-47). For more details the reader is referred to the proof of (6-75) in the next subsection, which differs by cosmetic changes from the proof for the preceding identity.

Let $Z_*^\lambda = Z_0^\lambda \otimes \jmath + Z_1^\lambda \otimes y$ denote the version of Z_* constructed using λ.

(ii) Now suppose that p and p' belong to different connected components of $M_\sqcup$. Let $p \in M_1$ and $p' \in M_2$, $\lambda_1 \subset M_1$ be a path from p_1 to p and $\lambda_2 \subset M_1$ be a path from p_2 to p'. Then, by the discussion in case (i) above,

$$S_{U_\sqcup}(U_{p'}) - S_{U_\sqcup}(U_p) = S_{U_\sqcup}(U_\sqcup) + [Z_*^{\lambda_2} - Z_*^{\lambda_1}, D_\sqcup].$$

Meanwhile, according our construction of K_{U_p}, we have $K_{U_\sqcup} = 0$ and

$$S_{U_\sqcup}(U_\sqcup) = \widehat{U}_\sqcup \otimes 1 = [1 \otimes \partial_y, D_\sqcup].$$

So we have

$$S_{U_\sqcup}(U_{p'}) - S_{U_\sqcup}(U_p) = [Z_*^{\lambda_2} - Z_*^{\lambda_1} + 1 \otimes \partial_y, D_\sqcup]$$

and we take $Z_* = Z_*^{\lambda_2} - Z_*^{\lambda_1} + 1 \otimes \partial_y$ in this case. □

Corollary 6.5 *The $A_\dagger(M_\sqcup)$–action on $S_{U_\sqcup}(\hat{C}(M_\sqcup))$, as defined above, induces an $A_\dagger(M_\sqcup)$–action on $H_*\big(S_{U_\sqcup}(\hat{C}(M_\sqcup))\big)$ that is independent of choices of p, $\{\gamma_i^{[1]}\}_i$ and $\{\gamma_j^{[2]}\}_j$.*

Proof The assertion regarding the U–action follows directly from the previous lemma. To verify the assertion for $H_1(M_\sqcup;\mathbb{Z})/\mathrm{Tors}$–actions, take two embedded circles γ and γ' in $M_\sqcup$ representing the same $[\gamma]\in H_1(M_\sqcup;\mathbb{Z})/\mathrm{Tors}$. Note that they must lie in the same connected components of $M_\sqcup$, and therefore there exists an embedded surface in $\mathbb{R}\times M_\sqcup$ from γ to γ'. We wish to show that there exists a map T_* from $S_{U_\sqcup}(\hat{C}(M_\sqcup))$ back to itself, satisfying

$$S_{U_\sqcup}(\hat{\mathfrak{m}}_{\gamma'}) - S_{U_\sqcup}(\hat{\mathfrak{m}}_{\gamma}) = [T_*, D_\sqcup].$$

Assume this time that T_* is of the form $T_* = T_0\otimes 1 + T_1\otimes y$, where T_0 and T_1 are maps from $\hat{C}(M_\sqcup)$ to itself, respectively of degree 0 and -1. Then the preceding identity now reads

$$(\hat{\mathfrak{m}}_{\gamma'} - \hat{\mathfrak{m}}_{\gamma}) = [T_0, \hat{\partial}], \quad (\hat{K}_{\mathfrak{m}_{\gamma'}} - \hat{K}_{\mathfrak{m}_{\gamma}}) = [T_1, \hat{\partial}] + [\hat{U}_\sqcup, T_0]. \tag{6-8}$$

By (2-55), $\hat{\mathfrak{m}}_{\gamma'} - \hat{\mathfrak{m}}_{\gamma} = [\hat{m}[\mathrm{F}_\Sigma], \hat{\partial}_\sqcup]$; so we set

$$T_0 = \hat{m}[\mathrm{F}_\Sigma](\mathbb{R}\times M_\sqcup),$$

so that the first line of (6-8) holds for this T_0. To verify the second line, we choose

$$T_1 = \hat{m}[u_\sqcup \mathrm{F}_\Sigma](\mathbb{R}\times M_\sqcup).$$

With this choice and our construction of $\hat{K}_{\mathfrak{m}_\gamma}$, the second line of (6-8) says

$$\hat{n}[\mathrm{u}_{\gamma'}\mathrm{u}_\sqcup] - \hat{n}[\mathrm{u}_{\gamma}\mathrm{u}_\sqcup] = [\hat{m}[u_\sqcup \mathrm{F}_\Sigma](\mathbb{R}\times M_\sqcup), \hat{\partial}_\sqcup] + [\hat{n}[\mathrm{u}_\sqcup], \hat{m}[\mathrm{F}_\Sigma](\mathbb{R}\times M_\sqcup)].$$

The proof of this identity is virtually identical to that for the second line of (6-75), and the reader is referred to the next subsection for details. □

Part 4: the cobordisms $\mathcal{V}$ and $\overline{\mathcal{V}}$, and the cobordism maps V_* and $V_*^\dagger$ Let $\mathcal{V} := (X, s)$ denote a cobordism as described in (2-8) and (2-9), with $Y_- = M_\#$ and $Y_+ = M_\sqcup$. Assume that s has a unique critical point of index 3 with critical value 0.

There is a unique Spin^c–structure $\mathfrak{s}_X$ on such X with $c_1(\mathfrak{s}_X)|_{s^{-1}(-c)} = c_1(\mathfrak{s}_\#)$ and $c_1(\mathfrak{s}_X)|_{s^{-1}(c)} = c_1(\mathfrak{s}_\sqcup)$ for $c \gg 0$. Meanwhile, given $[\varpi_i]\in H^2(M_i)$, there is a unique $[\varpi_\#]\in H^2(M_\#)$ and a $[\omega]\in H^2(X)$ that restricts to $[\varpi_1]$, $[\varpi_2]$ and $[\varpi_\#]$

respectively on the M_1–, M_2– and $M_\#$–ends of X. Suppose as before that $[\varpi_1]$ is nonbalanced with respect to $c_1(\mathfrak{s}_1)$ (and therefore $[\varpi_\#]$ is also nonbalanced with respect to $c_1(\mathfrak{s}_\#)$). Let ω be a closed 2–form on X representing $[\omega]$ above, so that $\varpi_X = 2\omega^+$ satisfies (2-11). In particular, ω restricts respectively to (pullbacks) of closed 2–forms ϖ_1, ϖ_2 and $\varpi_\#$ on the M_1–, M_2– and $M_\#$–ends of X. Let $\varpi_\sqcup$ denote the 2–form on $Y_+ = M_\sqcup$ that restricts respectively to ϖ_1 and ϖ_2 on the M_1 and M_2 component of $M_\sqcup$.

Let $\overline{\mathcal{V}} := (X, -s)$ denote the "time reversal" of $\mathcal{V}$. Given local systems Γ_i on $\mathcal{B}^\sigma(Y_i)$ for $i = 1, 2$, let $\Gamma_\sqcup = \Gamma_1 \otimes \Gamma_2$ denote the local system on $\mathcal{B}^\sigma(M_1) \times \mathcal{B}^\sigma(M_2) \simeq \mathcal{B}^\sigma(M_\sqcup)$. Note that X satisfies the condition that $\delta_\pm$ are both isomorphisms in the first bullet of Remark 2.3, and thus there is a unique $\mathcal{V}$–morphism $\Gamma_{\mathcal{V}}$, which together with its inverse $\Gamma_{\overline{\mathcal{V}}}$, gives an 1–1 correspondence between local systems on $\mathcal{B}^\sigma(M_\sqcup)$ and $\mathcal{B}^\sigma(M_\#)$. Let $\Gamma_\#$ denote the local system on $\mathcal{B}^\sigma(M_\#)$ corresponding to $\Gamma_\sqcup$. Meanwhile, by the second bullet of Remark 2.3, $\Gamma_\sqcup$ is (strongly) $(\mathfrak{s}_\sqcup, \varpi_\sqcup)$–complete if and only if $\Gamma_\#$ is (strongly) $(\mathfrak{s}_\#, \varpi_\#)$–complete, and in this case $\widehat{m}[u](X, \mathfrak{s}_X, \varpi_X; \Gamma_X)$ is well defined through (2-23).

In what follows, take

$$\widehat{C}_*(M_\sqcup) = \widehat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup, \varpi_\sqcup; \Gamma_\sqcup), \quad \mathrm{CM}_*(M_\#) = \mathrm{CM}_*(M_\#, \mathfrak{s}_\#, \varpi_\#; \Gamma_\#).$$

The statement of the upcoming connected sum theorem involves certain maps

$$V_*\colon {}_*(M_\#) \to S_{U_\sqcup}\widehat{C}_*(M_\sqcup) \quad \text{and} \quad V_*^\dagger\colon S_{U_\sqcup}\widehat{C}_*(M_\sqcup) \to \mathrm{CM}_*(M_\#).$$

These are constructed using the moduli spaces $\mathcal{M}_k(\mathcal{V}, \mathfrak{c}_\#, \mathfrak{c}_\sqcup) = \mathcal{M}_k(\overline{\mathcal{V}}, \mathfrak{c}_\sqcup, \mathfrak{c}_\#)$ of solutions to (2-10) associated to the Spin^c 4–manifold $(X, \mathfrak{s}_X)$ and the perturbation form ϖ_X described above.

Here is how they are defined. Use the shorthand $(\widehat{C}_\sqcup, \widehat{\partial}_\sqcup) = (\widehat{C}(M_\sqcup), \widehat{\partial}(M_\sqcup))$, $(C_\#, \partial_\#) = (\mathrm{CM}(M_\#), \partial_{M_\#})$ etc below. Write the chain module of $S_{U_\sqcup}(\widehat{C}_\sqcup)$, $\widehat{C}_\sqcup \otimes \mathbb{Z}[y]$, as the direct sum

$$S_{U_\sqcup}(\widehat{C}_\sqcup) = \widehat{C}_\sqcup \oplus y\widehat{C}_\sqcup. \tag{6-9}$$

With respect to this decomposition, its differential takes the block form

$$D_\sqcup = \begin{bmatrix} \widehat{\partial}_\sqcup & 0 \\ \widehat{U}_\sqcup & -\widehat{\partial}_\sqcup \end{bmatrix}. \tag{6-10}$$

Correspondingly, write the maps V_* and $V_*^\dagger$ in block form with respect to the decomposition (6-9) as

$$V_* = \begin{bmatrix} V_0 \\ V_1 \end{bmatrix}, \quad V_*^\dagger = \begin{bmatrix} V_1^\dagger & V_0^\dagger \end{bmatrix}, \tag{6-11}$$

where $V_i\colon \widehat{C}_\sqcup \to \widehat{C}_\#$ and $V_i^\dagger\colon \widehat{C}_\# \to \widehat{C}_\sqcup$ for $i = 0, 1$ are defined through cobordism maps of the form $\widehat{m}[u](X, \mathfrak{s}_X, \varpi_X; \Gamma_X)$ for $X = \mathcal{V}$ or $\overline{\mathcal{V}}$. These cobordism maps are defined as in Part 4 of Section 2.4, noting that $\mathcal{V}$ and $\overline{\mathcal{V}}$ satisfy the condition (2-26), and assuming for the rest of this subsection the same completeness condition for Γ_X alluded to in the end of Section 2.4. Meanwhile, cochains u involved in the definition of these maps are of the type introduced in Section 2.5's Part 3(a), with the relevant arc λ chosen as follows. In the present section, let λ denote the ascending manifold of the unique critical point of s; it is a path in X asymptotic to $(p_1, p_2) \in M_\sqcup = Y_+$. We orient it so that it begins from $p_1 \in M_1$ and ends at $p_2 \in M_2$. Meanwhile, the descending manifold of this critical point will be denoted by B; it is an embedded 3–ball in X that intersects each $s^{-1}(c) \simeq M_\#$ in a 2–sphere for all $c \ll 0$. We orient it so that it intersects with λ positively. Let $\overline{\lambda}$ and $\overline{\mathrm{B}}$, respectively, denote the descending and ascending manifold from the unique critical point of $-s$. These are the same submanifolds in X as λ and B, but equipped with the opposite orientation.

With the above said, we are ready to write down the formulas for V_i and $V_i^\dagger$ for $i = 0, 1$:

$$\begin{aligned} V_0 &= \widehat{m}[1](\mathcal{V}, \mathfrak{s}_X, \varpi_X; \Gamma_{\mathcal{V}}), \\ V_1 &= \widehat{\mathrm{K}}_\lambda(\mathcal{V}, \mathfrak{s}_X, \varpi_X; \Gamma_{\mathcal{V}}) = \widehat{m}[u_\lambda](\mathcal{V}, \mathfrak{s}_X, \varpi_X; \Gamma_{\mathcal{V}}) \\ &= \widehat{m}[\theta_\lambda](\mathcal{V}, \mathfrak{s}_X, \varpi_X; \Gamma_{\mathcal{V}}) + \Theta_\sqcup V_0, \\ V_0^\dagger &= \widehat{m}[1](\overline{\mathcal{V}}, \mathfrak{s}_X, \varpi_X; \Gamma_{\mathcal{V}}), \\ V_1^\dagger &= \widehat{\mathrm{K}}_{\overline{\lambda}}(\overline{\mathcal{V}}, \mathfrak{s}_X, \varpi_X; \Gamma_{\overline{\mathcal{V}}}) = \widehat{m}[u_{\overline{\lambda}}](\overline{\mathcal{V}}, \mathfrak{s}_X, \varpi_X; \Gamma_{\overline{\mathcal{V}}}) \\ &= \widehat{m}[\theta_{\overline{\lambda}}](\overline{\mathcal{V}}, \mathfrak{s}_X, \varpi_X; \Gamma_{\overline{\mathcal{V}}}) - V_0^\dagger \Theta_\sqcup, \end{aligned} \tag{6-12}$$

where $\Theta_\sqcup$ denotes the map from $\widehat{C}_\sqcup = \widehat{C}(M_1) \otimes \widehat{C}(M_2)$ to itself, $1 \otimes \Theta_{p_2} - \Theta_{p_1} \otimes 1$. See (2-50) for the definition of $\widehat{\mathrm{K}}_\lambda$.

Remark 6.6 With [17]'s notion of canonical $\mathbb{Z}/2$–gradings suitably generalized, the maps V_* and $V_*^\dagger$ are of degree 0 with respect to this canonical $\mathbb{Z}/2$–grading. Recall the characteristic number $\iota(X)$ for a cobordism X from Y_- and Y_+, with $Y_\pm$ both connected, from Definition 25.4.1 in [17]. When $Y_\pm$ are allowed to be disconnected, we generalize the formula in [17] as

$$\iota(X) := \tfrac{1}{2}(\chi(X)+\sigma(X)+b_1(Y_+)-b_1(Y_-)-b_0(Y_+)+b_0(Y_-)).$$

With this generalized $\iota(X)$, the statements of [17, Lemma 25.4.2 and Proposition 25.4.3] remain valid: $\iota(X)\in\mathbb{Z}$ and is additive under composition of cobordisms, and a map (if well defined) of the form $\mathring{m}[u](X)$ is of even or odd degree with respect to this canonical $\mathbb{Z}/2$–grading depending on the parity of

$$\deg(u)-\iota(X).$$

For $X=\mathcal{V},\overline{\mathcal{V}}$, $\iota(\mathcal{V})=0$ and $\iota(\overline{\mathcal{V}})=1$; hence V_0 and $V_1^\dagger$ are of even degree, while V_1 and $V_0^\dagger$ are of odd degree. Hence $V_*=V_0+yV_1\colon C_\#\to\widehat{C}_\sqcup\otimes\mathbb{K}[y]$ and $V_*^\dagger=V_1^\dagger+V_0^\dagger\partial_y\colon\widehat{C}_\sqcup\otimes\mathbb{K}[y]\to C_\#$ are both of even *degree* with respect to the canonical grading. (In fact, they are both of degree 0 when the canonical $\mathbb{Z}/2$–grading lifts to an absolute grading; see Section 28.3 in [17].) This is not to be confused with the notion of an *even or odd map* in the sense of signs when it appears in commutators. In the latter sense V_* is even, while $V_*^\dagger$ is odd, since V_0 and $V_0^\dagger$ are even and V_1 and $V_1^\dagger$ are odd. The parity of a map $\mathring{m}[u](X)$ (in the sense of commutators) is determined *purely by* $\deg(u)$, independent of X. This is because only $\deg(u)$ contributes to the signs in gluing formulas.

6.2 A connected sum formula for nonbalanced perturbations

Adopt the notation and assumptions from the previous subsection.

Proposition 6.7 *Under the above assumptions*:

(a) *Suppose that* $[\varpi_\#]$ *is negative monotone, nonbalanced with respect to* $\mathfrak{s}_\#$. *Let* $\Gamma_\#$ *be arbitrary, and* $\Gamma_\sqcup$ *be determined by* $\Gamma_\#$ *via* Γ_X. *Then the maps* V_* *and* $V_*^\dagger$ *given in the previous subsection are well-defined chain maps, and* V_* *defines a chain homotopy equivalence*

$$V_*\colon C_*(M_\#,\mathfrak{s}_\#,[\varpi_\#],\Gamma_\#)\to S_{U_\sqcup}(\widehat{C}_*(M_\sqcup,\mathfrak{s}_\sqcup,[\varpi_\sqcup];\Gamma_\sqcup)) \tag{6-13}$$

respecting the (relative) $\mathbb{Z}/c_\#$*–grading on both sides. Moreover, the map* V_* *intertwines with the*

$$\boldsymbol{A}_\dagger(M_\#)\overset{(6\text{-}2)}{\simeq}\textstyle\bigwedge^*(H_1(M_1)/\mathrm{Tors})\otimes\bigwedge^*(H_1(M_2)/\mathrm{Tors})\otimes\mathbb{K}[u]=\boldsymbol{A}_\dagger(M_\sqcup)$$

actions on the two sides, defined in the previous subsection's Parts 2 *and* 3 *using* p *and* $\{\gamma_i^{[1]}\}_i\cup\{\gamma_j^{[2]}\}_j$.

(b) *Suppose that* $[\varpi_1]$ *is nonbalanced with respect to* $\mathfrak{s}_1$, *and that* Γ_i *is strongly* $(\mathfrak{s}_i, [\varpi_i])$*–complete for* $i = 1, 2$. *Then the maps* V_* *and* $V_*^\dagger$ *are well-defined chain maps, and* V_* *defines a chain homotopy equivalence*

$$V_*\colon C_\bullet(M_\#, \mathfrak{s}_\#, [\varpi_\#], \Gamma_\#) \to S_{U_\sqcup}(\hat{C}_\bullet(M_\sqcup, \mathfrak{s}_\sqcup, [\varpi_\sqcup]; \Gamma_\sqcup))$$

respecting the (relative) $\mathbb{Z}/c_\#$*–grading on both sides. Moreover, the map* V_* *above intertwines with the*

$$A_\dagger(M_\#) \overset{(6\text{-}2)}{\simeq} \textstyle\bigwedge^*(H_1(M_1)/\mathrm{Tors}) \otimes \bigwedge^*(H_1(M_2)/\mathrm{Tors}) \otimes \mathbb{K}[u] = A_\dagger(M_\sqcup)$$

actions on the two sides defined using p *and* $\{\gamma_i^{[1]}\}_i \cup \{\gamma_j^{[2]}\}_j$.

Proof (a) The proof has six steps.

Step 1 In this part we show that the assumption on Γ_X of part (a) ensures that Γ_X satisfies the completeness conditions alluded to in Remark 2.3, so that the maps V_i and $V_i^\dagger$ for $i = 0, 1$ are well defined. More precisely, we show that the sum

$$\sum_{\mathfrak{c}_\# \in \mathfrak{C}(M_\#)} \sum_{(\mathfrak{c}_1, \mathfrak{c}_2) \in \mathfrak{C}(M_1) \times \mathfrak{C}(M_2)} \sum_{z \in \pi_0(\mathcal{B}^\sigma(\mathcal{V}; \mathfrak{c}_\#, (\mathfrak{c}_1, \mathfrak{c}_2)))} \langle u, \mathcal{M}_{k,z}(\mathfrak{c}_\#, (\mathfrak{c}_1, \mathfrak{c}_2)) \rangle \tag{6-14}$$

has finitely many nonvanishing terms, and therefore V_* is a well-defined map between the (*precompleted*) chain complexes $\mathrm{CM}_*(M_\#)$ and $S_{U_\sqcup}(\hat{C}_*(M_\sqcup))$ for any coefficients $\Gamma_\#$ and its twin $\Gamma_\sqcup$. To see this, observe that by the well-known compactness property of spaces of 3–dimensional Seiberg–Witten solutions, $\mathrm{CM}(M_1) = C^o(M_1)$, $C^o(M_2)$ and $\mathrm{CM}(M_\#) = C^o(M_\#)$ are all finitely generated over $\mathbb{K}$, while $C^u(M_2)$ is finitely generated over $\mathbb{K}[u]$, with u having degree -2. Write the generating sets of these free $\mathbb{K}$–modules respectively as $\mathfrak{C}(M_1) = \{\mathfrak{a}_i\}_i$, $\mathfrak{C}^o(M_2) = \{\mathfrak{b}_j^o\}_j$, $\mathfrak{C}(M_\#) = \{\mathfrak{c}_k\}_k$ and $\mathfrak{C}^u(M_2) = \{\mathfrak{b}_q^u u^n\}_{q,n}$, where there are finitely many indices i, j, k and q, and n runs through all nonnegative integers. Let $\pi^\sigma\colon \mathcal{B}^\sigma \to \mathcal{B}$ denote the projection of the blown-up space. The index $\iota_\mathfrak{d}$ and the topological energy (see [17, Definition 4.5.4 and page 593] in the case of nonexact perturbations) of an element $\mathfrak{d} \in \mathcal{M}(\mathcal{V})$ depends only its relative homotopy class under π^σ, and the former is controlled via $c_1(\mathfrak{s}_X)$, the latter through $[\omega] - 2\pi[c_1(\mathfrak{s}_X)]$. The monotonicity condition and the compactness property of $\mathcal{M}(\mathcal{V})$ under bounds on the topological energy then ensures that only finitely $\mathfrak{a}_i$, $\mathfrak{b}_j^o$, $\mathfrak{b}_q^u$ and z appear in the sum on the right-hand side of (6-14). Meanwhile, since $\mathrm{gr}(\mathfrak{b}_q^u u^n) - \mathrm{gr}(\mathfrak{b}_q^u u^m) = -2(n-m)$, the index bound $\iota_\mathfrak{d} = k$ on the right-hand side of (6-14) implies that for each q, only finitely many $\mathfrak{b}_q^u u^n$ appear on the right-hand side of (6-14). (The aforementioned compactness result follows from a straightforward generalization of Theorem 24.5.2 in [17] to include nonexact perturbations.)

The $\widehat{m}(\overline{\mathcal{V}})$ analog of (6-14) involves sum over $\mathfrak{C}(M_\#)$ instead, which consists of finitely many elements. The finiteness of the relevant sum then follows from the monotonicity assumption alone.

Step 2 In this step, we show that V_* and $V_*^\dagger$ are (respectively even and odd) chain maps. This amounts to verifying the identities

$$
\begin{aligned}
\widehat{\partial}_\sqcup V_0 - V_0\partial_\# = 0, &\qquad \widehat{\partial}_\sqcup V_1 + V_1\partial_\# - \widehat{U}_\sqcup V_0 = 0,\\
\partial_\# V_0^\dagger - V_0^\dagger\widehat{\partial}_\sqcup = 0, &\qquad V_1^\dagger\widehat{\partial}_\sqcup + \partial_\# V_1^\dagger + V_0^\dagger\widehat{U}_\sqcup = 0.
\end{aligned} \tag{6-15}
$$

In view of (2-51), these would have followed directly from [17, Proposition 25.3.4] if the latter's assumption on the connectedness of $Y_\pm$ could be removed. In the specific setting under discussion, such generalization requires only simple modifications of what was in [17]. To do so, write the identities in full in terms of $m^o_{o\natural}$, $m_o^{o\sharp}$, $\partial^o_o(M_1)$, $\partial^o_o(M_\#)$, $\widehat{\partial}(M_2)$ and $\overline{\partial}{}^s_u(M_2)$ as given by (2-17), (2-27) and (6-11). These can be reduced to the identities in Lemma 25.3.6 in [17] (with many vanishing terms), with these substitutions:

- Drop the o's from the double superscript or subscripts $o*$ of m.
- Replace the entries of $\widehat{\partial}(M_\sqcup) = (1\otimes\partial^\sharp_\natural(M_2) + \partial^o_o(M_1)\otimes 1))$ by $\partial^\sharp_\natural$.

Theorem 24.7.2 in [17] conveniently supplies us with the general gluing theorem required for verifying these formulas. (We have at worst rank 1 boundary-obstruction.)

Step 3 In the upcoming three steps, we show that the two chain complexes in (6-13) are chain homotopy equivalent via V_* and $V_*^\dagger$. More precisely, we shall show that their compositions satisfy the identities

$$V_*^\dagger\circ V_* - [H'_\#, \partial_\#]_{\mathrm{even}} = V_1^\dagger\circ V_0 + V_0^\dagger\circ V_1 = \mathrm{Id}_\# - [\mathrm{Z}_\#, \partial_\#]_{\mathrm{odd}}, \tag{6-16}$$

$$V_*\circ V_*^\dagger - [H'_\sqcup, D_\sqcup]_{\mathrm{even}} = \mathrm{Id}_\sqcup\otimes 1 - [\mathrm{Z}_\sqcup\otimes \jmath + \mathrm{X}\otimes y, D_\sqcup]_{\mathrm{odd}}, \tag{6-17}$$

or, in block form,

$$
\begin{aligned}
&\begin{bmatrix} V_0\circ V_1^\dagger & V_0\circ V_0^\dagger\\ V_1\circ V_1^\dagger & V_1\circ V_0^\dagger\end{bmatrix} - \left[\begin{bmatrix} A' & B'\\ C' & D'\end{bmatrix}, \begin{bmatrix}\widehat{\partial}_\sqcup & 0\\ \widehat{U}_\sqcup & -\widehat{\partial}_\sqcup\end{bmatrix}\right]_{\mathrm{even}}\\
&\qquad\qquad = \begin{bmatrix}\mathrm{Id}_\sqcup & 0\\ 0 & \mathrm{Id}_\sqcup\end{bmatrix} - \left[\begin{bmatrix}\mathrm{Z}_\sqcup & 0\\ \mathrm{X} & \mathrm{Z}_\sqcup\end{bmatrix}, \begin{bmatrix}\widehat{\partial}_\sqcup & 0\\ \widehat{U}_\sqcup & -\widehat{\partial}_\sqcup\end{bmatrix}\right]_{\mathrm{odd}}
\end{aligned}
$$

for certain maps $H'_{\#}$ and $\mathrm{Z}_{\#}$ from $CM(M_{\#})$ back to itself, and maps A', B', C', D' and $\mathrm{Z}_{\sqcup}, \mathrm{X}$ from $\widehat{C}(M_{\sqcup})$ back to itself. Here $\mathrm{Id}_{\#}$ and $\mathrm{Id}_{\sqcup}$, respectively, denote the identity maps from $CM(M_{\#})$ and $\widehat{C}(M_{\sqcup})$ back to themselves.

The verification of the first identity (6-16) involves the cobordism $W_{\#}$ obtained from composing $\mathcal{V}$ with $\overline{\mathcal{V}}$. This cobordism goes from $M_{\#}$ to $M_{\#}$, and contains the circle $\lambda_{\#} = \lambda \cup \overline{\lambda}$ in its interior. A surgery along $\lambda_{\#}$ replacing a tubular neighborhood $S^1 \times B^3$ of $\lambda_{\#}$ with $D^2 \times S^2$ yields $\mathbb{R} \times M_{\#}$. On the other hand, to verify the second identity (6-17), one composes in the opposite order to get the cobordism $W_{\sqcup}$ from $M_{\sqcup}$ to $M_{\sqcup}$. There is an embedded 3–sphere $S_{\sqcup} \subset W_{\sqcup}$ obtained by joining the 3–balls B and $\overline{\mathrm{B}}$ from Part 4 of the previous subsection. Doing a surgery along $S_{\sqcup}$ — namely, replace a tubular neighborhood of it, $I \times S_{\sqcup}$, by a disjoint union of two 3–balls $B_1 \sqcup B_2$ — turns $W_{\sqcup}$ into the product cobordism $\mathbb{R} \times M_{\sqcup}$. One may find arcs $\gamma_1 \subset B_1$ and $\gamma_2 \subset B_2$ so that under this surgery they join with $(\lambda \cup \overline{\lambda}) - I \times S_{\sqcup}$ to yield $\mathbb{R} \times \{-p_1, p_2\} \subset \mathbb{R} \times M_{\sqcup}$. The cobordisms $W_{\#}$ and $W_{\sqcup}$ are equipped with metrics and Spin^c–structures $\mathfrak{s}_{W_{\#}}$ and $\mathfrak{s}_{W_{\sqcup}}$ determined by the metric and Spin^c–structure, $\mathfrak{s}_X$, on $X = \mathcal{V}$. The closed 2–form ω on X likewise defines, via concatenation, closed 2–forms $\omega_{\#}$ and $\omega_{\sqcup}$, respectively, on $W_{\#}$ and $W_{\sqcup}$. Let $\varpi_{W_{\#}} := 2\omega_{\#}^{+}$ and $\varpi_{W_{\sqcup}} := 2\omega_{\#}^{-}$.

Note that like $\mathcal{V}$ and $\overline{\mathcal{V}}$, the composite cobordisms $W_{\#}$ and $W_{\sqcup}$ also satisfy the assumption in the first bullet of Remark 2.3. Therefore, given any $\Gamma_{\#}$, there is a unique $\Gamma_{W_{\#}}$–morphism which is an isomorphism from $\Gamma_{\#}$ to itself. In fact, $\Gamma_{W_{\#}} = \Gamma_{\overline{\mathcal{V}}} \circ \Gamma_{\mathcal{V}}$. Similarly, given any $\Gamma_{\sqcup}$, there is a unique $\Gamma_{W_{\sqcup}}$–morphism from $\Gamma_{\sqcup}$ which is an isomorphism from $\Gamma_{\sqcup}$ to itself, and $\Gamma_{W_{\sqcup}} = \Gamma_{\mathcal{V}} \circ \Gamma_{\overline{\mathcal{V}}}$.

The proofs of (6-16) and (6-17) make use of cobordism maps of the form

$$\widehat{m}[u](W_{\#}, \mathfrak{s}_{\#}, \varpi_{\#}; \Gamma_{W_{\#}}), \quad \widehat{m}[u](W_{\sqcup}, \mathfrak{s}_{\sqcup}, \varpi_{W_{\sqcup}}; \Gamma_{W_{\sqcup}}),$$

as well as their parametrized variants. (We often abbreviate these maps as $\widehat{m}[u](W_{\#})$ and $\widehat{m}[u](W_{\sqcup})$ below.) The manifold $W_{\sqcup}$ does not satisfy the condition (2-26), but the formula for $\widehat{m}$ in (2-27) has a straightforward adaptation in this context: simply replace terms of the form $m^{\#}_{\flat}$ in (2-20) by $m^{o\#}_{o\flat}$, and drop all the terms $\overline{m}^{\#}_{\flat}$. Replace $\partial^{\#}_{\flat}$ and $\overline{\partial}^{\#}_{\flat}$, respectively, by $\partial^{o}_{0}(M_1) \otimes 1 + 1 \otimes \partial^{\#}_{\flat}(M_2)$ and $1 \otimes \overline{\partial}^{\#}_{\flat}(M_2)$.

We next describe the relevant cochains u. Let $u_{\lambda_{\#}} \in C^{1;\mathbb{Z}}_{\mathcal{M}(W_{\#})}$ be the 1–cocycle associated to the circle $\lambda_{\#} \subset W_{\#}$, as defined in Section 2.5's Part 1(b). Let $\lambda_{\sqcup}$ denote the union $\lambda \cup \overline{\lambda} \subset W_{\sqcup}$, and use $\lambda_{\sqcup -}$ and $\lambda_{\sqcup +}$ to denote respectively the arcs $\overline{\lambda}$ and λ in $W_{\sqcup}$. Let $u_{\lambda_{\sqcup +}}$ and $u_{\lambda_{\sqcup -}} \in C^{1;\mathbb{Z}}_{\mathcal{M}(W_{\sqcup})}$ be respectively the 1–cochains defined in Section 2.5's

Part 3(a). (The notation u_λ and $u_{\overline{\lambda}}$ is usually reserved for the 1–cochains on $\mathcal{B}^\sigma(\mathcal{V})$ and $\mathcal{B}^\sigma(\overline{\mathcal{V}})$ associated to the arcs λ and $\overline{\lambda}$ in $\mathcal{V}$, which appeared previously in (6-12).) Define the 2–cochain $u_{\lambda_\sqcup} := u_{\lambda_{\sqcup-}} u_{\lambda_{\sqcup+}} \in C^{2;\mathbb{Z}}_{\mathcal{M}(W_\sqcup)}$. Concretely,

$$\begin{aligned}
\widehat{m}[u_{\lambda_{\sqcup-}}](W_\sqcup) &= \widehat{\mathrm{K}}_{\lambda_{\sqcup-}}(W_\sqcup) = \widehat{m}[\theta_{\lambda_{\sqcup-}}](W_\sqcup) - \widehat{m}[1](W_\sqcup)\Theta_\sqcup,\\
\widehat{m}[u_{\lambda_{\sqcup+}}](W_\sqcup) &= \widehat{\mathrm{K}}_{\lambda_{\sqcup+}}(W_\sqcup) = \widehat{m}[\theta_{\lambda_{\sqcup+}}](W_\sqcup) + \Theta_\sqcup\widehat{m}[1](W_\sqcup),\\
\widehat{m}[u_{\lambda_\sqcup}](W_\sqcup) &:= \widehat{\mathrm{K}}_{\lambda_\sqcup}(W_\sqcup)\\
&= \widehat{m}[\theta_{\lambda_{\sqcup-}} \wedge \theta_{\lambda_{\sqcup+}}](W_\sqcup) - \widehat{\mathrm{K}}_{\lambda_{\sqcup+}}(W_\sqcup)\Theta_\sqcup + \Theta_\sqcup\widehat{m}[\theta_{\lambda_{\sqcup-}}](W_\sqcup)\\
&= \widehat{m}[\theta_{\lambda_{\sqcup-}} \wedge \theta_{\lambda_{\sqcup+}}](W_\sqcup) + \Theta_\sqcup\widehat{\mathrm{K}}_{\lambda_{\sqcup-}}(W_\sqcup) - \widehat{m}[\theta_{\lambda_{\sqcup+}}](W_\sqcup)\Theta_\sqcup,
\end{aligned} \tag{6-18}$$

where $\Theta_\sqcup := 1 \otimes \Theta_{p_2} - 1 \otimes \Theta_{p_1}$. It will prove useful to denote the 0–cocycle $1 \in C^{0;\mathbb{Z}}_{\mathcal{M}(W_\sqcup)}$ on $\mathcal{B}^\sigma(W_\sqcup)$ by $u_\varnothing$.

The proof of (6-16)–(6-17) involves two ingredients. The first is a set of gluing identities:

$$V_1^\dagger V_0 + V_0^\dagger V_1 = m[u_{\lambda_\#}](W_\#) + [H_\#, \partial_\#]_{\text{even}}. \tag{6-19}$$

(6-20) (1) The map $V_0 V_1^\dagger = \widehat{m}[u_{\lambda_{\sqcup-}}](W_\sqcup) + [A, \widehat{\partial}_\sqcup]_{\text{even}} + B\widehat{U}_\sqcup$.

(2) The map $V_0 V_0^\dagger = \widehat{m}[u_\varnothing](W_\sqcup) - [\widehat{\partial}_\sqcup, B]_{\text{odd}}$.

(3) The map $V_1 V_1^\dagger = \widehat{m}[u_{\lambda_\sqcup}](W_\sqcup) + [\widehat{\partial}_\sqcup, C]_{\text{odd}} - \widehat{U}_\sqcup A + D\widehat{U}_\sqcup$.

(4) The map $V_1 V_0^\dagger = \widehat{m}[u_{\lambda_{\sqcup+}}](W_\sqcup) + [\widehat{\partial}_\sqcup, D]_{\text{even}} - \widehat{U}_\sqcup B$.

The definition of the maps $H_\#$, A, B, C and D and the verification of these identities occupy the remainder of this step and Step 4 below. In short, they all follow from an adaption of [17, Lemma 26.2.2], together with a parametrized variant of the identity (2-51). Rephrased in our language, the composition identity in [17], which was stated for the check version of monopole Floer homology, has the following companion version in for the hat version: Let W_1 be a connected cobordism from Y_- to Y_0, and W_2 a connected cobordism from Y_0 to Y_+. Let $W = W_2 \circ W_1$ denote the composite cobordism of W_1 and W_2. For $u_1 \in \mathrm{C}(\mathcal{B}^\sigma((W_1)_c); \mathbb{K})$ and $u_2 \in \mathrm{C}(\mathcal{B}^\sigma((W_2)_c); \mathbb{K})$, Kronheimer and Mrowka [17] defined an "inner product" of u_1 and u_2, denoted by $u := c(u_1 \otimes u_2) \in \mathrm{C}(\mathcal{B}^\sigma(W_c); \mathbb{K})$ (see [17, equation (26.9)]). We have

$$\widehat{m}[u_2](W_2)\widehat{m}[u_1](W_1) = \widehat{m}[u](W) + [\widehat{\mathrm{K}}[\boldsymbol{u}](W), \widehat{\partial}] + \widehat{\mathrm{K}}[\delta\boldsymbol{u}](W), \tag{6-21}$$

where the maps $\widehat{\mathrm{K}}$ are defined via integrations on a certain parametrized moduli space, and $\boldsymbol{u}$ is a parametrized version of u. (Though cobordism maps $\mathring{m}[u](W)$ were previously defined for cochains on $\mathcal{B}^\sigma_{\text{loc}}(W)$ instead of those on $\mathcal{B}^\sigma(W_c)$, there is a restriction map, s, from an open dense subset of the former to the latter.) As explained in [17], because of unique continuation, it makes no practical difference to work with either $\mathcal{B}^\sigma(W_c)$ or $\mathcal{B}^\sigma_{\mathrm{loc}}(W)$, or the aforementioned open dense subset of $\mathcal{B}^\sigma_{\mathrm{loc}}(W)$. The identity (6-21) is the consequence of applying a Stokes' theorem to the compactification of the aforementioned parametrized moduli space. See [17, (26.2)–(26.3)] for the definition of the aforementioned parametrized moduli space, equations (26.11)–(26.12) therein for the definition of the associated maps $\widehat{\mathrm{K}}$ (denoted by $\check{K}$ in [17]), and Lemma 26.2.2 and its siblings in [17] for proofs of the key gluing identities.

Roughly speaking, the proofs of (6-19) and (6-20) follow from applying variants of (6-21) to $W = \overline{\mathcal{V}} \circ \mathcal{V}$ and $W = \mathcal{V} \circ \overline{\mathcal{V}}$, respectively, with u taken to be $u_{\lambda_\#}$ in (6-19), and u set to be $u_{\lambda_{\sqcup -}}$, $u_\varnothing$, $u_{\lambda_\sqcup}$ and $u_{\lambda_{\sqcup +}}$, respectively, in items (1)–(4) of (6-20). The maps $H_\#$, A, B, C and D are then given by

$$
\begin{aligned}
H_\# &= \widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_\#}](W_\#),\\
A &= \widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_{\sqcup -}}](W_\sqcup),\\
-B &= \widehat{\mathrm{K}}[\boldsymbol{u}_\varnothing](W_\sqcup) = \widehat{\mathrm{K}}[1](W_\sqcup),\\
C &= \widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_\sqcup}](W_\sqcup),\\
-D &= \widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_{\sqcup +}}](W_\sqcup).
\end{aligned}
\tag{6-22}
$$

A couple of issues need to be addressed to be able to apply (6-21) in our setting. Firstly, in [17], $Y_\pm$ and Y_0 are assumed to be connected. As previously explained, there is no problem adapting to the case when $Y_\pm$ is the disconnected manifold $M_\sqcup$. In the case of (6-19), $W_\# = \overline{\mathcal{V}} \circ \mathcal{V}$ is glued along $Y_0 = M_\sqcup$. This creates no new troubles: The assumption that only the M_2 component of $M_\sqcup$ can be associated with balanced perturbations implies that the straightforward sort of gluing argument applies with gluing along M_1, leaving the more delicate analysis described in [17] required for M_2 alone. The second issue is related to the fact that, recalling the discussion in Section 2.5, the cochains u_γ and their associated maps $\widehat{m}[u_\gamma]$ and $\widehat{\mathrm{K}}[\boldsymbol{u}_\gamma]$ relevant to our discussion are of a more general sort. In particular, when γ is noncompact, unlike those cochains on $\mathcal{B}^\sigma(W_c)$ considered in [17], our $u_\gamma \in \mathrm{C}(\mathcal{B}^\sigma_{\mathrm{loc}}(W); \mathbb{K})$ are sensitive to the behavior of the Seiberg–Witten configurations over the ends $W \setminus W_c$. To explain this issue in more detail, as well as to describe the modification to generalize (6-21) to this context, some preliminary discussions are required.

Here are some key ingredients of [17]'s derivation of (6-21). Let $W(S)_c$ denote the variants of [17]'s $W(S)$ (see [17, (26.2)] and thereabouts). We write it as

$$
W(S)_c = (W_1)_c \cup \left(\left[-\tfrac{1}{2}S, \tfrac{1}{2}S\right] \times Y_0\right) \cup (W_2)_c. \tag{6-23}
$$

Let $W(S)$ be the (complete) manifold with cylindrical ends containing $W(S)_c$ as its "compact piece". (Recall the notation from Section 2.2; they were denoted by $W(S)^*$ in [17].) For example, the cobordisms $W_\#(S)$ and $W_\sqcup(S)$ are illustrated respectively in (6-43) and (6-44) below, where the shaded regions represent the "necks" of length S. The parametrized moduli spaces involved in the proof of (6-21) are of the following sort:

$$\begin{aligned}
\boldsymbol{M}_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+) &:= \bigcup_{S\in[0,\infty)} \{S\}\times\mathcal{M}_{k,z}(W(S),\mathfrak{c}_-,\mathfrak{c}_+),\\
\boldsymbol{M}_{k+1}(W,\mathfrak{c}_-,\mathfrak{c}_+) &:= \bigcup_{z} \boldsymbol{M}_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+),\\
\boldsymbol{M}^+_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+) &:= \bigcup_{S\in[0,\infty]} \{S\}\times\mathcal{M}^+_{k,z}(W(S),\mathfrak{c}_-,\mathfrak{c}_+),\\
\boldsymbol{M}^+_{k+1}(W,\mathfrak{c}_-,\mathfrak{c}_+) &:= \bigcup_{z} \boldsymbol{M}^+_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+),
\end{aligned}$$

with the "fiber at ∞", $\mathcal{M}^+_{k,z}(W(\infty),\mathfrak{c}_-,\mathfrak{c}_+) = \mathcal{M}_{k,z}(W(\infty),\mathfrak{c}_-,\mathfrak{c}_+)$, as given in [17, (26.4)]. Their reducible variants are defined similarly. The compactified moduli space $\boldsymbol{M}^+_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+)$ maps to a smaller compactification, $\overline{\boldsymbol{M}}_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+)$ embedded in

$$[0,\infty]\times\mathcal{B}^\sigma((W_1)_c)\times\mathcal{B}^\sigma((W_2)_c), \tag{6-24}$$

in a way similar to the map $\mathfrak{r}$ in Section 2.4. See [17, (26.7)]. This map preserves the fibration (over $[0,\infty]$) structure on both spaces, and over the fiber

$$\{S\}\times\mathcal{M}^+_{k,z}(W(S),\mathfrak{c}_-,\mathfrak{c}_+)\subset\boldsymbol{M}^+_{k+1,z}(W,\mathfrak{c}_-,\mathfrak{c}_+),\qquad S\in[0,\infty),$$

this map factors through

$$\begin{aligned}\mathcal{M}^+_{k,z}(W(S),\mathfrak{c}_-,\mathfrak{c}_+) \xrightarrow{\mathfrak{r}} \mathcal{B}^\sigma_{\mathrm{loc}}(W(S))^\circ \xrightarrow{s} \mathcal{B}^\sigma\big((W(S))_c\big)^\circ&\\ \xrightarrow{s_1\times s_2} \mathcal{B}^\sigma((W_1)_c)\times\mathcal{B}^\sigma((W_2)_c)&,\end{aligned} \tag{6-25}$$

where $\mathfrak{r}$ is as in Section 2.4, and, for each $i=1,2$, $s_i\colon \mathcal{B}^\sigma((W(S))_c)^\circ\to\mathcal{B}^\sigma((W_1)_c)$ denotes the map of restricting to $(W_i)_c\subset W(S)$. Here, $\mathcal{B}^\sigma_{\mathrm{loc}}(W(S))^\circ$ denotes a certain open dense subset of $\mathcal{B}^\sigma_{\mathrm{loc}}(W(S))$, and similarly for $\mathcal{B}^\sigma(W_c)^\circ$. The cochains $u_1\in \mathrm{C}\big(\mathcal{B}^\sigma((W_i)_c)\big)$ from (6-21) thus defines a cochain

$$u = c(u_1\otimes u_2) := (s_1\times s_2)^*(u_1\times u_2)$$

in $\mathrm{C}(\mathcal{B}^\sigma(W_c))$, and $\boldsymbol{u}$ in (6-21) refers to the cochain on (6-24) induced from u_1 and u_2.

Use $\mathfrak{r}$ to denote the aforementioned map from $\boldsymbol{M}^+_{k+1}(W, \mathfrak{c}_-, \mathfrak{c}_+)$ to (6-24), and $\boldsymbol{\varsigma}$ for the embedding of $\overline{\boldsymbol{M}}_{k+1}(W, \mathfrak{c}_-, \mathfrak{c}_+)$ into (6-24). Use $\mathfrak{r}|_S$ and $\boldsymbol{\varsigma}|_S$, respectively, to denote the restriction of $\mathfrak{r}$ and $\boldsymbol{\varsigma}$ from the fiber over S of $\boldsymbol{M}^+_{k+1}(W, \mathfrak{c}_-, \mathfrak{c}_+) \to [0,\infty]$ or $\overline{\boldsymbol{M}}_{k+1}(W, \mathfrak{c}_-, \mathfrak{c}_+) \to [0,\infty]$ to $\mathcal{B}^\sigma((W_1)_c) \times \mathcal{B}^\sigma((W_2)_c)$.

The map $\widehat{\mathrm{K}}[\boldsymbol{u}]$ is constructed from $\mathrm{K}[\boldsymbol{u}]^\#_\flat$ and $\overline{\mathrm{K}}[\boldsymbol{u}]^\#_\flat$, where $\mathrm{K}[\boldsymbol{u}]^\#_\flat$ is a sum of terms with coefficients taking the form

$$\langle \mathfrak{r}^*\boldsymbol{u}, \boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)\rangle = \langle \boldsymbol{\varsigma}^*\boldsymbol{u}, \boldsymbol{M}_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)\rangle, \tag{6-26}$$

where k is the degree of u, and similarly for $\overline{\mathrm{K}}[\boldsymbol{u}]^\#_\flat$. Proposition 26.1.6 of [17] shows that for any k and z, $\boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)$ is a stratified manifold where Stokes' theorem (in the sense of [17, Lemma 21.3.1]) is applicable. Equation (6-21) is then the consequence of applying this Stokes' theorem to integrals of the form

$$\langle \mathfrak{r}^*(\delta\boldsymbol{u}), \boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)\rangle = \langle \mathfrak{r}^*\boldsymbol{u}, \partial[\boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)]\rangle, \tag{6-27}$$

together with an analysis of the structure of $(\boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+))_k$. That is, the codimension one stratified submanifold, $(\boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+))_k \subset \boldsymbol{M}^+_{k+1,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)$, is described as a union of the form

$$\begin{aligned}(\boldsymbol{M}^+_{k+1,z}&(W, \mathfrak{c}_-, \mathfrak{c}_+))_k \\ &= \big(\{\infty\} \times \mathcal{M}^+_{k,z}(W(\infty), \mathfrak{c}_-, \mathfrak{c}_+)\big) \cup \big(\{0\} \times \mathcal{M}^+_{k,z}(W(0), \mathfrak{c}_-, \mathfrak{c}_+)\big) \\ &\qquad \cup \bigcup_{S\in(0,\infty)} \{S\} \times \big(\mathcal{M}^+_{k,z}(W_\#(S), \mathfrak{c}_-, \mathfrak{c}_+)\big)_{k-1}.\end{aligned} \tag{6-28}$$

The first two terms on the right-hand side of (6-28) contribute respectively

$$\begin{gathered}\langle (\mathfrak{r}|_\infty)^*(u_1 \times u_2), \mathcal{M}^+_{k,z}(W(\infty), \mathfrak{c}_-, \mathfrak{c}_+)\rangle, \\ -\langle (\mathfrak{r}|_0)^*(u_1 \times u_2), \mathcal{M}^+_{k,z}(W(0), \mathfrak{c}_-, \mathfrak{c}_+)\rangle = -\langle \mathfrak{r}^* c(u_1 \otimes u_2), \mathcal{M}^+_{k,z}(W, \mathfrak{c}_-, \mathfrak{c}_+)\rangle\end{gathered} \tag{6-29}$$

to the right-hand side of (6-27), resulting respectively in the left-hand side of (6-21) and the first term on the right-hand side of (6-21). The last term in (6-21) arises from the left-hand side of (6-27). The contribution from the last term of (6-28) to the right-hand side of (6-27) leads to the penultimate term in (6-21), based on the straightforward adaptation of [17, Proposition 25.3.4] to the parametrized context.

A simple reformulation of [17]'s work suffices to make (6-21) applicable to general $u_i \in \mathrm{C}(\mathcal{B}^\sigma_{\mathrm{loc}}(W_i); \mathbb{K})$. Let $W_1(S) \supset (W_1)_c$ and $W_2(S) \supset (W_2)_c$ be (the closure of) the two halves of $W(S)$ when divided in the middle of "the neck", namely, at the

3–manifold $\{0\}\times Y_0$ in (6-23). For $i=1,2$, define $W_i(\infty)$ to be the previously introduced complete manifold, W_i. Instead of (6-24), consider another space fibering over $[0,\infty]$, whose fiber over $S\in[0,\infty]$ is

$$\mathcal{B}^\sigma(W_1(S))\times\mathcal{B}^\sigma(W_2(S)) =: \mathcal{B}^\sigma(W_2\circ_S W_1),$$

where $\mathcal{B}^\sigma(W_i(S))$ for $i=1,2$ are both equipped with the topology inherited from its embedding to $\mathcal{B}^\sigma_{\mathrm{loc}}(W_i)$. For all $S\in[0,\infty]$, $\mathcal{B}^\sigma(W_1(S))$ admits a well-defined $(-\infty)$–limit map by construction, $\Pi^{-\infty}:=\Pi^{-\infty}_{W_1}\colon \mathcal{B}^\sigma(W_1(S))\to\mathcal{B}^\sigma(Y_-)$; likewise, $\mathcal{B}^\sigma(W_2(S))$ has a well-defined $(+\infty)$–limit map, $\Pi^\infty:=\Pi^\infty_{W_2}\colon\mathcal{B}^\sigma(W_2(S))\to\mathcal{B}^\sigma(Y_+)$. (Recall that these maps played important roles in the construction of the cochains in Section 2.5. These are not available with the spaces $\mathcal{B}^\sigma((W_i)_c)$ used in [17].) Denote the fibered space by

$$\boldsymbol{\mathcal{B}}^\sigma(W_2\circ W_1) := \bigcup_{S\in[0,\infty]}\{S\}\times\mathcal{B}^\sigma(W_2\circ_S W_1).$$

(This space is homeomorphic to (6-24) if endowed with the stronger Banach topology.) When S is finite, let $\Pi^{Y_0}_{W_*}\colon\mathcal{B}^\sigma(W_*(S))^\circ\to\mathcal{B}^\sigma(Y_0)$ denote the map of restricting to the 3–manifold $\{0\}\times Y_0\subset W_*(S)$ for $W_*=W,W_1,W_2$. (Again, the superscript $\circ$ is used to denote an appropriate open dense subspace. It is sometimes dropped to make the notation less cumbersome. As previously mentioned, this makes no practical difference.) As was done in Section 2, when $S=\infty$, let $\Pi^{Y_0}_{W_i}\colon\mathcal{B}^\sigma(W_i)^\circ\to\mathcal{B}^\sigma(Y_0)$ denote the map of taking $(+\infty)$–limits for $i=1$, and that of taking the $(-\infty)$–limits for $i=2$. Slightly abusing notation, we now let $s_i\colon\mathcal{B}^\sigma(W(S))^\circ\to\mathcal{B}^\sigma(W_i(S))$ for $i=1,2$ denote the map of restricting to $W_i(S)\subset W(S)$. Equation (6-25) has straightforward analog here: For finite S, the map $s_1\times s_2$ factors as

$$\begin{aligned}\text{(6-30)}\quad \mathcal{B}^\sigma(W(S))^\circ \xrightarrow{s_1\times s_2} \mathcal{B}^\sigma(W_1(S))\times_{\mathcal{B}^\sigma(Y_0)}\mathcal{B}^\sigma(W_2(S))&\\ \hookrightarrow \mathcal{B}^\sigma(W_1(S))\times\mathcal{B}^\sigma(W_2(S)),&\end{aligned}$$

In the above, the fiber product $\mathcal{B}^\sigma(W_1(S))\times_{\mathcal{B}^\sigma(Y_0)}\mathcal{B}^\sigma(W_2(S))$ is regarded as a subspace of the product $\mathcal{B}^\sigma(W_1(S))\times\mathcal{B}^\sigma(W_2(S))$, where the maps $\Pi^{Y_0}_{W_i}\colon\mathcal{B}^\sigma(W_i)\to\mathcal{B}(Y_0)$ for $i=1,2$ take the same value. The previously introduced maps $\mathfrak{r}$, ς, $\mathfrak{r}|_S$ and $\varsigma|_S$, as well as the inner products $c(u_1\otimes u_2)$, also admit straightforward adaptions, which we denote by the same notation. For finite S, the restriction of the maps $\mathfrak{r}|_S$ and $\varsigma|_S$ to $\mathcal{M}_{k,z}(W(S),\mathfrak{c}_-,\mathfrak{c}_+)$ are respectively the composition of $\mathfrak{r}$ and ς with (6-30). The arguments of [17] still apply with this modification to establish (6-21) in the context of more general $u_i\in\mathrm{C}(\mathcal{B}^\sigma_{\mathrm{loc}}(W_i);\mathbb{K})$.

Even in this (slightly) generalized form, specific applications of (6-21) in our context are often complicated by the fact that the cochains u constructed in Section 2.5 do not necessarily take the form of, or have no obvious interpretation as, an inner product $c(u_1 \otimes u_2)$. In view of the roles played by bundles over $\mathcal{B}^\sigma(W)$ (such as $\mathcal{B}^\sigma_x(W)$ or $\mathcal{B}^\sigma_\lambda(W)$) in the construction of the cochains from Section 2.5, we typically deal with this problem by going through various bundles over $\boldsymbol{\mathcal{B}}^\sigma(W_2 \circ W_1)$. They are constructed in a manner similar to what was done in Section 2.5. For example, what will be called $\widetilde{\boldsymbol{\mathcal{B}}}^\sigma_x(W_2 \circ W_1)$ is defined as follows.

Take a point $x \in W_c$ that lies in the 3–manifold $Y_0 \subset W_c$ that separates $(W_1)_c$ and $(W_2)_c$. (The point $x \in W$ is denoted by $\underline{x}$ when regarded as a point in the 3–manifold Y_0.) Recall the $U(1)$–bundles $\pi_{\underline{x}}\colon \widetilde{\mathcal{B}}^\sigma_{\underline{x}}(Y_0) \to \mathcal{B}^\sigma(Y_0)$ and $\pi_x\colon \widetilde{\mathcal{B}}^\sigma_x(W) \to \mathcal{B}^\sigma(W)$ from Section 2.5. For finite S, the map $\Pi^{Y_0}_W$ lifts to a map $\widetilde{\Pi}^{Y_0}_W\colon \widetilde{\mathcal{B}}^\sigma_x(W(S)) \to \widetilde{\mathcal{B}}_{\underline{x}}(Y_0)^\sigma$ by construction. Meanwhile, for $i = 1, 2$ and any $S \in [0, \infty]$, one may define $\widetilde{\Pi}^{Y_0}_{W_i}$, $\pi^{W_i}_x$ and $\widetilde{\mathcal{B}}^\sigma_x(W_i)$ through the commutative diagram

$$
\begin{array}{ccc}
\widetilde{\mathcal{B}}^\sigma_x(W_i(S)) & \xrightarrow{\widetilde{\Pi}^{Y_0}_{W_i}} & \widetilde{\mathcal{B}}^\sigma_{\underline{x}}(Y_0) \\
\Big\downarrow{\scriptstyle \pi^{W_i}_x} & & \Big\downarrow{\scriptstyle \pi_{\underline{x}}} \\
\mathcal{B}^\sigma(W_i(S)) & \xrightarrow{\Pi^{Y_0}_{W_i}} & \mathcal{B}^\sigma(Y_0)
\end{array}
\tag{6-31}
$$

Now let

$$
\begin{aligned}
\widetilde{\mathcal{B}}^\sigma_x(W_2 \circ_S W_1) &:= \widetilde{\mathcal{B}}^\sigma_x(W_1(S)) \times \widetilde{\mathcal{B}}^\sigma_x(W_2(S)), \\
\widetilde{\boldsymbol{\mathcal{B}}}^\sigma_x(W_2 \circ W_1) &:= \bigcup_{S \in [0,\infty]} \{S\} \times \widetilde{\mathcal{B}}^\sigma_x(W_2 \circ_S W_1).
\end{aligned}
$$

By construction, these are $U(1) \times U(1)$–bundles respectively over $\mathcal{B}^\sigma(W_2 \circ_S W_1)$ and $\boldsymbol{\mathcal{B}}^\sigma(W_2 \circ W_1)$. The fibered product $\widetilde{\mathcal{B}}^\sigma_x(W_1(S)) \times_{\widetilde{\mathcal{B}}_x(Y_0)} \widetilde{\mathcal{B}}^\sigma_x(W_2(S))$, as a subspace of $\widetilde{\mathcal{B}}^\sigma_x(W_1(S)) \times \widetilde{\mathcal{B}}^\sigma_x(W_2(S))$, is preserved under the diagonal $U(1)$–action. The quotienting by this action is

$$
\begin{aligned}
\big(\widetilde{\mathcal{B}}^\sigma_x(W_1(S)) \times_{\widetilde{\mathcal{B}}_x(Y_0)} \widetilde{\mathcal{B}}^\sigma_x(W_2(S))\big)/U(1)_\Delta &\simeq \mathcal{B}^\sigma(W_1(S)) \times_{\mathcal{B}^\sigma(Y_0)} \mathcal{B}^\sigma(W_2(S)) \\
&\hookrightarrow \mathcal{B}^\sigma(W_1(S)) \times \mathcal{B}^\sigma(W_2(S)) =: \mathcal{B}^\sigma(W_2 \circ_S W_1),
\end{aligned}
$$

where $U(1)_\Delta \subset U(1) \times U(1)$ denotes the diagonal.

The previously introduced restriction maps $s_i\colon \mathcal{B}^\sigma(W(S))^\circ \to \mathcal{B}^\sigma(W_i(S))$ lift to define maps $\tilde{s}_i\colon \widetilde{\mathcal{B}}^\sigma_x(W(S))^\circ \to \widetilde{\mathcal{B}}^\sigma_x(W_i(S))$ for $i = 1, 2$. With them we have the following variant of (6-30) for *finite* S:

$$\begin{aligned}(6\text{-}32)\quad \widetilde{\mathcal{B}}^{\sigma}_{x}(W(S)) &\xrightarrow{\tilde{s}_1\times\tilde{s}_2} \widetilde{\mathcal{B}}^{\sigma}_{x}(W_1(S))\times_{\widetilde{\mathcal{B}}_{\underline{x}}(Y_0)}\widetilde{\mathcal{B}}^{\sigma}_{x}(W_2(S))\\ &\hookrightarrow \widetilde{\mathcal{B}}^{\sigma}_{x}(W_1(S))\times\widetilde{\mathcal{B}}^{\sigma}_{x}(W_2(S)) := \widetilde{\mathcal{B}}^{\sigma}_{x}(W_2\circ_S W_1);\end{aligned}$$

and together they form a commutative diagram

$$(6\text{-}33)\quad \begin{tikzcd} \widetilde{\mathcal{B}}^{\sigma}_{x}(W(S)) \arrow[r, "\tilde{s}_1\times\tilde{s}_2"] \arrow[d, "\pi^W_x"'] & \widetilde{\mathcal{B}}^{\sigma}_{x}(W_1(S))\times_{\widetilde{\mathcal{B}}_{\underline{x}}(Y_0)}\widetilde{\mathcal{B}}^{\sigma}_{x}(W_2(S)) \arrow[r, "\text{embeds}"] \arrow[d, "\pi^{W_1}_x\times\pi^{W_2}_x"'] & \widetilde{\mathcal{B}}^{\sigma}_{x}(W_2\circ_S W_1) \arrow[d, "\pi^{W_2\circ_S W_1}_x:=\pi^{W_1}_x\times\pi^{W_2}_x"'] \\ \mathcal{B}^{\sigma}(W(S)) \arrow[r, "s_1\times s_2"] & \mathcal{B}^{\sigma}(W_1(S))\times_{\mathcal{B}^{\sigma}(Y_0)}\mathcal{B}^{\sigma}(W_2(S)) \arrow[r, "\text{embeds}"] & \mathcal{B}^{\sigma}(W_2\circ_S W_1) \end{tikzcd}$$

The pair of horizontal maps $\tilde{s}_1\times\tilde{s}_2$ and $s_1\times s_2$ in the left square above define a map between $U(1)$–bundles (but not the right square), and the map $(\widetilde{\Pi}^{Y_0}_W, \Pi^{Y_0}_W)$ between the $U(1)$–bundles $\pi^W_x\colon \widetilde{\mathcal{B}}^{\sigma}_{x}(W(S))\to\mathcal{B}^{\sigma}(W(S))$ and $\pi_{\underline{x}}\colon \widetilde{\mathcal{B}}^{\sigma}_{\underline{x}}(Y_0)\to\mathcal{B}^{\sigma}(Y_0)$ factors through the bundle map

$$(6\text{-}34)\quad \begin{tikzcd} \widetilde{\mathcal{B}}^{\sigma}_{x}(W(S)) \arrow[r, "\pi^W_x"] \arrow[d, "\tilde{s}_1\times\tilde{s}_2"] \arrow[dd, bend right=60, "\widetilde{\Pi}^{Y_0}_W"'] & \mathcal{B}^{\sigma}(W(S)) \arrow[d, "s_1\times s_2"] \arrow[dd, bend left=60, "\Pi^{Y_0}_W"] \\ \widetilde{\mathcal{B}}^{\sigma}_{x}(W_1(S))\times_{\widetilde{\mathcal{B}}_{\underline{x}}(Y_0)}\widetilde{\mathcal{B}}^{\sigma}_{x}(W_2(S)) \arrow[r, "\pi^{W_1}_x\times\pi^{W_2}_x"] \arrow[d, "\widetilde{\Pi}^{Y_0}_{W_1}=\widetilde{\Pi}^{Y_0}_{W_2}"] & \mathcal{B}^{\sigma}(W_1(S))\times_{\mathcal{B}^{\sigma}(Y_0)}\mathcal{B}^{\sigma}(W_2(S)) \arrow[d, "\Pi^{Y_0}_{W_1}=\Pi^{Y_0}_{W_2}"'] \\ \widetilde{\mathcal{B}}^{\sigma}_{\underline{x}}(Y_0) \arrow[r, "\pi_{\underline{x}}"] & \mathcal{B}^{\sigma}(Y_0) \end{tikzcd}$$

As a general rule, in what follows we adopt the convention of adding subscripts or superscripts W in notation previously introduced in Section 2 to emphasize the cobordism referred to. For example, $\Pi^{\pm\infty}_W$ denote the version of the $(\pm\infty)$–limit map $\Pi^{\pm\infty}$ for the cobordism W, and π^W_x is the W version of the projection map π_x in Section 2.

We shall also use other variants of the bundle $\pi^{W_2\circ W_1}_x\colon \widetilde{\boldsymbol{\mathcal{B}}}^{\sigma}_{x}(W_2\circ W_1)\to\boldsymbol{\mathcal{B}}^{\sigma}(W_2\circ W_1)$. These are constructed in a similar fashion, with the role of $\widetilde{\mathcal{B}}^{\sigma}_{x}(Y_0)$ replaced by other bundles over $\mathcal{B}^{\sigma}(Y_0)$, say $\widetilde{\mathcal{B}}^{\sigma}_{\nu}(Y_0)$, ν being a 0–chain in Y_0. The composition formula (6-21) is verified for a cochain u_γ from Section 2.5 by applying the trick already used repeatedly in Section 2; see eg the diagrams (2-34) and (2-46). That is, we choose an appropriate lift of the embedding $\boldsymbol{\varsigma}\colon \overline{\boldsymbol{M}}\to\boldsymbol{\mathcal{B}}^{\sigma}(W_2\circ W_1)$ to $\tilde{\boldsymbol{\varsigma}}\colon \boldsymbol{M}^+\to\widetilde{\boldsymbol{\mathcal{B}}}^{\sigma}(W_2\circ W_1)$

that fit in a commutative diagram of the form

$$\begin{tikzcd} \boldsymbol{M}^+ \arrow[r, "\tilde{\varsigma}"] \arrow[d, "\mathfrak{r}"'] & \tilde{\mathcal{B}}^\sigma(W_2 \circ W_1) \arrow[d, "\pi"'] \\ \overline{\boldsymbol{M}} \arrow[r, "\varsigma"] & \mathcal{B}^\sigma(W_2 \circ W_1) \end{tikzcd} \tag{6-35}$$

and apply Stokes' theorem over the top row of the preceding diagram. The cochain $u \in \mathrm{C}(\mathcal{B}^\sigma(W); \mathbb{K})$ is typically interpreted in terms of inner products by considering a variant of the diagram (6-33) for finite S. Once in the inner product form, the cochain u extends to be defined on the fiber at infinity, $\mathcal{M}^+(W(\infty))$, and consequently also a cochain $\boldsymbol{u}$ suitable for applying the arguments of (6-21). As outlined previously, the term on the left-hand side of (6-21), $\widehat{m}[u_2](W_2)\widehat{m}[u_1](W_1)$, arises from integrals over strata in $\mathcal{M}^+(W(\infty))$. To put the integrals in a suitable product form, we must factor the strata of $\mathcal{M}^+(W(\infty), \mathfrak{c}_-, \mathfrak{c}_+)$ as products of two spaces, provisionally written as $\mathcal{M}^+_{W_1}(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}^+_{W_2}(\mathfrak{c}, \mathfrak{c}_+)$, with $\mathfrak{c} \in \mathcal{B}^\sigma(Y)$. Here, $\mathcal{M}^+_{W_1}(\mathfrak{c}_-, \mathfrak{c})$ consists of "broken W_1–paths" from $\mathfrak{c}_-$ to $\mathfrak{c}$, and $\mathcal{M}^+_{W_2}(\mathfrak{c}, \mathfrak{c}_+)$ consists of "broken W_2–paths" from $\mathfrak{c}$ to $\mathfrak{c}_+$. Recall from [17, (26.4)] that a general element of $\mathcal{M}^+(W(\infty))$ is defined to be an element in a product space

$$\begin{aligned} \mathcal{N}^+(Y_-, \mathfrak{c}_-, \mathfrak{c}'_-) \times \mathcal{M}(W_1, \mathfrak{c}'_-, \mathfrak{c}_{0-}) \times \mathcal{N}^+(Y_0, \mathfrak{c}_{0-}, \mathfrak{c}_{0+}) \\ \times \mathcal{M}(W_2, \mathfrak{c}_{0+}, \mathfrak{c}'_+) \times \mathcal{N}^+(Y_+, \mathfrak{c}'_+, \mathfrak{c}_+). \end{aligned} \tag{6-36}$$

There are different ways of organizing this space in the form $\mathcal{M}^+_{W_1}(\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}^+_{W_2}(\mathfrak{c}, \mathfrak{c}_+)$. When deriving the hat version of composition formula, we take $\mathcal{M}^+_{W_1}(\mathfrak{c}_-, \mathfrak{c})$ and $\mathcal{M}^+_{W_2}(\mathfrak{c}, \mathfrak{c}_+)$, respectively, to be the first and the second line of the preceding expression. (For the check version, one takes $\mathcal{M}^+_{W_1}(\mathfrak{c}_-, \mathfrak{c})$ to be the product of the first two factors in (6-36), and $\mathcal{M}^+_{W_2}(\mathfrak{c}, \mathfrak{c}_+)$ the product of the remaining factors.) Applying [17, Proposition 26.1.6] to write out each entry of the identity from Stokes' theorem as a sum in the manner of [17, (26.13)] leads to a variant of the composition formula (6-21) (but with our generalized definition of the cobordism maps)

$$\widehat{m}[u](W(\infty)) = \widehat{m}[u](W) + [\widehat{\mathrm{K}}[\boldsymbol{u}](W), \widehat{\partial}] + \widehat{\mathrm{K}}[\delta \boldsymbol{u}](W). \tag{6-37}$$

Depending on how u is expressed in terms of inner products, the left-hand side, $\widehat{m}[u](W(\infty))$, will expressed in terms of products of the form $\widehat{m}[u_2](W_2)\widehat{m}[u_1](W_1)$. Note though that, compared with the long sum in the expression following [17, (26.13)], in our case there will be additional terms involving boundary-obstructed maps of the form $\overline{n}^s_u[\mathrm{u}](Y_0)$ (including the boundary-obstructed differential $\overline{\partial}^s_u(Y_0)$ that appears in [17]'s formula). These additional terms are absorbed in our generalized definition of $\widehat{m}[u]$ and $\widehat{\mathrm{K}}[\boldsymbol{u}]$,

$$\widehat{m}[u](W(\infty)) = \widehat{m}[u](W) + [\widehat{\mathrm{K}}[\boldsymbol{u}](W), \widehat{\partial}] + \widehat{\mathrm{K}}[\delta\boldsymbol{u}](W). \tag{6-38}$$

Step 4 We now apply the general discussion above to derive the identities (6-19) and (6-20). What follows describes the formulas in (6-22) in a more explicit manner. The degree -1 map $H_\#\colon C_\# \to C_\#$ is given by

$$H_\# = \sum_{\mathfrak{c}_1,\mathfrak{c}_2\in\mathfrak{C}_\#}\ \sum_{z\in\pi_0(\mathcal{B}^\sigma(M_\#,\mathfrak{c}_1,\mathfrak{c}_2))} \langle \varsigma^* \boldsymbol{u}_{\lambda_\#}, \boldsymbol{M}_{1,z}(W_\#, \mathfrak{c}_1, \mathfrak{c}_2)\rangle \Gamma_\#(z).$$

For practical purposes, it is often more convenient to work with the more concrete variant of $H_\#$; this is denoted by $\dot{H}_\# = \widehat{\mathrm{K}}[\boldsymbol{\theta}_{\lambda_\#}](W_\#)$ below and is defined by replacing $\boldsymbol{u}_{\lambda_\#}$ in the preceding formula by $\boldsymbol{\theta}_{\lambda_\#}$. The two maps $H_\#$ and $\dot{H}_\#$ are related by

$$H_\# = \dot{H}_\# + [\widehat{\partial}_\#, \widehat{\mathrm{K}}[\boldsymbol{\varepsilon}_{\lambda_\#}]],$$

where $\boldsymbol{\varepsilon}_{\lambda_\#}$ is the parametrized variant of the 0–cochain $\varepsilon_{\lambda_\#}$ defined in Section 2.5's Part 1(b).

Likewise, the maps $A, B, C, D\colon \widehat{C}_\sqcup \to \widehat{C}_\sqcup$, are assembled respectively from the constituents $A^{o\sharp}_{o\flat}, B^{o\sharp}_{o\flat}, C^{o\sharp}_{o\flat}, D^{o\sharp}_{o\flat}\colon C^{o\sharp}_\sqcup \to C^{o\flat}_\sqcup$:

$$\begin{aligned}
A^{o\sharp}_{o\flat} &= \sum_{\mathfrak{c}_1\in\mathfrak{C}^{o\sharp}_\sqcup}\ \sum_{\mathfrak{c}_2\in\mathfrak{C}^{o\flat}_\sqcup}\ \sum_{z\in\pi_0(\mathcal{B}^\sigma(M_\sqcup,\mathfrak{c}_1,\mathfrak{c}_2))} \langle \varsigma^*\boldsymbol{u}_{\lambda_{\sqcup -}}, \boldsymbol{M}_{1,z}(W_\sqcup, \mathfrak{c}_1, \mathfrak{c}_2)\rangle \Gamma_\sqcup(z),\\
B^{o\sharp}_{o\flat} &= \sum_{\mathfrak{c}_1\in\mathfrak{C}^{o\sharp}_\sqcup}\ \sum_{\mathfrak{c}_2\in\mathfrak{C}^{o\flat}_\sqcup}\ \sum_{z\in\pi_0(\mathcal{B}^\sigma(M_\sqcup,\mathfrak{c}_1,\mathfrak{c}_2))} \langle \varsigma^*\boldsymbol{u}_{\varnothing}, \boldsymbol{M}_{0,z}(W_\sqcup, \mathfrak{c}_1, \mathfrak{c}_2)\rangle \Gamma_\sqcup(z),\\
C^{o\sharp}_{o\flat} &= \sum_{\mathfrak{c}_1\in\mathfrak{C}^{o\sharp}_\sqcup}\ \sum_{\mathfrak{c}_2\in\mathfrak{C}^{o\flat}_\sqcup}\ \sum_{z\in\pi_0(\mathcal{B}^\sigma(M_\sqcup,\mathfrak{c}_1,\mathfrak{c}_2))} \langle \varsigma^*\boldsymbol{u}_{\lambda_\sqcup}, \boldsymbol{M}_{2,z}(W_\sqcup, \mathfrak{c}_1, \mathfrak{c}_2)\rangle \Gamma_\sqcup(z),\\
D^{o\sharp}_{o\flat} &= \sum_{\mathfrak{c}_1\in\mathfrak{C}^{o\sharp}_\sqcup}\ \sum_{\mathfrak{c}_2\in\mathfrak{C}^{o\flat}_\sqcup}\ \sum_{z\in\pi_0(\mathcal{B}^\sigma(M_\sqcup,\mathfrak{c}_1,\mathfrak{c}_2))} \langle \varsigma^*\boldsymbol{u}_{\lambda_{\sqcup +}}, \boldsymbol{M}_{1,z}(W_\sqcup, \mathfrak{c}_1, \mathfrak{c}_2)\rangle \Gamma_\sqcup(z).
\end{aligned} \tag{6-39}$$

The reducible variants of the above, $\overline{\mathrm{K}}[\boldsymbol{u}](W_\sqcup)$, do not appear in the formulas for A, B, C and D, as by assumption $W_\sqcup$ is equipped with nonbalanced perturbations. However, keep in mind that while B, the simplest map among the four, is assembled from the above according to the rule (2-20) (substituting $m^\sharp_\flat[u]$ therein by $B^{o\sharp}_{o\flat}$), the more general rule of Remark 2.2 must be applied to construct the more complicated maps A, C and D from their constituents above. Being a hat version of a cobordism map, the formula for A remains the same as that given in (2-20) since both endpoints of $\overline{\lambda}$ fall in Y_-. The formulas for C and D contain additional terms from the endpoints of λ in Y_+, similar to those in (2-44) and (2-45): for $\mathrm{K} = C$ or D, the explicit formula for the cobordism map C or D is

$$\mathrm{K} = \begin{bmatrix} \mathrm{K}^{oo}_{oo} & \mathrm{K}^{ou}_{oo} \\ \hat{\mathrm{K}}^{oo}_{ou} & \hat{\mathrm{K}}^{ou}_{ou} \end{bmatrix}, \tag{6-40}$$

where (as the $\overline{\mathrm{K}}$–terms vanish)

$$\begin{aligned} \hat{\mathrm{K}}^{oo}_{ou} &:= -(\overline{\partial}_{\sqcup})^{os}_{ou}\mathrm{K}^{oo}_{os} + (\overline{U}_{\sqcup})^{os}_{ou}B^{oo}_{os}, \\ \hat{\mathrm{K}}^{ou}_{ou} &:= -\overline{\mathrm{K}}^{ou}_{ou} - (\overline{\partial}_{\sqcup})^{os}_{ou}\mathrm{K}^{ou}_{os} + (\overline{U}_{\sqcup})^{os}_{ou}B^{ou}_{os}. \end{aligned} \tag{6-41}$$

The maps A, C and D above also each has a companion version, denoted respectively by $\dot{A}$, $\dot{C}$ and $\dot{D}$. They are defined from constituents given by the same formulas as in (6-39), with the cochains $u_{\lambda_{\sqcup -}}$, $u_{\lambda_{\sqcup}}$ and $u_{\lambda_{\sqcup +}}$ therein replaced by their more concrete variants, that is, respectively by $\theta_{\lambda_{\sqcup -}}$, $\theta_{\lambda_{\sqcup -}} \wedge \theta_{\lambda_{\sqcup +}}$ and $\theta_{\lambda_{\sqcup +}}$. The maps $\dot{A}$, $\dot{C}$ and $\dot{D}$ are built from these constituents in manners parallel to their sister versions above, but in the case of $\dot{C}$ and $\dot{D}$, all appearances of the boundary-obstructed map $(\overline{U}_{\sqcup})^{os}_{ou}$ in (6-41) are replaced by its companion version,

$$1 \otimes \overline{n}^{s}_{u}[d\underline{\mathrm{h}}_{\hat{p}_2}](M_2) - n^{o}_{o}[d\underline{\mathrm{h}}_{\hat{p}_1}](M_1) \otimes 1 =: \overline{n}^{os}_{ou}[d\underline{\mathrm{h}}_{\hat{\sqcup}}](M_{\sqcup}).$$

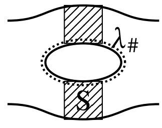
The aforementioned pairs of maps are related by the formulas

$$A = \dot{A} - B\Theta_{\sqcup}, \quad C = \dot{C} + \Theta_{\sqcup}A - \dot{D}\Theta_{\sqcup} = \dot{C} + \Theta_{\sqcup}\dot{A} - D\Theta_{\sqcup}, \quad D = \dot{D} + \Theta_{\sqcup}B, \tag{6-42}$$

where

$$\Theta_{\sqcup} = 1 \otimes \Theta_{p_2} - \Theta_{p_1} \otimes 1 = \hat{n}[\underline{h}_{\hat{p}_1}] \otimes 1 - 1 \otimes \hat{n}[\underline{h}_{\hat{p}_2}] =: -\hat{n}[\underline{h}_{\sqcup}].$$

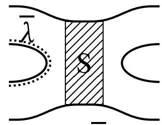
We shall make use of the following figure, which illustrates the construction of $H_\#$ schematically:

(6-43) $-\infty$–end at $M_\#$ 

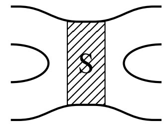
$+\infty$–end at $M_\# \rightsquigarrow$ the map $H_\#$.

$W_\#(S) \supset \lambda_\#$

The construction of the maps $A, B, C, D\colon \hat{C}_{\sqcup} \to \hat{C}_{\sqcup}$ (or their companions $\dot{A}$, $\dot{B} := B$, $\dot{C}$ and $\dot{D}$) are illustrated in a similar fashion in the next set of pictures:

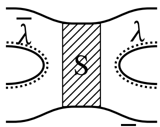
$-\infty$–end at $M_\sqcup$  $+\infty$–end at $M_\sqcup \rightsquigarrow$ the map A or $\dot{A}$,

$W_\sqcup(S) \supset \overline{\lambda} = \lambda_{\sqcup -}$

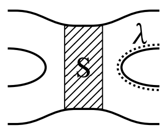
$-\infty$–end at $M_\sqcup$  $+\infty$–end at $M_\sqcup \rightsquigarrow$ the map $B = \dot{B}$,

$W_\sqcup(S) \supset \varnothing$

(6-44)

$-\infty$–end at $M_\sqcup$  $+\infty$–end at $M_\sqcup \rightsquigarrow$ the map C or $\dot{C}$,

$W_\sqcup(S) \supset \lambda \cup \overline{\lambda} = \lambda_\sqcup$

$-\infty$–end at $M_\sqcup$  $+\infty$–end at $M_\sqcup \rightsquigarrow$ the map D or $\dot{D}$.

$W_\sqcup(S) \supset \lambda = \lambda_{\sqcup +}$

The dotted 1–submanifold γ (possibly empty or disconnected) in each of the cobordisms W in (6-43) and (6-44) is there to indicate that the map on the right of the picture is constructed via coefficients given by evaluating the cochain $\boldsymbol{u}_\gamma$ (or $\boldsymbol{\theta}_\gamma$) associated to $u_\gamma \in \mathrm{C}^*(\mathcal{B}^\sigma(W);\mathbb{K})$ (or θ_γ) on relevant parametrized moduli spaces $\boldsymbol{M}$ associated to W ($\gamma = \lambda_\#$ in (6-43) and $\gamma = \lambda_{\sqcup -}, \varnothing, \lambda_\sqcup, \lambda_{\sqcup +}$ respectively in the four lines in (6-44)).

We now proceed with:

(i) (verifying (6-19)) Reexpressed using the more concrete companion, $\theta_{\lambda_\#}$, of $u_{\lambda_\#}$, this identity is equivalent to

$$\widehat{m}[\theta_{\overline{\lambda}}](\overline{\mathcal{V}}) \circ \widehat{m}[1](\mathcal{V}) + \widehat{m}[1](\overline{\mathcal{V}}) \circ \widehat{m}[\theta_\lambda](\mathcal{V}) = m[\theta_{\lambda_\#}](W_\#) + [\widehat{\mathrm{K}}[\boldsymbol{\theta}_{\lambda_\#}](W_\#), \partial_\#] = m[\theta_{\lambda_\#}](W_\#) + [\dot{H}_\#, \partial_\#]. \tag{6-45}$$

To verify (6-45), we shall apply (6-38) to $W = W_\#$, $W_1 = \mathcal{V}$, $W_2 = \overline{\mathcal{V}}$ and $u = \theta_{\lambda_\#}$. Note that $\delta\theta_{\lambda_\#} = 0$, and thus the last term of (6-38) vanishes. The right-hand side of (6-45) then coincides term by term with the first two terms of the right-hand side of (6-38). To compute the left-hand side of (6-38), namely $\widehat{m}[\theta_{\lambda_\#}](W(\infty))$, we claim that

$$\theta_{\lambda_\#} = c(\theta_\lambda \otimes 1) + c(1 \otimes \theta_{\overline{\lambda}}). \tag{6-46}$$

This would then imply that

$$\widehat{m}[\theta_{\lambda_\#}](W(\infty)) = \widehat{m}[\theta_{\bar\lambda}](\bar{\mathcal{V}}) \circ \widehat{m}[1](\mathcal{V}) + \widehat{m}[1](\bar{V}) \circ \widehat{m}[\theta_\lambda],$$

thus establishing (6-45).

To verify (6-46), recall the definitions $\theta_{\lambda_\#} = d\,\mathrm{hol}_{\lambda_\#}$ and $\mathrm{hol}_{\lambda_\#}\colon \mathcal{B}^\sigma(W_\#) \to \mathbb{R}/\mathbb{Z}$ from Section 2.5's Part 1(b). Recall also the bundles and maps in (2-40) and (2-41). Use $\widetilde{\widetilde{\mathcal{B}}}{}^\sigma_{p_1,p_2}(M_\sqcup)$, $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$, $\mathcal{B}^\sigma(M_\sqcup)$ and $\mathfrak{C}(M_\sqcup)$ to denote the terms, top to bottom, in the right column of (2-40). ($\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ and $\widetilde{\mathcal{B}}^\sigma_{p_1-p_2}(M_\sqcup)$ denote the same space, but our convention is to use the notation $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ when it is equipped with the $U(1)$–action associated to the Thom form $\vartheta'_{p_2-p_1}$ given in (2-39). Thus, $\widetilde{\mathcal{B}}^\sigma_{p_1-p_2}(M_\sqcup)$ is endowed with the dual $U(1)$–action.) Recall the maps $\mathrm{hol}_\lambda\colon \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \to \widetilde{\widetilde{\mathcal{B}}}_{p_1,p_2}(M_\sqcup)$ and $\mathrm{hol}_{\bar\lambda}\colon \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}) \to \widetilde{\widetilde{\mathcal{B}}}_{p_1,p_2}(M_\sqcup)$ from (2-41) and fix ϑ'_{p_1}, ϑ'_{p_2}, $\rho_{\vartheta'_{p_1}}$ and $\rho_{\vartheta'_{p_2}}$ as was done there, using them to define *both* $\mathrm{h}_\lambda\colon \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \to \mathbb{R}/\mathbb{Z}$ and $\mathrm{h}_{\bar\lambda}\colon \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}) \to \mathbb{R}/\mathbb{Z}$, as prescribed in Section 2.5's Part 3(a).

To interpret $u_{\lambda_\#}$ in terms of inner products, consider now the analog of (6-34): Let $\widetilde{\Pi}^{M_\sqcup}_{W_\#}\colon \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(W_\#(S)) \to \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)$ denote the pullback of $\Pi^{M_\sqcup}_{W_\#}\colon \mathcal{B}^\sigma(W_\#(S)) \to \mathcal{B}^\sigma(M_\sqcup)$ under the map $\pi_{p_2-p_1}\colon \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup) \to \mathcal{B}^\sigma(M_\sqcup)$. Let

$$\tilde{s}_1\colon \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(W_\#(S)) \to \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}(S)) \quad \text{and} \quad \tilde{s}_2\colon \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(W_\#(S)) \to \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}(S))$$

be the direct analogs of their counterparts in (6-34). The analog of (6-33) in the present context reads

$$\begin{tikzcd}[column sep=small] \widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(W_\#(S)) \arrow[r, "\tilde{s}_1\times\tilde{s}_2"] \arrow[d, "\pi^W_{\lambda_\#}"'] & \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times_{\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)} \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}) \arrow[r, "\text{embeds}"] \arrow[d, "\pi^{\mathcal{V}}_\lambda\times\pi^{\bar{\mathcal{V}}}_{\bar\lambda}"'] & \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}) \arrow[d, "\pi^{\mathcal{V}}_\lambda\times\pi^{\bar{\mathcal{V}}}_{\bar\lambda}"'] \\ \mathcal{B}^\sigma(W_\#(S)) \arrow[r, "s_1\times s_2"] & \mathcal{B}^\sigma(\mathcal{V}(S)) \times_{\mathcal{B}^\sigma(M_\sqcup)} \mathcal{B}^\sigma(\bar{\mathcal{V}}(S)) \arrow[r, "\text{embeds}"] & \mathcal{B}^\sigma(\mathcal{V}(S)) \times \mathcal{B}^\sigma(\bar{\mathcal{V}}(S)) \end{tikzcd} \tag{6-47}$$

Now observe that:

- On the top row, the pullback of the $U(1) = \mathbb{R}/\mathbb{Z}$–valued function

$$\mathrm{h}_\lambda \times 1 + 1 \times \mathrm{h}_{\bar\lambda} \tag{6-48}$$

  on $\widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}})$ to $\widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times_{\widetilde{\mathcal{B}}_{p_2-p_1}(M_\sqcup)} \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}}) \subset \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}})$ does not depend on the choices of ϑ'_{p_1}, ϑ'_{p_2}, $\rho_{\vartheta'_{p_1}}$ and $\rho_{\vartheta'_{p_2}}$.
- The preceding function is also invariant under the diagonal $U(1)$–action on $\widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \times_{\widetilde{\mathcal{B}}_{p_2-p_1}(M_\sqcup)} \widetilde{\mathcal{B}}^\sigma_{\bar\lambda}(\bar{\mathcal{V}})$, and hence descends to define an $\mathbb{R}/\mathbb{Z}$–valued

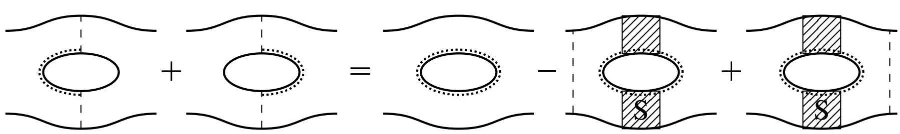
function on the space in the middle of the bottom row of the diagram, that is, $\mathcal{B}^\sigma(\mathcal{V}(S))\times_{\mathcal{B}^\sigma(M_\sqcup)}\mathcal{B}^\sigma(\bar{\mathcal{V}}(S))$.

- The $\mathbb{R}/\mathbb{Z}$–valued function $\mathrm{hol}_{\lambda_\#}$ on $\mathcal{B}^\sigma(W_\#)$ agrees with the pullback of the preceding function under the left arrow in the bottom row of the diagram, and we have
$$(\pi^{W_\#})^*\,\mathrm{hol}_{\lambda_\#}=(\tilde{s}_1\times\tilde{s}_2)^*(\mathrm{h}_\lambda\times 1+1\times\mathrm{h}_{\bar{\lambda}}).$$
Taking the differential on both sides, we have
$$(\pi^{W_\#})^*\theta_{\lambda_\#}=(\tilde{s}_1\times\tilde{s}_2)^*(\vartheta_\lambda\times 1+1\times\vartheta_{\bar{\lambda}})=(\tilde{s}_1)^*\vartheta_\lambda+(\tilde{s}_2)^*\vartheta_{\bar{\lambda}}$$
on $\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(W_\#(S))$.
- Recalling (2-43), we then have
$$\theta_{\lambda_\#}=(s_1\times s_2)^*(\theta_\lambda\times 1+1\times\theta_{\bar{\lambda}}),$$
since $(\tilde{s}_1)^*(\tilde{\Pi}_\lambda)^*\vartheta'_{p_2-p_1}=-(\tilde{s}_2)^*(\tilde{\Pi}_{\bar{\lambda}})^*\vartheta'_{p_1-p_2}$ on $\widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V})\times_{\widetilde{\mathcal{B}}^\sigma_{p_2-p_1}(M_\sqcup)}\widetilde{\mathcal{B}}^\sigma_{\bar{\lambda}}(\bar{\mathcal{V}})$. Meanwhile, the 1–form $\theta_\lambda\times 1+1\times\theta_{\bar{\lambda}}$ on $\mathcal{B}^\sigma(\mathcal{V})\times\mathcal{B}^\sigma(\bar{\mathcal{V}})$ is nothing but $c(\theta_\lambda\otimes 1)+c(1\otimes\theta_{\bar{\lambda}})$.

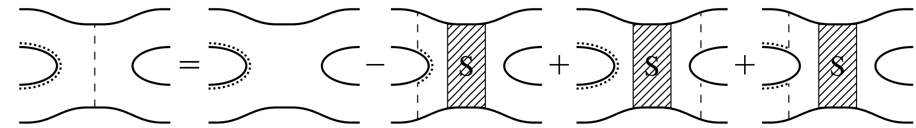
(ii) (verifying (6-20)) These identities follow directly from applying (6-21) to $W=W_\sqcup$, $W_1=\bar{\mathcal{V}}$ and $W_2=\mathcal{V}$, with u taken to be respectively to be $u_{\lambda_{\sqcup-}}$, $u_\varnothing=1$, $u_{\lambda_\sqcup}$ and $u_{\lambda_{\sqcup+}}$. These cochains have natural interpretations as inner products:
$$u_{\lambda_{\sqcup-}}=c(u_{\bar{\lambda}}\otimes 1),\quad u_\varnothing=c(1\otimes 1),\quad u_{\lambda_\sqcup}=c(u_{\bar{\lambda}}\otimes u_\lambda),\quad u_{\lambda_{\sqcup+}}=c(1\otimes u_\lambda).$$
In the case of item (2) of (6-20), $\delta u_\varnothing=0$ and therefore the last term of (6-21) vanishes. Meanwhile, a straightforward adaptation of (2-51) to the parametrized setting identifies $\hat{\mathrm{K}}[\boldsymbol{u}]$ in the cases of $u=u_{\lambda_{\sqcup-}}$, $u_{\lambda_\sqcup}$ and $u_{\lambda_{\sqcup+}}$ respectively with the last terms of (6-20)'s items (1), (2) and (4).

The figures below illustrate the identities in (6-19) and (6-20), as well as hint on their origins.

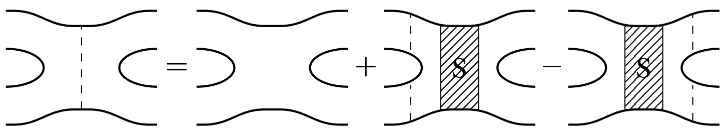
- For (6-19):

(6-49) 

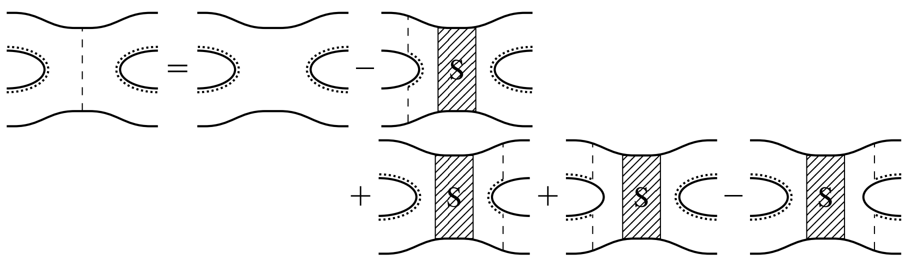
- The identity (6-20)(1):

(6-50) 

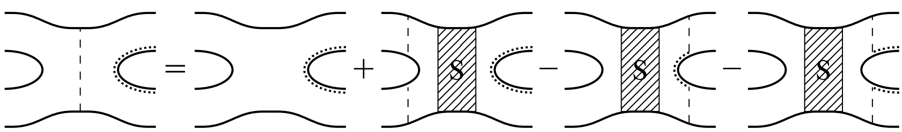
- The identity (6-20)(2):

(6-51) 

- The identity (6-20)(3):

(6-52) 

- The identity (6-20)(4):

(6-53) 

In each cobordism in the pictures, s increases from left to right. They are read as follows: The dashed lines (if present) in the cobordisms stand for 3–manifolds that split the cobordisms into a composition of what we call "factor cobordisms". Each factor cobordism (or the cobordism itself, if it is not split) in the pictures is associated with a pair (W, γ), where W is a cobordism and γ is a 1–submanifold (possibly empty) of W, the latter being represented by dotted arcs or circles. This pair is associated with a cobordism map of the form

(i) $\widehat{m}[u_\gamma]$ (resp. $\widehat{m}[\theta_\gamma]$) when (W, γ) is not cylindrical;

(ii) $\widehat{n}[\mathrm{u}_\gamma]$ (resp. $\widehat{m}[d\underline{\mathrm{h}}_\gamma]$) when (W, γ) is cylindrical, namely, it is of the form $\mathbb{R} \times (Y, p)$, p being a (possibly empty) 0–submanifold in Y;

(iii) $\widehat{\mathrm{K}}[\boldsymbol{u}_\gamma]$ (resp. $\widehat{\mathrm{K}}[\boldsymbol{\theta}_\gamma]$) when there is a shaded region in the cobordism.

Composition of cobordisms along the dashed lines correspond to compositions of maps associated to the factor cobordisms. For example, the dashed line in the first term of (6-49) splits the composite cobordism $W_\#$ into $\mathcal{V} \supset \lambda$ on the left and $\overline{\mathcal{V}} \supset \varnothing$ on the right. The left part $\mathcal{V} \supset \lambda$ corresponds to the map $V_1 = \widehat{m}[u_\lambda](\mathcal{V})$, and the right part corresponds to the map $V_0^\dagger = \widehat{m}[u_\varnothing](\overline{\mathcal{V}})$; therefore this term stands for $V_0^\dagger V_1$. The dashed line in the last term splits $W_\#(S)$ into $W_\#(S) \supset \lambda_\#$ on the left and the product cobordism $\mathbb{R} \times M_\# \supset \varnothing$ on the right. The former corresponds to the map $H_\#$ according to (6-43), and the latter corresponds to $\partial_\#$. Thus this term corresponds to the term $\partial_\# H_\#$ in (6-19). With γ again standing respectively for $\lambda_\#$, $\lambda_{\sqcup -}$, $\varnothing$, $\lambda_\sqcup$ and $\lambda_{\sqcup +}$ in (6-49), (6-50), (6-51), (6-52) and (6-53), the pictures suggest how each term of the identity arises from Stokes' theorem, that is, as the integral $\boldsymbol{u}_\gamma$ over a constituent stratum of the "boundary" (to be more precise, see (6-28)) of the relevant compactified parametrized moduli space. Each such constituent stratum corresponds to the moduli space of a particular type of "broken W–paths" (in keeping with [17]'s terminology; see Definition 23.3.2 therein). The type for each term is specified by the corresponding picture, with dashed lines signifying "breaking points" of the broken W–path. The integrands in the identities, being defined from differentials of holonomy maps along γ, take the simple form of an inner product under the decomposition when the dotted arc/circle γ does not intersect the dashed line. When they do intersect, the dashed line splits γ into two arcs γ_1 and γ_2, each lying in a factor cobordism under the decomposition. The holonomy along γ being the product of the holonomy along γ_1 and that along γ_2 (see eg (6-48)), the integral of θ_γ over the spaces of such broken W–paths is thus a sum of two terms, each involving integrating over one of the θ_{γ_i}. For example, this accounts for the two terms on the left-hand side of (6-19), as well as the last two terms of (6-53).

Step 5 The identities (6-19) and (6-20) reduce the proof of (6-16) and (6-17) to the next lemma, with the maps H', A', B', C' and D' from (6-16) and (6-17) taken to be

$$H' = H - \textsc{h}, \quad A' = A - \textsc{a}, \quad B' = B - \textsc{b}, \quad C' = C - \textsc{c}, \quad D' = D - \textsc{d},$$

H, A, B, C and D being the maps from (6-19) and (6-20), and $\textsc{h}$, $\textsc{a}$, $\textsc{b}$, $\textsc{c}$ and $\textsc{d}$ being as in the lemma below.

Lemma 6.8 *There exist maps* $\textsc{h}$, $\textsc{a}$, $\textsc{b}$, $\textsc{c}$ *and* $\textsc{d}$, *and* $\textsc{z}_\#$ *and* $\textsc{z}_\sqcup$, *such that*

$$\mathrm{Id}_\# - [\textsc{z}_\#, \partial_\#]_{\mathrm{odd}} = \widehat{m}[u_{\lambda_\#}](W_\#) + [\textsc{h}, \partial_\#]_{\mathrm{even}}, \tag{6-54}$$

$$\mathrm{Id}_\sqcup - [\textsc{z}_\sqcup, \partial_\sqcup]_{\mathrm{odd}} = \widehat{m}[u_{\lambda_{\sqcup -}}](W_\sqcup) + [\textsc{a}, \widehat{\partial}_\sqcup]_{\mathrm{even}} + \textsc{b}\widehat{U}_\sqcup, \tag{6-55)(1}$$

$$0 = \widehat{m}[1](W_\sqcup) - [\textsc{b}, \widehat{\partial}_\sqcup]_{\mathrm{odd}}, \tag{6-55)(2}$$

$$\text{(6-55)(3)}\qquad [\hat\partial_\sqcup, \mathrm{X}]_{\mathrm{even}} - [\hat U_\sqcup, \mathrm{Z}_\sqcup]_{\mathrm{even}} = \hat m[u_{\lambda_\sqcup}](W_\sqcup) + [\mathrm{C}, \hat\partial_\sqcup]_{\mathrm{odd}} - \hat U_\sqcup \mathrm{A} + \mathrm{D}\hat U_\sqcup,$$

$$\text{(6-55)(4)}\qquad \mathrm{Id}_\sqcup - [\mathrm{Z}_\sqcup, \partial_\sqcup]_{\mathrm{odd}} = \hat m[u_{\lambda_{\sqcup +}}](W_\sqcup) + [\hat\partial_\sqcup, \mathrm{D}]_{\mathrm{even}} - \hat U_\sqcup \mathrm{B}.$$

Proof These are also consequences of (6-38), taking $W = W_\#, W_\sqcup$ for (6-54) and (6-55), respectively, and with the same choices of u as in the previous step. The splitting 3–manifolds $Y_0 \subset W$ however are chosen differently from those in the previous step.

In the case of $W = W_\#$, we take $(W_1)_c$ to be a tubular neighborhood of $\lambda_\#$, $U(\lambda_\#)$, and so in this case $Y_0 = \partial(W_1)_c \simeq S^1 \times S^2$, and $(W_2)_c = W_\# \setminus U(\lambda_\#)$. There is a diffeomorphism taking the pair $((W_1)_c, \lambda_\#) = (U(\lambda_\#), \lambda_\#)$ to $(S^1 \times B^3, S^1 \times \{0\})$, $\{0\} \in B^3$ denoting the center of the 3–ball B^3. We denote the embedded circle $S^1 \times \{0\} \subset S^1 \times B^3$ by γ_0. In the case when $W = W_\sqcup$, we take Y_0 to be the 3–sphere $S_\sqcup \subset W_\sqcup$ described in Step 3. This 3–sphere decomposes $W_\sqcup$ as a connected sum of $\mathbb{R} \times M_1$ and $\mathbb{R} \times M_2$, and for both $i = 1, 2$, $(W_i)_c \subset (W_\sqcup)_c$ is a manifold with boundary diffeomorphic to a product $[-1, 1] \times M_i$ with an interior 4–ball removed. The rest of the proof is divided into several parts, (i)–(viii) below.

(i) (alternative metrics and perturbations) A preliminary issue needs to be addressed before we are ready to apply (6-38). Recall that in the statement of the lemma, the cobordism maps $\hat m[u](W_\#)$ and $\hat m[u](W_\sqcup)$ refer respectively to $\hat m[u](W_\#, \mathfrak{s}_{W_\#}, \varpi_{W_\#}; \Gamma_{W_\#})$ and $\hat m[u](W_\sqcup, \mathfrak{s}_{W_\sqcup}, \varpi_{W_\sqcup}; \Gamma_{W_\sqcup})$, where the metrics and the closed 2–forms $\omega_{W_\#}$ and $\omega_{W_\sqcup}$ are defined via the decompositions of $W_\#$ and $W_\sqcup$ along $M_\#$ and $M_\sqcup$. To apply the composition formula (6-21) or (6-38) to the alternative decomposition described in the preceding paragraph, we need to work with cobordism maps associated to different choices of metrics and perturbation forms, which are compatible with the aforementioned alternative decomposition of $W_\#$ and $W_\sqcup$. However, we claim that the identities (6-54) and (6-55) are equivalent to identities of the same form for $\hat m[u](W, \mathfrak{s}_W, 2\omega^+; \Gamma_W)$, $W = W_\#$ or $W_\sqcup$, with the latter endowed with different metrics and perturbation forms ω, as long as:

(6-56) The differences are supported on compact regions in W, and in the case of the perturbation form ω, the difference is exact.

(The maps H, A, B C and D will be altered, but that is inconsequential.) This claim follows from [17, Proposition 25.3.8] (extended in the manner previously described, and with changes in perturbation forms incorporated).

Slightly reformulated, the hat version of the identity in [17, Proposition 25.3.8] takes the following form: Suppose there is a path of pairs consisting of a metric and a perturbation form on W such that (6-56) holds for the entire path. Denote by $\widehat{m}^+[u](W)$ and $\widehat{m}[u](W)$, respectively, the version of $\widehat{m}^-[u](W)$ associated to the pair at the end and at the beginning of the path. Then

$$\widehat{m}^+[u](W) - \widehat{m}^-[u](W) = [\widehat{\mathrm{Z}}[\boldsymbol{u}], \widehat{\partial}] + \widehat{\mathrm{Z}}[\delta \boldsymbol{u}], \tag{6-57}$$

where the $\widehat{\mathrm{Z}}$–maps are defined using parametrized moduli spaces associated to this path of metrics and perturbations parallel to the definition of the previously introduced $\widehat{\mathrm{K}}$–maps, and $\boldsymbol{u}$ is the parametrized variant of u as before. (The Z–maps are analogs of the $\check{K}$–maps in [17, Proposition 25.3.8].)

Remark Our signs differ from those in [17, Proposition 25.3.8] because we adopt the "fiber last" convention of orienting the parametrized moduli spaces, as opposed to [17]'s "fiber first" convention. This is preferred as it is more consistent with the orientation convention used for (6-38).

The preceding identity in hand, suppose identities of the form (6-54) and (6-55) are established for a particular pair of metric and perturbation form. Use $\widehat{m}^-[u](W)$ for the version of cobordism maps associated to this pair, and use H^-, A^-, B^- C^- and D^- to denote the version of maps H, A, B C and D in this version of (6-54) and (6-55). On the other hand, use $\widehat{m}^+[u](W)$ to denote the version of cobordism maps associated to the pair of metric and perturbation appearing in the statement of the lemma. Then combining the $-$ versions of the identities (6-54) and (6-55) with (6-57), one would have a $+$ version of the identities (6-54) and (6-55) with respect to a new set of maps H, A, B C and D, if the latter is set to be

$$\begin{aligned}
\mathrm{H}^+ &= \mathrm{H}^- - \widehat{\mathrm{Z}}[\mathbf{1}](W_\#),\\
\mathrm{A}^+ &= \mathrm{A}^- - \widehat{\mathrm{Z}}[\boldsymbol{u}_{\lambda_{\sqcup -}}](W_\sqcup),\\
\mathrm{B}^+ &= \mathrm{B}^- + \widehat{\mathrm{Z}}[\mathbf{1}](W_\sqcup),\\
\mathrm{C}^+ &= \mathrm{C}^- - \widehat{\mathrm{Z}}[\boldsymbol{u}_{\lambda_\sqcup}](W_\sqcup),\\
\mathrm{D}^+ &= \mathrm{D}^- + \widehat{\mathrm{Z}}[\boldsymbol{u}_{\lambda_{\sqcup +}}](W_\sqcup).
\end{aligned}$$

To reach the preceding conclusion, we made use of the identities

$$\begin{aligned}
\widehat{\mathrm{Z}}[\delta\boldsymbol{u}_{\lambda_{\sqcup -}}](W_\sqcup) &= \widehat{U}_\sqcup \widehat{\mathrm{Z}}[\mathbf{1}](W_\sqcup),\\
\widehat{\mathrm{Z}}[\delta\boldsymbol{u}_{\lambda_{\sqcup +}}](W_\sqcup) &= -\widehat{\mathrm{Z}}[\mathbf{1}](W_\sqcup)\widehat{U}_\sqcup,\\
\widehat{\mathrm{Z}}[\delta\boldsymbol{u}_{\lambda_\sqcup}](W_\sqcup) &= -\widehat{\mathrm{Z}}[\boldsymbol{u}_{\lambda_{\sqcup +}}](W_\sqcup)\widehat{U}_\sqcup - \widehat{U}_\sqcup\widehat{\mathrm{Z}}[\boldsymbol{u}_{\lambda_{\sqcup -}}](W_\sqcup).
\end{aligned} \tag{6-58}$$

In complete parallel to the $\widehat{\mathrm{K}}$–analogs mentioned in the paragraph preceding (6-49), these identities are also straightforward adaptations of (2-51).

Now permitted to work with alternative metrics and perturbation forms by the preceding arguments, we endow $W_\#$ and $W_\sqcup$ with the following sort of metrics and perturbations for the rest of this proof. For $W = W_\sqcup$ or $W_\#$, we require the metric:

- To agree with a product metric on a tubular neighborhood $U(Y_0) \simeq [-1,1] \times Y_0$ of $Y_0 \subset W$, where Y_0 is $S^1 \times S^2$ in the case of $W_\#$ and S^3 in the case of $W_\sqcup$. These are endowed with the standard metrics. (In particular, these metrics on Y_0 have positive scalar curvature.)
- To agree with the original metric on the complements of $(W_\#)_c$ and $(W_\sqcup)_c$.
- In the case of $W_\#$, have restriction to $(W_1)_c \simeq S^1 \times B^3$ that has nonnegative scalar curvature and is invariant under rotation along the S^1 factor.

The abbreviated notation $\widehat{m}[u](W_\#)$ and $\widehat{m}[u](W_\sqcup)$ also take on different meanings for the rest of this proof: they will stand respectively for $\widehat{m}[u](W_\#, \mathfrak{s}_{W_\#}, 2\dot{\omega}_\#^+; \Gamma_{W_\#})$ and $\widehat{m}[u](W_\sqcup, \mathfrak{s}_{W_\sqcup}, 2\dot{\omega}_\sqcup^+; \Gamma_{W_\sqcup})$, where the metrics are as previously mentioned, and $\dot{\omega}_\#$ and $\dot{\omega}_\#$ are closed 2–forms that

- vanish over $U(Y_0)$;
- are cohomologous respectively to $\omega_\#$ and $\omega_\sqcup$;
- agree respectively with $\omega_\#$ and $\omega_\sqcup$ on the complements of $(W_\#)_c$ and $(W_\sqcup)_c$;
- are such that $\dot{\omega}_\#$ vanishes over $(W_1)_c \simeq S^1 \times B^3 \subset W_\#$.

Such $\dot{\omega}_\#$ and $\dot{\omega}_\#$ exist as $\omega_{W_\#}$ and $\omega_{W_\sqcup}$ both restrict to exact forms on Y_0, and $\omega_\#$ restricts to an exact form on $(W_1)_c \simeq S^1 \times B^3 \subset W_\#$.

Now write $W = W_2 \circ W_1$ and define $W(S)$ according to the recipe (6-23). In the case $W = W_\#$, W_1 is regarded as a cobordism from the empty set to $S^1 \times S^2$; the $-\infty$–end of W_2 consists of two connected components, $S^1 \times S^2$ and $M_\#$, but only the $S^1 \times S^2$ component is "glued to" W_1 to form $W(S)$. See Figure 1, top left, for an illustration. In the case $W = W_\sqcup$, the $+\infty$–end of W_1 consists of two connected components, M_1 and S^3, and the $-\infty$–end of W_2 consists of two connected components as well, S^3 and M_2, but only the S^3–ends from both sides are "glued" to form $W(S)$. See Figure 1, top right. To indicate the 3–manifold where gluing take place in the composition, we write $W_\# = W_2 \circ_{S^1 \times S^2} W_1$ and $W_\sqcup = W_2 \circ_{S^3} W_1$.

(ii) (surgered cobordisms) Recall also from Step 3 above that surgery along $\lambda_\# \subset W_\#$ and $S_\sqcup \subset W_\sqcup$ gives respectively the cobordisms $W'_\# \simeq \mathbb{R}\times M_\#$ and $W'_\sqcup \simeq \mathbb{R}\times M_\sqcup$. We decompose these surgered cobordisms in a way compatible with the decomposition of $W_\#$ and $W_\sqcup$ above. In the case when $W = W_\#$, the corresponding surgered manifold is decomposed as $W'_\# = W_2 \circ_{S^1\times S^2} W'_1$, with $(W'_1)_c \simeq D^2\times S^2$ and W_2 being as in the decomposition of $W_\#$, $W_\# = W_2 \circ_{S^1\times S^2} W_1$. Like $(W_1)_c \subset (W_\#)_c$, we also equip $(W'_1)_c$ with a metric with nonnegative scalar curvature. See Figure 1, bottom left, for an illustration. In the case when $W = W_\sqcup$, the surgered manifold has two connected components, $W'_\sqcup = \widehat{W}_1 \sqcup \widehat{W}_2$. Each connected component $\widehat{W}_i \simeq \mathbb{R}\times M_i$ for $i = 1,2$ is obtained from $W_i \subset W_\sqcup$ by filling in a 4–ball at the boundary 3–sphere of W_i. Let $\mathfrak{B}_c$ denote a closed 4–ball equipped with a metric which has nonnegative scalar curvature, and that is cylindrical on a collar of the boundary. Let $\mathfrak{B}$ be the corresponding manifold with one cylindrical end, regarded as a cobordism from the empty set to S^3, and let $\overline{\mathfrak{B}}$ denote the reverse cobordism. We decompose $\widehat{W}_1$ and $\widehat{W}_2$, respectively, as $\widehat{W}_1 = \overline{\mathfrak{B}} \circ_{S^3} W_1$ and $\widehat{W}_2 = W_2 \circ_{S^3} \mathfrak{B}$, where W_1 and W_2 are as in the decomposition of $W_\sqcup$. See Figure 1, bottom right, for an illustration.

We now apply the decomposition theorem (6-21) and (6-38) to the composite cobordisms $W = W_\#$, $W'_\#$, $W_\sqcup$ and $W'_\sqcup$ described above, and illustrated schematically in Figure 1. In each of the pictures, the dashed line represents Y_0, the 3–manifold where composition takes place. The shaded regions in each picture, $W_2 \subset W_\#$ and $W_2 \subset W'_\#$, and $W_1 \subset W_\sqcup$ and $W_1 \subset W'_\sqcup$, are associated with nonbalanced perturbation forms in the relevant Seiberg–Witten equation, implying that the corresponding moduli spaces of Seiberg–Witten solutions contain no reducible elements.

(iii) (some useful facts about $\mathring{C}(Y_0)$) In all four pictures, the Spin^c–structure $\mathfrak{s}_W$ on W restricts to the trivial Spin^c–structure on Y_0, denoted by $\mathfrak{s}_0$ below ($\mathfrak{s}_0$ is characterized by the condition $c_1(\mathfrak{s}_0) = 0$). The 3–manifolds Y_0 also all carry positive scalar curvatures, so that $\mathfrak{C}^o(Y_0)$ is trivial and the Floer complex is $\widehat{C}(Y_0,\mathfrak{s}_0) = C^u(Y_0,\mathfrak{s}_0)$. In fact, $Y_0 = S^1\times S^2, S^3$ respectively in the cases of $W_\#$ and $W_\sqcup$, and in both cases $\mathring{C}(Y_0,\mathfrak{s}_0)$ are explicitly described in [17] (see eg Chapter 36 therein). We write

$$(\widehat{C}(S^1\times S^2,\mathfrak{s}_0),\hat{\partial}) = (\mathbb{K}[u,y]\hat{1}_Z, 0), \tag{6-59}$$

where $\mathbb{K}[u,y]$ is the (graded) polynomial algebra with variables u and y, $\deg(u) = -2$ and $\deg(y) = 1$, and $\hat{1}_Z$ denotes an element with degree -1. Similarly,

$$(\widehat{C}(S^3,\mathfrak{s}_0),\hat{\partial}) = (\mathbb{K}[u]\hat{1}, 0), \quad (\check{C}(S^3,\mathfrak{s}_0),\hat{\partial}) = (\mathbb{K}[u^{-1}]\check{1}, 0),$$

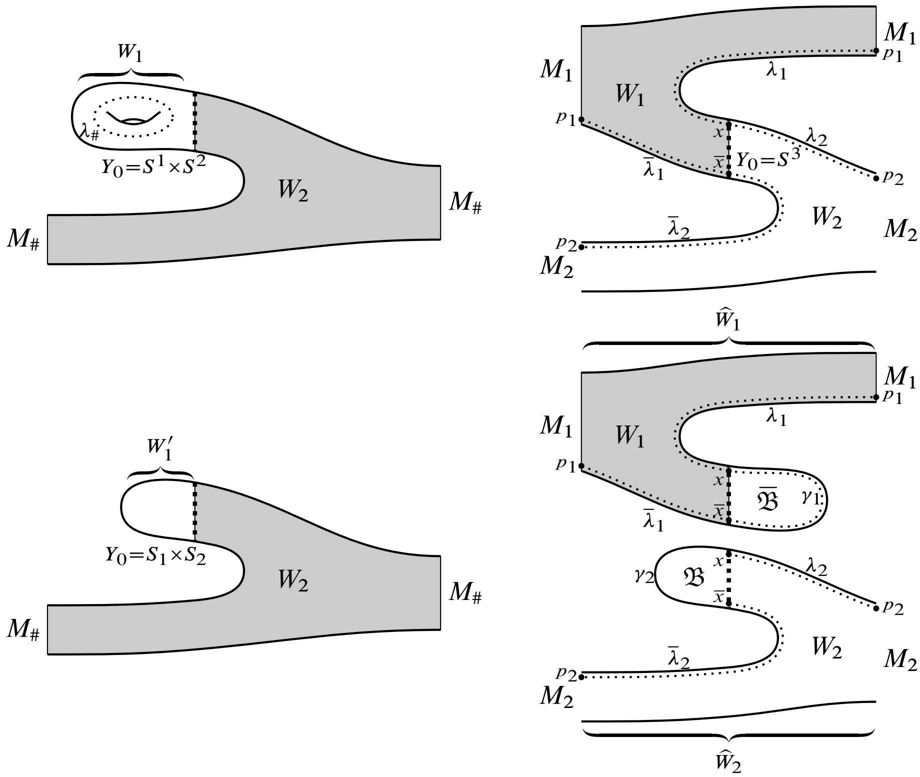


Figure 1: Top left: $W_\# = W_2 \circ W_1$; $(W_1, \lambda_\#) \simeq (S^1 \times B^3, \gamma_0)$; s increases from left to right. Top right: $W_\sqcup = W_2 \circ W_1$; $\lambda = \lambda_1 \cup_x \lambda_2$; $\bar{\lambda} = \bar{\lambda}_1 \cup_{\bar{x}} \bar{\lambda}_2$. Bottom left: $W_\#' = W_2 \circ W_1' \simeq \mathbb{R} \times M_\#$; $W_1' \simeq D^2 \times S^2$. Bottom right: $W_\sqcup' = \hat{W}_1 \sqcup \widehat{W}_2$; $\widehat{W}_i = W_i \cup_{S^3} B^4 \simeq \mathbb{R} \times M_i$ for $i = 1, 2$; $(-1)^i \hat{p}_i = \lambda_i \cup \gamma_i \cup \bar{\lambda}_i \simeq \mathbb{R} \times \{(-1)^i p_i\} \subset \mathbb{R} \times M_i$.

$\hat{1} \in \widehat{C}(S^3, \mathfrak{s}_0)$ and $\check{1} \in \check{C}(S^3, \mathfrak{s}_0)$ being respectively the elements of degree -1 and 0 explicitly described in [17]. (In our convention, the plane field on S^3 denoted by $[\xi_-]$ and the plane field on $S^1 \times S^2$ denoted by $[\xi_0]$ in [17] both have degree 0.) We also use the notation $u^n \hat{1}$ and $u^{-n} \check{1}$ for $n \in \mathbb{Z}^{\geq 0}$ to denote respectively the element in $\mathfrak{C}^u(S^3)$ with gr–grading $[\xi_0] - 1 - 2n$ (equivalently, $\overline{\mathrm{gr}}$–grading $[\xi_0] - 2 - 2n$), and the element in $\mathfrak{C}^u(S^3)$ with gr–grading $[\xi_0] + 2n$ (equivalently, $\overline{\mathrm{gr}}$–grading $[\xi_0] + 2n$). The $\bar{U}$–action on $\bar{C}(S^3)$ is also well known: $\langle \mathfrak{c}', \bar{U}(S^3)\mathfrak{c} \rangle = 1$ for any pair $\mathfrak{c}, \mathfrak{c}' \in \mathfrak{C}(S^3)$ with $\overline{\mathrm{gr}}(\mathfrak{c}) - \overline{\mathrm{gr}}(\mathfrak{c}') = 2$.

We are now ready to proceed with:

(iv) (verifying (6-54)) To compute $\widehat{m}[u_{\lambda_\#}](W_\#)$, first note that the 1–cochain $u_{\lambda_\#}$ has the simple form of an inner product, $u_{\lambda_\#} = c(u_{\gamma_0} \otimes 1)$ with respect to the decomposition of $W_\#$ shown in Figure 1, top left. Thus, (6-21) is directly applicable. Noting that $\delta u_{\lambda_\#}=0$, this gives us

$$\widehat{m}[1](W_2)\widehat{m}[u_{\gamma_0}](S^1\times B^3)=\widehat{m}[u_{\lambda_\#}](W_\#)+[\widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_\#}](W_\#),\widehat{\partial}_\#]. \tag{6-60}$$

This is compared to the formula obtained by applying (6-21) to $W'_\#$, decomposed as shown in Figure 1, bottom left, and with the 1–cochain taken to be $u=1=c(1\otimes 1)$. Here we have

$$\widehat{m}[1](W_2)\widehat{m}[1](D^2\times S^2)=\widehat{m}[1](W'_\#)+[\widehat{\mathrm{K}}[\mathbf{1}](W'_\#),\partial_\#]. \tag{6-61}$$

If the manifolds $W_1\simeq S^1\times B^3$ and $W'_1\simeq D^2\times S^2$ above are regarded cobordisms from the empty set to $(S^1\times S^2,\mathfrak{s}_0)$, then $\widehat{m}[u](W_1)$ and $\widehat{m}[u](W'_1)$ are both elements of $\widehat{C}(S^1\times S^2,\mathfrak{s}_0)$. Alternatively (in line with the definition of closed 4–manifold invariants in [17]), for a cobordism W from the empty set to Y, $\widehat{m}[u](W)\in\widehat{C}(Y)$ can be defined as

$$\widehat{m}[u](W)=\widehat{m}[u](\dot{W})\hat{1},$$

where $\dot{W}$ is a cobordism from S^3 to Y obtained by removing a 3–ball from the interior of W, and $u\in \mathrm{C}(\mathcal{B}^\sigma(\dot{W});\mathbb{K})$ is used to denote the cochain induced from that in $\mathrm{C}(\mathcal{B}^\sigma(W);\mathbb{K})$. With $W_1\simeq S^1\times B^3$ and $W'_1\simeq D^2\times S^2$ endowed with the metrics prescribed above, the values $\widehat{m}[u_{\gamma_0}](S^1\times B^3)$ and $\widehat{m}[1](D^2\times S^2)$ are also well known (and follow from simple computations): in the notation of (6-59),

$$\begin{aligned}\widehat{m}[u_{\gamma_0}](S^1\times B^3)&=\hat{1}_Z\in\widehat{C}(S^1\times S^2),\\ \widehat{m}[1](S^1\times B^3)&=y\hat{1}_Z\in\widehat{C}(S^1\times S^2),\\ \widehat{m}[1](D^2\times S^2)&=\hat{1}_Z\in\widehat{C}(S^1\times S^2).\end{aligned}$$

Inserting these into (6-60) and (6-61), we have:

$$\widehat{m}[u_{\lambda_\#}](W_\#)=\widehat{m}[1](W'_\#)+[\widehat{\mathrm{K}}[\mathbf{1}](W'_\#)-\widehat{\mathrm{K}}[\boldsymbol{u}_{\lambda_\#}](W_\#),\partial_\#]. \tag{6-62}$$

As observed before, $W'_\#\simeq\mathbb{R}\times M_\#$. When the latter is endowed with cylindrical metric and perturbation, $\widehat{m}[1](\mathbb{R}\times M_\#)=\mathrm{Id}$. Thus, since $\delta(1)=0$, by (6-57) again,

$$\widehat{m}[1](W'_\#)=\mathrm{Id}_\#-[\widehat{\mathrm{Z}}[\mathbf{1}](W'_\#),\partial_\#],$$

where $\widehat{\mathrm{Z}}[\mathbf{1}](W'_\#)$ is defined using a path of metrics/perturbations from the original version to the cylindrical one. Combining this with (6-62), we arrive at (6-54).

(v) (verifying (6-55): preparations) Consider Figure 1, top right, and write $\widehat{m}[u](W_1)$ as a map from $\widehat{C}(M_1)$ to $\widehat{C}(M_1)\otimes\widehat{C}(S^3)$; $\widehat{m}[u](W_2)$ as a map from $\widehat{C}(S^3)\otimes\widehat{C}(M_2)$

to $\widehat{C}(M_2)$. To simplify notation, we denote $\lambda_{\sqcup+}, \lambda_{\sqcup-} \subset W_\sqcup$ respectively as λ and $\overline{\lambda}$ below. Recall that the point $x = \lambda \cap Y_0$ (which is in $W_\sqcup$) separates λ into $\lambda_1 \cup \lambda_2$, with $\lambda_i \subset W_i$ for each $i = 1, 2$. We denote the point of intersection of $\overline{\lambda}$ with Y_0 as $\overline{x}$, and $\overline{\lambda} = \overline{\lambda}_1 \cup \overline{\lambda}_2$. Recall also the arcs $\gamma_1 \subset \overline{\mathfrak{B}}_c$ and $\gamma_2 \subset \mathfrak{B}_c$ from Step 3. (In Step 3, $\overline{\mathfrak{B}}_c$ and $\mathfrak{B}_c$ were respectively denoted by B_1 and B_2.) In the surgered manifold $W'_\sqcup = \widehat{W}_1 \sqcup \widehat{W}_2$, for each $i = 1, 2$, γ_i join with $\lambda_i \cup \overline{\lambda}_i$ at $\{x, \overline{x}\} \subset S^3$ to form paths in $\widehat{W}_i \simeq \mathbb{R} \times M_i$. We denote the path in $\widehat{W}_1$ by $-\widehat{p}_1$ and that in $\widehat{W}_2$ by $\widehat{p}_2$, as they are diffeomorphic respectively to the paths $\mathbb{R} \times \{-p_1\} \subset \mathbb{R} \times M_1$ and $\mathbb{R} \times \{p_2\} \subset \mathbb{R} \times M_2$ under suitable diffeomorphisms taking $\widehat{W}_i$ to $\mathbb{R} \times M_i$. See Figure 1, bottom right.

We begin with some computations of $\widehat{m}[u](W_1)$. Express $\widehat{m}[u](W_1)$ in block form as in (2-44). First, note the following facts: $\mathfrak{C}(M_1) = \mathfrak{C}^o(M_1)$ and $\mathfrak{C}^o(S^3) = \varnothing$; $\overline{m}^{\#}_{\flat}(W_1)$ vanishes for all # and $\flat$ while $m^{\#}_{\flat}(W_1)$ vanishes except when $\# = o$ and $\flat = os$. This means that, when $\widehat{m}[u](W_1)$ is given by the simpler formula (2-20) (for example when $u = 1$), only one term, $-(\partial^o_o(M_1) \otimes \overline{\partial}^s_u(S^3)) m^o_{os}[u](W_1)$ on the lower left, can be nonvanishing. However, $\overline{\partial}^s_u(S^3) = 0$. Thus, $\widehat{m}[u](W_1) = 0$ for such u.

More generally, the lower row of (2-44) contains additional terms as explained in Remark 2.2. These correspond to the last terms in both lines of (2-45). According to the discussion following (2-45), in the cases of $\widehat{m}[\theta_{\lambda_1}](W_1)$ and $\widehat{m}[\theta_{\overline{\lambda}_1}](W_1)$, these terms are of the same form as those in (2-45), but with the map $\overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{p}_2}]$ therein replaced respectively by

$$1 \otimes \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{x}}](S^3) - \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{p}_1}](M_1) \otimes 1 = \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{x}}](S^3) \quad \text{for } \widehat{m}[\theta_{\lambda_1}](W_1),$$
$$1 \otimes \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{\overline{x}}}](S^3) \quad \text{for } \widehat{m}[\theta_{\overline{\lambda}_1}](W_1).$$

In the present context, the additional term in the lower right corner of (2-44), being a product of one of the expressions above with $m^u_{os}[1](W_1)$, vanishes because the latter does. Thus, for both $\widehat{m}[\theta_{\lambda_1}](W_1)$ and $\widehat{m}[\theta_{\overline{\lambda}_1}](W_1)$, all entries in (2-45) vanish except possibly for the lower left entry, which is respectively

$$\widehat{m}^o_{ou}[\theta_{\lambda_1}](W_1) = \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{x}}](S^3) m^o_{os}[1](W_1),$$
$$\widehat{m}^o_{ou}[\theta_{\overline{\lambda}_1}](W_1) = \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{\overline{x}}}](S^3) m^o_{os}[1](W_1).$$

Now, for any $p \in S^3$,

$$\langle \mathfrak{c}', \overline{n}^s_u[d\underline{\mathrm{h}}_{\widehat{p}}](S^3)\mathfrak{c}\rangle = \langle \mathfrak{c}', (\overline{U}_p)^s_u(S^3)\mathfrak{c}\rangle = \begin{cases} 1 & \text{if } \mathfrak{c} = \check{1} \in \mathfrak{C}^s(S^3) \text{ and } \mathfrak{c}' = \hat{1} \in \mathfrak{C}^u(S^3), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\langle \mathfrak{c}_{+,1}\otimes\mathfrak{c}, \widehat{m}[\theta_{\lambda_1}](W_1), \mathfrak{c}_{-,1}\rangle = \langle \mathfrak{c}_{+,1}\otimes\mathfrak{c}, \widehat{m}[\theta_{\bar\lambda_1}](W_1), \mathfrak{c}_{-,1}\rangle$ vanishes for all $\mathfrak{c}\in\mathfrak{C}(S^3)$ and $\mathfrak{c}_{\pm,1}\in\mathfrak{C}(M_1)=\mathfrak{C}^o(M_1)$ except when $\mathfrak{c}=\hat{1}$, in which case

$$\begin{aligned}\langle \mathfrak{c}_{+,1}\otimes\hat{1}, \widehat{m}[\theta_{\lambda_1}](W_1), \mathfrak{c}_{-,1}\rangle &= \langle \mathfrak{c}_{+,1}\otimes\hat{1}, \widehat{m}[\theta_{\bar\lambda_1}](W_1), \mathfrak{c}_{-,1}\rangle\\ &= \langle \mathfrak{c}_{+,1}\otimes\check{1}, \check{m}[1](W_1), \mathfrak{c}_{-,1}\rangle.\end{aligned}$$

Note that $\bar{n}[\mathrm{u}](S^3)$ vanishes for all u of odd degrees, since all pairs of $\mathfrak{c},\mathfrak{c}'\in\mathfrak{C}(S^3)$, the difference $\overline{\mathrm{gr}}(\mathfrak{c}')-\overline{\mathrm{gr}}(\mathfrak{c})$ is even. Together with the preceding arguments, this implies that in the block form (2-44), all entries of $\widehat{m}[\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}](W_1)$ also vanish except possibly for the lower left entry, which is

$$\begin{aligned}\widehat{m}^o_{ou}[\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}](W_1) &= \bar{n}^s_u[d\underline{\mathrm{h}}_{\hat{\bar{x}}}](S^3)m^o_{os}[\theta_{\lambda_1}](W_1)+\bar{n}^s_u[d\underline{\mathrm{h}}_{\hat{x}}](S^3)m^o_{os}[\theta_{\bar\lambda_1}](W_1)\\ &= m^o_{os}[\theta_{\lambda_1}](W_1)+m^o_{os}[\theta_{\bar\lambda_1}](W_1),\end{aligned}$$

and $\langle \mathfrak{c}_{+,1}\otimes\mathfrak{c}, \widehat{m}[\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}](W_1), \mathfrak{c}_{-,1}\rangle$ vanish for all $\mathfrak{c}\in\mathfrak{C}(S^3)$ and $\mathfrak{c}_{\pm,1}\in\mathfrak{C}(M_1)=\mathfrak{C}^o(M_1)$ except when $\mathfrak{c}=\hat{1}$, in which case

$$\langle \mathfrak{c}_{+,1}\otimes\hat{1}, \widehat{m}[\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}](W_1)\mathfrak{c}_{-,1}\rangle = \langle \mathfrak{c}_{+,1}\otimes\check{1}, \check{m}[\theta_{\lambda_1}+\theta_{\bar\lambda_1}](W_1)\mathfrak{c}_{-,1}\rangle.$$

Imitating physicists' notation, we use $\widehat{m}[u](W_1)|\mathfrak{c}\rangle$ for $\mathfrak{c}\in\mathfrak{C}^u(S^3)$ to denote the map from $\widehat{C}(M_1)=\check{C}(M_1)$ to itself defined by

$$\big\langle \mathfrak{c}_{+,1}, \widehat{m}[u](W_1)|\mathfrak{c}\rangle(\mathfrak{c}_{-,1})\big\rangle = \langle \mathfrak{c}_{+,1}\otimes\mathfrak{c}, \widehat{m}[u](W_1)(\mathfrak{c}_{-,1})\rangle.$$

Similarly, $\check{m}[u](W_1)|\mathfrak{c}\rangle$ for $\mathfrak{c}\in\mathfrak{C}^s(S^3)$ will denote the map from $\widehat{C}(M_1)=\check{C}(M_1)$ to itself defined by

$$\big\langle \mathfrak{c}_{+,1}, \check{m}[u](W_1)|\mathfrak{c}\rangle(\mathfrak{c}_{-,1})\big\rangle = \langle \mathfrak{c}_{+,1}\otimes\mathfrak{c}, \check{m}[u](W_1)(\mathfrak{c}_{-,1})\rangle.$$

Also, $\langle\mathfrak{c}|\widehat{m}[u](W_2)$ will denote a map from $\widehat{C}(M_2)$ to itself given by

$$\big\langle \mathfrak{c}_{+,2}, \langle\mathfrak{c}|\widehat{m}[u](W_2)(\mathfrak{c}_{-,2})\big\rangle = \langle \mathfrak{c}_{+,2}, \widehat{m}[u](W_2)(\mathfrak{c}\otimes\mathfrak{c}_{-,2})\rangle.$$

(vi) (verifying (6-55)(2)) In this case, the cochain $u=1\in\mathrm{C}(\mathcal{B}^\sigma_{\mathrm{loc}}(W_\sqcup);\mathbb{K})$ can be written as an inner product, $1=c(1\otimes 1)$, and (6-21) applies. Noting that $\delta(1)=0$, in this context (6-21) gives, with respect to the factorization $\widehat{C}(M_\sqcup)=\widehat{C}(M_1)\otimes\widehat{C}(M_2)$,

$$\sum_{\mathfrak{c}\in\mathfrak{C}^u(S^3)}\big(\widehat{m}[1](W_1)|\mathfrak{c}\rangle\big)\otimes\big(\langle\mathfrak{c}|\widehat{m}[1](W_2)\big) = \widehat{m}[1](W_\sqcup)+[\widehat{\mathrm{K}}[\mathbf{1}](W_\sqcup),\widehat{\partial}_\sqcup].$$

The left-hand side of the preceding formula vanishes, since we saw that $\widehat{m}[1](W_1)=0$. This directly leads to (6-55)(2), with B set to be

$$\mathrm{B}=-\widehat{\mathrm{K}}[\mathbf{1}](W_\sqcup).$$

(vii) (verifying (6-55)(1), (4)) As with in the proof of (6-20), in cases (1), (4) and (3), the relevant cochains do not take the simple form as an inner product under the decomposition, and instead of (6-21), the more delicate formula (6-38) is required. To begin, We again reexpress the formulas (6-55)(1), (3), (4) of in terms of the more concrete θ_λ and $\theta_{\bar{\lambda}}$, making use of the identities (6-18) as well as the previously established (6-55)(2):

$$\text{(6-63)(1')}\quad \mathrm{Id}_\sqcup - [\hat{\mathrm{Z}}_\sqcup, \hat{\partial}_\sqcup]_{\mathrm{odd}} = \hat{m}[\theta_{\bar{\lambda}}](W_\sqcup) + [\dot{\mathrm{A}}, \hat{\partial}_\sqcup]_{\mathrm{even}} + \mathrm{B}\hat{n}[d\underline{\mathrm{h}}_\sqcup],$$

$$\text{(6-63)(3')}\quad [\hat{\partial}_\sqcup, \mathrm{X}]_{\mathrm{even}} - [\hat{n}[d\underline{\mathrm{h}}_\sqcup], \mathrm{Z}_\sqcup]_{\mathrm{even}} = \hat{m}[\theta_{\bar{\lambda}} \wedge \theta_\lambda](W_\sqcup) + [\dot{\mathrm{C}}, \hat{\partial}_\sqcup]_{\mathrm{odd}} - \hat{n}[d\underline{\mathrm{h}}_\sqcup]\dot{\mathrm{A}} + \dot{\mathrm{D}}\hat{n}[d\underline{\mathrm{h}}_\sqcup],$$

$$\text{(6-63)(4')}\quad \mathrm{Id}_\sqcup - [\hat{\mathrm{Z}}_\sqcup, \hat{\partial}_\sqcup]_{\mathrm{odd}} = \hat{m}[\theta_\lambda](W_\sqcup) + [\hat{\partial}_\sqcup, \dot{\mathrm{D}}]_{\mathrm{even}} - \hat{n}[d\underline{\mathrm{h}}_\sqcup]\mathrm{B},$$

where

$$\hat{n}[d\underline{\mathrm{h}}_\sqcup](M_\sqcup) := 1 \otimes \hat{n}[d\underline{\mathrm{h}}_{\hat{p}_2}](M_2) - \hat{n}[d\underline{\mathrm{h}}_{\hat{p}_1}](M_1) \otimes 1,$$

and $\dot{\mathrm{A}}$, $\dot{\mathrm{C}}$ and $\dot{\mathrm{D}}$ are related to A, B, C and D via formulas parallel to those in (6-42) relating $\dot{A}$, $\dot{C}$ and $\dot{D}$ to A, B, C and D.

We omit the proof of (1′) above as its proof is entirely parallel to that for (4′). To proceed with the proof of the latter, we first express θ_λ as a sum of inner products in parallel to (6-46). Namely, we claim that, in this context,

$$\theta_\lambda = c(\theta_{\lambda_1} \otimes 1) + c(1 \otimes \theta_{\lambda_2}). \tag{6-64}$$

Once this is established, applying (6-38) in this case would yield

$$\begin{aligned}\sum_{\mathfrak{c}\in\mathfrak{C}^u(S^3)} \big(\hat{m}[\theta_{\lambda_1}](W_1)|\mathfrak{c}\rangle \otimes \langle\mathfrak{c}|\hat{m}[1](W_2) + \hat{m}[1](W_1)|\mathfrak{c}\rangle \otimes \langle\mathfrak{c}|\hat{m}[\theta_{\lambda_2}](W_2)\big) &= \hat{m}[\theta_\lambda](W_\sqcup) + [\hat{\mathrm{K}}[\boldsymbol{\theta}_\lambda](W_\sqcup), \hat{\partial}_\sqcup]_{\mathrm{even}} + \hat{n}[d\underline{\mathrm{h}}_\sqcup]\hat{\mathrm{K}}[\mathbf{1}](W_\sqcup)\\ &= \hat{m}[\theta_\lambda](W_\sqcup) + [\hat{\mathrm{K}}[\boldsymbol{\theta}_\lambda](W_\sqcup), \hat{\partial}_\sqcup]_{\mathrm{even}} - \mathrm{B}\hat{\mathrm{K}}[\mathbf{1}](W_\sqcup).\end{aligned} \tag{6-65}$$

According to the computation of $\hat{m}[u](W_1)$ in part (v) above, the second term on the left-hand side of the above vanishes, while the first term is given by

$$\hat{m}[\theta_{\lambda_1}](W_1)|\hat{1}\rangle \otimes \langle\hat{1}|\hat{m}[1](W_2) = \check{m}[1](W_1)|\check{1}\rangle \otimes \langle\hat{1}|\hat{m}[1](\widehat{W}_2).$$

Filling 3–balls at the S^3–end of W_1 and W_2 to get $\widehat{W}_1$ and $\widehat{W}_2$ as shown in Figure 1, bottom right, we see that:

$$\begin{aligned}\check{m}[1](W_1)|\check{1}\rangle &= \check{m}[1](\widehat{W}_1) + [\check{\mathrm{K}}[\mathbf{1}](\widehat{W}_1), \check{\partial}(M_1)]_{\mathrm{odd}} \\ &= \widehat{m}[1](\widehat{W}_1) + [\hat{\mathrm{K}}[\mathbf{1}](\widehat{W}_1), \widehat{\partial}(M_1)]_{\mathrm{odd}}, \\ \langle\hat{1}|\widehat{m}[1](W_2) &= \widehat{m}[1](\widehat{W}_2) + [\hat{\mathrm{K}}[\mathbf{1}](\widehat{W}_2), \widehat{\partial}(M_2)]_{\mathrm{odd}}.\end{aligned} \tag{6-66}$$

Combining these with (6-65), we have

$$\begin{aligned}\widehat{m}[1](W'_\sqcup) &= \widehat{m}[1](\widehat{W}_1) \otimes \widehat{m}[1](\widehat{W}_2) \\ &= \widehat{m}[\theta_\lambda](W_\sqcup) + [\hat{\mathrm{K}}[\boldsymbol{\theta}_\lambda](W_\sqcup), \widehat{\partial}_\sqcup]_{\mathrm{even}} - \mathrm{B}\hat{\mathrm{K}}[\mathbf{1}](W_\sqcup) - [\hat{\mathrm{K}}[\mathbf{1}](W'_\sqcup), \widehat{\partial}_\sqcup]_{\mathrm{odd}}.\end{aligned} \tag{6-67}$$

Since for both $i = 1,2$ $\widehat{W}_i \simeq \mathbb{R} \times M_i$, and when equipped with cylindrical metric and perturbation, $\widehat{m}[1](\mathbb{R} \times M_i) = \mathrm{Id}$, by (6-57) we then have that

$$\widehat{m}[1](W'_\sqcup) = \mathrm{Id}_\sqcup - [\hat{\mathrm{Z}}[\mathbf{1}](W'_\sqcup), \widehat{\partial}_\sqcup]_{\mathrm{odd}},$$

where $\hat{\mathrm{Z}}[\mathbf{1}](W'_\sqcup)$ is defined via a path of metrics/perturbations from the original version on $W'_\sqcup \simeq \mathbb{R} \times M_\sqcup$ to the cylindrical version. Combining this with (6-67), *modulo the proof of* (6-64), we have verified (6-63)(4′), with $\dot{\mathrm{D}}$ and $\mathrm{Z}_\sqcup$ therein set respectively to be

$$\dot{\mathrm{D}} = -\hat{\mathrm{K}}[\boldsymbol{\theta}_\lambda](W_\sqcup), \quad \mathrm{Z}_\sqcup = \hat{\mathrm{Z}}[\mathbf{1}](W'_\sqcup) - \hat{\mathrm{K}}[\mathbf{1}](W'_\sqcup). \tag{6-68}$$

Item (6-63)(1′) is derived using the same arguments, with $\dot{\mathrm{A}}$ set to be

$$\dot{\mathrm{A}} = \hat{\mathrm{K}}[\boldsymbol{\theta}_{\bar{\lambda}}](W_\sqcup).$$

We now return to the task of verifying (6-64). This is again done following the strategy outlined in the end of Step 3 above. Recall the definitions of the bundles

$$\tilde{\pi}_\lambda\colon \tilde{\tilde{\mathcal{B}}}{}^\sigma_\lambda(W) \to \mathcal{B}^\sigma(W) \quad \text{and} \quad \pi_\lambda\colon \widetilde{\mathcal{B}}{}^\sigma_\lambda(W) \to \mathcal{B}^\sigma(W)$$

from the diagram (2-41). Let $\tilde{\pi}_{\lambda_1}\colon \tilde{\tilde{\mathcal{B}}}{}^\sigma_{\lambda_1}(W_1) \to \mathcal{B}^\sigma(W_1)$ and $\tilde{\pi}_{\lambda_1}\colon \tilde{\tilde{\mathcal{B}}}{}^\sigma_{\lambda_2}(W_2) \to \mathcal{B}^\sigma(W_2)$ be $U(1)\times U(1)$–bundles defined in a similar manner, namely by the commutative diagrams

$$\begin{tikzcd} \tilde{\tilde{\mathcal{B}}}{}^\sigma_{\lambda_1}(W_1(S)) \arrow[r, "\tilde{\tilde{\Pi}}{}^\partial_{\lambda_1}"] \arrow[d, "\tilde{\pi}_{\lambda_1}"] & \widetilde{\mathcal{B}}{}^\sigma_{-p_1}(M_1)\times\widetilde{\mathcal{B}}{}^\sigma_{\underline{x}}(S^3) \arrow[d, "\pi_{p_1}\times\pi_{\underline{x}}"] \\ \mathcal{B}^\sigma(W_1(S)) \arrow[r, "\Pi^\partial_{\lambda_1}"] & \mathcal{B}^\sigma(M_1)\times\mathcal{B}^\sigma(S^3) \end{tikzcd} \qquad \begin{tikzcd} \tilde{\tilde{\mathcal{B}}}{}^\sigma_{\lambda_2}(W_2(S)) \arrow[r, "\tilde{\tilde{\Pi}}{}^\partial_{\lambda_2}"] \arrow[d, "\tilde{\pi}_{\lambda_2}"] & \widetilde{\mathcal{B}}{}^\sigma_{-\underline{x}}(S^3)\times\widetilde{\mathcal{B}}{}^\sigma_{p_2}(M_2) \arrow[d, "\pi_{\underline{x}}\times\pi_{p_2}"] \\ \mathcal{B}^\sigma(W_2(S)) \arrow[r, "\Pi^\partial_{\lambda_2}"] & \mathcal{B}^\sigma(S^3)\times\mathcal{B}^\sigma(M_2) \end{tikzcd}$$

where $\Pi^\partial_{\lambda_i}$ for $i = 1,2$ are defined similarly to their cousins in Section 2: $\Pi^\partial_{\lambda_1} = \Pi^{+M_1}_{W_1} \times \Pi^{S^3}_{W_i}$ and $\Pi^\partial_{\lambda_2} = \Pi^{S^3}_{W_2} \times \Pi^{+M_2}_{W_2}$, with $\Pi^{S^3}_{W_i}$ now as defined in (6-31). In the above, the sign $\pm$ in $\Pi^{\pm M_i}_{W_i}$ was introduced to distinguish between the two M_i–ends of W_i, that is, $\Pi^{\pm M_i}_{W_i}$ respectively denote the maps of taking limits to the M_i–end at $s \to \pm\infty$. Factor $\tilde{\tilde{\Pi}}^{\partial}_{\lambda_1}$ and $\tilde{\tilde{\Pi}}^{\partial}_{\lambda_2}$ respectively as $\tilde{\tilde{\Pi}}^{\partial}_{\lambda_1} = \tilde{\tilde{\Pi}}^{+M_1}_{W_1} \times \tilde{\tilde{\Pi}}^{S^3}_{W_1}$ and $\tilde{\tilde{\Pi}}^{\partial}_{\lambda_2} = \tilde{\tilde{\Pi}}^{S^3}_{W_2} \times \tilde{\tilde{\Pi}}^{+M_2}_{W_2}$.

Let $\tilde{\tilde{\tilde{\mathcal{B}}}}^{\sigma}_{x,\lambda}(W_{\sqcup}(S))$ be the $U(1)^{\times 3}$–bundle over $\mathcal{B}^{\sigma}(W_{\sqcup}(S))$ defined by the commutative diagram

$$\begin{tikzcd}
\tilde{\tilde{\tilde{\mathcal{B}}}}^{\sigma}_{x,\lambda}(W_{\sqcup}(S)) \arrow[r, "\tilde{\tilde{\pi}}_{\lambda}"] \arrow[d, "\tilde{\tilde{\pi}}_{x}"] \arrow[dr, "\tilde{\tilde{\tilde{\pi}}}_{x,\lambda}"] & \tilde{\mathcal{B}}^{\sigma}_{x}(W_{\sqcup}(S)) \arrow[d, "\tilde{\pi}_{x}"] \\
\tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda}(W_{\sqcup}(S)) \arrow[r, "\tilde{\pi}_{\lambda}"] \arrow[d, "\tilde{\pi}"] & \mathcal{B}^{\sigma}(W_{\sqcup}(S)) \\
\tilde{\mathcal{B}}^{\sigma}_{\lambda}(W_{\sqcup}(S)) \arrow[ur, "\pi_{\lambda}"] &
\end{tikzcd}$$

We have the following variant of (6-33) in the present context:

$$\begin{tikzcd}
\tilde{\tilde{\tilde{\mathcal{B}}}}^{\sigma}_{x,\lambda}(W_{\sqcup}(S)) \arrow[r, "\tilde{\tilde{s}}_1\times\tilde{\tilde{s}}_2"] \arrow[d, "\tilde{\tilde{\tilde{\pi}}}_{x,\lambda}"] & \tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_1}(W_1(S))\times_{\tilde{\mathcal{B}}_{\underline{x}}(S^3)}\tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_2}(W_2(S)) \arrow[r] \arrow[d, "\tilde{\pi}_{\lambda_1}\times\tilde{\pi}_{\lambda_2}"] & \tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_1}(W_1(S))\times\tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_2}(W_2(S)) \arrow[d, "\tilde{\pi}_{\lambda_1}\times\tilde{\pi}_{\lambda_2}"] \\
\mathcal{B}^{\sigma}(W_{\sqcup}(S)) \arrow[r, "s_1\times s_2"] & \mathcal{B}^{\sigma}(W_1(S))\times_{\mathcal{B}^{\sigma}(S^3)}\mathcal{B}^{\sigma}(W_2(S)) \arrow[r] & \mathcal{B}^{\sigma}(W_1(S))\times\mathcal{B}^{\sigma}(W_2(S))
\end{tikzcd} \tag{6-69}$$

Fix now ϑ'_{p_1}, ϑ'_{p_2} and $\vartheta'_{\underline{x}}$, together with compatible trivializations $\rho_{\vartheta'_{p_1}}$, $\rho_{\vartheta'_{p_2}}$ and $\rho_{\vartheta'_{\underline{x}}}$, and use them to define $\mathbb{R}/\mathbb{Z}$–valued functions h_{λ}, h_{λ_1} and h_{λ_2} respectively on $\tilde{\mathcal{B}}^{\sigma}_{\lambda}(W_{\sqcup}(S))$, $\tilde{\mathcal{B}}^{\sigma}_{\lambda_1}(W_1(S))$ and $\tilde{\mathcal{B}}^{\sigma}_{\lambda_2}(W_2(S))$. Let $\tilde{\tilde{\mathrm{h}}}_{\lambda}$, $\tilde{\mathrm{h}}_{\lambda_1}$ and $\tilde{\mathrm{h}}_{\lambda_2}$ respectively denote their pullbacks to

$$\tilde{\tilde{\tilde{\mathcal{B}}}}^{\sigma}_{x,\lambda}(W_{\sqcup}(S)), \quad \tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_1}(W_1(S)) \quad \text{and} \quad \tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_2}(W_2(S)).$$

Then, arguing as in the paragraph following (6-48), we see that $\tilde{\tilde{\mathrm{h}}}_{\lambda}$ agrees with the pullback of the function $(\tilde{\mathrm{h}}_{\lambda_1}\times 1 + 1\times\tilde{\mathrm{h}}_{\lambda_2})\colon \tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_1}(W_1(S))\times\tilde{\tilde{\mathcal{B}}}^{\sigma}_{\lambda_2}(W_2(S)) \to \mathbb{R}/\mathbb{Z}$ via $\tilde{\tilde{s}}_1\times\tilde{\tilde{s}}_2$, and we have

$$\begin{aligned}
\tilde{\tilde{\pi}}^{*}_{x}\tilde{\pi}^{*}\vartheta_{\lambda} = d\tilde{\tilde{\mathrm{h}}}_{\lambda} &= (\tilde{\tilde{s}}_1\times\tilde{\tilde{s}}_2)^{*}(d\tilde{\mathrm{h}}_{\lambda_1}\times 1 + 1\times d\tilde{\mathrm{h}}_{\lambda_2}) \\
&= (\tilde{\tilde{s}}_1\times\tilde{\tilde{s}}_2)^{*}(\tilde{\pi}^{*}_{W_1}\vartheta_{\lambda_1}\times 1 + 1\times\tilde{\pi}^{*}_{W_2}\vartheta_{\lambda_2}).
\end{aligned}$$

Note that both sides of the preceding equation depends only on the choices of ϑ'_{p_1} and ϑ'_{p_2}, independent of all other choices made to (simultaneously) define h_{λ}, h_{λ_1}

and h_{λ_2}. Meanwhile, note that

$$\begin{aligned}
\tilde{\tilde{\tilde{\pi}}}^*_{x,\lambda}\theta_\lambda &= \tilde{\tilde{\pi}}^*_x\tilde{\pi}^*\vartheta_\lambda - \tilde{\tilde{\pi}}^*_x(\tilde{\Pi}^{+M_2}_{W_\sqcup})^*\vartheta'_{p_2} + \tilde{\tilde{\pi}}^*_x(\tilde{\Pi}^{+M_1}_{W_\sqcup})^*\vartheta'_{p_1},\\
\tilde{\pi}^*_{\lambda_1}\theta_{\lambda_1} &= \tilde{\pi}^*_{W_1}\vartheta_{\lambda_1} - (\tilde{\tilde{\Pi}}^{S^3}_{W_1})^*\vartheta'_{\underline{x}} + (\tilde{\tilde{\Pi}}^{+M_1}_{W_1})^*\vartheta'_{p_1},\\
\tilde{\pi}^*_{\lambda_2}\theta_{\lambda_2} &= \tilde{\pi}^*_{W_2}\vartheta_{\lambda_2} + (\tilde{\tilde{\Pi}}^{S^3}_{W_2})^*\vartheta'_{\underline{x}} - (\tilde{\tilde{\Pi}}^{+M_2}_{W_2})^*\vartheta'_{p_2},
\end{aligned}$$

and, over $\tilde{\tilde{\mathcal{B}}}^\sigma_{\lambda_1}(W_1(S))\times_{\tilde{\mathcal{B}}_{\underline{x}}(S^3)}\tilde{\tilde{\mathcal{B}}}^\sigma_{\lambda_2}(W_2(S))\hookrightarrow\tilde{\tilde{\mathcal{B}}}^\sigma_{\lambda_1}(W_1(S))\times\tilde{\tilde{\mathcal{B}}}^\sigma_{\lambda_2}(W_2(S))$,

$$\begin{aligned}
(-(\tilde{\tilde{\Pi}}^{S^3}_{W_1})^*\vartheta'_{\underline{x}} + (\tilde{\tilde{\Pi}}^{+M_1}_{W_1})^*\vartheta'_{p_1})\times 1 + 1\times((\tilde{\tilde{\Pi}}^{S^3}_{W_2})^*\vartheta'_{\underline{x}} - (\tilde{\tilde{\Pi}}^{+M_2}_{W_2})^*\vartheta'_{p_2})&\\
= (\tilde{\tilde{\Pi}}^{+M_1}_{W_1})^*\vartheta'_{p_1}\times 1 - 1\times(\tilde{\tilde{\Pi}}^{+M_2}_{W_2})^*\vartheta'_{p_2}&.
\end{aligned}$$

Thus,

$$\tilde{\tilde{\tilde{\pi}}}^*_{x,\lambda}\theta_\lambda = (\tilde{\tilde{s}}_1\times\tilde{\tilde{s}}_2)^*(\tilde{\pi}^*_{\lambda_1}\theta_{\lambda_1}\times 1 + 1\times\tilde{\pi}^*_{\lambda_2}\theta_{\lambda_2}),$$

and hence, through (6-69),

$$\theta_\lambda = (s_1\times s_2)^*(\theta_{\lambda_1}\times 1 + 1\times\theta_{\lambda_2}),$$

which means (6-64).

(viii) (verifying (6-55)(3)) The composition formula (6-38) in this case gives

$$\begin{aligned}
\text{(6-70)}\quad \widehat{m}[\theta_{\bar\lambda}\wedge\theta_\lambda](W_\sqcup(\infty)) &= \widehat{m}[\theta_{\bar\lambda}\wedge\theta_\lambda](W_\sqcup) + [\widehat{\mathrm{K}}[\boldsymbol{\theta}_{\bar\lambda}\wedge\boldsymbol{\theta}_\lambda](W_\sqcup),\widehat{\partial}_\sqcup]_{\mathrm{odd}}\\
&\quad -\widehat{\mathrm{K}}[\boldsymbol{\theta}_\lambda](W_\sqcup)\widehat{n}[d\,\underline{\mathrm{h}}_\sqcup](M_\sqcup) - \widehat{n}[d\,\underline{\mathrm{h}}_\sqcup](M_\sqcup)\widehat{\mathrm{K}}[\boldsymbol{\theta}_{\bar\lambda}](W_\sqcup)\\
&= \widehat{m}[\theta_{\bar\lambda}\wedge\theta_\lambda](W_\sqcup) + [\widehat{\mathrm{K}}[\boldsymbol{\theta}_{\bar\lambda}\wedge\boldsymbol{\theta}_\lambda](W_\sqcup),\widehat{\partial}_\sqcup]_{\mathrm{odd}}\\
&\quad -\dot{\mathrm{A}}\widehat{n}[d\,\underline{\mathrm{h}}_\sqcup](M_\sqcup) + \widehat{n}[d\,\underline{\mathrm{h}}_\sqcup](M_\sqcup)\dot{\mathrm{D}}.
\end{aligned}$$

To compute the left-hand side of the preceding formula, first use (6-64) and its sister version for $\theta_{\bar\lambda}$ to write

$$\theta_{\bar\lambda}\wedge\theta_\lambda = s_1^*(\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}) + s_1^*\theta_{\lambda_1}\wedge s_2^*\theta_{\bar\lambda_2} + s_1^*\theta_{\bar\lambda_1}\wedge s_2^*\theta_{\lambda_2} + s_2^*(\theta_{\bar\lambda_2}\wedge\theta_{\lambda_2}).$$

Combining this with the computation of $\widehat{m}[u](W_1)$ in part (v), we get

$$\begin{aligned}
\text{(6-71)}\quad &\widehat{m}[\theta_{\bar\lambda}\wedge\theta_\lambda](W_\sqcup(\infty))\\
&= \widehat{m}[\theta_{\bar\lambda_1}\wedge\theta_{\lambda_1}](W_1)|\hat{1}\rangle\otimes\langle\hat{1}|\widehat{m}[1](W_2)\\
&\quad + \widehat{m}[\theta_{\bar\lambda_1}](W_1)|\hat{1}\rangle\otimes\langle\hat{1}|\widehat{m}[\theta_{\lambda_2}](W_2) + \widehat{m}[\theta_{\lambda_1}](W_1)|\hat{1}\rangle\otimes\langle\hat{1}|\widehat{m}[\theta_{\bar\lambda_2}](W_2)\\
&\quad + \widehat{m}[1](W_1)|\hat{1}\rangle\otimes\langle\hat{1}|\widehat{m}[\theta_{\bar\lambda_2}\wedge\theta_{\lambda_2}](W_2)\\
&= \check{m}[\theta_{\bar\lambda_1}+\theta_{\lambda_1}](W_1)|\check{1}\rangle\otimes\langle\hat{1}|\widehat{m}[1](W_2) + \check{m}[1](\widehat{W}_1)|\check{1}\rangle\otimes\langle\hat{1}|\widehat{m}[\theta_{\bar\lambda_2}+\theta_{\lambda_2}](W_2).
\end{aligned}$$

In comparison to the identities from decomposing $W_\sqcup$, we have the following identity, obtained by applying (6-38) (and its check version) to the decomposition of $W'_\sqcup$ described in Figure 1, bottom right. Recall that for the underlying decomposition of $W'_\sqcup$, the paths $(-1)^i\,\hat p_i$ split as $(-1)^i\,\hat p_i=(\bar\lambda_i\cup\lambda_i)\cup_{x,\bar x}(\gamma_i)$. The arguments leading to (6-46) imply that in the present setting,

$$\theta_{(-1)^i\,\hat p_i}=c(\theta_{\lambda_i}\otimes 1)+c(\theta_{\bar\lambda_i}\otimes 1)+c(1\otimes\theta_{\gamma_i}).$$

Then

$$\begin{aligned}\widehat m[\theta_{-\hat p_1}](\widehat W_1(\infty))&=\check m[\theta_{-\hat p_1}](\widehat W_1(\infty))\\&=\check m[\theta_{\lambda_1}+\theta_{\bar\lambda_1}](W_1)|\check 1\rangle+\sum_{\mathfrak c\in\mathfrak C^s(S^3)}\check m[\theta_{\gamma_1}](\overline{\mathfrak B})(\mathfrak c)\check m[1](W_1)|\mathfrak c\rangle,\\ \widehat m[\theta_{\hat p_2}](\widehat W_2(\infty))&=\langle\hat 1|\widehat m[\theta_{\lambda_2}+\theta_{\bar\lambda_2}](W_2)+\langle\widehat m[\theta_{\gamma_2}](\mathfrak B)|\widehat m[1](W_2).\end{aligned}$$

Remember that $\check m[u](\overline{\mathfrak B})$ is a map from $\check C(S^3)$ to $\mathbb K$, while $\widehat m[u](\mathfrak B)\in\widehat C(S^3)$. However, when $\deg(u)$ is odd, both $\check m[u](\overline{\mathfrak B})$ and $\widehat m[u](\mathfrak B)$ must vanish, because all generators of $\check C(S^3)$ (resp. $\widehat C(S^3)$) are of even (resp. odd) degree. Thus, the last terms of both lines in the preceding expression vanish. Combining these with (6-71), we have

$$\begin{aligned}&\widehat m[\theta_{\hat p_\sqcup}](W'_\sqcup(\infty))\\&\quad=\widehat m[\theta_{-\hat p_1}](\widehat W_1(\infty))\otimes\widehat m[1](\widehat W_2(\infty))+\widehat m[1](\widehat W_1(\infty))\otimes\widehat m[\theta_{\hat p_2}](\widehat W_2(\infty))\\&\quad=\check m[\theta_{\lambda_1}+\theta_{\bar\lambda_1}](W_1)|\check 1\rangle\otimes\langle\hat 1|\widehat m[1](W_2)+\check m[1](W_1)|\check 1\rangle\otimes\langle\hat 1|\widehat m[\theta_{\lambda_2}+\theta_{\bar\lambda_2}](W_2)\\&\quad=\widehat m[\theta_{\lambda_\sqcup}](W_\sqcup(\infty)),\end{aligned}\tag{6-72}$$

where $\hat p_\sqcup$ denotes the 1–chain $\hat p_2-\hat p_1$ in $W'_\sqcup=\widehat W_1\sqcup\widehat W_2$, and $\theta_{\hat p_\sqcup}:=\theta_{\hat p_2}-\theta_{\hat p_1}$. Now appy (6-38) and (6-57) to $W'_\sqcup$ with u therein set to be $\theta_{\hat p_\sqcup}$; we get

$$\begin{aligned}&\widehat m[\theta_{\hat p_\sqcup}](W'_\sqcup(\infty))\\&\quad=\widehat m[\theta_{\hat p_\sqcup}](W'_\sqcup)+[\hat{\mathrm K}[\boldsymbol\theta_{\hat p_\sqcup}](W'_\sqcup),\hat\partial_\sqcup]_{\mathrm{even}}+[\hat{\mathrm K}[\mathbf 1](W'_\sqcup),\hat n[d\,\underline{\mathrm h}_\sqcup]]_{\mathrm{even}}\\&\quad=\widehat m[\theta_{\hat p_\sqcup}](\mathbb R\times M_\sqcup)+[\hat{\mathrm K}[\boldsymbol\theta_{\hat p_\sqcup}](W'_\sqcup)-\hat{\mathrm Z}[\boldsymbol\theta_{\hat p_\sqcup}](W'_\sqcup),\hat\partial_\sqcup]_{\mathrm{even}}\\&\qquad\qquad+[\hat{\mathrm K}[\mathbf 1](W'_\sqcup)-\hat{\mathrm K}[\mathbf 1](W'_\sqcup),\hat n[d\,\underline{\mathrm h}_\sqcup]]_{\mathrm{even}}\\&\quad=\widehat m[\theta_{\hat p_\sqcup}](\mathbb R\times M_\sqcup)+[\hat{\mathrm K}[\boldsymbol\theta_{\hat p_\sqcup}](W'_\sqcup)-\hat{\mathrm Z}[\boldsymbol\theta_{\hat p_\sqcup}](W'_\sqcup),\hat\partial_\sqcup]_{\mathrm{even}}-[\mathrm Z_\sqcup,\hat n[d\,\underline{\mathrm h}_\sqcup]]_{\mathrm{even}},\end{aligned}$$

where $\mathrm Z_\sqcup$ is as in (6-68), and $\widehat m[\theta_{\hat p_\sqcup}](\mathbb R\times M_\sqcup)=\widehat m[\theta_{\hat p_1}](\mathbb R\times M_1)\otimes\widehat m[\theta_{\hat p_2}](\mathbb R\times M_2)$ denotes the version of the cobordism map when the metric and perturbation form on $\mathbb R\times M_\sqcup$, as well as $\hat p_i\subset\mathbb R\times M_\sqcup$, are invariant under the $\mathbb R$–action. However, $\widehat m[\theta_{\hat p_i}](\mathbb R\times M_i)=0$ by construction. (Recall (2-43) and (2-30).) Thus, the first term in the last line of the preceding formula vanishes. Putting all these together with (6-70) and (6-72), we have

$$
\begin{aligned}
&[\hat{\mathrm{K}}[\boldsymbol{\theta}_{\hat{p}_\sqcup}](W'_\sqcup) - \hat{\mathrm{Z}}[\boldsymbol{\theta}_{\hat{p}_\sqcup}](W'_\sqcup), \hat{\partial}_\sqcup]_{\mathrm{even}} - [\mathrm{Z}_\sqcup, \hat{n}[d\underline{\mathrm{h}}_\sqcup]]_{\mathrm{even}} \\
&\quad = \hat{m}[\theta_{\bar{\lambda}} \wedge \theta_\lambda](W_\sqcup) + [\hat{\mathrm{K}}[\boldsymbol{\theta}_{\bar{\lambda}} \wedge \boldsymbol{\theta}_\lambda](W_\sqcup), \hat{\partial}_\sqcup]_{\mathrm{odd}} - \dot{\mathrm{A}}\hat{n}[d\underline{\mathrm{h}}_\sqcup](M_\sqcup) + \hat{n}[d\underline{\mathrm{h}}_\sqcup](M_\sqcup)\dot{\mathrm{D}}.
\end{aligned}
$$

This implies (6-63)(3′), and hence also (6-55)(3), if we set

$$
\dot{\mathrm{C}} = \hat{\mathrm{K}}[\boldsymbol{\theta}_{\bar{\lambda}} \wedge \boldsymbol{\theta}_\lambda](W_\sqcup), \quad \mathrm{X} = \hat{\mathrm{Z}}[\boldsymbol{\theta}_{\hat{p}_\sqcup}](W'_\sqcup) - \hat{\mathrm{K}}[\boldsymbol{\theta}_{\hat{p}_\sqcup}](W'_\sqcup)
$$

in these formulas. This finishes the proof of the lemma. □

Remark 6.9 The preceding lemma has Yang–Mills analogs; see Theorem 7.16 and Corollary 7.21 of [7]. A previous version of this article (`arXiv:1204.0115v1`) contains sketches of an alternative proof, where the underlying geometric meanings of computations done here are clearer.

We have now shown that V_* defines a chain homotopy equivalence.

Step 6 We now verify the claim that V_* and $V_*^\dagger$ intertwine with the $\boldsymbol{A}_\dagger(M_\sqcup)$–action on $S_{U_\sqcup}(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup, \mathrm{r}[w]_\sqcup; \Gamma_\sqcup))$ and the $\boldsymbol{A}_\dagger(M_\#)$–action on $\mathrm{CM}_*(M_\#, \mathfrak{s}_\#, \mathrm{r}[w]_\#; \Gamma_\#)$ described in Parts 2 and 3 of the previous subsection. More precisely, for each $Q = U_p$ and $\mathfrak{t}_\gamma$ for $\gamma \in \{\gamma_i^{[1]}\}_i \cup \{\gamma_j^{[2]}\}_j$, we shall show that there exist homomorphisms

$$
\begin{aligned}
Z_{Q*}&: \mathrm{CM}_*(M_\#, \mathfrak{s}_\#, \mathrm{r}[w_\#]; \Gamma_\#) \to S_{U_\sqcup}(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup, \mathrm{r}[w_\sqcup]; \Gamma_\sqcup)), \\
Z_{Q*}^\dagger&: S_{U_\sqcup}(\hat{C}_*(M_\sqcup, \mathfrak{s}_\sqcup, \mathrm{r}[w_\sqcup]; \Gamma_\sqcup)) \to \mathrm{CM}_*(M_\#, \mathfrak{s}_\#, \mathrm{r}[w_\#]; \Gamma_\#)
\end{aligned}
$$

satisfying

$$
\begin{aligned}
V_*\mathfrak{m}_Q - S_{U_\sqcup}(\mathfrak{m}_Q)V_* &= Z_{Q*}\partial_\# + (-1)^{\deg Q} D_\sqcup Z_{Q*}, \\
V_*^\dagger S_{U_\sqcup}(\mathfrak{m}_Q) - (-1)^{\deg Q}\mathfrak{m}_Q V_*^\dagger &= \partial_\# Z_{Q*}^\dagger + (-1)^{\deg Q} Z_{Q*}^\dagger D_\sqcup.
\end{aligned}
$$

We shall only verify the first line above, since the second line is basically the adjoint of the first. Stated in terms of the decomposition (6-9), this amounts to verifying the set of identities

(6-73)(1) $$V_0 U_p - \hat{U}_p V_0 = \hat{\partial}_\sqcup Z_{U,0} + Z_{U,0}\partial_\#,$$

(6-73)(2) $$V_1 U_p - \hat{U}_p V_1 - \hat{K}_{U_p} V_0 = -\hat{\partial}_\sqcup Z_{U,1} + Z_{U,1}\partial_\# + \hat{U}_\sqcup Z_{U,0},$$

(6-73)(3) $$V_0\mathfrak{m}_\gamma - \mathfrak{m}_\gamma V_0 = -\hat{\partial}_\sqcup Z_{\gamma,0} + Z_{\gamma,0}\partial_\#,$$

(6-73)(4) $$V_1\mathfrak{m}_\gamma + \mathfrak{m}_\gamma V_1 - \hat{K}_\gamma V_0 = \hat{\partial}_\sqcup Z_{\gamma,1} + Z_{\gamma,1}\partial_\# - \hat{U}_\sqcup Z_{\gamma,0},$$

where
$$Z_{U*} = \begin{bmatrix} Z_{U,0} \\ Z_{U,1} \end{bmatrix}, \quad Z_{\gamma*} = \begin{bmatrix} Z_{\gamma,0} \\ Z_{\gamma,1} \end{bmatrix},$$
and $\widehat{K}_{U_p}$ and $\widehat{K}_\gamma$ are as defined in Part 3 of the last subsection. These are established by arguments similar to those used to verify (6-3), (2-51) and (2-55).

To proceed, we define Z_{U*} and $Z_{\gamma*}$ as follows. Let $\boldsymbol{p} \subset \mathcal{V}$ be a path such that on $\mathcal{V} - \mathcal{V}_c \simeq (\mathbb{R}^- \times M_\#) \cup (\mathbb{R}^+ \times M_\sqcup)$, $\boldsymbol{p} \cap (\mathcal{V} - \mathcal{V}_c)$ agrees with $\mathbb{R}^\pm \times \{p\}$ under the diffeomorphisms in (2-8). Suppose also that the path $\boldsymbol{p} \cup \bar{\boldsymbol{p}} \subset \mathcal{V} \cup_{M_\sqcup} \overline{\mathcal{V}} = W_\#$ becomes the line $\mathbb{R} \times \{p\} \subset \mathbb{R} \times M_\# \simeq W'_\#$ after the surgery of $W_\#$ along $\lambda_\#$. (Equivalently, $\bar{\boldsymbol{p}} \cup \boldsymbol{p} \subset \overline{\mathcal{V}} \cup_{M_\sqcup} \mathcal{V} = W_\sqcup$ also becomes $\mathbb{R} \times \{p\} \subset \mathbb{R} \times M_\sqcup \simeq W'_\sqcup$ after the surgery of $W_\sqcup$ along $S_\sqcup$.) For each $\gamma \in \{\gamma_i^{[1]}\}_i \cup \{\gamma_j^{[2]}\}_j$, define in a similar fashion an embedded cylinder $\Upsilon \subset \mathcal{V}$ that ends at circles $\gamma \subset Y_\pm$ on both ends of $\mathcal{V}$. Now set

$$\begin{aligned} Z_{U,0} &:= \widehat{\mathrm{K}}_{\boldsymbol{p}}(\mathcal{V}; \Gamma_{\mathcal{V}}) = \widehat{m}[\theta_{\boldsymbol{p}}](\mathcal{V}; \Gamma_{\mathcal{V}}) + \widehat{\Theta}_\sqcup V_0, \\ Z_{U,1} &:= \widehat{m}[u_{\boldsymbol{p}} u_\lambda](\mathcal{V}; \Gamma_{\mathcal{V}}) = \widehat{m}[\theta_{\boldsymbol{p}} \theta_\lambda](\mathcal{V}; \Gamma_{\mathcal{V}}) + [\widehat{m}[\theta_\lambda](\mathcal{V}; \Gamma_{\mathcal{V}}), \widehat{\Theta}_\sqcup] - \widehat{\Theta}_\sqcup Z_{U,0}, \\ Z_{\gamma,0} &:= \widehat{m}[\mathrm{F}\Upsilon](\mathcal{V}; \Gamma_{\mathcal{V}}), \\ Z_{\gamma,1} &:= \widehat{m}[\mathrm{F}\Upsilon u_\lambda](\mathcal{V}; \Gamma_{\mathcal{V}}) = \widehat{m}[\mathrm{F}\Upsilon \theta_\lambda](\mathcal{V}; \Gamma_{\mathcal{V}}) + \Theta_\sqcup Z_{\gamma,0}, \end{aligned} \tag{6-74}$$

where $\widehat{\mathrm{K}}_{\boldsymbol{p}}(\mathcal{V}; \Gamma_{\mathcal{V}})$ is as defined in (2-50) for $X = \mathcal{V}$ and $\lambda = \boldsymbol{p}$ with respect to the X–morphism $\Gamma_{\mathcal{V}}$, and $\mathrm{F}\Upsilon$ is as defined in (2-53).

With the preceding definitions, items (1) and (3) of (6-73) are direct consequences of (2-51) and (2-56). To derive items (2) and (4), first rewrite them in terms of the more concrete cochains, $\theta_{\boldsymbol{p}}$, θ_λ and $F\Upsilon$, using the now-established items (1) and (3) and the identities (6-15):

$$\begin{aligned} (2')\quad & [\widehat{m}[\theta_\lambda](\mathcal{V}), \hat{n}[d\underline{\mathrm{h}}_{\hat{p}}]]_{\text{even}} - \hat{n}[d\underline{\mathrm{h}}_{\hat{p}} \wedge d\underline{\mathrm{h}}_\sqcup]\widehat{m}[1](\mathcal{V}) \\ &\qquad = [\widehat{m}[\theta_{\boldsymbol{p}} \wedge \theta_\lambda](\mathcal{V}), \hat{\partial}]_{\text{even}} + \hat{n}[d\underline{\mathrm{h}}_\sqcup](\mathcal{V})\widehat{m}[\theta_{\boldsymbol{p}}](\mathcal{V}), \\ (4')\quad & [\widehat{m}[\theta_\lambda](\mathcal{V}), \hat{n}[\mathrm{u}_\gamma]]_{\text{odd}} - \hat{n}[\mathrm{u}_\gamma \wedge d\underline{\mathrm{h}}_\sqcup]\widehat{m}[1](\mathcal{V}) \\ &\qquad = [\widehat{m}[\mathrm{F}\Upsilon\theta_\lambda](\mathcal{V}), \hat{\partial}]_{\text{odd}} - \hat{n}[d\underline{\mathrm{h}}_\sqcup](\mathcal{V})\widehat{m}[\mathrm{F}\Upsilon](\mathcal{V}), \end{aligned}$$

where u_γ is the 0–cochain on $\mathcal{B}^\sigma_{\text{loc}}(Y_\pm)$ introduced in Part 2(b) of Section 2.5. Recall that in our case γ is used to denote both embedded circles in Y_+ and Y_-; we use the same notation u_γ for the corresponding cochains on $\mathcal{B}^\sigma_{\text{loc}}(Y_+)$ and $\mathcal{B}^\sigma_{\text{loc}}(Y_-)$. The same convention will be applied to the Y_+ and Y_- versions of other cochains associated to γ that were constructed in Section 2.

According to Proposition 25.3.4 of [17], verifying the identities ($2'$) and ($4'$) above is equivalent to verifying that

$$
\begin{aligned}
\widehat{m}[d(\theta_{\boldsymbol{p}}\wedge\theta_\lambda)](\mathcal{V}) &= [\widehat{m}[\theta_\lambda](\mathcal{V}),\hat{n}[d\,\underline{\mathrm{h}}_{\hat{p}}]]_{\mathrm{even}} - \hat{n}[d\,\underline{\mathrm{h}}_{\hat{p}}\wedge d\,\underline{\mathrm{h}}_{\sqcup}]\widehat{m}[1](\mathcal{V}) \\
&\qquad - \hat{n}[d\,\underline{\mathrm{h}}_{\sqcup}](\mathcal{V})\widehat{m}[\theta_{\boldsymbol{p}}](\mathcal{V}), \\
\widehat{m}[d(\mathrm{F}\Upsilon\theta_\lambda)](\mathcal{V}) &= [\widehat{m}[\theta_\lambda](\mathcal{V}),\hat{n}[\mathrm{u}_\gamma]]_{\mathrm{odd}} - \hat{n}[\mathrm{u}_\gamma\wedge d\,\underline{\mathrm{h}}_{\sqcup}]\widehat{m}[1](\mathcal{V}) \\
&\qquad + \hat{n}[d\,\underline{\mathrm{h}}_{\sqcup}](\mathcal{V})\widehat{m}[\mathrm{F}\Upsilon](\mathcal{V}).
\end{aligned}
\tag{6-75}
$$

The rest of the this step is devoted to verifying the preceding identities.

(i) (verifying the first line in (6-75)) We argue similarly to (2-47). Let $\overline{\mathcal{M}}$ denote a 3–dimensional moduli space of the form $\overline{\mathcal{M}}_{3,z}(\mathcal{V};\mathfrak{c}_-,\mathfrak{c}_+)$. Let $\varsigma\colon\overline{\mathcal{M}}\to\mathcal{B}^\sigma(\mathcal{V})\subset\mathcal{B}^\sigma_{\mathrm{loc}}(\mathcal{V})$ denote the embedding. Let $\mathfrak{M}$ and $\mathcal{M}^+$ denote respectively the top-dimensional strata of $\overline{\mathcal{M}}$ and $\mathfrak{r}^{-1}\overline{\mathcal{M}}$ as in Section 2.5. The coefficients of the map on the left-hand side are given by integrals of the form $\langle d(\theta_{\boldsymbol{p}}\wedge\theta_\lambda),\overline{\mathcal{M}}\rangle$. To compute them, let $\widetilde{\widetilde{\mathcal{B}}}{}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V})\to\mathcal{B}^\sigma(\mathcal{V})$ be the $U(1)\times U(1)$–bundle defined by the commutative diagram

$$
\begin{tikzcd}
\widetilde{\widetilde{\mathcal{B}}}{}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V}) \arrow[r, "\pi'_{\boldsymbol{p}}"] \arrow[d, "\pi'_\lambda"] \arrow[dr, "\pi_{\boldsymbol{p}\cup\lambda}"] & \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \arrow[d, "\pi_\lambda"] \\
\widetilde{\mathcal{B}}^\sigma_{\boldsymbol{p}}(\mathcal{V}) \arrow[r, "\pi_{\boldsymbol{p}}"] & \mathcal{B}^\sigma(\mathcal{V})
\end{tikzcd}
$$

Similarly to (2-34), we shall choose a map $\tilde{\tilde{\varsigma}}\colon\mathcal{M}^+\to\widetilde{\widetilde{\mathcal{B}}}{}^\tau(\mathcal{V})$ so that the diagram below commutes:

$$
\begin{tikzcd}
\mathcal{M}^+ \arrow[rr, "\tilde{\tilde{\varsigma}}"] \arrow[dr, "\tilde{\varsigma}_\lambda"] \arrow[rrr, "\tilde{\varsigma}_{\boldsymbol{p}}"'] \arrow[dd, "\mathfrak{r}"] & & \widetilde{\widetilde{\mathcal{B}}}{}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V}) \arrow[dl, "\pi'_{\boldsymbol{p}}"] \arrow[dr, "\pi'_\lambda"] \arrow[dd, "\pi_{\boldsymbol{p}\cup\lambda}"] & \\
& \widetilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V}) \arrow[dr, "\pi_\lambda"] & & \widetilde{\mathcal{B}}^\sigma_{\boldsymbol{p}}(\mathcal{V}) \arrow[dl, "\pi_{\boldsymbol{p}}"] \\
\overline{\mathcal{M}} \arrow[rr, "\varsigma"] & & \mathcal{B}^\sigma(\mathcal{V}) &
\end{tikzcd}
\tag{6-76}
$$

Consider the form $((\pi'_\lambda)^*\vartheta_{\boldsymbol{p}})\wedge((\pi'_{\boldsymbol{p}})^*\vartheta_\lambda)$ on $\widetilde{\widetilde{\mathcal{B}}}{}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V})$. Use the identities $\vartheta_{\boldsymbol{p}} = \pi^*_{\boldsymbol{p}}\theta_{\boldsymbol{p}} + (\widetilde{\Pi}^\infty)^*\vartheta'_p - (\widetilde{\Pi}^{-\infty})^*\vartheta'_p$ and $\vartheta_\lambda = \pi^*_\lambda\theta_\lambda + (\Pi^\partial_\lambda)^*\vartheta'_{p_2-p_1}$ to rewrite it as

$$
\begin{aligned}
&((\pi'_\lambda)^*\vartheta_{\boldsymbol{p}})\wedge((\pi'_{\boldsymbol{p}})^*\vartheta_\lambda) \\
&\quad = \pi^*_{\boldsymbol{p}\cup\lambda}(\theta_{\boldsymbol{p}}\wedge\theta_\lambda) + ((\pi'_\lambda)^*(\widetilde{\Pi}^\infty)^*\vartheta'_p - (\pi'_\lambda)^*(\widetilde{\Pi}^{-\infty})^*\vartheta'_p)\wedge\pi^*_{\boldsymbol{p}\cup\lambda}\theta_\lambda \\
&\qquad\qquad + (\pi^*_{\boldsymbol{p}\cup\lambda}\theta_{\boldsymbol{p}})\wedge(\pi'_{\boldsymbol{p}})^*(\widetilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1} \\
&\qquad\qquad\qquad + ((\pi'_\lambda)^*(\widetilde{\Pi}^\infty)^*\vartheta'_p - (\pi'_\lambda)^*(\widetilde{\Pi}^{-\infty})^*\vartheta'_p)\wedge(\pi'_{\boldsymbol{p}})^*(\widetilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}.
\end{aligned}
$$

Recall that the same notation p is used to denote corresponding points in both Y_- and Y_+. In the above, the same notation ϑ'_p is used to denote either the Y_- or the Y_+ version.

Now pull back the preceding identity by $\tilde{\tilde{\varsigma}}$ and integrate over $[(\mathcal{M}^+)_2] = \partial[\mathcal{M}]$. (Here we again used [17, Theorem 24.7.2 and Lemma 31.3.1].) The integral over the left-hand side vanishes, because both ϑ_λ and $\vartheta_{\boldsymbol{p}}$ are exact. Meanwhile, by way of the commutative diagram (6-76) and Stokes' theorem, the integral over the first term on the right-hand side is

$$\langle \tilde{\tilde{\varsigma}}^* \pi^*_{\boldsymbol{p}\cup\lambda}(\theta_{\boldsymbol{p}} \wedge \theta_\lambda), \partial[\mathcal{M}^+]\rangle = \langle \tilde{\tilde{\varsigma}}^* \pi^*_{\boldsymbol{p}\cup\lambda} d(\theta_{\boldsymbol{p}} \wedge \theta_\lambda), [\mathcal{M}^+]\rangle = \langle d(\theta_{\boldsymbol{p}} \wedge \theta_\lambda), \overline{\mathcal{M}}\rangle.$$

This is exactly the coefficients of the map $\widehat{m}[\delta(\theta_{\boldsymbol{p}} \wedge \theta_\lambda)](\mathcal{V})$ that we aim to compute. With a bit of diagram chasing, the aforementioned integral identity then becomes

$$\begin{aligned}\langle d(\theta_{\boldsymbol{p}} \wedge \theta_\lambda), \overline{\mathcal{M}}\rangle &\\ = -\big\langle \tilde{\varsigma}^*_{\boldsymbol{p}}\big(((\tilde{\Pi}^{\infty})^*\vartheta'_p - (\tilde{\Pi}^{-\infty})^*\vartheta'_p) \wedge \pi^*_{\boldsymbol{p}}\theta_\lambda\big), (\mathcal{M}^+)_2\big\rangle &\\ - \langle \tilde{\varsigma}^*_\lambda((\pi^*_\lambda\theta_{\boldsymbol{p}}) \wedge (\tilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}), (\mathcal{M}^+)_2\rangle &\\ - \big\langle \tilde{\tilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\tilde{\Pi}^{\infty})^*\vartheta'_p - (\pi'_\lambda)^*(\tilde{\Pi}^{-\infty})^*\vartheta'_p) &\\ \wedge (\pi'_{\boldsymbol{p}})^*(\tilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}\big), (\mathcal{M}^+)_2\big\rangle.&\end{aligned} \tag{6-77}$$

According to [17], $(\mathcal{M}^+)_2$ is a union of product spaces of the forms

$$\begin{aligned}&\mathcal{N}^+_0(Y_-\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}_2(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+), \quad \mathcal{M}_2(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}^+_0(Y_+; \mathfrak{c}, \mathfrak{c}_+),\\ &\mathcal{N}^+_1(Y_-; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}_1(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+), \quad \mathcal{M}_1(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}^+_1(Y_+; \mathfrak{c}, \mathfrak{c}_+),\\ &\mathcal{N}^+_2(Y_-; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}_0(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+), \quad \mathcal{M}_0(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}^+_2(Y_+; \mathfrak{c}, \mathfrak{c}_+).\end{aligned} \tag{6-78}$$

The diagram requires that each $\mathcal{N}^+$ factor of the preceding product spaces must map to fibers of the bundles $\tilde{\mathcal{B}}^\sigma_{\boldsymbol{p}}(\mathcal{V})$, $\tilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V})$ or $\tilde{\tilde{\mathcal{B}}}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V})$, respectively, under $\tilde{\varsigma}_{\boldsymbol{p}}$, $\tilde{\varsigma}_\lambda$ and $\tilde{\tilde{\varsigma}}$. Meanwhile, observe that on the right-hand side of (6-77), the first, second and third term has respectively a factor of $(\tilde{\Pi}^{\pm\infty})^*\vartheta'_p$, $(\tilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}$ and

$$(\pi'_\lambda)^*(\tilde{\Pi}^{\pm\infty})^*\vartheta'_p \wedge (\pi'_{\boldsymbol{p}})^*(\tilde{\Pi}^\partial_\lambda)^*\vartheta'_{p_2-p_1}.$$

They restrict respectively to the volume forms on the fibers of, respectively, $\tilde{\mathcal{B}}^\sigma_{\boldsymbol{p}}(\mathcal{V})$, $\tilde{\mathcal{B}}^\sigma_\lambda(\mathcal{V})$ and $\tilde{\tilde{\mathcal{B}}}^\sigma_{\boldsymbol{p}\cup\lambda}(\mathcal{V})$. This means that product spaces of the types in the first line of (6-78) never contribute to the integrals on the right-hand side of (6-77); those of the types in the second line of (6-78) contribute only to the integrals in the first and second terms on the right-hand side of (6-77); those of the types in the third line of (6-78) contribute only to the integrals in the last term on the right-hand side of (6-77). Consequently,

$$
\begin{aligned}
\text{(6-79)}\quad \langle d(\theta_{\boldsymbol p}\wedge\theta_\lambda),\overline{\mathcal M}\rangle = &-\sum_{\mathfrak c}\langle\theta_\lambda,\mathcal M_1(\mathcal V;\mathfrak c_-,\mathfrak c)\rangle\langle d\,\underline{\mathfrak h}_p,\mathcal N_1^+(Y_+,\mathfrak c,\mathfrak c_+)\rangle\\
&+\sum_{\mathfrak c}\langle d\,\underline{\mathfrak h}_p,\mathcal N_1^+(Y_-;\mathfrak c_-,\mathfrak c)\rangle\langle\theta_\lambda,\mathcal M_1(\mathcal V;\mathfrak c,\mathfrak c_+)\rangle\\
&-\sum_{\mathfrak c}\langle\theta_{\boldsymbol p},\mathcal M_1(\mathcal V;\mathfrak c_-,\mathfrak c)\rangle\langle d\,\underline{\mathfrak h}_\sqcup,\mathcal N_1^+(Y_+,\mathfrak c,\mathfrak c_+)\rangle\\
&-\sum_{\mathfrak c}\langle 1,\mathcal M_0(\mathcal V;\mathfrak c_-,\mathfrak c)\rangle\langle d\,\underline{\mathfrak h}_p\wedge d\,\underline{\mathfrak h}_\sqcup,\mathcal N_2^+(Y_+;\mathfrak c,\mathfrak c_+)\rangle.
\end{aligned}
$$

This identity leads directly to the identity in the first line of (6-75).

(ii) (verifying the second line in (6-75)) We proceed similarly, but now take $\overline{\mathcal M}=(\mathcal M)_2$ to be a 2–dimensional moduli space of the form $\overline{\mathcal M}_{2,z}(\mathcal V;\mathfrak c_-,\mathfrak c_+)$. To compute $\langle\delta(\mathrm{F}\Upsilon\theta_\lambda),\overline{\mathcal M}\rangle=\langle\mathrm{F}\Upsilon\theta_\lambda,(\overline{\mathcal M})_1\rangle$, consider the bundles $\pi_\gamma\colon\hat{\mathcal B}^\sigma_\gamma(\mathcal V)\to\mathcal B^\sigma(\mathcal V)$, $\hat{\tilde\pi}\colon\hat{\tilde{\mathcal B}}^\sigma(\mathcal V)\to\mathcal B^\sigma(V)$ defined by the commutative diagram

$$
\begin{tikzcd}
\hat{\tilde{\mathcal B}}^\sigma_{\gamma\cup\lambda}(\mathcal V) \arrow[r, "\pi'_\gamma"] \arrow[d, "\pi'_\lambda"] \arrow[dr, "\pi_{\gamma\cup\lambda}"] & \tilde{\mathcal B}^\sigma_\lambda(\mathcal V) \arrow[d, "\pi_\lambda"] \\
\hat{\mathcal B}^\sigma_\gamma(\mathcal V) \arrow[r, "\pi_\gamma"] \arrow[d, "\hat\Pi^{\pm\infty}"] & \mathcal B^\sigma(\mathcal V) \arrow[d, "\Pi^{\pm\infty}"] \\
\hat{\mathcal B}^\sigma_{\mathfrak t}(Y_\pm) \arrow[r, "\mathrm{P}_{\mathfrak t}"] & \mathcal B^\sigma(Y_\pm)
\end{tikzcd}
$$

where $\mathrm{P}_{\mathfrak t}\colon\hat{\mathcal B}^\sigma_{\mathfrak t}(Y_\pm)\to\mathcal B^\sigma(Y_\pm)$ was defined in Section 2.5's Part 2(b). Note that $\hat{\mathcal B}^\sigma_\gamma(\mathcal V)=(\Pi^{+\infty})^*\hat{\mathcal B}^\sigma_{\mathfrak t}(Y_+)\simeq(\Pi^{-\infty})^*\hat{\mathcal B}^\sigma_{\mathfrak t}(Y_-)$. Choose liftings $\hat{\tilde\varsigma}$, $\hat\varsigma_\gamma$ and $\tilde\varsigma_\lambda$ of the embedding $\varsigma\colon\overline{\mathcal M}\to\mathcal B^\sigma(\mathcal V)$ that fit into the commutative diagram

$$
\text{(6-80)}\qquad
\begin{tikzcd}
\mathcal M^+ \arrow[rr, "\hat{\tilde\varsigma}"] \arrow[dr, "\tilde\varsigma_\lambda"] \arrow[drrr, "\hat\varsigma_\gamma"] \arrow[dd, "\mathfrak r"] & & \hat{\tilde{\mathcal B}}^\sigma_{\gamma\cup\lambda}(\mathcal V) \arrow[dl, "\pi'_\gamma"] \arrow[dr, "\pi'_\lambda"] \arrow[dd, "\pi_{\gamma\cup\lambda}"] & \\
& \tilde{\mathcal B}^\sigma_\lambda(\mathcal V) \arrow[dr, "\pi_\lambda"] & & \hat{\mathcal B}^\sigma_\gamma(\mathcal V) \arrow[dl, "\pi_\gamma"] \\
\overline{\mathcal M} \arrow[rr, "\varsigma"] & & \mathcal B^\sigma(\mathcal V) &
\end{tikzcd}
$$

Let $\widetilde{\mathrm{F}} := (\pi'_\lambda)^* \mathrm{F}\Upsilon$. Noting that ϑ_λ is closed, we have by (2-56) that

$$d(\widetilde{\mathrm{F}}\vartheta_\lambda) = ((\pi_\lambda)^*(\Pi^\infty)^*\mu_\gamma) \wedge \vartheta_\lambda - ((\pi_\lambda)^*(\Pi^{-\infty})^*\mu_\gamma) \wedge \vartheta_\lambda.$$

Pull back by π'_γ on both sides of the preceding identity. Using the fact that $\mathrm{P}_{\mathfrak{t}}^*\mu_\gamma = d\mathrm{x}_\gamma$, a bit of diagram chasing then yields

$$d(\widehat{\widetilde{\mathrm{F}}}\widehat{\vartheta}_\lambda) = d\big(((\pi'_\lambda)^*(\widehat{\Pi}^\infty)^*\,\mathrm{x}_\gamma\,\widehat{\vartheta}_\lambda)\big) - d\big(((\pi'_\lambda)^*(\widehat{\Pi}^{-\infty})^*\,\mathrm{x}_\gamma\,\widehat{\vartheta}_\lambda)\big),$$

where $\widehat{\widetilde{\mathrm{F}}} := (\pi_{\gamma\cup\lambda})^*\mathrm{F}\Upsilon$ and $\widehat{\vartheta}_\lambda := (\pi'_\gamma)^*\vartheta_\lambda$. Pull back both sides by $\widehat{\widetilde{\varsigma}}$ and integrate over $\mathcal{M}^+$. Then apply the Stokes' theorem [17, Theorem 24.7.2 and Lemma 31.3.1] to get

$$\begin{aligned}
&\langle \widehat{\widetilde{\varsigma}}^*(\widehat{\widetilde{\mathrm{F}}}\widehat{\vartheta}_\lambda), (\mathcal{M}^+)_1\rangle \\
&\quad = \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^\infty)^*\,\mathrm{x}_\gamma\,\widehat{\vartheta}_\lambda)\big), (\mathcal{M}^+)_1\big\rangle - \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^{-\infty})^*\,\mathrm{x}_\gamma\,\widehat{\vartheta}_\lambda)\big), (\mathcal{M}^+)_1\big\rangle.
\end{aligned}$$

Recall that $\vartheta_\lambda = \pi_\lambda^*\theta_\lambda + (\widetilde{\Pi}_\lambda^\partial)^*\vartheta'_{p_2-p_1}$. With a bit more diagram chasing, the preceding formula can be rewritten as

$$\begin{aligned}
\text{(6-81)}\quad &\langle \widehat{\widetilde{\varsigma}}^*\pi_{\gamma\cup\lambda}^*(\mathrm{F}\Upsilon\theta_\lambda), (\mathcal{M}^+)_1\rangle + \langle \widehat{\widetilde{\varsigma}}^*((\pi_{\gamma\cup\lambda}^*\mathrm{F}\Upsilon)(\pi'_\gamma)^*(\widetilde{\Pi}_\lambda^\partial)^*\vartheta'_{p_2-p_1}), (\mathcal{M}^+)_1\rangle \\
&\quad = \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^\infty)^*\,\mathrm{x}_\gamma)(\pi_{\gamma\cup\lambda}^*\theta_\lambda)\big), (\mathcal{M}^+)_1\big\rangle \\
&\qquad\quad - \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^{-\infty})^*\,\mathrm{x}_\gamma)(\pi_{\gamma\cup\lambda}^*\theta_\lambda)\big), (\mathcal{M}^+)_1\big\rangle \\
&\qquad\qquad + \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^\infty)^*\,\mathrm{x}_\gamma)(\pi'_\gamma)^*(\widetilde{\Pi}_\lambda^\partial)^*\vartheta'_{p_2-p_1}\big), (\mathcal{M}^+)_1\big\rangle \\
&\qquad\qquad\quad - \big\langle \widehat{\widetilde{\varsigma}}^*\big(((\pi'_\lambda)^*(\widehat{\Pi}^{-\infty})^*\,\mathrm{x}_\gamma)(\pi'_\gamma)^*(\widetilde{\Pi}_\lambda^\partial)^*\vartheta'_{p_2-p_1}\big), (\mathcal{M}^+)_1\big\rangle.
\end{aligned}$$

By the diagram (6-80), the leftmost term in the preceding formula is

$$\langle \widehat{\widetilde{\varsigma}}^*\pi_{\gamma\cup\lambda}^*(\mathrm{F}\Upsilon\theta_\lambda), (\mathcal{M}^+)_1\rangle = \langle \varsigma^*(\mathrm{F}\Upsilon\theta_\lambda), \partial[\mathcal{M}^+]\rangle = \langle d(\mathrm{F}\Upsilon\theta_\lambda), \overline{\mathcal{M}}\rangle,$$

namely, it is precisely the typical coefficient of $\widehat{m}[\delta(\mathrm{F}\Upsilon\theta_\lambda)](\mathcal{V})$ that we seek to compute. To compute the other terms in (6-81), recall that according to [17], $(\mathcal{M}^+)_1$ is a union of product spaces of the forms

$$\text{(6-82)}\qquad \begin{aligned}
&\mathcal{N}_0^+(Y_-\mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}_1(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+), \quad \mathcal{M}_1(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_0^+(Y_+; \mathfrak{c}, \mathfrak{c}_+), \\
&\mathcal{N}_1^+(Y_-; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{M}_0(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+), \quad \mathcal{M}_0(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c}) \times \mathcal{N}_1^+(Y_+; \mathfrak{c}, \mathfrak{c}_+).
\end{aligned}$$

The map $\mathfrak{r}$ is a diffeomorphism when restricted to spaces described by the first line of the preceding expression while, according to (6-80), spaces described by the second line lie in fibers of the $U(1)$–bundle $\pi_{\gamma\cup\lambda}\colon \widehat{\widetilde{\mathcal{B}}}^\sigma(\mathcal{V}) \to \mathcal{B}^\sigma(\mathcal{V})$. Note that $(\pi'_\gamma)^*(\widetilde{\Pi}_\lambda^\partial)^*\vartheta'_{p_2-p_1}$ restricts to Thom forms on fibers of $\widehat{\mathcal{B}}^\sigma(\mathcal{V})$, and both p_1 and p_2 lie in the Y_+–end of $\mathcal{V}$. These imply that only spaces of the last type described in (6-82) contribute to the integrals in the second term on the left-hand side of (6-81), as well as to the integrals in the last two terms on the right-hand side. Meanwhile, for the first two terms on the right-hand side of (6-81), only spaces described in the first line of (6-82) contribute. Make use of these observations to rewrite (6-81) as

$$\begin{aligned}\langle d(\mathrm{F}_\Upsilon\theta_\lambda), \overline{\mathcal{M}}\rangle = \sum_{\mathfrak{c}} \langle\theta_\lambda, \mathcal{M}_1(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c})\rangle\langle \mathrm{u}_\gamma, \mathcal{N}_0^+(Y_+, \mathfrak{c}, \mathfrak{c}_+)\rangle \\ + \sum_{\mathfrak{c}} \langle \mathrm{u}_\gamma, \mathcal{N}_0^+(Y_-; \mathfrak{c}_-, \mathfrak{c})\rangle\langle\theta_\lambda, \mathcal{M}_1(\mathcal{V}; \mathfrak{c}, \mathfrak{c}_+)\rangle \\ - \sum_{\mathfrak{c}} \langle 1, \mathcal{M}_0(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c})\rangle\langle \mathrm{u}_\gamma\, d\underline{\mathrm{h}}_\sqcup, \mathcal{N}_1^+(Y_+; \mathfrak{c}, \mathfrak{c}_+)\rangle \\ + \sum_{\mathfrak{c}} \langle \mathrm{F}_\Upsilon, \mathcal{M}_0(\mathcal{V})(\mathfrak{c}_-, \mathfrak{c})\rangle\langle d\underline{\mathrm{h}}_\sqcup, \mathcal{N}_1(M_\sqcup)(\mathfrak{c}, \mathfrak{c}_+)\rangle.\end{aligned}$$

Note that the last term in (6-81) is zero, because integrals of the form

$$\langle \hat{\varsigma}^*(\hat{\Pi}^{-\infty})^*\, \mathrm{x}_\gamma, \mathcal{M}_0(\mathcal{V}; \mathfrak{c}_-, \mathfrak{c})\rangle$$

vanish. Now, the preceding identity relates the coefficients in the identity of maps in the second line of (6-75), directly establishing the latter identity.

(b) The proof for assertion (b) of Proposition 6.7 differs from part (a) only in the mechanism to ensure that the right-hand side of (2-27) and its analog are well defined. Instead of monotonicity, this is now justified by the completeness condition on the local coefficients, and by working with the grading-completed version of monopole Floer complexes $C_\bullet$. The relevant compactness theorem here is Theorem 24.5.2 of [17]. □

Remark 6.10 (a) Recall from Section 2.1 that when $c_1(\mathfrak{s})$ is torsion, the following types of perturbations are all equivalent: positive monotone, negative monotone, balanced, exact. Thus, the assumption in part (a) implies that $c_1(\mathfrak{s}_\#)$ is nontorsion. On the other hand, the assumption that $[w_\#]$ is monotone with respect to $c_\#$ in part (a) implies that both $[w_1]$ and $[w_2]$ are respectively monotone with respect to $c_1(\mathfrak{s}_1)$ and $c_1(\mathfrak{s}_2)$ with the same monotonicity constant. Combined with the assumption that $[w_\#]$ is nonbalanced with respect to $c_1(\mathfrak{s})$, this implies that $[w_i]$ is nonbalanced with respect to $c_1(\mathfrak{s}_i)$ for at least one of $i = 1$ or 2. Keep in mind that we always choose M_1 to be the one endowed with a nonbalanced perturbation.

(b) Our proof follows the "standard" cobordism argument that appeared in [9; 7, Section 7.4] in the Yang–Mills setting. Bloom, Mrowka and Ozsváth [2] proved a connected sum formula for the case of exact perturbations, using a different approach that involves surgery exact sequences. The use of the latter necessitates the use of the completed version of monopole Floer homologies $\mathrm{HM}_\bullet$.

6.3 Filtered monopole Floer homology and handle addition

Continue to work with the same settings and notation from earlier parts of the section, but now specialize to the 3–manifolds and cobordisms described in Sections 3.5 and 3.7. More specifically, the following two cases are considered; fix an $\mathrm{r} \gg \pi$:

(6-83) (1) Let $M_1 = Y_i$ for $i = 0, \dots, \mathrm{G}-1$. Equip Y_i with the nontorsion Spin^c–structure and a metric from the set *Met* in Proposition 3.9. Let w_1 be the corresponding harmonic 2–form w in Proposition 3.9. Let $M_2 = S^1 \times S^2$, $\mathfrak{s}_2$ be the trivial Spin^c–structure, and $w_2 \equiv 0$. Then $M_\# \simeq Y_{i+1}$, and $[w_\#] = c_1(\mathfrak{s}_\#)$ is nontorsion. Choose the metric on Y_{i+1} to be from the set *Met* from Proposition 3.9.

(2) Let $M_1 = S^1 \times S^2$, with the nontorsion Spin^c–structure $\mathfrak{s}_1$, closed 2–form w_1, and metric as described in Part 1 of Section 3.5. Let $(M_2, \mathfrak{s}_2) = (M, \mathfrak{s})$ be a connected Spin^c 3–manifold, with $\varpi_2 = \mathrm{r} w_2$ for a closed 2–form w_2 in the cohomology class $c_1(\mathfrak{s}_2)$. Choose a metric on M with respect to which w_2 is harmonic, and, in the case when $c_1(\mathfrak{s})$ is nontorsion, having nondegenerate zeros. (When $c_1(\mathfrak{s})$ is torsion, w_2 is necessarily 0.) In other words, $M_\sqcup$ is the Y_Z in Part 1 of Section 3.5. Thus $M_\# \simeq Y_0$, and $[w_\#] = c_1(\mathfrak{s}_\#)$ is nontorsion. Choose the metric on Y_0 to be from the set *Met* from Proposition 3.9.

In both cases above, M_1 is of the type Y_Z in Section 3.2, and hence contains a special 1–cycle γ. We denote this by γ_1. Consequently, assuming that p_1 is disjoint from γ_1, both $M_\sqcup$ and $M_\#$ inherit a 1–cycle from $\gamma_1 \subset M_1$. They are respectively denoted by $\gamma_\sqcup$ and $\gamma_\#$. According to Section 3.8, the filtered monopole Floer homologies $\mathrm{HM}^\circ(M_1, \langle w_1 \rangle; \Lambda_{\gamma_1})$ and $\mathrm{HM}^\circ(M_\#, \langle w_\# \rangle; \Lambda_{\gamma_\#})$ are well defined. In parallel to what was done in Section 2.4, define $\mathrm{CM}^\circ(M_\sqcup, \langle w_\sqcup \rangle; \Lambda_{\gamma_\#})$ to be the product complex of $\mathrm{CM}^\circ(M_1, \langle w_1 \rangle; \Lambda_{\gamma_1})$ and $\widehat{C}(M_2, \mathrm{r} w_2)$. The map $U_\sqcup = \widehat{U}_{M_\sqcup}$, as given in (2-57), acts on $\mathrm{CM}^\infty(M_\sqcup)$ and maps

$$\mathrm{CM}^-(M_\sqcup) \subset \mathrm{CM}^\infty(M_\sqcup)$$

into itself. The same notation is used to denote its induced maps on $\mathrm{CM}^+(M_\sqcup)$ and $\mathrm{CM}^-(M_\sqcup)$. By construction, the four flavors of $\mathrm{CM}^\circ(M_\sqcup)$ are related by short exact sequences of the form (3-18). Thus by Lemma 4.2, the $H_*(S^1)$–modules $S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup))$ are related by short exact sequences of the same form. The long exact sequences induced are also called the fundamental exact sequences.

The remainder of this subsection consists of three parts. The first part contains a filtered analog of Proposition 6.7. The second part analyzes the filtered connected sum formula from Part 1. The last part derives Theorem 1.1 from this computation.

Part 1 A filtered variant of Proposition 6.7 states:

Proposition 6.11 *Let $M_{\sqcup}$ and $M_{\#}$ be as in either case of* (6-83). *Then there is a system of isomorphisms from* $\mathrm{HM}^{\circ}(M_{\#}, \langle w_{\#}\rangle; \Lambda_{\gamma_{\#}})$ *to* $H_*\big(S_{U_{\sqcup}}(\mathrm{CM}^{\circ}(M_{\sqcup}))\big)$ *for* $\circ = -, \infty, +, \wedge$ *as graded* $\boldsymbol{A}_{\dagger}(M_{\#}) \simeq \boldsymbol{A}_{\dagger}(M_{\sqcup})$*–modules, which is natural with respect to the fundamental exact sequences on both sides.*

Proof Both cases in (6-83) satisfy the conditions of Proposition 6.7(a). Take $\Gamma_1 = \Lambda_{\gamma_1}$ and $\Gamma_2 = \mathbb{K}$ (the constant local coefficients). Then $\Gamma_{\sqcup} = \Lambda_{\gamma_{\sqcup}}$ and $\Gamma_{\#} = \Lambda_{\gamma_{\#}}$. Repeat the proof of Proposition 6.7 using cobordisms (X, w_X) constructed from Proposition 3.13 for case (1) of (6-83), and Proposition 3.11 for case (2). Like in the previous section, we denote this by the shorthand $\mathcal{V}$ when $Y_- = M_{\#}$, and by $\overline{\mathcal{V}}$ when $Y_- = M_{\sqcup}$. By construction, there is a cylinder $C \subset X$ ending at $\gamma_{\sqcup} \subset Y_{\sqcup}$, and $\gamma_{\#} \subset Y_{\#}$ satisfying the constraints in Section 3.7. According to Section 3.8, this gives us chain maps

$$m^{\infty}[u](X, \langle w_X\rangle; \Lambda_C)\colon \mathrm{CM}^{\infty}(Y_-) \to \mathrm{CM}^{\infty}(Y_+),$$
$$m^{-}[u](X, \langle w_X\rangle; \Lambda_C)\colon \mathrm{CM}^{-}(Y_-) \to \mathrm{CM}^{-}(Y_+).$$

In parallel to (6-12), let

$$V_0^{\circ} = m^{\circ}[1](\mathcal{V}; \Lambda_C), \qquad V_1^{\circ} = m^{\circ}[u_{\lambda}](\mathcal{V}; \Lambda_C),$$
$$V_1^{\dagger,\circ} = m^{\circ}[u_{\bar{\lambda}}](\overline{\mathcal{V}}; \Lambda_C), \quad V_0^{\dagger,\circ} = m^{\circ}[1](\overline{\mathcal{V}}; \Lambda_C)$$

for $\circ = -, \infty$, and use them to define V_*° and $V_*^{\dagger,\circ}$ as in (6-11). Keeping in mind the nonnegativity of the integers $n(\mathfrak{d})$ entering the definitions of ∂^{∞} and m^{∞}, the rest of the proof of Proposition 6.7 may be repeated with only cosmetic changes to see that V_*° and $V_*^{\dagger,\circ}$ induce chain homotopy equivalences between $\mathrm{CM}^{\circ}(M_{\#}, \langle w_{\#}\rangle; \Lambda_{\gamma_{\#}})$ and $S_{U_{\sqcup}}(\mathrm{CM}^{\circ}(M_{\sqcup}))$ for $\circ = -, \infty$. These fit into commutative diagrams with the fundamental exact sequences (3-18) on both sides of V_*° and $V_*^{\dagger,\circ}$. This understood, the rest of the proposition follows from the five lemma. $\square$

Part 2 We next analyze the homologies $H_*\big(S_{U_{\sqcup}}(\mathrm{CM}^{\circ}(M_{\sqcup}))\big)$ in the two cases of (6-83).

Case 1 Choose a product metric with constant curvature on $M_2 = S^1 \times S^2$. The moduli space of Seiberg–Witten solutions over it is a circle of flat connections. Choose a real Morse function on this circle with a pair of index 1 and index 0 critical points, and two gradient flow lines between them. Perform a perturbation to the Seiberg–Witten equations adapted to this Morse function, as described in Chapter 33 of [17]. In this context, $\widehat{C}(M_2) = C^u = \mathbb{K}[u_2, y_2]$ and $\partial_{M_2} = 0$, where:

- The unit $1 \in \mathbb{K}[u_2, y_2]$ has grading $[\xi_+]$ in the notation of [17, page 57].
- u_2 has degree -2, and the U_2–map acts by multiplication by u_2.
- y_2 has degree 1 and represents a generator of $H_1(S^1;\mathbb{Z})$ cooriented with the moduli spaces. In particular, $y_2^2 = 0$.

Thus,

$$\begin{aligned} S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup)) &= \mathrm{CM}^\circ(M_1)[u_2, y_2] \otimes \mathbb{K}[y], \\ D_\sqcup &= \partial_{M_1} \otimes \jmath + (U_1 \otimes -1 \otimes u_2) \otimes y. \end{aligned} \tag{6-84}$$

Write a generic element $a \in S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup))$ as

$$a_0 + a_1 y, \quad \text{where } a_0, a_1 \in \mathrm{CM}^\circ(M_1)[u_2, y_2].$$

Then

$$D_\sqcup a = \partial_{M_1} a_0 - (\partial_{M_1} a_1) y + (U_1 - u_2)(a_0) y.$$

Thus,

$$\begin{aligned} H_*\big(S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup))\big) &= \{a_0 + a_1 y \mid \partial_{M_1} a_0 = 0,\ (U_1 - u_2) a_0 = \partial_{M_1} a_1\} \otimes \mathbb{K}[y] \\ &\qquad \bmod (\partial_{M_1} b_1 y \sim 0, u_2 b_0 y \sim U_1 b_0 y - \partial_{M_1} b_0) \otimes \mathbb{K}[y_2] \\ &\simeq \mathrm{HM}^\circ(M_1) y \otimes \mathbb{K}[y_2]. \end{aligned}$$

Consequently,

$$H_*(S^\circ_{U_\sqcup}(M_\sqcup)) \simeq \mathrm{HM}^\circ(M_1)[y_2].$$

(Alternatively, one may use a spectral sequence computation, filtrating (6-84) first by degree in y, then by degree in u_2.)

Case 2 Now $\mathfrak{C}(M_1) = \mathfrak{C}^o(M_1)$ consists of a single irreducible point, $(A, (\alpha, \beta)) = \big(0, ((2\mathrm{r})^{-1/2}, 0)\big)$. (See eg [6] for this well-known fact.) Thus, $\mathrm{CM}^\circ(M_1)$ and the fundamental short exact sequences relating them are simply the modules V° and the sequences in (4-7), (4-8) and (4-9). Write the variable u in (4-7) as u_1 below. As pointed out in Remark 3.16(a), u_1 stands both for the deck transformation and U–map on $\mathrm{CM}^\circ(M_1)$.

This said, we have, in this case,

$$S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup)) = V^\circ(u_1) \otimes \mathrm{CM}(M, c_-) \otimes \mathbb{K}[y],\tag{6-85}$$
$$D_\sqcup = 1 \otimes \partial_M \otimes \jmath + (u_1 \otimes 1 - 1 \otimes U_2) \otimes y.$$

This can alternatively be written as

$$E^\circ(\mathrm{CM}(M, c_-) \otimes \mathbb{K}[y], \partial_M \otimes \jmath - U_2 \otimes y) = E^\circ\big(S_{U_2}(\mathrm{CM}(M, c_-))\big).\tag{6-86}$$

By Proposition 5.9 and Remark 5.10, the homology of the latter is $\mathring{\mathrm{HM}}(M, c_b)$, and the isomorphisms from $H_*\big(S_{U_\sqcup}(\mathrm{CM}^\circ(M_\sqcup))\big)$ to the latter preserves the $\mathbb{K}[u]$–module structure and are natural with respect to the fundamental exact sequences. Since the U–map commutes with the $\bigwedge^* H_1(M;\mathbb{Z})/\mathrm{Tors}$–actions on both sides, These are isomorphisms as $\boldsymbol{A}_\dagger(M)$–modules.

To conclude, combining the above computation with Proposition 6.11, we have:

Corollary 6.12 (1) *There is a system of isomorphisms of $\boldsymbol{A}_\dagger(M)$–modules*

$$\mathrm{HM}^\circ(Y_i, \langle w\rangle; \Lambda_\gamma) \simeq \mathrm{HM}^\circ(Y_{i-1}, \langle w\rangle; \Lambda_\gamma) \otimes H_*(S^1) \quad \text{for } i = 1, \dots, \mathrm{G}$$

preserving the relative gradings and natural with respect to the fundamental exact sequences.

(2) *There is a system of isomorphisms of $\boldsymbol{A}_\dagger(M)$–modules*

$$\mathrm{HM}^\circ(Y_0, \langle w\rangle; \Lambda_\gamma) \simeq \mathring{\mathrm{HM}}(M, c_b)$$

preserving the relative gradings and natural with respect to the fundamental exact sequences, respectively for $\circ = -, \infty, +, \wedge$ on the left-hand side and $\circ = \wedge, -, \vee, \sim$ on the right-hand side.

Proof of Theorem 1.1 (1) This follows from an iteration of Corollary 6.12(1) and Lemma 6.13 below, in terms of the alternative notation (3-19).

(2) This is a restatement of Corollary 6.12(2) in alternative notation, according to the second bullet of (3-19). □

Lemma 6.13 *There is a system of isomorphisms of $\boldsymbol{A}_\dagger(Y)$–modules*

$$\mathrm{HM}^\circ(Y, \langle w\rangle; \Lambda_\gamma) \xrightarrow{\simeq} \mathrm{HM}^\circ(Y_{\mathrm{G}}, \langle w\rangle; \Lambda_\gamma)$$

preserving the relative gradings and natural with respect to the fundamental exact sequences.

Proof Y and Y_{G} stand for the same manifold with different metrics and associated 2–form w. As mentioned in Section 2.4, chain homotopies between the corresponding monopole Floer complexes are provided by chain maps induced from cobordisms $X = \mathbb{R} \times Y$ equipped with metrics and self-dual 2–forms interpolating those associated to Y_- and Y_+. (See eg Section IV.7.3 for this type of argument.) In our setting, choose X with the metrics and self-dual 2–forms over it to be those constructed in Proposition 3.14. This construction also provides a cylinder $C \subset X$ ending at the Y and Y_{G} versions of γ. which induces X–morphisms Λ_C between the Y and Y_{G} versions of Γ_γ. The positivity result in Proposition 3.5 guarantees that these chain maps are filtration-preserving, namely they map the Y_- version of $\mathrm{CM}^- \subset \mathrm{CM}^\infty$ to the Y_+ version of $\mathrm{CM}^- \subset \mathrm{CM}^\infty$. As in the end of the proof of Proposition 6.11, their induced maps on homology together with the five lemma supply the isomorphisms asserted in the lemma. □

7 Properties of solutions to (2-5)

This section supplies proofs for Lemma 3.2 and Proposition 3.7. Even so, much of what is done here is either used in Section 8 or has analogs in Section 8. Section 7.3 has the proof of Lemma 3.2 and Section 7.8 has the proof of Proposition 3.7.

By way of a convention, the manifold Z is assumed implicitly to be connected except in Section 7.8's proof of Proposition 3.7.

What follows is a brief outline of this section.

Section 7.1 Lemmas 7.1–7.3 in this section establish pointwise bounds on the norms of ψ, $\nabla_A\psi$ and B_A when (A, ψ) is a solution to some (r, μ) version of (2-5) in the case when r is large.

Section 7.2 Supposing that r is large and (A, ψ) is a solution to an (r, μ) version of (2-5), this section depicts length scales that are $O(\mathrm{r}^{-1/2})$. This is the content of Lemma 7.4.

Section 7.3 This section introduces the notion of holomorphic domain. The principal examples are $\mathcal{H}_0$ and suitable neighborhoods of the special curve γ that is described in Section 3.2. Lemmas 7.4 and 7.5 establish some very strong a priori bounds for solutions on holomorphic domains to (r, μ) versions of (2-5) when r is large. This section has the proof of Lemma 3.2.

Section 7.4 Lemma 7.7 in this section establishes very strong a priori bounds on the 1–form B_A for a solution (A, ψ) to an (r, μ) version of (2-5) where the w is harmonic.

Section 7.5 Supposing that (A, ψ) is a solution to some (r, μ) version of (2-5), there is a dichotomy between its behavior where $|\psi| \sim |w|$ and where $|\psi| \ll |w|$. In the former case, the ψ is nearly A–covariantly constant and A is nearly flat. This section and Lemma 7.8 in particular describes (A, ψ) where $|\psi|$ is significantly less than $|w|$.

Section 7.6 This section gives a precise definition of the spectral flow function $\mathfrak{f}_s$ (see (7-37)) and summarizes some of its basic properties.

Section 7.7 Lemma 7.9 in this section gives a priori, r– and $\mathfrak{f}_s$–dependent bounds for the functions $\mathfrak{cs}$, W and $\mathfrak{a}$ that appear in (2-6) and (2-7).

Section 7.8 This section has the proof of Proposition 3.7.

7.1 Pointwise bounds

Fix a Riemannian metric on Y_Z and a closed 2–form, denoted by w, whose de Rham class is that of $c_1(\det \mathbb{S})$. The four parts of this subsection assume such data so as to supply a priori pointwise bounds for the $C^\infty(Y_Z; \mathbb{S})$ component of any given pair in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ that obeys (2-5).

Part 1 The first lemma asserts relatively crude bounds which are subsequently refined.

Lemma 7.1 *There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let (A, ψ) denote a solution to the (r, μ) version of* (2-5). *Then $|\psi| + \mathrm{r}^{-1/2}|\nabla_A \psi| + \mathrm{r}^{-1}|\nabla_A \nabla_A \psi| \leq \kappa(\sup_{Y_Z} |w|^{1/2} + \mathrm{r}^{-1/2})$.*

Proof If w is identically zero, write $\psi = \mathrm{r}^{-1/2}\lambda$. The pair (A, λ) obeys the $\mathrm{r} = 1$ version of (2-5). In this case, the standard differential equation techniques give the desired bounds. See for example what is said in Chapter 5 of [17].

Granted that w is not identically zero, assume for what follows that $w \neq 0$ at points on Y_Z. The bound on $|\psi|$ follows by first using the Weitzenböck formula for the square of the Dirac operator to see that $|\psi|^2$ obeys a differential inequality that has the schematic form

(7-1) $$d^\dagger d|\psi|^2 + 2|\nabla_A \psi|^2 + 2\mathrm{r}(|\psi|^2 - |w| - c_0\mathrm{r}^{-1})|\psi|^2 \leq 0.$$

The maximum principle is now used with (7-1) to see that $|\psi|^2 \le c_0 \sup_{Y_Z} |w|$ when $\mathrm{r} \ge c_0^{-1}$. To say more about this, note that (7-1) in turn implies that

$$d^\dagger d|\psi|^2 + 2\mathrm{r}\big(|\psi|^2 - \sup_{Y_Z} |w| - c_0\mathrm{r}^{-1}\big)|\psi|^2 \le 0. \tag{7-2}$$

Now suppose that $p \in Y_Z$ is a point where $|\psi|^2$ achieves its maximum. The term $d^\dagger d|\psi|^2$ in (7-2) is nonnegative at p since it is -1 times the trace of the Hessian of $|\psi|^2$ and the Hessian of $|\psi|^2$ at p is nonpositive because p is a point where $|\psi|^2$ is maximal. It follows as a consequence that the term $|\psi|^2 - \sup_{Y_Z} |w| - c_0\mathrm{r}^{-1}$ must be nonpositive at p, and this requires that $|\psi|^2$ at p be less than $\sup_{Y_Z} |w| + c_0\mathrm{r}^{-1}$. The asserted bound follows from this.

To see about the norm of $|\nabla_A\psi|$, we digress for a moment and fix a point $p \in Y_Z$ and a number ρ that is positive but less than c_0^{-1}. We use χ to construct a function on Y_Z that is equal to 1 on the ball of radius ρ centered at p and is equal to zero outside the ball of radius 2ρ centered at p. This function can and should be constructed so that the norm of its differential is nowhere larger than $c_0\rho^{-1}$ and so that the norm of the covariant derivative of its differential is nowhere larger than $c_0\rho^{-2}$. Denote this function by χ_ρ. Now let B denote the ball of radius 2ρ centered at p. Multiply both sides of (7-1) by χ_ρ and then integrate the resulting inequality over B. An integration by parts and an appeal to the bounds for $|d\chi_\rho|$ and $|\nabla d\chi_\rho|$ and the bounds for $|\psi|$ leads to the bound

$$\int_B \chi_\rho |\nabla_A\psi|^2 \le c_0\big(\sup_{Y_Z} |w| + \mathrm{r}^{-1}\big)(\rho + \rho^3\mathrm{r}). \tag{7-3}$$

To continue, let G_p denote the Dirichlet Green's function for the operator $d^\dagger d$ on B with pole at p. This is a smooth, nonnegative function on $B - p$ that vanishes on ∂B and obeys the pointwise bound

$$G_p(\cdot) \le c_0 \operatorname{dist}(p,\cdot)^{-1} \quad \text{and} \quad |dG_p| \le c_0 \operatorname{dist}(p,\cdot)^{-2} \tag{7-4}$$

at any given point $q \in B - p$. Multiply both sides of (7-1) by $\chi_\rho G_p$ and then integrate the resulting inequality over the ball B. Use an integration by parts, the bounds in (7-4), the a priori bounds on $d\chi_\rho$ and $\nabla d\chi_\rho$, the a priori bound on $|\psi|$ and (7-3) (with ρ replaced by 2ρ) to see that

$$\int_B \chi_\rho G_p |\nabla_A\psi|^2 \le c_0\big(\sup_{Y_Z} |w| + \mathrm{r}^{-1}\big)(1 + \rho^2\mathrm{r}). \tag{7-5}$$

One last step is needed to obtain the asserted pointwise bound for $|\nabla_A\psi|^2$. To start this step, differentiate the equation $D_A^2\psi=0$, commute covariant derivatives and then use the Bochner–Weitzenböck formula again to obtain a differential inequality for $|\nabla_A\psi|^2$ that has the form

$$d^\dagger d|\nabla_A\psi|^2+2\int_B|\nabla_A\nabla_A\psi|^2\le c_0\mathrm{r}(|\nabla_A\psi|^2+1). \tag{7-6}$$

Multiply both sides of (7-6) by $\chi_\rho G_p$ and then integrate the resulting inequality over B. An integration by parts (for the left-hand integral) leads to an inequality that reads

$$\begin{aligned}|\nabla_A\psi|^2(p)+2\int_B\chi_\rho G_p|\nabla_A\nabla_A\psi|^2 \\ \le c_0\rho^3\int_B|\nabla_A\psi|^2+c_0\mathrm{r}\int_B\chi_\rho G_p(|\nabla_A\psi|^2+1).\end{aligned} \tag{7-7}$$

Granted (7-7), take $\rho=c_0^{-1}\mathrm{r}^{-1/2}$ and then the desired bound for $|\nabla_A\psi|^2(p)$ follows from (7-7) with appeals to (7-3) and (7-5). Much the same sort of argument using G_p and χ_ρ can be used to obtain the asserted bounds for $|\nabla_A\nabla_A\psi|^2$. Here is an outline of the argument: Multiplying the inequality in (7-6) by χ_ρ, integrating the result over B, then integrating by parts and using the now-derived bounds for $|\psi|^2$ and $|\nabla_A\psi|^2$ leads to a $c_0(\sup_{Y_Z}|w|+\mathrm{r}^{-1})(1+\rho^2\mathrm{r})$ bound for the integral of $\chi_\rho|\nabla_A\nabla_A\psi|^2$. Multiplying (7-6) by $\chi_\rho G_p$ and integrating the result over B leads to a $c_0(\sup_{Y_Z}|w|+\mathrm{r}^{-1})(1+\rho^2\mathrm{r})$ bound for the integral of $\chi_\rho G_p|\nabla_A\nabla_A\psi|^2$. Meanwhile, differentiating the equation $D_A^2\psi=0$ twice leads to an inequality much like (7-6) for $d^\dagger d|\nabla_A\psi|^2$. Multiplying the latter equation by $\chi_\rho G_p$ and integrating the result over B leads to the desired bound on $|\nabla_A\nabla_A\psi|^2(p)$ with the help of the previously derived bounds. □

Part 2 This part of the subsection sets the notation for what is to come in Part 3 and in the subsequent sections. To start, introduce K_*^{-1} to denote the 2–plane subbundle of the tangent bundle over the $|w|>0$ part of Y_Z given by the kernel of $*w$. Orient K_*^{-1} by the restriction of w and use the induced metric with this orientation to view K_*^{-1} as a complex line bundle. Clifford multiplication by the 1–form $*w$ on the $|w|>0$ part of Y_Z writes $\mathbb{S}$ as a direct sum of eigenbundles $E_*\oplus(E_*\otimes K_*^{-1})$ with E_* being the $+i|w|$–eigenbundle.

Use $\mathbb{I}_\mathbb{C}$ to denote the product complex line bundle and θ_0 to denote the product connection on $\mathbb{I}_\mathbb{C}$. Let $1_\mathbb{C}$ denote the θ_0–constant section of $\mathbb{I}_\mathbb{C}$ with value 1 at all points. Fix a unitary identification between $E_*^{-1}\otimes_\mathbb{C}E_*$ and $\mathbb{I}_\mathbb{C}$ and use the latter to write $E_*^{-1}\otimes_\mathbb{C}\mathbb{S}$ as $\mathbb{I}_\mathbb{C}\oplus K_*^{-1}$. The bundle K_*^{-1} has a canonical connection, which we denote by A_{K_*}, such that the section $(1_{\mathbb{C}}, 0)$ of the bundle $\mathbb{I}_{\mathbb{C}} \oplus K_*^{-1}$ obeys the Dirac equation as defined using the connection $A_{K_*} + 2\theta_0$ on its determinant line bundle. The norm of the curvature of A_{K_*} is bounded by $c_0|w|^{-2}$ and the norm of the k^{th} derivative of A_{K_*}'s curvature is bounded by $c_k|w|^{-2-k}$ with c_k being a constant. A section ψ of $\mathbb{S}$ over U is written with respect to this splitting as $|w|^{1/2}(\alpha, \beta)$. Meanwhile, the connection A on E defines a corresponding connection on E_*, that is, the connection $A_* = A - \frac{1}{2}(A_K - A_{K_*})$. To keep the notation under control in what follows, the A_*–covariant derivative on E_* is also denoted by ∇_A, as is the $A_*{+}A_{K_*}$–covariant derivative on $E_* \otimes K_*^{-1}$.

Part 3 The next lemma refines Lemma 7.1's bound on the $|w| > 0$ part of Y_Z.

Lemma 7.2 *There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let (A, ψ) denote a solution to the (r, μ) version of (2-5). Fix $m \in (\kappa, \kappa \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa})$ and let U_m denote the $|w| > m^{-1}$ part of Y_Z. Write ψ on U_m as $|w|^{1/2}(\alpha, \beta)$. Then the pair (α, β) obeys the following on U_m:*

- $|\alpha|^2 \leq 1 + \kappa m^3 \mathrm{r}^{-1}$.
- $|\beta|^2 \leq \kappa m^3 \mathrm{r}^{-1}(1 - |\alpha|^2) + \kappa^3 m^6 \mathrm{r}^{-2}$.
- $|\nabla_A \alpha|^2 + m^{-3}\mathrm{r}|\nabla_A \beta|^2 \leq \kappa m^{-1}\mathrm{r}(1 - |\alpha|^2) + \kappa^2 m^2$.
- *Denote by U_* the $1 - |\alpha|^2 \geq \kappa^{-1}$ part of U_m. Then*
$$\big|1 - |\alpha|^2\big| \leq (m^2 e^{-\sqrt{\mathrm{r}/m}\,\mathrm{dist}(\cdot, U_*)/\kappa} + \kappa m^3 \mathrm{r}^{-1}).$$

Proof Write $\psi = |w|^{1/2}\eta$ on U_{2m}. The section η on U_{2m} obeys an equation having the schematic form $D_A \eta + \mathfrak{R} \cdot \eta = 0$ with $\mathfrak{R}$ being Clifford multiplication by the 1–form $\frac{1}{2}d(\ln|w|)$. Note in particular that $|\mathfrak{R}| \leq c_0 m$ and the absolute value of the covariant derivative of $\mathfrak{R}$ is bounded by $c_0 m^2$. Use the Weitzenböck formula for the operator $D_A + \mathfrak{R}$ to see that η obeys an equation that has the schematic form

$$\nabla_A^\dagger \nabla_A \eta - \mathrm{cl}(B_A) \cdot \eta + \mathfrak{R}_1 \cdot \nabla_A \eta + \mathfrak{R}_0 \cdot \eta = 0, \tag{7-8}$$

where $\mathrm{cl}(\cdot)$ denotes the Clifford multiplication endomorphism from T^*M to $\mathrm{End}(\mathbb{S})$ and where $\mathfrak{R}_1$ and $\mathfrak{R}_0$ are linear and obey $|\mathfrak{R}_1| \leq c_0 m$ and $|\mathfrak{R}_0| \leq c_0 m^2$. Let q denote the maximum of 0 and $|\eta|^2 - 1 - c_0 m^3 \mathrm{r}^{-1}$. It follows from (7-8) that q on U_{2m} obeys

$$d^\dagger dq + 2\mathrm{r}m^{-1} q \leq 0. \tag{7-9}$$

As Lemma 7.1 bounds q by $c_0 m$ on U_{2m}, the comparison principle with the Green's function for the operator $d^\dagger d + \mathrm{r}m^{-1}$ to see that $q \le c_0 m^3 \mathrm{r}^{-1}$ on $U_{3m/2}$. This implies the claim in the first bullet. It also implies that $|\beta|^2$ is less than $1 + c_0 m^3 \mathrm{r}^{-1}$ on $U_{3m/2}$.

To see about the second bullet, project (7-8) onto the $E_* \otimes K_*^{-1}$–summand of $\mathbb{S}$ and take the fiberwise inner product of the resulting equation with β to obtain a differential inequality that has the form

$$d^\dagger d|\beta|^2 + 2\mathrm{r}m^{-1}|\beta|^2 \le -|\nabla_A\beta|^2 + c_0 \mathrm{r}^{-1} m^3 |\nabla_A \alpha|^2 + c_0 \mathrm{r}^{-1} m^5. \tag{7-10}$$

Fix for the moment $\varepsilon > 0$. Project (7-8) next onto the E_*–summand and take the pointwise inner product with α to obtain an equation for the function $\mathrm{w} = 1 - |\alpha|^2$ that has the form

$$d^\dagger d\mathrm{w} + 2\mathrm{r}m^{-1}\mathrm{w} = 2|\nabla_A\alpha|^2 + \mathrm{r}m^{-1}\mathrm{w}^2 + \mathfrak{e}, \tag{7-11}$$

where $|\mathfrak{e}| \le c_0 \varepsilon |\nabla_A\beta|^2 + c_0(1+\varepsilon^{-1})m^2 + c_0 m|\nabla_A\alpha|$.

It follows from (7-10) and (7-11) that there exist constants z_1 and z_2 that are both bounded by c_0 and $\varepsilon > c_0^{-1}$ such that the function $q = |\beta|^2 - z_1 \mathrm{r}^{-1} m^3 \mathrm{w} - z_2 \mathrm{r}^{-2} m^6$ obeys the equation

$$d^\dagger dq + 2\mathrm{r}m^{-1} q \le 0 \tag{7-12}$$

on $U_{3m/2}$. Granted this inequality, use the Green's function for $d^\dagger d + \mathrm{r}m^{-1}$ as before to see that $|\beta|^2 \le z_1 m^3 \mathrm{r}^{-1}(1-|\alpha|^2) + z_2 m^6 \mathrm{r}^{-2}$ on U_m.

The proofs of the third and fourth bullets start by differentiating (7-8) to obtain an equation for the components of $\nabla_A\eta$ and it then copies the manipulations done in Step 2 of Section 4d in [30] to obtain a differential inequality on $U_{3m/2}$ for the function $\mathfrak{h} := |\nabla_A\eta|^2$ that has the form

$$d^\dagger d\mathfrak{h} + 2\mathrm{r}m^{-1}\mathfrak{h} \le c_0(\mathrm{r}m^{-1}\mathrm{w}\mathfrak{h} + m^2\mathfrak{h} + m^4 + \mathrm{r}^2 m^{-2}\mathrm{w}^2). \tag{7-13}$$

To prove the third bullet, use (7-10), (7-11) and (7-13) to find constants z_1, $z_2 > 0$ and z_3, all with absolute value less than c_0, such that the function

$$q := \mathfrak{h} - z_1 \mathrm{r}m^{-1}(1-|\alpha|^2) - z_2 m^2 + z_3 \mathrm{r}m^{-1}|\beta|^2$$

obeys (7-12) on U_{2m} when $m < c_0^{-1}\mathrm{r}^{1/3}$. Meanwhile, Lemma 7.1 implies that $\mathfrak{h}$ is no larger than $c_0 m\mathrm{r}$ on U_{2m}. Given this last bound, the comparison argument that uses the Green's function for $d^\dagger d + c_0^{-1}\mathrm{r}m^{-1}$ says that $|\nabla_A\eta|^2$ is bounded by $c_0 m^{-1}\mathrm{r}(1-|\alpha|^2) + c_0^2 m^2$ on U_m when $m \le c_0^{-1}\mathrm{r}^{1/3}$. This gives Lemma 7.2's bound for $|\nabla_A\alpha|^2$. The refinement that gives the asserted bound for $|\nabla_A\beta|^2$ is obtained by the same sort of argument after first projecting (7-8) onto the $E_*\otimes K_*^{-1}$–summand of $\mathbb{S}$ before differentiating so as to get an elliptic equation for $\nabla_A\beta$. The details of this part of the story are straightforward and omitted.

To prove the fourth bullet, use the first bullet of the lemma with (7-11) and (7-13) to see that $q := \mathfrak{h} + c_0^{-1}\mathrm{r}m^{-1}\mathrm{w} - c_0 m^2$ obeys an equation on the $\mathrm{w} \le c_0^{-1}$ part of U_{2m} that has the form $d^\dagger dq + c_0^{-1}\mathrm{r}m^{-1}q \le 0$ when $m \le c_0^{-1}\mathrm{r}^{1/3}$. Granted the latter and granted the a priori bound $q \le c_0\mathrm{r}m$ from Lemma 7.1, then the comparison principle using the Green's function for $d^\dagger d + c_0^{-1}\mathrm{r}m^{-1}$ leads to the following: if $c > c_0$, then $q \le c_0\mathrm{r}me^{-\sqrt{\mathrm{r}/m}\,\mathrm{dist}(\cdot,U_c)/c_0}$ where U_c denotes the $\mathrm{w} \ge c^{-1}$ part of U_{2m}. This last inequality implies Lemma 7.2's fourth bullet. □

Part 4 The final lemma of this subsection refines what is said by Lemma 7.1 on the part of Y_Z where $|w|$ is positive but small.

Lemma 7.3 *There exists $\kappa > 1$ with the following property: Fix m in the interval $(\kappa, \kappa^{-1}\mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa})$. Fix $\mathrm{r} \ge \kappa$ and fix $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and let (A,ψ) be a solution to the (r,μ) version of (2-5). Then $|\psi| \le \kappa m^{-1/2}$ and $|\nabla_A\psi| \le \kappa m^{-1}\mathrm{r}^{1/2}$ on the $|w| < m$ part of Y_Z.*

Proof The maximum principle applied to (7-1) implies that $|\psi|^2$ cannot have a local maximum where $|\psi|^2 > |w| + c_0\mathrm{r}^{-1}$. Indeed, if $p \in Y_Z$ is a point where this condition holds, then the left-hand side of (7-1) at p is strictly greater than $d^\dagger d|\psi|^2$ at p. If p is a local maximum of $|\psi|^2$, then $d^\dagger d|\psi|^2 \ge 0$ at p and so the left-hand side of (7-1) would be positive which violates (7-1).

Since $|\psi|^2$ cannot have a local maximum where $|\psi|^2 > |w| + c_0\mathrm{r}^{-1}$, it follows that $|\psi|^2$ cannot have a local maximum where $|\psi|^2 > m^{-1} + c_0\mathrm{r}^{-1}$ on the set where $|w| < m^{-1}$. Meanwhile, Lemma 7.2 implies that $|\psi|^2 \le c_0 m^{-1}$ on the boundary of the set where $|w| < m^{-1}$ (which is the boundary of U_m). Therefore, $|\psi|^2$ cannot be greater than the maximum of $c_0 m^{-1}$ and $m^{-1} + c_0\mathrm{r}^{-1}$ on the set where $|w| < m^{-1}$. If $m \in (c_0, c_0^{-1}\mathrm{r}^{1/3}(\ln \mathrm{r})^{-c_0})$, then this maximum is $c_0 m^{-1}$.

To see about $|\nabla_A\psi|$, let $p \in Y_Z$ denote a given point where $w \le 2m^{-1}$. Fix Gaussian coordinates for a ball of radius c_0^{-1} centered at p and then rescale the coordinates so that the ball of radius $m^{-1/2}\mathrm{r}^{1/2}$ about the origin in $\mathbb{R}^3$ and the ball of radius 1 are identified. Let φ denote the corresponding map from the ball of radius 1 about the origin in $\mathbb{R}^3$ to the original ball in Y_Z. With this understood, the pullback $(\varphi^* A, m^{1/2}\varphi^*\psi)$ satisfies a version of (2-5) on the unit ball in $\mathbb{R}^3$ that is defined by the rescaled metric. It follows from the bound on $|\psi|$ that $|B_A| \le c_0 m^{-1}\mathrm{r}$, and this implies that $|\varphi^* B_A| \le c_0$. This understood, standard elliptic regularity techniques can be employed to see that the rescaled version of $m^{1/2}|\varphi^*(\nabla_A\psi)|$ has norm bounded by c_0 and so $|\nabla_A\psi|$ has norm bounded by $c_0 m^{-1}\mathrm{r}^{1/2}$. □

7.2 The microlocal structure of (A, ψ)

Part 3 of this section states and then proves Lemma 7.4, this being a lemma that describes solutions to (2-5) on the $|w| > 0$ part of Y_Z when viewed with microscope that magnifies by a factor of the order of $\mathrm{r}^{1/2}$. Parts 1–2 of the subsection set the notation that is used in particular for Lemma 7.4 but elsewhere as well.

Part 1 This part of the subsection introduces the *vortex equations* on $\mathbb{C}$. This is a system of equations that asks that a pair (A_0, α_0) of connection on a complex line bundle over $\mathbb{C}$ and section of this bundle obey

$$\begin{cases} *F_{A_0} = -i(1 - |\alpha_0|^2), \\ \bar\partial_{A_0}\bar\alpha_0 = 0, \\ |\alpha_0| \le 1. \end{cases} \tag{7-14}$$

The notation here is such that $*$ denotes the Euclidean Hodge dual on $\mathbb{C}$, while F_{A_0} and $\bar\partial_{A_0}$ denote the respective curvature 2–form of A_0 and the d-bar operator defined by A_0 on the space of sections of the given complex line bundle. Note that if (A_0, α_0) is a solution to (7-14), then so is $(A_0 - u^{-1}\,du, u\alpha_0)$ with u being any smooth map from $\mathbb{C}$ to S^1.

Solutions with $1 - |\alpha_0|^2$ integrable are discussed at length in Sections 1 and 2 of [36], Section IV.2.2 and Section IV.3.1. As noted in these references, if $1 - |\alpha_0|^2$ is integrable then its integral is 2π times a nonnegative integer. Fix $\mathrm{m} \in \{0, 1, \dots\}$. The space of $C^\infty(\mathbb{C}; S^1)$ equivalence classes of solutions to (7-14) with the integral of $1 - |\alpha_0|^2$ equal to $2\pi\mathrm{m}$ has the structure of a smooth, $2\mathrm{m}$–dimensional manifold. This manifold is denoted in what follows by $\mathfrak{C}_\mathrm{m}$. By way of a parenthetical remark, the space $\mathfrak{C}_\mathrm{m}$ has a natural complex structure that identifies it with $\mathbb{C}^\mathrm{m}$. A solution with $1 - |\alpha_0|^2$ integrable is said here to be a *finite-energy solution* to the vortex equation.

Part 2 Lemma 7.4 and some of the later subsections refer to the notion of a transverse disk with a given radius through a given $|w| > 0$ point in Y_Z. A *transverse disk* is the image via the metric's exponential map of the centered disk of the given radius in the 2–plane bundle $\mathrm{Ker}(*w)$ at the given point. There exists $c_0 > 100$ such that any transverse disk with radius c_0^{-1} is embedded with a priori bounds on the derivatives to any given order of its extrinsic curvature. If $D \subset Y_Z$ is a transverse disk centered at a point p, and if $c \geq c_0$, then $|w|$ will be greater than $\frac{1}{2}|w|(p)$ on the subdisk in D centered at p with radius $c^{-1}|w|(p)$. The constant c can be chosen so that the following is also true: Let v denote the vector field on the $|w| > 0$ part of Y_Z that generates the kernel of w and has pairing 1 with $*w$. Then v is orthogonal to D at p and the length of the projection to TD of v on the concentric disk in D of radius $c^{-1}|w|(p)$ is no greater than $c_0 c^{-1}$. Choose $c \geq c_0$ with this property and use D_p to denote the transverse disk through p of radius $c^{-1}|w|(p)$.

Reintroduce from Part 2 of Section 7.1 the complex line bundle K_*^{-1} defined over the $|w| > 0$ part of Y_Z. Recall that the underlying real bundle is the 2–plane bundle in TY_Z annihilated by $*w$. Let p again denote a point in the $|w| > 0$ part of Y_Z. Fix an isometric isomorphism from $K_*^{-1}|_p$ to $\mathbb{C}$. Use φ in what follows to denote the map from $\mathbb{C}$ to Y_Z that is obtained by composing first the isomorphism with $K_*|_p = \mathrm{Ker}(*w)|_p$ and then the metric's exponential map. With $\mathrm{r} \geq 1$ given, use φ_{r} to denote the composition of first multiplication by $\mathrm{r}^{-1/2}|w(p)|^{-1/2}$ on $\mathbb{C}$ and then applying φ.

To finish the notational preliminaries, let (A, ψ) be a pair in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$. Write ψ where $|w| > 0$ as $|w|^{1/2}(\alpha, \beta)$ to conform with Part 2 of Section 7.1's splitting of $\mathbb{S}$ as $E_* \oplus (E_* \otimes K_*^{-1})$. Likewise reintroduce from Part 2 of Section 7.1 the connection A_* on the bundle E_*. Given $p \in Y_Z$ with $|w(p)| > 0$, introduce $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ to denote the φ_{r}–pullback of the pair (A_*, α) to the radius $c^{-1}\mathrm{r}^{1/2}|w(p)|^{1/2}$ disk in $\mathbb{C}$.

Part 3 Lemma 7.4 below characterizes the pair $(A_{\mathrm{r}}, \psi_{\mathrm{r}})$.

Lemma 7.4 *There exists $\kappa > 10$ and given $\mathrm{R} > \kappa^2$, there exists $\kappa_{\mathrm{R}} > 1$ with the following property: Fix $\mathrm{r} \geq \kappa_{\mathrm{R}}$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Suppose that (A, ψ) is a solution to the (r, μ) version of (2-5). Fix a point in Y_Z where $|w| > \mathrm{r}^{-1/3}(\ln \mathrm{r})^\kappa$ and use the corresponding version of φ_{r} to obtain the pair $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ of connection and section of a complex line bundle over $\mathbb{C}$. There exists a solution to the vortex equation on $\mathbb{C}$ whose restriction to the radius R disk about the origin in $\mathbb{C}$ has C^1–distance less than R^{-4} from $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ on this same disk. Moreover, if $1 - |\alpha_{\mathrm{r}}|^2 < \frac{1}{2}$ at distances between $\mathrm{R} + \kappa(\ln \mathrm{R})^2$ and $\mathrm{R} - \kappa(\ln \mathrm{R})^2$ from the origin, then $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ has C^1–distance less than R^{-4} in the radius R disk about the origin in $\mathbb{C}$ from* *a finite-energy solution to the vortex equations that defines a point in some* $\mathrm{m} \le \pi \mathrm{R}^2$ *version of* $\mathfrak{C}_{\mathrm{m}}$.

Proof It follows from (2-5) and what is said by the first three bullets of Lemma 7.2 that the curvature of A_{r} and α_{r} are such that

$$(7\text{-}15) \qquad *F_{A_{\mathrm{r}}} = -i(1-|\alpha_{\mathrm{r}}|^2) + \mathfrak{e}_0 \quad \text{and} \quad \bar{\partial}_{A_{\mathrm{r}}}\alpha_{\mathrm{r}} = \mathfrak{e}_1,$$

where $|\mathfrak{e}_0| + |\mathfrak{e}_1| \le c_0(\ln \mathrm{r})^{-c_0}$ on the disk in $\mathbb{C}$ of radius less than $c^{-1}\mathrm{r}^{1/2}m^{-1/2}$. The third bullet in Lemma 7.2 also finds $|\nabla_{A_{\mathrm{r}}}\alpha_{\mathrm{r}}| \le c_0$. Granted (7-15), then the argument used to prove Lemma 6.1 in [33] can be used with only minor modifications to prove the assertion with C^1–distance replaced by the distance as measured by any $\upsilon < 1 - \mathrm{R}^{-1}$ Hölder norm. The convergence in the C^1–topology follows using the arguments from Section 6 in [33] given also the second derivative bound from Lemma 7.1. □

7.3 Holomorphic domains

What follows directly sets the notation for what is to come in this subsection. An open set $U \subset Y_Z$ is said to be a *holomorphic domain* when the following criteria are met:

- The metric has nonnegative Ricci curvature on U.
- The 2–form w is nonzero on U and covariantly constant.
- The curvature of A_K on U is a multiple of w.
- The 1–form μ on U and its derivatives to order 10 have norm less than $e^{-\mathrm{r}^2/2}$.

The following lemma strengthens the conclusions of Lemma 7.2 on a holomorphic domain:

Lemma 7.5 *Let $U \subset Y_Z$ denote a holomorphic domain and let $U_1 \subset U$ denote an open set with compact closure in U. Use D to denote the function on U that measures the distance to $Y_Z - U$. There exists $\kappa > \pi$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and a 1–form $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 whose norm on U and those of its first 10 derivatives is bounded by $e^{-\mathrm{r}^2/2}$. Suppose that (A, ψ) is a solution to the (r, μ) version of (2-5). Write ψ on U as $|w|^{1/2}(\alpha, \beta)$. Then β on U_1 obeys:*

- $|\beta| \le \kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$.
- *Given $q \ge 1$, there exists $\kappa_q \ge 1$ such that $|(\nabla_A)^q\beta| \le \kappa_q e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$ with κ_q depending only on the metric, A_K, U and U_1.*

Proof The proof that follows assumes that $\mu = 0$ on U. The proof in the general case differs little from what is said below and is left to the reader.

Keep in mind that the norm of $|w|$ is constant on U because w is covariantly constant. Project the Weitzenböck formula for D_A^2 onto the $E_*\otimes K_*^{-1}$–summand of $\mathbb{S}$ to obtain an equation for β on U that has the schematic form

$$\nabla_A^\dagger \nabla_A \beta + \mathrm{r}|w|(1+|\alpha|^2+|\beta|^2)\beta + \mathfrak{R}\beta = 0, \tag{7-16}$$

with $\mathfrak{R}$ determined solely by the metric and A_K. Granted this, then by the conditions on the metric and A_K over U, $|\beta|$ obeys an equation of the form $d^\dagger d|\beta| + \mathrm{r}|w||\beta| \leq 0$ on U when r is larger than a constant that depends only on U and U_1. The bound in the first bullet of the lemma follows from the latter equation using the comparison principle and the Green's function for the operator $d^\dagger d + \mathrm{r}|w|$. Given the bounds from Lemma 7.2, very much the same strategy leads to the bounds in the subsequent bullets after differentiating (7-1) to obtain an equation for $(\nabla_A)^q\beta$. □

Lemma 7.5 leads directly to the next lemma, which describes ψ on U_γ and $\mathcal{H}_0$.

Lemma 7.6 *Given $\varepsilon > 0$, there exists $\kappa \geq \pi$ with the following significance: Introduce U to denote $U_\gamma \cup \mathcal{H}_0$ and let D denote the function on U that measures the distance to $Y_Z - U$. Introduce $U_\varepsilon \subset U$ to denote the subset with $\mathrm{D} > \varepsilon$. Fix $\mathrm{r} \geq \kappa$ and a 1–form $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 whose norm on U and those of its first ten derivatives is bounded by $e^{-\mathrm{r}^2/2}$. Let (A,ψ) denote a solution to the (r,μ) version of (2-5). The following is true on U_ε:*

- *The conclusions of Lemma 7.5 hold with U_1 therein set to U_ε.*
- $-\kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa} \leq 1 - |\alpha|^2 \leq \kappa e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$.
- *Given $q \geq 1$, there exists $\kappa_q \geq 1$ such that $|(\nabla_A)^q\alpha| \leq \kappa_q e^{-\sqrt{\mathrm{r}}\mathrm{D}/\kappa}$ with κ_q depending only on the metric, A_K, U and ε.*

Proof The first bullet follows by virtue of the fact that $U_\gamma \cup \mathcal{H}_0$ is a holomorphic domain where the constraints in (3-5) and (3-6) are obeyed. To see about the other bullets of the lemma, suppose for the moment that $\delta > 0$, that $p \in \mathcal{H}_0 \cap U_\varepsilon$ and that $1 - |\alpha| > \delta$ at p. As is proved in what follows, this assumption leads to nonsense unless δ is very small. So, supposing that $1-|\alpha| > \delta$ at p, it follows from the second bullet of Lemma 7.2 and from Lemma 7.5 that the integral of $*B_A$ on the radius $c_0^{-1}\mathrm{r}^{-1/2}\delta$ disk in the constant u slice of $\mathcal{H}_0$ through p is greater than $c_0^{-1}\delta^3$. Lemma 7.5 implies that the pullback of $*B_A$ to the constant u sphere through p can be written as $\frac{i}{4\pi}\mathrm{F}\sin\theta\, d\theta \wedge d\phi$ and that $\mathrm{F} \geq -c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. This implies that the integral of $*B_A$ on this transverse sphere in $\mathcal{H}_0$ will be positive if $\delta > c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. But the integral of $*B_A$ on this transverse sphere is zero because E's first Chern class has zero pairing with the $H_2(\mathcal{H}_0;\mathbb{Z})$–summand in (3-4). Therefore, it must be the case that $1-|\alpha| < c_0 e^{-\sqrt{\mathrm{r}}/c_0}$ on $\mathcal{H}_0 \cap U_\varepsilon$. Now suppose that $p \in U_\gamma \cap U_\varepsilon$. The Dirac equation writes the $\frac{\partial}{\partial t}$–covariant derivative of α as a linear combination of covariant derivatives of β. This understood, Lemma 7.5 implies that the absolute value of the $\frac{\partial}{\partial t}$–covariant derivative of α in U_γ is bounded by $c_0 e^{-\sqrt{\mathrm{r}}/c_0}$. It follows as a consequence that if $1-|\alpha| > \delta$ at a point in $U_\gamma \cap U_\varepsilon$, then $|\alpha| > \frac{1}{2}\delta$ at points in $\mathcal{H}_0 \cap U_\varepsilon$ if $\delta > c_0 e^{-\sqrt{\mathrm{r}}/c_0}$, and as explained previously, this is not allowed if $\mathrm{r} \ge c_0$. Therefore, the conclusion is that $1-|\alpha| < c_0 e^{-\sqrt{\mathrm{r}}/c_0}$ on the whole of $(U_\gamma \cup \mathcal{H}_0) \cup U_\varepsilon$. Much the same sort of argument proves that $1-|\alpha| > -c_0 e^{-\sqrt{\mathrm{r}}/c_0}$ on this same domain.

The assertion in the third bullet is proved by writing $\psi = |w|^{1/2}\eta$ on U. Keeping in mind that $|w|$ is constant on U, project the Weitzenböck formula for $D_A^2\psi$ onto the E–summand of $\mathbb{S}$ and differentiating to obtain an equation for $(\nabla_A)^q\alpha$. Given the first bullet of Lemma 7.6 and given Lemma 7.5, the latter implies a differential inequality for the function $\sigma := |(\nabla_A)^q\alpha|$ of the form $d^\dagger d\sigma + \mathrm{r}|w|\sigma \le c_q e^{-\sqrt{\mathrm{r}}/c_0}$ when $q = 1$, and it implies an equality of this same sort for $q > 1$ if the second bullet holds for all $q' < q$. Here, c_q depends only on q. Use the Green's function for $d^\dagger d + \mathrm{r}|w|$ with this differential inequality for σ to prove the third bullet's assertion. □

Lemma 7.6 in turn leads to the:

Proof of Lemma 3.2 If $\mathrm{r} \ge c_0$, then Lemma 7.6 asserts that $|\alpha|$ is very close to 1 on a neighborhood of γ and so what is denoted in (3-8) by $\wp(|\alpha|)$ is equal to 1 on this neighborhood. With this in mind, note that $\alpha|\alpha|^{-1}$ is $\hat{A}$–covariantly constant where $\wp = 1$. This implies that $\hat{A}$ has holonomy 1 along γ. Since A_E has holonomy 1 on γ, it follows that $\hat{A} - A_E$ on γ can be written as $i\hat{u}(t)\,dt$ with $\hat{u}$ being a function on $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ whose integral is an integer multiple of 2π. □

7.4 The L^1–norm of B_A when w is harmonic

This section supplies a crucial bound for the integral of $|B_A|$ over Y_Z given an extra assumption about w.

Lemma 7.7 *Suppose that w is a harmonic 2–form and that the zeros of w are nondegenerate. There exists $\kappa \ge \pi$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and a 1–form $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that (A,ψ) is a solution to the (r,μ) version of (2-5). Then $\int_{Y_Z} |w||B_A| \le \kappa$ and $\int_{Y_Z} |B_A| \le \kappa \mathrm{r}^{1/5}$.*

By way of a look ahead, the lemma's bound of $\kappa\mathrm{r}^{1/5}$ for the L^1–norm of B_A is replaced in Lemma 7.9 by the bound $(\ln\mathrm{r})^{c_0}$.

Proof The proof has three steps. By way of an overview, the plan is to compare the integrals of $|B_A|$ and $|w||B_A|$ with the integral of $w\wedge iB_A$. The point being that the absolute value of the latter integral enjoys an (A,ψ)–, r– and μ–independent bound by virtue of the fact that w is harmonic; it computes the cup product pairing between the de Rham class of $*w$ and 2π times the first Chern class of the bundle E.

Step 1 Fix $m\in(c_0, c_0\mathrm{r}^{1/3}(\ln\mathrm{r})^{-c_0})$ so as to invoke Lemmas 7.2 and 7.3. Use U_m to again denote the part of Y_Z where $|w|>m^{-1}$. Since w has nondegenerate zeros, the volume of Y_Z-U_m is less than c_0m^{-3}. Since $|B_A|\le c_0\mathrm{r}(|\psi|^2+|w|)+c_0$, it follows from Lemma 7.3 that

$$\int_{Y_Z-U_m}|B_A|\le c_0\mathrm{r}m^{-4}\quad\text{and}\quad\int_{Y_Z-U_m}|w\wedge B_A|\le c_0\mathrm{r}m^{-5}. \tag{7-17}$$

Save these bounds for the moment.

Step 2 Fix $m\in(c_0, c_0\mathrm{r}^{1/3})$. Use the equations in (2-5) and Lemma 7.2 to see that $|B_A|$ on U_m obeys $|B_A|\le\mathrm{r}|w|\big(\big|1-|\alpha|^2\big|+|\beta|\big)+c_0$. This understood, the first and second bullets in Lemma 7.2 imply that

$$|B_A|\le c_0\mathrm{r}|w|(1-|\alpha|^2)+c_0|w|m^3 \tag{7-18}$$

at all points in U_m. Meanwhile, use the equations in (2-5) to see that

$$w\wedge iB_A\ge\mathrm{r}|w|^2(1-|\alpha|^2)-c_0|w| \tag{7-19}$$

on U_m. This lower bound and the upper bound in (7-18) imply that if $q\in\{0,1\}$, then

$$|w|^q|B_A|\le c_0m^{1-q}(w\wedge iB_A)+c_0m^{2-q} \tag{7-20}$$

at all points in U_m.

Step 3 Fix for the moment $m_0\ge c_0$ and a positive integer N with an upper bound such that $2^Nm_0<c_0^{-1}\mathrm{r}^{1/3}$. For $k\in\{1,2,\dots,N\}$, set $m_k:=2^km_0$. Noting that the volume of $U_{m_k}-U_{m_{k-1}}$ is bounded by c_02^{-3k}, it follows from (7-20) that

$$\int_{U_{m_k}-U_{m_{k-1}}}|w|^q|B_A|\le c_0m_N^{1-q}\int_{U_{m_k}-U_{m_{k-1}}}w\wedge iB_A+c_02^{-k}. \tag{7-21}$$

Sum the various $k\in\{1,\dots,N\}$ versions of (7-21) to see that

$$\int_{U_{m_N}}|w|^q|B_A|\le c_0m_N^{1-q}\int_{U_{m_N}}w\wedge iB_A+c_0. \tag{7-22}$$

This last inequality and the $m = m_N$ version of (7-17) imply that

$$\int_{Y_Z} |B_A| \le c_0 m_N^{1-q} \int_{Y_Z} w \wedge iB_A + c_0(\mathrm{r} m_N^{-4-q} + 1). \tag{7-23}$$

The integral on the right-hand side of (7-23) is in any event bounded by c_0 and so what is written in (7-23) leads to the bound

$$\int_{Y_Z} |w|^q |B_A| \le c_0(m_N^{1-q} + \mathrm{r} m_N^{-4-q}). \tag{7-24}$$

This understood, take N so that $\mathrm{r}^{1/5} \le m_N \le c_0 \mathrm{r}^{1/5}$ to obtain Lemma 7.7's assertion. □

7.5 Where $1 - |\alpha|^2$ is not small

Suppose that (A, ψ) is a solution to a given (r, μ) version of (2-5). Write ψ where $|w| > 0$ as $|w|^{1/2}(\alpha, \beta)$ and denote the version of κ that appears in Lemma 7.4 by $\kappa_\diamond$.

The lemma that follows in a moment characterizes the $|w| > \mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_\diamond}$ part of Y_Z where $1 - |\alpha|^2$ is not very small. To set the notation for the lemma, introduce v to denote the unit-length vector field on the part of Y_Z where $|w| > 0$ that generates the kernel of w and has positive pairing with $*w$. A final bit of notation concerns the version of κ that appears in Lemma 7.2. The latter is denoted in what follows by $\kappa_\diamond$.

Lemma 7.8 *Assume that w is a harmonic 2–form with nondegenerate zeros. There exists $\kappa > \kappa_\diamond$ and $\kappa_1 \gg \kappa$ with the following significance: Fix $\mathrm{r} \ge \kappa_1$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and let (A, ψ) denote a solution to the (r, μ) version of (2-5). Fix a positive integer k and set $m_k := (1 + \kappa^{-1})^k \kappa^2$. If $m_k < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa}$, then there exists a set Θ_k, of at most κ segments of integral curves of v with the following properties:*

- *Each segment from Θ_1 is properly embedded in the $|w| \ge m_2^{-1}$ part of Y_Z and has length at most κ. Moreover, the union of the radius $\kappa \mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Θ_1 contain all points in the $|w| > \kappa^{-2}$ part of Y_Z where $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$.*
- *If $k > 1$, then each segment from Θ_k is properly embedded in the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z and the union of the radius $\kappa m_k^{1/2} \mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Θ_k contain all $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$ points in the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z.*

Proof The proof has eight steps. By way of a parenthetical remark, the proof follows a strategy like that used in Section IV.2.3 to prove Proposition IV.2.4.

Step 1 This step states a fact about the finite-energy solutions to the vortex equations that plays a central role in the subsequent arguments. Keep in mind that a solution (A_0, α_0) is a finite-energy solution when $1-|\alpha_0|^2$ is an L^1–function. As noted in Part 1 of Section 7.2, if (A_0, α_0) is a finite-energy solution then the integral of $1-|\alpha_0|^2$ is 2π times a nonnegative integer. Use m to denote this integer. The function α_0 vanishes at precisely m points in $\mathbb{C}$ (with repetitions allowed). This set of zeros of α_0 is denoted by ϑ. As noted in Part 4 from Section 2a in [36],

$$1-|\alpha_0|^2 \le c_0 \sum_{z\in\vartheta} e^{-\operatorname{dist}(\cdot,z)}, \tag{7-25}$$

with the number c_0 in (7-25) being independent of (A_0, α_0) and m. The bound in (7-25) with Lemma 7.4 has a number of consequences with regards to the proof.

To say more, return to the context of Lemma 7.4. Let $\kappa_\diamond$ denote the version of the constant κ that appears in this lemma. Take $\mathrm{R} > \kappa_\diamond$ so as to apply Lemma 7.4 when r is greater than the corresponding κ_{R}. With $\mathrm{r} \ge \kappa_{\mathrm{R}}$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1, let (A, ψ) denote a solution to the (r, μ) version of (2-5). Fix $p \in Y_Z$ with $|w(p)| \ge \mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_\diamond}$ and use p to define the pair $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ as instructed in Part 2 of Section 7.2. Assume for what follows that $1-|\alpha_{\mathrm{r}}|^2 < \frac{1}{2}$ at distances between $\mathrm{R}+\kappa_\diamond(\ln \mathrm{R})^2$ and $\mathrm{R}-\kappa_\diamond(\ln \mathrm{R})^2$ from the origin in $\mathbb{C}$.

Lemma 7.4 asserts that $(A_{\mathrm{r}}, \alpha_{\mathrm{r}})$ has C^1–distance at most R^{-4} in the radius R disk about the origin in $\mathbb{C}$ from a finite-energy vortex that defines a point in some $\mathrm{m} \le \pi \mathrm{R}^2$ version of $\mathfrak{C}_{\mathrm{m}}$. Let (A_0, α_0) denote this solution. It follows from Lemma 7.4 that $1-|\alpha_0|^2$ can be no greater than $\frac{1}{2}+2\mathrm{R}^{-4}$ at all points in $\mathbb{C}$ with distance between $\mathrm{R}-\kappa_\diamond(\ln \mathrm{R})^2$ and R from the origin in $\mathbb{C}$ (since otherwise (A_0, α_0) would have C^0–distance greater than R^{-4} in the radius R disk about the origin in $\mathbb{C}$). This implies that each zero of α_0 (which are the points in the set ϑ) that appears in (7-25) has distance either less than $\mathrm{R}-\kappa_\diamond(\ln \mathrm{R})^2$ from the origin in $\mathbb{C}$ or it has distance greater than R from the origin in $\mathbb{C}$. This understood, then it follows as a consequence of (7-25) that $1-|\alpha_0|^2 \le \mathrm{R}^{-4}$ on the annulus about the origin in $\mathbb{C}$ with inner radius $\mathrm{R}-\kappa_\diamond(\ln \mathrm{R})^2 + c_0 \ln \mathrm{R}$ and outer radius $\mathrm{R}-c_0 \ln \mathrm{R}$. Indeed, at distance ρ from the set ϑ, the sum on the right-hand side of (7-25) is at most $c_0 \mathrm{m} e^{-\rho}$. Since $\mathrm{m} < \pi \mathrm{R}^2$, this is at most $c_0 \mathrm{R}^2 e^{-\rho}$. Thus, if $\rho > c_0 \ln \mathrm{R}$, then the sum on the right-hand side of (7-25) will be at most R^{-4}. Granted that $1-|\alpha_0|^2 \le \mathrm{R}^{-4}$ on the annulus in $\mathbb{C}$ centered at the origin with inner radius equal to $\mathrm{R} - \kappa_\diamond (\ln \mathrm{R})^2 + c_0 \ln \mathrm{R}$ and outer radius $\mathrm{R} - c_0 \ln \mathrm{R}$, it then follows from Lemma 7.4 that $1 - |\alpha_{\mathrm{r}}|^2 \le 2\mathrm{R}^{-4}$ on this same annulus.

If $\mathrm{R} > c_0$, then the preceding conclusion implies that $1 - |\alpha|^2$ is bounded by $2\mathrm{R}^{-4}$ on the annulus in transverse disk centered at p with respective outer and inner radii given by $(\mathrm{R} - c_0 \ln \mathrm{R})(\mathrm{r}|w|(p))^{-1/2}$ and inner radius $(\mathrm{R} - \kappa_\diamond (\ln \mathrm{R})^2 + c_0 \ln \mathrm{R})(\mathrm{r}|w|(p))^{-1/2}$. Since α is nowhere-vanishing on this annulus, the connection $\widehat{A}_*$ is defined on this annulus by the same formula (3-8), and the last observation implies in particular that the connection $\widehat{A}_*$ is flat and $\alpha|\alpha|^{-1}$ is $\widehat{A}_*$–covariantly constant at points on this same annulus.

In the applications to come, the integer m will be bounded by c_0. If this is the case, then (7-25) with Lemma 7.4 implies that $\widehat{A}_*$ is flat and $\alpha|\alpha|^{-1}$ is $\widehat{A}_*$–covariantly constant at all point on the radius $(\mathrm{R} - c_0 (\ln \mathrm{R})^2)(\mathrm{r}|w|(p))^{-1/2}$ transverse disk centered at p except at distance less than $c_0 (\mathrm{r}|w|(p))^{-1/2}$ from a set of at most c_0 points.

Step 2 Fix $m_0 > c_0$ so that the $|w| \le m_0^{-1}$ part of Y_Z is a disjoint union of components with each component lying in the radius $c_0 m_0^{-1}$ ball about a zero of w. Require in addition that each such component lie in a Gaussian coordinate chart centered on the nearby zero of w as the embedded image of a closed ball in $\mathbb{R}^3$.

Fix $z > m_0$ and let κ_0 denote the sum of the versions of κ that appear in Lemmas 7.1, 7.2 and 7.7; and let κ_{z_0} denote the sum of κ_0 and the $\mathrm{R} = z^{10}$ version of the constant κ_{R} that appears in Lemma 7.4. But for cosmetic changes, the arguments in Section 6.4 of [33] can be used with Lemmas 7.2, 7.4 and 7.6 plus what is said in Step 1 to find a z–independent $\kappa_1 \ge 100\kappa_0$ and a z–dependent $\kappa_z > \kappa_{z_0}$ such that the following is true:

Fix $\mathrm{r} \ge \kappa_z$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Suppose that (A, ψ) is a solution to the (r, μ) version of (2-5). There exists a positive integer $n_0 < \kappa_1$ and a set Θ_0, of at most n_0 pairs of the form (γ, m) with γ being a properly embedded segment of an integral curve of v in the $|w| \ge z^{-6}$ part of Y_Z with length less than κ_1. Meanwhile, m is a positive integer. The set Θ_0 has the following additional properties:

(7-26)
- $\sum_{(\gamma,\mathrm{m}) \in \Theta_0} \mathrm{m} \le \kappa_1$.
- *Distinct curves from Θ_0 are separated by distance at least $\kappa_1 z^4 \mathrm{r}^{-1/2}$.*
- *If $p \in Y_Z$ is such that $|w(p)| \ge z^{-6}$ and $1 - |\alpha|^2 > \kappa_\diamond^{-1}$, then p has distance less than $z^4 \mathrm{r}^{-1/2}$ from a curve in Θ_0.*

- *If* $(\gamma, \mathrm{m}) \in \Theta_0$, *then the integral of* $\frac{i}{2\pi} F_{\widehat{A}_*}$ *over the radius* $z^4 \mathrm{r}^{-1/2}$ *transverse disk centered at each point in* γ *is equal to* m.

What follows is a parenthetical remark concerning the fourth bullet. The condition in the third bullet of (7-26) implies that $\alpha|\alpha|^{-1}$ is $\widehat{A}_*$–covariantly constant near the boundary of the radius $z^4\mathrm{r}^{-1/2}$ transverse disk about each point in γ. It follows as a consequence that the integral of $\frac{i}{2\pi} F_{\widehat{A}_*}$ over this disk is an integer; and it follows from Lemma 7.2 that this integer is nonnegative. This being the case, the fourth bullet adds only that the integer is at least 1 and it is bounded a priori by a z–, (A, ψ)–, μ– and r–independent number.

Step 3 Fix a ball $B \subset Y_Z$ centered on a zero of w that contains a component of the $|w| \le m_0^{-1}$ part of Y_Z. Suppose that $\varepsilon \in (0, 1)$ and that $z > m_0$ have been specified. With κ_z as in Step 2, fix $\mathrm{r} \ge \kappa_z$, an element $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and a solution, (A, ψ), to the (r, μ) version of (2-5). Let k denote the largest integer with the properties listed below in (7-27). By way of notation, set $m_j := (1+\varepsilon)^j z^6$.

For each $j \in \{1, \dots, k\}$, there exists $c_j \in (100, (100)^{2^{\kappa_1}})$ and a set, Θ_j, that consists of data sets which have the form $(\gamma, \mathrm{m}, \mathrm{D})$ with γ being a properly embedded segment of an integral curve of v in the $|w| \in [m_{j+1}^{-1}, m_{j-1}^{-1}]$ part of B, with m being a positive integer and with $\mathrm{D} \in (1, c_j)$. The set Θ_j has the following additional properties:

(7-27)
- $\sum_{(\gamma,\mathrm{m},\mathrm{D}) \in \Theta_j} \mathrm{m} \le \kappa_1$.
- Curves from distinct data sets in Θ_j are separated by distance at least $\frac{1}{2} c_j^2 z m_j^{1/2} \mathrm{r}^{-1/2}$.
- If $p \in Y_Z$ is such that $|w(p)| \in [m_{j+1}^{-1}, m_{j-1}^{-1})$ and $1 - |\alpha|^2 > \frac{1}{4}\kappa_0^{-1}$, then p has distance at most $\mathrm{D} z m_j^{1/2} \mathrm{r}^{-1/2}$ from a point on a curve from a dataset in Θ_j.
- If $(\gamma, m, \mathrm{D}) \in \Theta_j$, then the integral of $\frac{i}{2\pi} F_{\widehat{A}_*}$ over the radius $\mathrm{D} z m_j^{1/2} \mathrm{r}^{-1/2}$ transverse disk centered at each point in γ is equal to m.

The next steps find (A, ψ)–, μ– and r–independent choices for ε and then z, and an (A, ψ)–, μ– and r–independent $\kappa_* \ge \kappa_z$ such that $m_k \ge \mathrm{r}^{1/3} (\ln \mathrm{r})^{-\kappa_*}$ when r is greater than κ_*. Lemma 7.8 follows if such ε, z and κ_* exist.

The upcoming steps find the desired conditions on ε and z and the lower bound for r such that the conditions of the integer $k+1$ version of (7-27) are met if they are met for an integer k with $m_k < \mathrm{r}^{1/3} (\ln \mathrm{r})^{-2\kappa_0}$. This being the strategy, assume in what follows that k is such that $m_k < \mathrm{r}^{1/3} (\ln \mathrm{r})^{-2\kappa_0}$ and (7-27) holds.

Step 4 The A_*–directional covariant derivative along the vector field v is used in a moment to analyze the behavior of α at points along v's integral curves. This directional derivative is denoted in what follows by $(\nabla_A\alpha)_v$. The equations in (2-5) identify the latter with a linear combination of A_*–covariant derivatives of β. This being the case, Lemma 7.2 finds $|(\nabla_A\alpha)_v| \le c_0 m[(1-|\alpha|^2)+c_0\mathrm{r}^{-1}m^3]^{1/2}$ on the $|w| > (2m)^{-1}$ part of Y_Z if $m \le \mathrm{r}^{1/3}(\ln\mathrm{r})^{-\kappa_0}$. By way of a comparison, Lemma 7.2 bounds the norm of the remaining components of $\nabla_A\alpha$ by $c_0 m^{-1/2}\mathrm{r}^{-1/2}[(1-|\alpha|^2)+c_0\mathrm{r}^{-1}m^3]^{1/2}$.

What was said in the preceding paragraph about the norm of $|(\nabla_A\alpha)_v|$ has the following consequences for a point $p \in Y_Z$ where $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$; let γ_p denote the integral curve of v through p and let p' denote a point on the segment of γ_p where the distance to p is less than $c_0^{-1}\kappa_\diamond^{-1}m_k^{-1}$:

(7-28)
- If $1-|\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$ at p, then $1-|\alpha|^2 > \frac{1}{8}\kappa_\diamond^{-1}$ at p'.
- If $1-|\alpha|^2 \le \frac{1}{4}\kappa_\diamond^{-1}$ at p, then $1-|\alpha|^2 < \frac{1}{2}\kappa_\diamond^{-1}$ at p'.

This segment of γ_p is said in what follows to be the *short* segment of γ_p.

Note that if $\varepsilon \le c_0^{-1}\kappa_\diamond^{-2}$, then γ_p's short segment has points with $|w| > m_{k-1}^{-1}$. Assume in what follows that $\varepsilon \le c_0^{-1}\kappa_\diamond^{-2}$ is satisfied so as to invoke this fact about the short segment.

Step 5 This step constitutes a digression to supply a coordinate chart for any given $|w| > 0$ point in Y_Z that is used to exploit what is said in Step 4. To this end, suppose that $m > 1$ has been specified. Use I_m to denote the interval $[-c_0^{-1}m^{-1}, c_0^{-1}m^{-1}]$ and use D_m to denote the centered disk in $\mathbb{C}$ with radius $c_0^{-1}m^{-1}$. Use t to denote the coordinate for the interval I_m and use z for the complex coordinate on D_m. As will be explained in a moment, there is a coordinate chart embedding from $I_m \times D_m$ to Y_Z with the following properties:

(7-29)
- The point $(0,0)$ is mapped to p and $I_m \times \{0\}$ is mapped to a segment of the integral curve of v through p.
- The image of any disk $\{t\}\times D_m$ is a transverse disk centered at the image of $(t,0)$.
- The function $\mathrm{z} \mapsto |\mathrm{z}|$ on $\{t\}\times D_m$ is the pullback of the distance along the image of $\{t\}\times D_m$ to the image of $\{t,0\}$.
- The vector field v appears in these coordinates as $\frac{\partial}{\partial t} + \mathfrak{e}$, with $|\mathfrak{e}| \le c_0 m|\mathrm{z}|$.

To construct such a coordinate chart, fix an isometric isomorphism between $K_*^{-1}|_p$ and $\mathbb{C}$. By way of a reminder, K_*^{-1} is used to denote the complex line bundle over the $|w(p)| > 0$ part of Y_Z whose underlying real bundle is the kernel of $*w$ with the complex structure defined using the metric and the restriction of the form w. Let γ_p again denote the integral curve of v through p. Parallel transport the resulting frame for K_*^{-1} along γ_p to identify K_*^{-1} along γ_p with $\gamma_p \times \mathbb{C}$. Fix a unit-length affine parameter, t, for the segment of γ_p consisting of points with distance $c_0^{-1}m^{-1}$ or less from p with $t = 0$ corresponding to p. This identifies this segment with I_m. Granted this identification, compose the metric's exponential map from the I_m part of γ_p with the identification between K_*^{-1} on this segment and the product $\mathbb{C}$–bundle to define a map from $I_m \times \mathbb{C}$ into Y_Z. The restriction of this map to $I_m \times D_m$ gives the desired coordinate embedding.

Step 6 Fix $p \in Y_Z$ such that $|w(p)| \in [m_{k+2}^{-1}, m_k^{-1}]$ and $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$. Let p' denote a chosen point on Step 4's short segment of γ_p with $|w(p')| = m_{k+1}^{-1}$. It follows from (7-28) that $1 - |\alpha|^2 > \frac{1}{8}\kappa_\diamond^{-1}$ at p'. This being the case, it follows from Lemma 7.4 and Lemma IV.2.8 that if $z > c_0$ and if $\mathrm{r} > c_0$, then there is a point, with distance at most $c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from p' where $1 - |\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$. It then follows from the third bullet of (7-27) that there exists $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ such that p' has distance at most $(\mathrm{D}z + c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from a point in γ. Let p_* denote the latter point. Use the coordinate chart in (7-29) to see that short segment of γ_p intersects the transverse disk through p_* at a point with distance at most $(1 + c_0\varepsilon)(\mathrm{D}z + c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from p_*.

Extend the curves from Θ_k into the $|w| \ge m_{k+2}^{-1}$ part of Y_Z by integrating the vector field v. Use Γ_{k+1} to denote this set of extended curves. Given $\gamma \in \Gamma_{k+1}$, fix a point $p_\gamma \in \gamma$ where $|w| = m_{k+1}^{-1}$. The point p_γ has its corresponding version of the coordinate chart in (7-29) with γ appearing as an interval in the $\mathrm{z} = 0$ locus that contains $(0, 0)$. Let I_γ denote this interval.

It follows from what was said in the preceding paragraph that the each point in B where $1-|\alpha|^2 > \frac{1}{4}\kappa_\diamond^{-1}$ and $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$ lies in the $|\mathrm{z}| \le (1+c_0\varepsilon)(\mathrm{D}z+c_0)m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of some $\gamma \in \Gamma_{k+1}$ version of $I_\gamma \times D_{m_{k+1}}$. In particular, if $\varepsilon < c_0^{-1}$ and $z > c_0$, then this subset is contained in the subset where $|\mathrm{z}| < \frac{3}{2}\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$. Assume that ε and z are such that this is the case.

Note in this regard that if $(\gamma, \mathrm{m}, \mathrm{D})$ and $(\gamma', \mathrm{m}', \mathrm{D}')$ are distinct elements in Θ_k, then the respective subsets of B that are parametrized via (7-29) by the $|\mathrm{z}| \le 2\mathrm{D}z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $I_\gamma \times D_{m_{k+1}}$ and the $|\mathrm{z}| \le 2\mathrm{D}'z m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $I_{\gamma'} \times D_{m_{k+1}}$ are disjoint. This is a consequence of the second bullet in (7-27).

Step 7 Fix $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$. It follows from what was said in Step 6 that $\alpha|\alpha|^{-1}$ is $\widehat{A}_*$–covariantly constant in the solid annulus in $I_\gamma \times D_{m_{k+1}}$ that intersects any constant t slice as the annulus with inner radius $\frac{3}{2}\mathrm{D}zm_{k+1}^{1/2}\mathrm{r}^{-1/2}$ and outer radius $2\mathrm{D}zm_{k+1}^{1/2}\mathrm{r}^{-1/2}$. Granted this, it then follows from the third bullet of (7-27) that the integral of $\frac{i}{2\pi}F_{\hat{A}_*}$ over the $|\mathrm{z}| < 2\mathrm{D}zm_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of any constant t disk in $I_\gamma \times D_{m_{k+1}}$ is the integer m.

To exploit the preceding observation, fix $t \in I_\gamma$ and let $p \in Y_Z$ denote the point that corresponds to $(t, 0) \in I_\gamma \times D_{m_{k+1}}$. Associate to p the pair $(A_\mathrm{r}, \alpha_\mathrm{r})$ as described in Part 2 of Section 7.2. Use c_z in what follows to denote a constant that is greater than 1 and depends only on z. It follows from Lemma 7.4 that if $z > c_0$ and if $\mathrm{r} > c_z$, then $(A_\mathrm{r}, \alpha_\mathrm{r})$ have C^1–distance less than z^{-10} on the radius $2\mathrm{D}z$ disk in $\mathbb{C}$ from a finite-energy solution to the vortex equations. Moreover, what is said by Lemma 7.4 implies that such a finite-energy solution must define a point in the space $\mathfrak{C}_\mathrm{m}$. Granted this, then (7-25) and Lemma 7.4 imply the following when $z > c_0$ and $\mathrm{r} > c_z$:

If $z > c_0$ and $\mathrm{r} \geq c_z$, then there is a set of at most n_0 points in the $|\mathrm{z}| < \frac{3}{2}\mathrm{D}m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ part of $\{t\} \times D_{m_{k+1}}$ such that

(7-30)
- *each point is a zero of α;*
- *if $1 - |\alpha|^2 \geq \frac{1}{8}\kappa_\diamond^{-1}$ at (t, z) and $|\mathrm{z}| \leq 2\mathrm{D}m_{k+1}\mathrm{r}^{-1/2}$, then z has distance at most $c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ from some point in this set.*

Use $\vartheta_{\gamma,t}$ to denote this set of points and let $\mathfrak{U}_{\gamma,t}$ denote the set of connected components of the union of the disks of radius $c_0 m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ about the points in $\vartheta_{\gamma,t}$. The next assertion is a $z > c_0$ and $\mathrm{r} > c_z$ consequence of (7-30) plus Lemma 7.4 and (7-25):

(7-31)
- The connection $\hat{A}_*$ is flat and $\alpha|\alpha|^{-1}$ is $\hat{A}_*$–covariantly constant on the complement of $\bigcup_{U \in \mathfrak{U}_{\gamma,t}} U$ in the radius $2\mathrm{D}m_{k+1}^{1/2}\mathrm{r}^{-1/2}$ disk about the origin in $\{t\} \times D_{m_{k+1}}$.
- The integral of $\frac{i}{2\pi}F_{\hat{A}_*}$ over any set $U \in \mathfrak{U}_{\gamma,t}$ is a positive integer; and the sum of these integers is equal to m.

The next step constructs Θ_{k+1} with the help of the various $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ versions of $\vartheta_{\gamma,0}$.

Step 8 To construct Θ_{k+1}, it is necessary to cluster the points from the various $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ versions of $\vartheta_{\gamma,0}$ so that points in the same cluster are pairwise much closer to each other than they are to any point in another cluster. This is necessary so as to find the desired constant c_{k+1} for the integer $k+1$ version of (7-27). An appropriate clustering can be found by invoking Lemma 2.12 in [38]. In particular, an appeal to this lemma finds $c_{k+1} \in (100, (100)^{2^{\kappa_1}})$ and a set of at most κ_1 pairs of the form (p, D), where $p \in B$ is such that $|w(p)| = m_{k+1}^{-1}$ and where $\mathrm{D} \in (1, c_{k+1})$. This set is denoted by ϑ and it has the properties in the list that follows:

(7-32)
- If (p, D) and (p', D') are distinct elements in ϑ, then
$$\operatorname{dist}(p, p') > c_{k+1}^2 z m_{k+1}^{1/2} \mathrm{r}^{-1/2}.$$
- If p corresponds via (7-29) to a point in some $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ version of $\vartheta_{k,0}$, then p has distance at most $\frac{1}{4}\mathrm{D} z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ from a point of some pair from ϑ.

Note for future reference that the bound in the first bullet of (7-32) has the following implication when $z > c_0$ and $\mathrm{r} > c_z$:

(7-33) If (p, D) and (p', D') are distinct elements in ϑ, then the distance between any two points on the respective short segments γ_p and $\gamma_{p'}$ is greater than $\frac{1}{2} c_{k+1}^2 z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$.

It follows from (7-31) and (7-32) that if $(\gamma, \mathrm{m}, \mathrm{D}) \in \Theta_k$ and if $U \in \mathfrak{U}_{\gamma,0}$, then U is in the transverse disk of radius $\frac{1}{2}\mathrm{D} z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ centered at a point of some pair in ϑ. Granted this last conclusion, then the next assertion is a direct consequence of what is said in Step 4 if $z > c_0$ and $\mathrm{r} > c_z$.

(7-34) If $(\gamma, m, \mathrm{D}) \in \Theta_k$ and $t \in I_\gamma$, then each $U \in \mathfrak{U}_{\gamma,t}$ is contained in the radius $\mathrm{D} z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ tubular neighborhood of the integral curve of v through a point of some pair from ϑ.

Let $(p, \mathrm{D}) \in \vartheta$. What is said in (7-33) and (7-34) has the following consequence:

(7-35) The integral of $\frac{i}{2\pi} F_{\hat{A}_*}$ on the radius $\mathrm{D} z m_{k+1}^{1/2} \mathrm{r}^{-1/2}$ transverse disk about any point in the $|w| \in [m_{k+2}^{-1}, m_k^{-1}]$ part of γ_p is a positive integer.

Let m denote now this integer.

Define Θ_{k+1} to be the set $\{(p, m, \mathrm{D}) \mid (p, \mathrm{D}) \in \vartheta\}$. It follows from (7-31) and (7-33)–(7-35) that the requirements for the integer $k+1$ version of (7-27) are met using c_{k+1} and the set Θ_{k+1} if $\varepsilon < c_0$, $z > c_0$ and $\mathrm{r} > c_z$. □

7.6 The spectral flow function

This subsection constitutes a digression to say more about the definition of $\mathfrak{f}_s$. Each pair $\mathfrak{c} = (A, \psi)$ in $\operatorname{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ and a given real number z determine an associated, unbounded, self-adjoint operator on $L^2(Y_Z; iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R})$. This operator is denoted by $\mathfrak{L}_{\mathfrak{c},z}$ and it is defined as follows: a given smooth section $\mathfrak{h} = (b, \eta, \phi)$ of $iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R}$ is sent by $\mathfrak{L}_{\mathfrak{c},z}$ to the section whose respective iT^*Y_Z–, $\mathbb{S}$– and $i\mathbb{R}$–summands are

$$\begin{cases} *db - d\phi - 2^{-1/2} z^{1/2} (\psi^\dagger \tau \eta + \eta^\dagger \tau \psi), \\ D_A \eta + 2^{1/2} z^{1/2} (\mathrm{cl}(b)\psi + \phi\psi), \\ *d * b - 2^{-1/2} z^{1/2} (\eta^\dagger \psi - \psi^\dagger \eta). \end{cases} \tag{7-36}$$

The spectrum of this operator is discrete with no accumulation points and has finite multiplicity. The spectrum is also unbounded from above and unbounded from below.

The section ψ_E of $\mathbb{S}$ is chosen so that the (A_E, ψ_E) and $z = 1$ version of (7-36) has trivial kernel. If the $z = \mathrm{r}$ and $\mathfrak{c} = (A, \psi)$ version of (7-36) has trivial kernel, then the value of the spectral flow function $\mathfrak{f}_s(\mathfrak{c})$ is a certain algebraic count of the number of zero eigenvalues that appear along a continuous path $\mathfrak{d}$ of operators that start at the $z = 1$ and (A_E, ψ_E) version of (7-36) and end at the $z = \mathrm{r}$ and (A, ψ) version and such that each member of the path differs from $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ by a bounded operator on $L^2(Y_Z; iT^*Y_Z \oplus \mathbb{S} \oplus i\mathbb{R})$. For the purposes of the definition, it is sufficient to consider paths that are parametrized by $[0, 1]$ such that the following conditions are met: Let $\vartheta \subset [0, 1]$ denote the parameters that label an operator with zero as an eigenvalue. Then ϑ is finite and in each case, the zero eigenvalue has multiplicity 1 and the zero eigenvalue crossing is transversal as the parameter varies in a small neighborhood of the given point in $[0, 1]$. Having chosen such a path, a given point in the corresponding version of ϑ contributes either $+1$ or -1 to $\mathfrak{f}_s(\mathfrak{c})$. The contribution is $+1$ when the eigenvalue crosses zero from negative value to positive value as the parameter in $[0, 1]$ varies near the given point in ϑ; and it contributes -1 to $\mathfrak{f}_s(\mathfrak{c})$ if the eigenvalue crosses zero from a positive value to negative value near the given point.

If $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ has nontrivial kernel, then $\mathfrak{f}_s(\mathfrak{c})$ is defined in the upcoming (7-37). The definition uses the following terminology: Given $\varepsilon > 0$, and $\mathfrak{c} \in \mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{C})$, the definition uses $\mathfrak{N}_\varepsilon(\mathfrak{c})$ to denote the subset of pairs in $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{C})$ with the following two properties: a pair $\mathfrak{c}'$ is in $\mathfrak{N}_\varepsilon(\mathfrak{c})$ if it has C^1–distance less than ε from $\mathfrak{c}$, and if $\mathfrak{L}_{\mathfrak{c}',\mathrm{r}}$ has trivial kernel. Standard perturbation theory for elliptic operators proves that $\mathfrak{N}_\varepsilon(\mathfrak{c})$ is nonempty for any $\varepsilon > 0$. With the notation set, define $\mathfrak{f}_s(\mathfrak{c})$ by the rule

$$\mathfrak{f}_s(\mathfrak{c}) = \limsup_{\varepsilon \to 0} \{\mathfrak{f}_s(\mathfrak{c}') \mid \mathfrak{c}' \in \mathfrak{N}_\varepsilon(\mathfrak{c})\}. \tag{7-37}$$

Note by the way that the lim sup in (7-37) differs from the corresponding lim inf by the dimension of the kernel of $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$.

7.7 The L^1–norm of B_A, the spectral flow and the functions $\mathfrak{cs}^{\mathfrak{f}}$, $\mathrm{W}^{\mathfrak{f}}$, $\mathfrak{a}^{\mathfrak{f}}$

The functions

$$\mathfrak{cs}^{\mathfrak{f}} = \mathfrak{cs} - 4\pi^2 \mathfrak{f}_s, \quad \mathrm{W}^{f} = \mathrm{W} - 2\pi \mathfrak{f}_s \quad \text{and} \quad \mathfrak{a}^{\mathfrak{f}} = \mathfrak{a} + 2\pi(\mathrm{r} - \pi)\mathfrak{f}_s \tag{7-38}$$

are invariant under the $C^\infty(Y_Z; S^1)$–action on $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ that has $\hat{u} \in C^\infty(Y_Z; S^1)$ sending (A, ψ) to $(A - \hat{u}^{-1} d\hat{u}, \hat{u}\psi)$. The upcoming Lemma 7.9 supplies a priori bounds on the values of these functions when evaluated on solutions to a given (r, μ) version of (2-5). It also gives a better bound for the L^1–norm of the curvature of the connection component of a solution than the bound in Lemma 7.6.

Lemma 7.9 *Suppose that w is a harmonic 2–form with nondegenerate zeros. There exists $\kappa > \pi$ and $\kappa_1 \gg \kappa$ with the following significance: Fix $\mathrm{r} \ge \kappa_1$ and a 1–form $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that (A, ψ) is a solution to the (r, μ) version of (2-5). Then:*

- *The L^1–norm of B_A is no greater than $\kappa(\ln \mathrm{r})^4$.*
- $|\mathfrak{cs}^{\mathfrak{f}}| < \mathrm{r}^{6/7}$.
- $|\mathrm{W}^{\mathfrak{f}}| < \mathrm{r}^{6/7}$.
- $|\mathfrak{a}^{\mathfrak{f}}| < \mathrm{r}^{13/14}$.

As a parenthetical remark, the precise powers of r that appear in the last three bullets are significant with regards to the applications to come only to the extent that the power is less than 1 in the second and third bullets and so less than 2 in the final bullet.

Proof By way of a look ahead, what is said in Lemma 7.8 plays a vital role in the proof of all four bullets. The proof of Lemma 7.9 has 10 parts.

Part 1 The proof of Lemma 7.9's first bullet has four steps. To set the notation for the proof, introduce κ_* to denote the version of the constant κ that appears in Lemma 7.8. As in Lemma 7.8, set $m_k = (1 + \kappa_*^{-1})^k \kappa_*^2$ for $k \in \{1, 2, \dots\}$. Assume in what follows that k is such that $m_k < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa_*}$.

Step 1 Use the first bullet of Lemma 7.8 and the fourth bullet of Lemma 7.2 to see that $|B_A| < c_0$ at points in the $|w| > m_1^{-1}$ part of Y_Z where the distance to all segments in Θ_1 is greater than $c_0(\ln \mathrm{r})^2 \mathrm{r}^{-1/2}$. This understood, this part of Y_Z contributes at most c_0 to the L^1–norm of B_A. Meanwhile, the $|w| > m_1^{-1}$ part of Y_Z of the union of the radius $c_0(\ln \mathrm{r})^2 \mathrm{r}^{-1/2}$ tubular neighborhoods of the segments in Y_Z contributes at most $c_0(\ln \mathrm{r})^4$ to the L^1–norm of B_A.

Step 2 Fix $k > 1$. Use the integer k version of the second bullet of Lemma 7.8 with the fourth bullet of Lemma 7.2 to see that $|B_A|$ is bounded by $c_0(1+m_k^2)$ at points in the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ part of Y_Z where the distance to all segments in Θ_k is greater than $c_0 m_k^{1/2}(\ln \mathrm{r})^2 \mathrm{r}^{-1/2}$. Since this subset of Y_Z has volume at most $c_0 m_k^{-3}$, so this portion of the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ subset in Y_Z contributes at most $c_0 m_k^{-1}$ to the L^1–norm of B_A. The volume of the remaining part of the $|w| \in [m_k^{-1}, m_{k-1}^{-1}]$ subset in Y_Z is at most $c_0 \mathrm{r}^{-1}(\ln \mathrm{r})^4$. Indeed, this can be seen from (7-29) using the fact that each segment in Θ_k has length at most $c_0 m_k^{-1}$. As $|B_A|$ is no greater than $c_0 m_k^{-1}\mathrm{r}$ on this part of Y_Z, so this part of Y_Z contributes at most $c_0 m_k^{-1}(\ln \mathrm{r})^4$ to the L^1–norm of B_A.

Step 3 Lemma 7.3 implies that $|B_A|$ is bounded by $c_0 \mathrm{r}^{2/3}(\ln \mathrm{r})^{\kappa_*}$ on the subset of Y_Z where $|w| \le c_0 \mathrm{r}^{-1/3}(\ln \mathrm{r})^{\kappa_*}$. The volume of this subset is at most $\mathrm{r}^{-1}(\ln \mathrm{r})^{3\kappa_*}$ and so the contribution from this part of Y_Z to the L^1–norm of B_A is no greater than $c_0 \mathrm{r}^{-1/4}$.

Step 4 Sum the bounds in Steps 1–3 to see that the L^1–norm of B_A is no greater than $c_0(\ln \mathrm{r})^4 \sum_{k=0,1,\dots}(1+1/\kappa_*)^{-k}$. This sum is bounded by $c_0\kappa_*(\ln \mathrm{r})^4$.

Part 2 The proof of the last three bullets of the lemma starts with the following observation: There is a smooth map, $\hat{u}\colon Y_Z \to S^1$, such that the connection $A' = A - \hat{u}^{-1}d\hat{u}$ can be written as $A' = A_E + \hat{a}_{A'}$ where $\hat{a}_{A'}$ is a coclosed, $i\mathbb{R}$–valued 1–form whose L^2–orthogonal projection to the space of harmonic 1–forms on Y_Z is bounded by c_0. The upcoming Lemma 7.10 asserts the pointwise bound $|\hat{a}_{A'}| \le c_0 \mathrm{r}^{1/3}(\ln \mathrm{r})^{c_0}$. Assume this bound for the time being.

Introduce $\mathfrak{c}'$ to denote $(A - \hat{u}^{-1}d\hat{u}, \hat{u}\psi)$. The supremum bound for $|\hat{a}_{A'}|$ and the L^1–bound for B_A from Lemma 7.9's first bullet imply directly that $|\mathfrak{cs}(\mathfrak{c}')| \le c_0 \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. The L^1–bound for B_A also implies that $|\mathrm{W}(\mathfrak{c}')| \le c_0(\ln \mathrm{r})^{c_0}$. Thus, $|\mathfrak{a}(\mathfrak{c}')| \le c_0 \mathrm{r}(\ln \mathrm{r})^{c_0}$. Granted these bounds, then the last three bullets of Lemma 7.9 follow if

$$|\mathfrak{f}_s(\mathfrak{c}')| \le \mathrm{r}^{6/7}. \tag{7-39}$$

The fact that (7-39) holds given the assumptions of the lemma is proved in the remaining parts of this subsection.

Part 3 The proof of the last three bullets of Lemma 7.9 invoked a pointwise bound for $|\hat{a}_{A'}|$. The lemma that follows supplies the asserted bound:

Lemma 7.10 *There exists $\kappa > \pi$ and $\kappa_1 \gg \kappa$ with the following significance: Fix $\mathrm{r} \ge \kappa_1$ and an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let (A, ψ) denote a solution* *to the* (r, μ) *version of* (2-5). *Write* A *as* $A_E + \hat{a}_A$ *and assume that* $\hat{a}_A$ *is a coclosed* 1*–form. Use* c *to denote the* L^2*–norm of the* L^2*–orthogonal projection of* $\hat{a}_A$ *to the space of harmonic* 1*–forms. Then* $|\hat{a}_A| \le \mathrm{r}^{1/2}(\ln \mathrm{r})^{\kappa} + \kappa c$.

Proof The proof that follows has three steps.

Step 1 Write $\hat{a}_A$ as $\hat{a}^{\perp} + \mathfrak{p}$, where $\hat{a}^{\perp}$ is L^2–orthogonal to the space of harmonic 1–forms and where $\mathfrak{p}$ is a harmonic 1–form. The norm of $\mathfrak{p}$ is bounded by $c_0 c$. To bound $\hat{a}^{\perp}$, let $\mathcal{C}^{\perp} \subset C^{\infty}(Y_Z; T^*Y_Z)$ denote the subspace of coclosed 1–forms that are L^2–orthogonal to the space of harmonic 1–forms. The operator $*d$ maps $\mathcal{C}^{\perp}$ to itself and Hodge theory gives a Green's function inverse. Given $p \in M$, the corresponding Green's function with pole at p is denoted by $G_p^{\perp}(\cdot)$. This function is smooth on the complement of p and it obeys the pointwise bound $|G_p^{\perp}(\cdot)| \le c_0 \operatorname{dist}(\cdot, p)^{-2}$.

Step 2 Introduce κ_* to denote Lemma 7.8's version of κ. Reintroduce from Lemma 7.8 the sequence $\{m_k = (1+\kappa_*^{-1})^k \kappa_*^2\}_{k=1,2\dots,N}$ with N being the greatest integer such that $m_N < \mathrm{r}^{1/3}(\ln \mathrm{r})^{-\kappa_*}$. Let $\mathcal{U}_1$ denote the $|w| > m_2^{-1}$ part of Y_Z. For $k \in \{1, \dots, N-1\}$, use $\mathcal{U}_k$ to denote the $|w| \in [m_{k+1}^{-1}, m_{k-1}^{-1}]$ part of Y_Z, and use $\mathcal{U}_N$ to denote the part of Y_Z where $|w| \le m_{N-1}^{-1}$. Given $k \in \{1, \dots, N-1\}$, let Γ_k denote the set of curves from Θ_k's data sets. By way of a reminder, there are at most κ_* curves in Γ_k and each is a properly embedded segment of an integral curve of v in $\mathcal{U}_k$.

Lemmas 7.2 and 7.8 supply $c_* \in (1, c_0)$ with the following property: If $p \in \mathcal{U}_k$ has distance greater than $c_* m_k \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ to any curve from Γ_k, then $1 - |\alpha|^2 \le c_0 m_k^3 \mathrm{r}^{-1}$. Denote by $\mathcal{T}_{k1}$ the union of the radius $c_* m_k \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ tubular neighborhoods of the curves from Γ_k. Since $*d\hat{a}^{\perp} = B_A$, it follows from Lemmas 7.1 and 7.2 that $|B_A| \le c_0 m_k^2$ on $\mathcal{U}_k - \mathcal{T}_{k1}$, and it follows from Lemmas 7.2 and 7.3 that $|B_A| \le c_0 m_k^{-1} \mathrm{r}$ on $\mathcal{T}_{k1}$. Note also that the volume of $\mathcal{U}_k$ is at most $c_0 m_k^{-3}$ and that of $\mathcal{T}_{k1}$ at most $c_0 m_k^{-1} \mathrm{r}^{-1} (\ln \mathrm{r})^4$.

Step 3 Suppose that $k \in \{1, \dots, N-1\}$ and that $p \in \mathcal{U}_k$. Keeping in mind that the volume of $\mathcal{U}_k$ is bounded by $c_0 m_k^{-3}$, it follows from what is said about $G_p^{\perp}$ in Step 1 and what is said about $|B_A|$ in Step 2 that

$$|\hat{a}^{\perp}|(p) \le c_0 \int_{\mathcal{T}_k} \operatorname{dist}(\cdot, p)^{-2} |B_A| + c_0 (m_k + (\ln \mathrm{r})^{c_0}). \tag{7-40}$$

Use the various $\gamma \in \Gamma_k$ versions of (7-29) to see that the integral on the right-hand side of (7-40) is no greater than $c_0 m_k^{-1/2} \mathrm{r}^{1/2} (\ln \mathrm{r})^{c_0}$.

Suppose that $p \in \mathcal{U}_N$. In this case, what is said about $G_p^{\perp}$ in Step 1 and what is said in Step 2 about $|B_A|$ imply that $|\hat{a}^{\perp}|(p) \le c_0 \mathrm{r}^{1/3} (\ln \mathrm{r})^{c_0}$. □

Part 4 Fix $c > 1$ and suppose that $\mathfrak{c} = (A, \psi)$ solves (2-5) and is such that the $i\mathbb{R}$–valued 1–form $\hat{a}_A = A - A_E$ is coclosed and that the L^2–norm of its L^2–orthogonal projection to the space of harmonic 1–forms on Y_Z is less than c. The value of $\mathfrak{f}_s$ will be computed by choosing a convenient, piecewise continuous path of self-adjoint operator from the (A_E, ψ_E) and $z = 1$ version of (7-36) to $\mathfrak{L}_{c,\mathrm{r}}$. This path is the concatenation of the three real analytic segments that are described below. The absolute value of $\mathfrak{f}_s(\mathfrak{c})$ is no greater than the absolute value of the sum of the absolute values of the spectral flow along the three segments.

By way of notation, each segment is parametrized by $[0, 1]$ and the operator labeled by a given $s \in [0, 1]$ in the k^{th} segment is denoted by $\mathcal{L}_{k,s}$. The first segment's operator $\mathcal{L}_{1,s}$ for $s \in [0, 1]$ is the (A_E, ψ_E) and $z = 1 - s$ version of (7-36). This path has no dependence on (A, ψ) or r, and so the absolute value of the spectral flow along this path is no greater than c_0. The remaining two segments are:

(7-41)
- The second segment's operator $\mathcal{L}_{2,s}$ for $s \in [0, 1]$ is the $(A_E + s\,\hat{a}_A, 0)$ version of (7-36).
- The third segment's operator $\mathcal{L}_{3,s}$ for $s \in [0, 1]$ is the (A, ψ) and $z = s^2\mathrm{r}$ version of (7-36).

The strategy for bounding the absolute value of the spectral flow along (7-41)'s two segments borrows heavily from Section 3 of [34]. To say more about this, suppose that $\mathcal{L}$ is an unbounded, self-adjoint operator on a given separable Hilbert space with discrete spectrum with no accumulation points and finite multiplicities. Let $\{e_s\}_{s\in[0,1]}$ denote a real analytic family of bounded, self-adjoint operators on this same Hilbert space. Of interest is the spectral flow between the $s = 0$ and $s = 1$ members of the family $\{\mathcal{L}_s = \mathcal{L} + e_s\}_{s\in[0,1]}$. To obtain a bound, fix for the moment $T > 0$ and let $\mathfrak{n}_{T,s}$ denote the number of linearly independent eigenvectors of $\mathcal{L}_s$ whose eigenvalue has absolute value no greater than T. Set $\mathfrak{n}_T = \sup\{\mathfrak{n}_{T,s}\}_{s\in[0,1]}$. As explained in [32], the spectral flow for the family $\{\mathcal{L}_s\}_{s\in[0,1]}$ has absolute value no greater than

$$\frac{1}{2T}\mathfrak{n}_T \sup\Big\{\Big\|\frac{d}{ds}e_s\Big\|_{\mathrm{op}}\Big\}_{s\in[0,1]}, \tag{7-42}$$

where the norm $\|\cdot\|_{\mathrm{op}}$ here denotes the operator norm.

The supremum in (7-42) for the family $\{\mathcal{L}_{2,s}\}_{s\in[0,1]}$ is bounded by $c_0|\hat{a}_{A'}|$, and thus by $c_0\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. It follows from Lemma 7.3 that the supremum that appears in (7-42) for the family $\{\mathcal{L}_{3,s}\}_{s\in[0,1]}$ is $c_0\mathrm{r}^{1/2}$. This understood, then (7-42) in either case leads to:

(7-43) The absolute value of the spectral flow along the families $\{\mathcal{L}_{2,s}\}_{s\in[0,1]}$ and $\{\mathcal{L}_{3,s}\}_{s\in[0,1]}$ is no greater than $c_0\mathrm{r}^{1/2}(\ln\mathrm{r})^{c_0}\frac{1}{T}\mathfrak{n}_T$.

The next part of the subsection describes the strategy that is used to bound $\mathfrak{n}_T$ for a suitable choice of T.

Part 5 A bound for $\mathfrak{n}_T$ is obtained with the help of the Weitzenböck formula in (IV.A-12) for a given $z\geq 0$ version of $\mathfrak{L}_{\mathfrak{c},z}^2$. This formula writes $\mathfrak{L}_{\mathfrak{c},z}^2$ as $\nabla_A^\dagger\nabla_A+\mathcal{Q}$, where $\mathcal{Q}$ denotes an endomorphism of $iT^*Y_Z\oplus\mathbb{S}\oplus i\mathbb{R}$ and ∇_A denotes here the connection on the bundle $iT^*Y_Z\oplus\mathbb{S}\oplus i\mathbb{R}$ given by the Levi-Civita connection on the iT^*Y_Z–summand, the Levi-Civita connection and A on the $\mathbb{S}$–summand, and the product connection on the $i\mathbb{R}$–summand. This rewriting of $\mathfrak{L}_{\mathfrak{c},z}^2$ is used to write the square of the L^2–norm of $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{q}$ as

$$\int_{Y_Z}|\mathfrak{L}_{\mathfrak{c},z}\mathfrak{q}|^2=\int_{Y_Z}(|\nabla_A\mathfrak{q}|^2+\langle\mathfrak{q},\mathcal{Q}\mathfrak{q}\rangle),\tag{7-44}$$

with $\langle\cdot,\cdot\rangle$ denoting here the Hermitian inner product on $iT*Y_Z\oplus\mathbb{S}\oplus i\mathbb{R}$. If $\mathfrak{q}$ is a linear combination of eigenvectors of $\mathfrak{L}_{\mathfrak{c},z}$ with the norm of the eigenvalue bounded by T, then what is written in (7-44) is no greater than T^2 times the square of the L^2–norm of $\mathfrak{q}$.

The formula in (7-44) is exploited to bound $\mathfrak{n}_T$ using the following observation: Suppose that $\mathfrak{U}$ is an open cover of Y_Z such that no point is contained in more than c_0 sets from $\mathfrak{U}$. Let h denote for the moment a given function on Y_Z. Then

$$\int_{Y_Z}h^2\leq\sum_{U\in\mathfrak{U}}\int_U h^2\leq c_0\int_{Y_Z}h^2.\tag{7-45}$$

Hold onto this last observation for the moment. Use $c_\diamond$ to denote the version of c_0 that appears in this last inequality.

The endomorphism $\mathcal{Q}$ is self-adjoint, so it can be written at any given point as a sum $\mathcal{Q}^++\mathcal{Q}^-$ with $\mathcal{Q}^+$ being positive semidefinite and $\mathcal{Q}^-$ being negative definite. With this fact in mind, suppose now that each set $U\in\mathfrak{U}$ has an assigned, finite-dimensional vector subspace $V_U\in C^\infty(U;iT^*M\oplus\mathbb{S}\oplus i\mathbb{R})$ with the following property:

(7-46) If $q\in C^\infty(U;iT^*M\oplus\mathbb{S}\oplus i\mathbb{R})$ is L^2–orthogonal to V_U, then

$$\int_U(|\nabla_A\mathfrak{q}|^2+\langle\mathfrak{q},\mathcal{Q}^+\mathfrak{q}\rangle)>2c_\diamond\bigl(T^2+\sup_U|\mathcal{Q}^-|\bigr)\int_U|\mathfrak{q}|^2.$$

Given V_U, define $\Phi_U\colon C^\infty(Y_Z; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}) \to V_U$ to be the composition of first restriction to U and then the L^2–orthogonal projection. Set $\mathcal{V} = \bigoplus_{U \in \mathfrak{U}} V_U$ and denote by Φ the linear map from $C^\infty(Y_Z; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ to $\mathcal{V}$ given by $\bigoplus_{U \in \mathfrak{U}} \Phi_U$.

The inequalities in (7-45) and (7-46) have the following immediate consequence: If $\mathfrak{q} \in \operatorname{Ker}(\Phi)$, then the L^2–norm of $\mathfrak{L}_{\mathfrak{c},z}$ is greater than T. Given that such is the case, it then follows directly that $\mathfrak{n}_T \le \sum_{U \in \mathfrak{U}} \dim(V_U)$.

The subsequent parts of the proof define a version of $\mathfrak{U}$ for suitable T with associated vector spaces $\{V_U\}_{U \in \mathfrak{U}}$ such that (7-46) holds. The resulting bound for $\mathfrak{n}_T$ leads via (7-43) to the bound in (7-39) for $|\mathfrak{f}_s|$.

Part 6 Part 5 alludes to a certain open cover of Y_Z. This part of the subsection defines this cover. To this end, reintroduce from Step 2 of the proof of Lemma 7.10 the sets $\{\mathcal{U}_k\}_{1 \le k \le N}$. The cover in question is given as $\mathfrak{U} = \bigcup_{k=1,2,\dots N} \mathfrak{U}_k$ where all $U \in \mathfrak{U}_k$ are subsets of $\mathcal{U}_{k-1} \cup \mathcal{U}_k \cup \mathcal{U}_{k+1}$. The definition requires the choice of a constant $c > 1$. Part 10 of the proof gives a lower bound for c by c_0. Any choice above this bound suffices.

To define a given $k \in \{1, \dots, N-1\}$ version of $\mathfrak{U}_k$, reintroduce from Step 2 of the proof of Lemma 7.10 the set Γ_k, this being the set of curves from Θ_k's data sets. By way of a reminder, there are at most κ_* curves in Γ_k and each is a properly embedded segment of an integral curve of v in $\mathcal{U}_k$. This same step in the proof of Lemma 7.10 introduced a constant c_* such that $1 - |\alpha|^2 < c_0 m_k^3 \mathrm{r}^{-1}$ at points with distance $c_* m_k^{1/2} \mathrm{r}^{-1/2} (\ln \mathrm{r})^2$ or more to all curves from Γ_k. The discussion that follows uses R_k to denote $c_* m_k^{1/2} \mathrm{r}^{-1/2} (\ln \mathrm{r})^2$ and ρ_k to denote $c^{-1} \min(T, m_k^{-1})$.

The collection $\mathfrak{U}_k$ for $k \in \{1, \dots, N-1\}$ is written as $\mathfrak{U}_{k-} \cup \mathfrak{U}_{k0} \cup \mathfrak{U}_{k+}$. The sets from $\mathfrak{U}_{k-}$ are balls of radius ρ_k whose centers have distance at least ρ_k to all curves from Γ_k. These balls cover the complement in $\mathcal{U}_k$ of the union of the radius $2\rho_k$ tubular neighborhoods of the curves from Γ_k. A cover as just described can be found with less than $c_0 \rho_k^{-3} m_k^{-3}$ balls, and such is the case with the cover $\mathfrak{U}_{k-}$.

The sets from $\mathfrak{U}_{k0}$ are balls with distance between $2\rho_k$ and R_k to at least one curve from Γ_k. Let U denote a give ball from $\mathfrak{U}_{k0}$ and let D denote its distance to the union of the curves from Γ_k. The radius of U is equal to $\frac{1}{8}\mathrm{D}$. The various $\gamma \in \Gamma_k$ versions of (7-29) can be used to see that a collection of $c_0 \ln(\rho_k/\mathrm{R}_k)(\mathrm{R}_k m_k)^{-1}$ balls of this sort can be found whose union contains every point in $\mathcal{U}_k$ with distance between ρ_k and $2\mathrm{R}_k$ to at least one curve from Γ_k. The set $\mathfrak{U}_{k0}$ is such a collection of balls.

The set $\mathfrak{U}_{k+}$ consists of balls of radius $c^{-1}m_k^{1/2}\mathrm{r}^{-1/2}$ whose centers have distance at most R_k to some curve from Γ_k. The balls from $\mathfrak{U}_{k+}$ cover the set of points with distance R_k or less to some curve from Γ_k. The collection $\mathfrak{U}_{k+}$ has at most $c_0 c^3(\ln \mathrm{r})^4 m_k^{-3/2}\mathrm{r}^{1/2}$ balls.

The sets that form $\mathfrak{U}_N$ are balls of radius $\mathrm{r}^{-1/3}(\ln \mathrm{r})^{-c}$ with centers in $\mathcal{U}_N$. These sets define an open cover of $\mathcal{U}_N$. A cover of this sort can be found with less than $c_0(\ln \mathrm{r})^{c_0 c}$ elements, and such is the case for $\mathfrak{U}_N$.

Part 7 This part of the subsection defines the vector spaces $\{V_U\}_{U\in\mathfrak{U}}$. The next lemma is needed for the definition.

Lemma 7.11 *There exists $\kappa \geq 1$ with the following significance: Let $U \subset Y_Z$ denote a ball of radius $\rho \in (0, \kappa^{-1})$. Fix an isometric isomorphism between $E|_U$ and $U \times \mathbb{C}$. Use the latter to view the product connection on $U \times \mathbb{C}$ as a connection on $E|_U$. Use ∇_0 to denote the corresponding covariant derivative on $C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$. There exists a κ–dimensional vector space $W_U \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus i\mathbb{R})$ such that if $\mathfrak{q}$ is a section over U of $iT^*M \oplus \mathbb{S} \oplus i\mathbb{R}$ which is L^2–orthogonal to W_U, then $\int_U |\nabla_0\mathfrak{q}|^2 \geq \kappa^{-1}\rho^{-2}\int_U |\mathfrak{q}|^2$.*

This lemma will be proved in a moment; assume it to be true for now.

Fix $U \subset \mathfrak{U}$. If $c \geq c_0$ then the radius of each ball from $\mathfrak{U}$ will be smaller than Lemma 7.12's version of κ^{-1} and each ball from $\mathfrak{U}$ will sit in the Gaussian coordinate chart about its center point. With this understood, fix $U \in \mathfrak{U}$ and let p denote U's center point. Fix an isometric isomorphism between $E|_p$ and $\mathbb{C}$ and use A's parallel transport along the radial geodesics from p to extend this identification to one between $E|_U$ and the product bundle $U \times \mathbb{C}$. Define V_U to be Lemma 7.11's vector space W_U.

Proof of Lemma 7.11 If $\rho < c_0^{-1}$, then U has a Gaussian coordinate chart centered at its center point. Fix an isometric identification between K^{-1} at the center point of U with $\mathbb{C}$ and use the A_K parallel transport along the radial geodesics through the center point to extend this isomorphism to one between $K^{-1}|_U$ and $U \times \mathbb{C}$. Use the coordinate basis with the identification $K^{-1}|_U = U \times \mathbb{C}$ and the chosen identification $E|_U = U \times \mathbb{C}$ to give a product structure to T^*M and $\mathbb{S}$ over U. Having done so, rescale the coordinates so the ball of radius ρ becomes the ball of radius 1; then invoke the next lemma. $\square$

Lemma 7.12 *Let $U \subset \mathbb{R}^3$ denote the ball of radius 1 centered on the origin. If $\mathfrak{h} \in C^\infty(U;\mathbb{C})$ is such that $\int_U \mathfrak{h} = 0$, then $\int_U |d\mathfrak{h}|^2 \geq \frac{1}{4}\int_U \mathfrak{h}^2$.*

Proof It is sufficient to prove the bound for functions that depend only on z through its absolute value. This understood, use ρ to denote $|\mathrm{z}|$ and let $\mathfrak{h}$ denote a function that depends only on ρ and has integral zero over the unit ball. Let $\mathfrak{h}_* = \mathfrak{h} - \mathfrak{h}(1)$. Use integration by parts to see that

$$\int_0^1 \mathfrak{h}_*^2 \rho^2\, d\rho \leq 2\int_0^1 \left|\frac{\partial}{\partial\rho}\mathfrak{h}\right| |\mathfrak{h}_*| \rho\, d\rho. \tag{7-47}$$

What is written in (7-47) implies that

$$\int_0^1 \mathfrak{h}_*^2\, d\rho \leq 4\int_0^1 |d\mathfrak{h}|^2 \rho^2\, d\rho. \tag{7-48}$$

Meanwhile, $\int_0^1 \mathfrak{h}_*^2\, d\rho \geq \int_0^1 \mathfrak{h}_*^2 \rho^2\, d\rho$, the latter being the integral of $\mathfrak{h}_*^2$ over the unit ball. This last integral is $\frac{1}{3}\mathfrak{h}(1)^2$ plus the integral of $\mathfrak{h}^2$ because the integral of $\mathfrak{h}$ is zero. □

Part 8 This step sets the stage for the specification of c and $\{\rho_k\}_{1\leq k\leq N-1}$ so as to guarantee (7-46). To start, let $U \subset \mathfrak{U}$ denote a given ball and let p denote the center point of U. Fix an isometric isomorphism between $E|_p$ and $\mathbb{C}$ and then use A's parallel transport along the radial geodesics from p to extend this isomorphism to give an isomorphism between $E|_U$ and $U \times \mathbb{C}$. Let θ_0 denote the product connection on $U \times \mathbb{C}$. Use the isomorphism just defined to view θ_0 as a connection on $E|_U$. Having done so, write A on U as $\theta_0 + \hat{a}_{A,U}$ with $\hat{a}_{A,U}$ being an $i\mathbb{R}$–valued 1–form on U. Let D_U denote the radius of U. The norm of $\hat{a}_{A,U}$ is bounded by $c_0 \mathrm{D}_U \sup_U |B_A|$.

Fix $k \in \{1,\dots,N-1\}$; let U denote a ball from either $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. It follows from what Lemma 7.2 that $|B_A| \leq c_0 m_k^2$ on U and so $|\hat{a}_{A,U}| \leq c_0 c^{-1}\rho_k m_k^2$. If $U \in \mathcal{U}_{k+}$, then it follows from Lemma 7.2 that $|B_A| \leq c_0 m_k^{-1}\mathrm{r}$ on U and so $|\hat{a}_{A,U}| \leq c_0 c^{-1} m_k^{-1/2}\mathrm{r}^{1/2}$ on U. If U is from $\mathfrak{U}_N$, then Lemma 7.3 finds $|B_A| \leq c_0 \mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}$ on U and so $|\hat{a}_{A,U}| \leq c_0 c^{-1}\mathrm{r}^{1/3}(\ln\mathrm{r})^{c_0}$.

Given $U \subset \mathfrak{U}$, use the isomorphism defined above between $E|_U$ and $U \times \mathbb{C}$ to again view θ_0 as a connection on $E|_U$. Use ∇_0 to denote the corresponding covariant derivative on $C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$. Since $|\hat{a}_{A,U}|^2 \leq c_0 \sup_U |B_A|$ in all cases, so

$$|\nabla_A \mathfrak{q}|^2 \geq \tfrac{1}{2}|\nabla_0 \mathfrak{q}|^2 - c_0\bigl(\sup_U |B_A|\bigr)|\mathfrak{q}|^2 \tag{7-49}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Consider next the endomorphism $\mathcal{Q}$ that appears in (7-46). A look at the formula in (IV.A-12) finds

$$|\mathcal{Q}^-| \le c_0(1+|B_A|+z^{1/2}|\nabla_A\psi|) \quad \text{and} \quad |\mathcal{Q}^+| \ge c_0^{-1}z|\psi|^2. \tag{7-50}$$

To say more about the bounds in (7-50) on the sets from $\mathfrak{U}$, fix first $k \in \{1,\dots,N-1\}$ and let U denote a ball from $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. Lemma 7.2 finds $|\nabla_A\psi| \le c_0 m_k^{1/2}$ and $|\psi|^2 \ge c_0 m_k^{-1}$ on U. Since $|B_A|$ on U is bounded by $c_0 m_k^2$, the inequalities in (7-49) and (7-50) imply that

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge |\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0 m_k^2|\mathfrak{q}|^2 \tag{7-51}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$. Meanwhile, if U is a ball from $\mathfrak{U}_{k+}$, then Lemma 7.3 finds $|\nabla_A\psi| \le c_0 m_k^{-1}\mathrm{r}^{1/2}$ and $|B_A| \le c_0 m_k^{-1}\mathrm{r}$. This being the case, then (7-49) and (7-50) find

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge \tfrac{1}{2}|\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0 m_k^{-1}\mathrm{r}|\mathfrak{q}|^2 \tag{7-52}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Suppose next that U is a ball from $\mathfrak{U}_N$. What is said in Lemma 7.3 implies that $|B_A| \le c_0\mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}$ and $|\nabla_A\psi| \le c_0\mathrm{r}^{1/6}$ on U, so (7-49) and (7-50) lead to the inequality

$$|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle \ge \tfrac{1}{2}|\nabla_0\mathfrak{q}|^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\,|\mathfrak{q}|^2 - c_0\mathrm{r}^{2/3}(\ln \mathrm{r})^{c_0}|\mathfrak{q}|^2 \tag{7-53}$$

for all $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$.

Part 9 This part of the subsection specifies c and $\{\rho_k\}_{1\le k\le N-1}$ so as to satisfy (7-46). To this end, suppose that $k \in \{1,\dots,N-1\}$. Suppose that U is from $\mathfrak{U}_{k-}$ or $\mathfrak{U}_{k0}$. If $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$ is L^2–orthogonal to the subspace V_U, then Lemma 7.11 and (7-51) find

$$\int_U (|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle) \ge \bigl(c_0^{-1}\rho_k^{-2} - c_0 m_k^2 + 2c_\diamond \sup_U |\mathcal{Q}^-|\bigr)\int_U |\mathfrak{q}|^2. \tag{7-54}$$

It follows as a consequence that (7-46) holds if $\rho_k^{-2} \ge c_0(T^2 + m_k^2)$ and this is so if $c > c_0$. Suppose next that U is from $\mathfrak{U}_{k+}$ and that $\mathfrak{q} \in C^\infty(U; iT^*M \oplus \mathbb{S} \oplus V)$ is L^2–orthogonal to V_U. Lemma 7.11 and (7-52) imply that

$$\int_U (|\nabla_A\mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+\mathfrak{q}\rangle) \ge \bigl((c_0^{-1}c^2 - c_0)m_k^{-1}\mathrm{r} + 2c_\diamond \sup_U |\mathcal{Q}^-|\bigr)\int_U |\mathfrak{q}|^2 \tag{7-55}$$

if q is L^2–orthogonal to V_U. Thus (7-46) holds if $c \geq c_0(1 + m_k \mathrm{r}^{-1} T^2)$; and in particular, (7-46) holds for $c > c_0$ if the eigenvalue bound T is less than $\mathrm{r}^{1/6}(\ln \mathrm{r})^{-c_0}$.

The last case to consider is that where U comes from $\mathfrak{U}_N$. Lemma 7.11 and (7-53) imply for such U that

$$(7\text{-}56)\quad \int_U (|\nabla_A \mathfrak{q}|^2 + \langle \mathfrak{q}, \mathcal{Q}^+ \mathfrak{q}\rangle) \geq \big(c_0^{-1}(\ln \mathrm{r})^{2c} - (\ln \mathrm{r})^{c_0}\mathrm{r}^{2/3} + 2c_\diamond \sup_U |\mathcal{Q}^-|\big) \int_U |\mathfrak{q}|^2$$

if $\mathfrak{q}$ is L^2–orthogonal to V. It follows as a consequence that (7-46) holds for such U if both $c > c_0$ and the eigenvalue bound T is less than $\mathrm{r}^{1/3}$.

Granted all of the above, and given that $T < \mathrm{r}^{1/6}(\ln \mathrm{r})^{-c}$, then (7-46) holds for all sets from $\mathfrak{U}$ if $c > c_0$. This understood, choose c to be twice this lower bound.

Part 10 The dimension of each $U \in \mathfrak{U}$ version of V_U is bounded by c_0, and so it follows from what is said at the end of Part 5 that $\mathfrak{n}_T$ is no greater than c_0 times the number of sets in the collection $\mathfrak{U}$.

An upper bound for size of $\mathfrak{U}$ is obtained by summing upper bounds for the sizes of the various $k \in \{1, \dots, N\}$ versions of $\mathfrak{U}_k$. Let N_T denote the largest value of k such that $T > m_k$ and suppose first that $k \in \{1, \dots, N_T\}$. It follows from what is said in Part 6 that $\mathfrak{U}_{k-}$ contains no more than $c_0 T^3 m_k^{-3}$ sets. Meanwhile, $\mathfrak{U}_{k0}$ and $\mathfrak{U}_{k+}$ together contain at most $c_0 m_k^{-3/2}\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. Thus $\bigcup_{1 \leq k \leq N_T} \mathfrak{U}_k$ contains at most $c_0(T^3 + \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0})$ balls. Suppose next that $k \in \{N_T + 1, \dots, N-1\}$. In this case, $\mathfrak{U}_{k-}$ has at most c_0 balls while $\mathfrak{U}_{k0}$ and $\mathfrak{U}_{k+}$ again have at most $c_0 m_k^{-3/2}\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. Thus, $\bigcup_{N_T < k \leq N-1} \mathfrak{U}_k$ contains at most $c_0 T^{-3/2}\mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$ balls. As noted in Part 6, the set $\mathfrak{U}_N$ has at most $c_0(\ln \mathrm{r})^{c_0}$ balls.

Given that $T \leq c_0 \mathrm{r}^{1/6}(\ln \mathrm{r})^{c_0}$, the bounds just stated imply that $\mathfrak{n}_T \leq c_0 \mathrm{r}^{1/2}(\ln \mathrm{r})^{c_0}$. Thus, (7-43) bounds the spectral flow along the families $\{\mathcal{L}_{2,s}\}_{s \in [0,1]}$ and $\{\mathcal{L}_{3,s}\}_{s \in [0,1]}$ by $c_0 T^{-1}\mathrm{r}(\ln \mathrm{r})^{c_0}$. This understood, take $T = \mathrm{r}^{1/7}(\ln \mathrm{r})^{c_0}$ to obtain the bound in (7-39). □

7.8 The proof of Proposition 3.7

If Y_Z has a single component, then the function $\mathfrak{f}_s$ is defined in Section 7.6. Proposition 3.7's assertion in this case is implied directly by Lemma 7.9's fourth bullet.

Suppose now that Y_Z has more than 1 component. To define $\mathfrak{f}_s$ in this case, introduce $\mathcal{Y}$ to denote the set of components of Y_Z. The space $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$ can be written as $\prod_{Y'\in\mathcal{Y}}(\mathrm{Conn}(E|_{Y'})\times C^\infty(Y';\mathbb{S}|_{Y'}))$. Section 7.6 defines any given $Y'\in\mathcal{Y}$ version of $\mathfrak{f}_s$ on $\mathrm{Conn}(E|_{Y'})\times C^\infty(Y';\mathbb{S}|_{Y'})$. Denote the latter by $\mathfrak{f}_{s;Y'}$. Set

$$\mathfrak{f}_s=\sum_{Y'\in\mathcal{Y}}\mathfrak{f}_{s;Y'}.$$

Each $Y'\in\mathcal{Y}$ has its version of the function $\mathfrak{a}$ on $\mathrm{Conn}(E|_{\mathcal{Y}'})\times C^\infty(Y';\mathbb{S}|_{Y'})$. Use $\mathfrak{a}_{Y'}$ to denote the latter. Then $\mathfrak{a}^{\mathfrak{f}}=\sum_{Y'}(\mathfrak{a}_{Y'}+2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'})$. This understood, it is enough to bound $|\mathfrak{a}_{Y'}+2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'}|$ for each $Y'\in\mathcal{Y}$. Lemma 7.9 supplies a suitable bound when $c_1(\det(\mathbb{S}|_{Y'}))$ is not torsion. This understood, suppose $Y'\in\mathcal{Y}$ and $c_1(\det(\mathbb{S})|_{Y'})$ is torsion. Thus, $w=0$ on Y'.

Write ψ on Y' as $\mathrm{r}^{-1/2}\lambda$ to see that the set of solutions to (2-5) on Y' is r–independent. It follows as a consequence of what is said in Chapter 5 of [17] that the space of $C^\infty(Y';S^1)$–orbits of solutions to (2-5) on Y' is compact. Hold on to this fact for the moment. Write ψ in the Y' version of (7-36) as $\mathrm{r}^{1/2}\lambda$ and write the sections b and ϕ as $(\mathrm{r}z)^{1/2}b'$ and $(\mathrm{r}z)^{1/2}\phi'$ to see that the spectrum of the operator in (7-36) depends neither on r nor z. What was just said about compactness and what was just said about the spectrum implies directly that $|\mathfrak{a}_{Y'}+2\pi(\mathrm{r}-\pi)\mathfrak{f}_{s;Y'}|\le c_0$.

8 Cobordisms and the Seiberg–Witten equations

This section proves Propositions 3.5 and 3.8. Here is an outline of what is to come.

Section 8.1 This section states three key lemmas (Lemmas 8.1–8.3) that are used in Section 8.2 to prove Proposition 3.5. These are used subsequently also. These lemmas establish a priori estimates on the norms of ψ and $\nabla_A\psi$ and the curvature F_A when (A,ψ) is an instanton solution to (2-10) and r is large. Lemmas 8.1–8.3 are proved in subsequent subsections of Section 8.

Section 8.2 This section uses the lemmas in Section 8.1 to prove Proposition 3.5.

Section 8.3 This section ties up a loose end by giving the proof of Lemma 8.1 from Section 8.1.

Section 8.4 This section ties up a loose end by giving the proof of Lemma 8.2 from Section 8.1.

Section 8.5 This section gives the proof of Lemma 8.3 from Section 8.1 modulo Lemma 8.5, which is an assertion about the behavior of ψ on certain domains in a cobordism.

Section 8.6 This section proves Lemma 8.5.

Section 8.7 This section uses the results in the previous sections of Section 8 and the results from Section 7 to prove Proposition 3.8.

8.1 The three key lemmas

The three parts of this subsection supply three lemmas that assert pointwise bounds for ψ, the curvature of A and for the covariant derivative of ψ. These bounds are used in the next subsection to prove Proposition 3.5. All three lemmas assume implicitly that the conditions in Section 3.3 are satisfied. Additional assumptions are stated when needed.

Part 1 The first lemma starts the story with a pointwise bound for $|\psi|$ and L^2–bounds on F_A and the covariant derivatives of ψ. With regards to notation, this lemma uses $(\nabla_{\mathbb{A}}\psi)_s$ to denote the section of $\mathbb{S}^+$ over the $|s| > 1$ part of X that gives the pairing between $\nabla_{\mathbb{A}}\psi$ and the vector field $\frac{\partial}{\partial s}$.

Lemma 8.1 *There exists $\kappa > 1$ such that given any $c \geq \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \geq \kappa_c$. If X is not the product cobordism, assume that the metric obeys* (2-9) *with $L \leq c$, that the norm of the Riemann curvature is bounded by $\mathrm{r}^{1/c}$ and that the norm of w_X is bounded by c. Fix μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1 and use this data to define the equations in* (2-10). *Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to these equations. Then $|\psi| \leq \kappa_c$. If X is not the product cobordism, assume in addition that the volume of the s–inverse image of any length 1 interval is bounded by c and that the metric's injectivity radius is greater than $\mathrm{r}^{-1/c}$. Also assume in this case that $L_{\mathrm{tor}} \leq c\mathrm{r}$ and that w_X obeys* (2-12) *plus item* (c) *of the fourth bullet of* (3-15). *Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote the respective $s \to -\infty$ and $s \to \infty$ limits of $\mathfrak{d}$ and suppose that $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq c\mathrm{r}^2$. Then*:

- *The L^2–norms of $\left|F_{\mathbb{A}}\left(\frac{\partial}{\partial s}, \cdot\right)\right|$ and $\mathrm{r}^{1/2}|(\nabla_{\mathbb{A}}\psi)_s|$ on the $|s| \geq L$ part of X are less than $\kappa_c\mathrm{r}$.*
- *The L^2–norms of $F_{\mathbb{A}}$ and $\mathrm{r}^{1/2}\nabla_{\mathbb{A}}\psi$ on the s–inverse image of any length 1 interval in $\mathbb{R}$ are no greater than $\kappa_c\mathrm{r}$.*

This lemma is proved in Section 8.3.

Part 2 The next lemma supplies a refined set of bounds for $|\alpha|$ and its covariant derivatives on U_C and U_0. This lemma and the subsequent lemma implicitly write $\mathbb{S}^+$

on U_C and U_0 as $E \oplus (E \otimes K^{-1})$. Having done so, they then write ψ with respect to this splitting as (α, β); and they write the connection $\mathbb{A}$ as $\mathbb{A} = A_K + 2A$ with A being a connection on E.

The notation in these upcoming lemmas refers to the complex structure on U_C and U_0 that is defined using the metric and the compatible symplectic form $ds \wedge *w + w$. The $(1,0)$–part of the complexified cotangent space for this complex structure is the direct sum of the span of $ds + i *w$ and dz on U_C and it is the direct sum of the span of $ds + i *w$ and the $(1,0)$–part of the tangent space to the constant-(s,u) spheres in U_0 with the complex structure on S^2 being the standard one. These lemmas write $\nabla_A \alpha$ with respect to the $(1,0)$– and $(0,1)$–splitting of the complexified cotangent bundle as $\partial_A \alpha + \bar\partial_A \alpha$ with $\partial_A \alpha$ denoting the $(1,0)$–part of $\nabla_A \alpha$ and with $\bar\partial_A \alpha$ denoting the $(0,1)$–part. The next two lemmas also introduce ρ_D to denote the diameter of the cross-sectional disk D that is used to define U_C.

Lemma 8.2 *There exists $\kappa > 100(1 + \rho_D^{-1})$ such that given any $c \geq \kappa$, there exists $\kappa_c \geq \kappa$ with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric obeys* (2-9), (3-14) *and the $(c, r = \mathrm{r})$ versions of the conditions in the first two bullets of* (3-15). *Assume that $|w_X| \leq c$ and that w_X obeys* (3-13). *Fix elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1. Assume in addition that their norms and those of their derivatives to order 10 on U_γ and $\mathcal{H}_0$ are bounded by $e^{-\mathrm{r}^2}$. Use this data to define the equations in* (2-10). *Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote respective solutions to the (r, μ_-) version of* (2-5) *on Y_- and the (r, μ_+) version of* (2-5) *with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq c\mathrm{r}^2$, and suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to* (2-10) *with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. If p is a point in one of the domains U_C or U_0 with distance greater than $\kappa^2 \mathrm{r}^{-1/2} (\ln \mathrm{r})^2$ from the domain's boundary, then the following holds at p:*

- $|\beta|^2 \leq e^{-\sqrt{\mathrm{r}}/\kappa^2}$ *and* $|\alpha|^2 \leq 1 + e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- $|\nabla_A \beta| + |\nabla_A \nabla_A \beta| \leq e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- $|\bar\partial_A \alpha| \leq e^{-\sqrt{\mathrm{r}}/\kappa^2}$.
- *If $|\alpha|^2 \in (\kappa^{-1}, 1 - \kappa^{-1})$ at p, then either $|\nabla_A \alpha|^2 \geq \kappa^{-3} \mathrm{r}$ at p or the Hessian $\nabla d |\alpha|^2$ at p has an eigenvalue with absolute value greater than $\kappa^{-3} \mathrm{r}$.*
- $|\nabla_A \alpha| + \mathrm{r}^{-1/2} |\nabla_A (\nabla_A \alpha)| \leq \kappa \mathrm{r}^{1/2}$ *if $|F_A| \leq c\mathrm{r}$ on the radius $\kappa \mathrm{r}^{-1/2}$–ball centered at p.*

This lemma is proved in Section 8.4.

Part 3 The final lemma here writes F_A on U_C and U_0 as $F_A = ds \wedge \mathcal{E}_A + *\mathcal{B}_A$ with $\mathcal{E}_A$ and $\mathcal{B}_A$ denoting s–dependent, $i\mathbb{R}$ valued 1–forms on either $\mathbb{R}/(\ell_\gamma\mathbb{Z}) \times D$ or $\mathcal{H}_0$ as the case may be. These 1–forms are written as

$$\text{(8-1)} \qquad \begin{cases} \mathcal{E}_A = -i(1-\sigma)(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A)\,dt + \mathfrak{r} + \mathfrak{X}, \\ \mathcal{B}_A = -i\sigma(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_B)\,dt + \mathfrak{r} - \mathfrak{X}, \end{cases}$$

where σ, $\mathfrak{z}_A$ and $\mathfrak{z}_B$ are functions, and where both $\mathfrak{r}$ and $\mathfrak{X}$ annihilate the vector field $\frac{\partial}{\partial t}$. Note that $\mathcal{E}_A + \mathcal{B}_A = -i\big(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A + \mathfrak{z}_B\big)\,ds + 2\mathfrak{r}$, which means that $\mathfrak{r}$ and the combination $\mathfrak{z}_A + \mathfrak{z}_B$ contain the terms with β that appear in the leftmost equation of (2-10).

Lemma 8.3 *There exists $\kappa > \pi$ such that given any $c \geq \kappa$, there exists $\kappa_c > 200(1+\rho_D^{-1})$ with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$–compatible. Fix elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1. Assume in addition that their norms and those of their derivatives up to order 10 on U_γ and $\mathcal{H}_0$ are bounded by $e^{-\mathrm{r}^2}$. Use all of these data to define the equations in (2-10). Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote the respective solutions to the (r, μ_-) version of (2-5) on Y_- and the (r, μ_+) version of (2-5) on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq \mathrm{r}^{2-1/c}$. Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is an instanton solution to (2-10) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Let p denote a point in either one of the domains U_C or U_0 with distance κ^{-1} or more from the domain's boundary. Then the following are true at p:*

- $-\mathrm{r}^{-100} < 1 - \sigma < 1 + \mathrm{r}^{-100}$.
- $|\mathfrak{z}_A| + |\mathfrak{z}_B| \leq \mathrm{r}^{-100}$.
- $|\mathfrak{r}| \leq \kappa\mathrm{r}^{-100}$.
- $|\mathfrak{X}|^2 \leq 2\mathrm{r}^2\sigma(1-\sigma)(1-|\alpha|^2) + \kappa\mathrm{r}^{-100}$.
- $|\nabla\mathcal{E}_A| + |\nabla\mathcal{B}_A| \leq \kappa\mathrm{r}^{3/2}$.

Lemma 8.3 is proved in Section 8.5 modulo a key lemma which is proved in Section 8.6.

8.2 Proof of Proposition 3.5

This part of the subsection uses what is said in Lemmas 8.1–8.3 to prove Proposition 3.5. The argument assumes that the integral of $iF_{\hat{A}}$ over C is negative so as to derive nonsense. This is done in the eight parts that follow. Before starting, note that the assumptions in this proposition allow Lemmas 8.1 and 8.3 to be invoked, and the conclusions of Lemma 8.3 imply in particular that Lemma 8.2 can be invoked as well.

Part 1 This first part of the proof sets the stage for what is to come by supplying two observations about the pullback of $iF_{\hat{A}}$ to C. What follows is the first observation:

(8-2) $$\text{The integral of } \tfrac{i}{2\pi}F_{\hat{A}} \text{ over } C \text{ is an integer.}$$

This follows from Lemma 7.6 since the latter implies that $\hat{A}$ is flat and $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant where $|s| \gg 1$ on C.

The second observation concerns the function F on C that is defined by writing the pullback to C of $iF_{\hat{A}}$ as $\mathrm{F}\, ds \wedge dt$:

(8-3) $$\text{The function } \mathrm{F} \text{ is nearly nonnegative in the sense that } \mathrm{F} \geq -c_0 \mathrm{r}^{-100}.$$

This follows directly from the formula given below for F using the second bullet of Lemma 8.2 and the first and second bullets of Lemma 8.3. The upcoming formula for F uses $(\partial_A\alpha)_0$ to denote the $ds + i*w$ component of $\partial_A\alpha$ and use $(\bar\partial_A\alpha)_0$ to denote the $ds - i*w$ component of $\bar\partial_A\alpha$. Here is the promised formula for F:

(8-4) $$\mathrm{F} = (1-\wp)(1-\sigma)\big(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A\big) + \wp'\big(|(\partial_A\alpha)_0|^2 - |(\bar\partial_A\alpha)_0|^2\big).$$

This formula follows directly from (3-9) and (8-1).

Part 2 Let $\mathbb{I} \subset \mathbb{R}$ denote the set characterized as follows: a point s is in $\mathbb{I}$ if the integral of F over the slice $\{s\} \times \gamma$ in C is negative. The following assertion is a direct consequence of (8-2) and (8-3):

(8-5) $$\text{If } \textstyle\int_C iF_{\hat{A}} < 0 \text{ then the measure of the set } \mathbb{I} \text{ is greater than } c_0^{-1}\mathrm{r}^{100}.$$

Granted (8-5), there are at least $c_0^{-1}\mathrm{r}^{100}$ disjoint open intervals of length 1 in $\mathbb{R}$ with center point in $\mathbb{I}$. This understood, use the first bullet of Lemma 8.1 to find an interval $I \subset \mathbb{R}$ of length 1 with center point in $\mathbb{I}$, with $|s| > L+2$ and such that

(8-6) $$\int_{I\times Y}\Big(\Big|F_A\Big(\frac{\partial}{\partial s},\cdot\Big)\Big|^2 + \mathrm{r}|(\nabla_A\psi)_s|^2\Big) < \mathrm{r}^{-97}.$$

This inequality enters the story in Parts 3 and 7.

Part 3 Supposing that $I \subset \mathbb{R}$ is given by Part 2, let s denote its center point, this being a point for which the integral of F over $\{s\}\times\gamma$ is negative. This part proves that $|\alpha|^2 \leq \frac{5}{8}$ on $\{s\}\times\gamma$. To see why this is true, suppose for the sake of argument that this condition is violated at $p \in \{s\}\times\gamma$. Since the integral of F on $\{s\}\times\gamma$ is negative, there must be some point where the function $\wp$ is less than 1 and thus $|\alpha|^2 \leq \frac{9}{16}$. As a consequence, the variation of $|\alpha|$ on $\{s\}\times\gamma$ must be greater than c_0^{-1}. As explained next, this variation is in fact no greater than $c_0\mathrm{r}^{-15}$ if $\mathrm{r} \ge c_0$. To start the explanation, suppose that $\varepsilon > 0$ and that there are points on $\{s\}\times\gamma$ with their respective values of $|\alpha|$ differing by more than ε. Let $\frac{\partial}{\partial t}$ denote the unit-length tangent vector to $\{s\}\times\gamma$ and let $(\nabla_A\alpha)_t$ denote the directional covariant derivative of α along $\frac{\partial}{\partial t}$. It follows as a consequence of the fundamental theorem of calculus that there is a point $q \in \{s\}\times\gamma$ where $|(\nabla_A\alpha)_t| > c_0^{-1}\varepsilon$. Let $(\nabla_A\alpha)_s$ denote the directional covariant derivative of α along the vector field $\frac{\partial}{\partial s}$. Granted this lower bound for $|(\nabla_A\alpha)_t|$ at q, then the inequality in the third bullet of Lemma 8.2 requires that $|(\nabla_A\alpha)_s| \ge c_0^{-1}\varepsilon$ at q also if ε is greater than $c_0e^{-\sqrt{\mathrm{r}}/c_0}$. Assuming $\mathrm{r} \ge c_0$, then this will be the case when $\varepsilon > \mathrm{r}^{-15}$. The $c_0^{-1}\varepsilon$ lower bound for $|(\nabla_A\alpha)_s|$ at q, what is said by the fifth bullet of Lemma 8.2 and what is said by Lemma 8.3 imply that $|(\nabla_A\alpha)_s| \ge c_0^{-1}\varepsilon$ in the ball in U_C of radius $c_0^{-1}\varepsilon\mathrm{r}^{-1}$ centered at q. The latter bound implies in turn that the integral of $|(\nabla_A\alpha)_s|^2$ on this same ball is greater than $c_0^{-1}\varepsilon^6\mathrm{r}^{-4}$, which is nonsense if $\varepsilon > \mathrm{r}^{-15}$ because it runs afoul of what is said in (8-6).

Part 4 Let I and $s \in I$ be as in Part 3. Keep in mind that the integral of F over $\{s\}\times\gamma$ is negative. As will be explained in a moment, the lower bound in (8-3) for F leads to the following observation:

(8-7) The variation of $\wp$ over $\{s\}\times\gamma$ is no greater than $c_0\mathrm{r}^{-50}$.

To prove this, first use the fundamental theorem of calculus to see that

$$\sup_{\{s\}\times\gamma}\wp - \inf_{\{s\}\times\gamma}\wp \le c_0\biggl(\int_{\{s\}\times\gamma}\wp'|(\nabla_A\alpha)_t|^2\biggr)^{1/2}. \tag{8-8}$$

The bound in (8-7) follows from (8-8) using the lower bound for F and the third bullet of Lemma 8.2.

Part 5 This part uses the conclusions of Part 3 to deduce the following:

(8-9) The function σ on the $|u| < 1$ part of $\{s\}\times(\gamma\cap\mathcal{H}_0)$ obeys $\sigma < c_0\mathrm{r}^{-33}$.

To see why this is the case, let (s,p) denote a given point in the $|u| < 1$ part of $\{s\}\times(\gamma\cap\mathcal{H}_0)$ where $\sigma > 0$. Let S denote the cross-sectional sphere in $\mathcal{H}_0$ that contains p. Use (3-9) to write the pullback of F_A to S as $\frac{1}{2}\mathrm{B}\,dz\wedge d\bar{z}$ with $\mathrm{B} = \sigma\bigl(\mathrm{r}(1-|\alpha|^2)+\mathfrak{z}_B\bigr)$. Use ε to denote the value of σ at (s,p). Invoke the first and second bullets of Lemma 8.3 to conclude $\bigl($using what is said in Part 3 to the effect that $|\alpha|^2 \le \frac{5}{8}\bigr)$ that value of B at (s,p) is greater than $\frac{3}{8}\mathrm{r}\varepsilon - c_0\mathrm{r}^{-100}$. The fifth bullet of Lemma 8.3 finds that $\mathrm{B} > c_0^{-1}\mathrm{r}\varepsilon$ on the radius $c_0^{-1}\mathrm{r}^{1/2}\varepsilon$ disk in the cross-sectional sphere $\{s\} \times S$ with center at (s, p). Meanwhile, the first bullets of Lemma 8.3 and Lemma 8.2 imply that $\mathrm{B} > -c_0\mathrm{r}^{-99}$ on the whole of $\{s\} \times S$, and so the integral of B over $\{s\} \times S$ is no less than $c_0^{-1}\varepsilon^3 - c_0\mathrm{r}^{-99}$. This integral must be zero because the first Chern class of E has zero pairing with the cross-sectional spheres in $\mathcal{H}_0$. Thus $\varepsilon \le c_0\mathrm{r}^{-33}$.

Part 6 What is said in Part 5 implies that $(1 - \wp) < c_0\mathrm{r}^{-50}$ on $\{s\} \times \gamma$. Indeed, if this bound is violated, then it follows from (8-7) and the formula for F in (8-4) that the integral of F over the $|u| < 1$ part of $\{s\} \times \gamma$ is greater than $c_0\mathrm{r}^{-49}$. Given the lower bound in (8-3), this last lower bound runs afoul of the assumption that F's integral over $\{s\} \times \gamma$ is negative. The small size of $1 - \wp$ implies in particular that $|\alpha|^2 > \frac{3}{8}$ on $\{s\} \times \gamma$.

Part 7 Granted the conclusions of Parts 5 and 6, then the fourth bullet of Lemma 8.2 asserts that one or the other of the following are true at each point in the $|u| < 1$ part of $\{s\} \times (\gamma \cap \mathcal{H}_0)$: either $|\nabla_A\alpha|^2 \ge c_0^{-1}\mathrm{r}$ or the Hessian matrix $\nabla d|\alpha|^2$ has an eigenvalue with absolute value greater than $c_0^{-1}\mathrm{r}$. As explained next, this has the following consequence:

(8-10) Let $(\partial_A\alpha)_1$ denote the component of $\partial_A\alpha$ that annihilates both $\frac{\partial}{\partial s}$ and the kernel of w. Then $|(\partial_A\alpha)_1|^2$ is greater than $c_0^{-1}\mathrm{r}^{1/2}$ at all points in a radius $c_0^{-1}\mathrm{r}^{-1/2}$ ball with center at distance less than $c_0\mathrm{r}^{-1/2}$ from each point in the $|u| < 1$ part of $\{s\} \times (\gamma \cap \mathcal{H}_0)$.

To prove this, suppose first that $|\nabla_A\alpha|^2 \ge c_0^{-1}\mathrm{r}$ at a given point. Use the third bullet of Lemma 8.2 to see that one or both of $|(\partial_A\alpha)_1|^2$ and $|(\partial_A\alpha)_0|^2$ are greater than $c_0^{-1}\mathrm{r}$. In the latter case, the third bullet of Lemma 8.2 implies that $|(\nabla_A\alpha)_s|^2$ is greater than $c_0^{-1}\mathrm{r}$ at the point, and the second derivative bound from the fifth bullet of Lemma 8.2 implies that $|(\nabla_A\alpha)_s|^2 \ge c_0^{-1}\mathrm{r}$ at all points in a radius $c_0^{-1}\mathrm{r}^{-1/2}$ ball centered on this point. This being the case, the integral of $|(\nabla_A\alpha)_s|^2$ over this ball is greater than $c_0^{-1}\mathrm{r}^{-1}$ and this violates (8-6). Granted that $|(\nabla_A\alpha)_1|^2 \ge c_0^{-1}\mathrm{r}$ at the given point, then the second derivative bound from the fifth bullet of Lemma 8.2 implies what is asserted by (8-10).

Now suppose that the Hessian matrix $\nabla d|\alpha|^2$ at the given point has an eigenvalue that is greater than $c_0^{-1}\mathrm{r}$. Let v denote a unit-length eigenvector at the point with such an eigenvalue. As will be explained directly, this vector must be such that

$|ds(v)| + |dt(v)| < \frac{1}{100}$. To see why this is the case, suppose to the contrary that the latter bound is violated at a given point. It then follows from the first and fifth bullets of Lemma 8.2 that $|(\nabla_A\alpha)_s| \ge c_0^{-1}\mathrm{r}^{1/2}$ at all points in some ball of radius $c_0^{-1}\mathrm{r}^{-1/2}$ whose center has distance at most $c_0\mathrm{r}^{1/2}$ from the given point. This implies in particular that the integral of $|(\nabla_A\alpha)_s|^2$ over this same ball is no less than $c_0\mathrm{r}^{-1}$. But this is nonsense as it runs afoul of (8-6).

The fact that v is a unit-length vector implies that $|dz(v)| > \frac{1}{2}$. Use this lower bound for $|dz(v)|$ with the third bullet of Lemma 8.2 and the second derivative bounds from the fifth bullet of Lemma 8.2 to see that $|(\partial_A\alpha)_1|^2 \ge c_0^{-1}\mathrm{r}^{1/2}$ at all points in a ball of radius $c_0^{-1}\mathrm{r}^{-1/2}$ whose center point has distance at most $c_0\mathrm{r}^{-1/2}$ from the given point.

Part 8 Introduce the connection $\hat{A}_1$ on E's restriction to $I \times \mathcal{H}_0$ that is obtained from (A, α) by the formula $\hat{A}_1 = A - \frac{1}{2}(\bar{\alpha}\nabla_A\alpha - \alpha\nabla_A\bar{\alpha})$. The curvature 2–form of $\hat{A}_1$ is

$$F_{\hat{A}_1} = (1-|\alpha|^2)F_A + \nabla_A\alpha \wedge \nabla_A\bar{\alpha}. \tag{8-11}$$

Let (s', p') denote the center point of a ball that is described by (8-10). Introduce $S \subset \mathcal{H}_0$ to denote the cross-sectional sphere that contains the point p'. Use (3-9) to write the pullback of the curvature of $\hat{A}_1$ to $\{s\} \times S$ as $\mathrm{B}_1\, dz \wedge d\bar{z}$ with B_1 given by

$$\mathrm{B}_1 = \sigma(1-|\alpha|^2)\big(\mathrm{r}(1-|\alpha|^2) + \mathfrak{z}_A\big) + |(\partial_A\alpha)_1|^2 - |(\bar{\partial}_A\alpha)_1|^2, \tag{8-12}$$

with $(\bar{\partial}_A\alpha)_1$ denoting here the $d\bar{z}$ component of $\bar{\partial}_A\alpha$. The function B_1 is also very nearly nonnegative in the sense that $\mathrm{B}_1 \ge -c_0\mathrm{r}^{-100}$, this being a consequence of what is said in the first and third bullets of Lemma 8.2 and the first and second bullets of Lemma 8.3. This understood, then it follows from (8-10) and this lower bound for B_1 that the integral of B_1 over $\{s'\} \times S$ is positive. But this is nonsense because the latter integral computes 2π times the pairing of the first Chern class of E with the homology class defined by S, and this pairing is equal to zero. □

8.3 Proof of Lemma 8.1

The bounds in the lemma constitute a particular case of bounds that are used in Chapter 24 of [17]. As most of the machinery behind what is done in [17] is not needed for the proofs, the argument for Lemma 8.1 is presented in a moment. What follows directly lays a convention that is invoked implicitly in the arguments for Lemma 8.1 and in some of the subsequent lemmas.

If X is the product $\mathbb{R} \times Y_Z$, the bundles E and K^{-1} over Y_Z pull back via the projection to define bundles over X; their connections A_E and A_K likewise pull back to define connections on these bundles. The bundle $\det \mathbb{S}^+$ is isomorphic to $E^2 \otimes K^{-1}$ and thus to the pullback of $\det \mathbb{S}$. Fix once and for all an isometric isomorphism.

Suppose now that X is not a product. Use the embedding in the second bullet of (2-8) to identify the $s \le -1$ part of X with $(-\infty, -1] \times Y_-$, and then use the projection to Y_- to view the Y_- version of the bundle $\mathbb{S}$ as a bundle over the $s \le -1$ part of X. The bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ are isometrically isomorphic to $\mathbb{S}$ via an isomorphism that covers the isomorphisms between both Λ^+ and Λ^- and T^*Y given by the interior product with $\frac{\partial}{\partial s}$. Fix such an isomorphism once and for all. This induces a Hermitian isomorphism between the bundle $\det \mathbb{S}^+$ over the $s < -1$ part of X and the Y_- version $\det \mathbb{S}$. Fix once and for all an isometric isomorphism between these bundles. Use this isomorphism with the pullback via the composition of the embedding from (2-8)'s second bullet and the projection to Y_- to view $A_K + 2A_E$ as a Hermitian connection on the $s \le -1$ part of $\det \mathbb{S}^+$. The analogous constructions can be made on the $s > 1$ part of X using the Y_+ version of $\mathbb{S}$ and so define an incarnation of the Y_+ version of $A_K + 2A_E$ as a Hermitian connection on $\det \mathbb{S}^+$.

Suppose for the moment that $\mathbb{A}$ is a given Hermitian connection on $\det \mathbb{S}^+ \to X$. If X is the product $\mathbb{R} \times Y_Z$, then $\mathbb{A}$ can be written as $A_K + 2A$ with A now a connection on the bundle $E \to X$. There is a map $\hat{u} : X \to S^1$ such that $A - \hat{u}^{-1} d\hat{u} = A_E + \hat{a}_A$, where $\hat{a}_A$ annihilates the vector field $\frac{\partial}{\partial s}$. If X is not the product, then $\mathbb{A}$ can be written as $A_K + 2A$ on the $s \le -1$ and $s \ge 1$ parts of X with A being a connection on the incarnation of E over the relevant part of X In this case, there exists a map $\hat{u}$ as just described but with domain the $s \le -1$ part of X, and likewise there exists such a map with domain the $s \ge 1$ part of X.

The map $\hat{u}$ in the case when $X = \mathbb{R} \times Y_Z$ is unique up to multiplication by an $s-$independent map from Y_Z to S^1, and in the other cases, it is unique up to a map from the either the $s \le -1$ or $s \ge 1$ part of X whose differential annihilates $\frac{\partial}{\partial s}$. The convention in each case is to take a map $\hat{u}$ whose restrictions to the constant s slices of its domain are homotopic to the constant map to S^1.

The connection $A_* = A - \hat{u}^{-1} d\hat{u}$ can be viewed as a map from $\mathbb{R}$ or $(-\infty, -1]$ or $[1, \infty)$ to $\mathrm{Conn}(E|_{Y_*})$ with Y_* either Y_Z or Y_- or Y_+ as the case may be. If ψ is a given section over X of $\mathbb{S}$, then $\psi_* = \hat{u}\psi$ can likewise be viewed as a map from $\mathbb{R}$ or $(-\infty, -1]$ or $[1, \infty)$ to $C^\infty(Y_*; \mathbb{S}|_{Y_*})$. When viewed in this light, the equations in (2-10) can be written as equations for (A_*, ψ_*) on the whole of X when X is the product cobordism, and on the $s \le -L$ and $s \ge L$ parts of X when X is not the product cobordism. These equations are

$$\text{(8-13)} \qquad \begin{cases} \frac{\partial}{\partial s} A_* + B_{A_*} - \mathrm{r}(\psi_*^\dagger \tau \psi_* - i *w_{X_*}) - \frac{1}{2} B_{A_K} - i\, d\mu_* = 0, \\ \frac{\partial}{\partial s}\psi_* + D_{A_*}\psi_* = 0. \end{cases}$$

The notation here uses w_{X*} to denote the 2–form w when X is the product cobordism. When X is not the product cobordism, w_{X*} denotes the s–dependent 2–form that is defined on the relevant constant s slices of X by the pullback of w_X. In particular, $w_{X*} = w$ on the components of the $s \le -L$ and $s > L$ parts of X where $c_1(\det \mathbb{S})$ is not torsion. What is denoted in (8-13) by μ_* is either μ, μ_- or μ_+ as the case may be.

Proof The proof has four steps.

Step 1 The assertion that $|\psi| \le \kappa$ is proved by using the Weitzenböck formula to write $\mathcal{D}_{\mathbb{A}}^- \mathcal{D}_{\mathbb{A}}^+ \psi$ as $\nabla_{\mathbb{A}}^\dagger \nabla_{\mathbb{A}} \psi + \mathrm{cl}(F_{\mathbb{A}}^+)\psi + \frac{1}{4}\mathrm{R}\psi$, where R denotes the scalar curvature of the Riemannian metric. Granted this rewriting, it then follows from (2-10) and from the assumed bound on the norm of Riemann curvature that the function $|\psi|$ obeys the differential inequality $d^\dagger d|\psi| + \mathrm{r}(|\psi|^2 - |w_X| - c_c)|\psi| \le 0$. Use the maximum principle with this last inequality and the large $|s|$ bounds on $|\psi|$ that follow from Lemma 7.1 to see that $|\psi| \le c + c_0$.

Step 2 Let L_* denote either L or L_{tor}. Then use $\mathbb{I} \subset \mathbb{R}$ to denote either $\mathbb{R}$, $(-\infty, -L]$ or $[L, \infty)$. Define Y_* to be Y_Z in the case when $\mathbb{I} = \mathbb{R}$. When $\mathbb{I} = (-\infty, L_*]$ or $[L_*, \infty)$ and $L_* = L$, define Y_* to be the union of the components of the constant $s \in \mathbb{I}$ slices of X where $c_1(\det \mathbb{S})$ is not torsion. In the case when $L_* = L_{\mathrm{tor}}$, define Y_* to be the union of the components of the constant $s \in \mathbb{I}$ slices of X where $c_1(\det \mathbb{S})$ is torsion. Write $\mathbb{A}$ on $\mathbb{I} \times Y_*$ as $A_K + 2A$ and introduce by way of notation $\mathfrak{d}|_s$ to denote the pullback to $\{s\} \times Y_*$ of (A, ψ). Also introduce $\mathfrak{B}_{(A,\psi)}$ to denote

$$\text{(8-14)} \qquad \mathfrak{B}_{(A,\psi)} = B_A - \mathrm{r}(\psi^\dagger \tau \psi - i *w) + i *d\mu_* + \tfrac{1}{2} B_{A_K},$$

with μ_* denoting either μ_- or μ_+ as the case may be. Use D_A in what follows to denote the Dirac operator on Y_* as defined using the connection $A_K + 2A$ for the Spin^c–structure with spinor bundle $\mathbb{S} = \mathbb{S}^+$. Suppose that $s' > s$ are two points in $\mathbb{I}$. Take the L^2–norm of the left-hand expressions in both equations of (8-13) over $[s, s'] \times Y_*$. The square of these norms are zero. This being the case, integration by parts in the square of these L^2–norms results in an identity of the form

$$\frac{1}{2}\int_{[s,s']\times Y_*}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{(A,\psi)}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)=\mathfrak{a}(\mathfrak{d}|_s)-\mathfrak{a}(\mathfrak{d}|_{s'}). \tag{8-15}$$

Taking limits in (8-15) as $s\to-\infty$ or as $s'\to\infty$ as the case may be leads to the identities

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{I}\times Y_*}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{(A,\psi)}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)&=\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{d}|_{s=-L_*}),\\ \frac{1}{2}\int_{\mathbb{I}\times Y_*}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{(A,\psi)}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)&=\mathfrak{a}(\mathfrak{d}|_{s=L_*})-\mathfrak{a}(\mathfrak{c}_+).\end{aligned} \tag{8-16}$$

Note that the identities in (8-15) and (8-16) hold with $\mathfrak{d}=(A_K+2A,\psi)$ on the right-hand side. By way of an explanation, the integration by parts proves the analogs that have $\mathfrak{d}_*=(A_K+2A_*,\psi_*)$ used on the right-hand side, and if they hold using $\mathfrak{d}_*$, then they hold using $\mathfrak{d}$ because the restriction of the map $\hat{u}$ to any slice $\{s\}\times Y_*$ in $\mathbb{I}\times Y_*$ is homotopic to the constant map to S^1.

Step 3 The assertion made by the first bullet of Lemma 8.1 follows directly from (8-16) when the data is such that X is the product cobordism. The proof in the general case and the proof of the second bullet of Lemma 8.1 use an integral version of the Weitzenböck formula for the operator $\mathcal{D}_{\mathbb{A}}^-\mathcal{D}_{\mathbb{A}}^+$. The details follow directly.

Integrate $\big|F_{\mathbb{A}}^+-\mathrm{r}\big(\psi^\dagger\tau\psi-\frac{i}{2}w_X\big)+i\mathfrak{w}_\mu^+\big|^2+2\mathrm{r}|\mathcal{D}_{\mathbb{A}}^+\psi|^2$ over $s^{-1}([-L-3,L+3])$ and denote the result by $\mathcal{I}$. Integrate this same expression over the respective $|s|\in[L,L_*-4]$ and $[L_*-5,L_*+1]$ parts of X_{tor}. Denote these integrals as $\mathcal{I}_{\mathrm{tor}0}$ and $\mathcal{I}_{\mathrm{tor}1}$. In each case, let X_* denote the region of integration and let ∂_-X_* and ∂_+X_* denote the two boundaries of the relevant region of integration with ∂_-X_* at the smaller value of s and ∂_+X_* at the larger value. Use the Weitzenböck formula for $\mathcal{D}_{\mathbb{A}}^-\mathcal{D}_{\mathbb{A}}^+\psi$ from Step 1 with Stokes' theorem to rewrite the identities $\mathcal{I}=0$, $\mathcal{I}_{\mathrm{tor}0}=0$ and $\mathcal{I}_{\mathrm{tor}1}=0$, respectively, as

$$\frac{1}{2}\int_{X_*}(|F_{\mathbb{A}}|^2+\mathrm{r}^2|\psi^\dagger\tau\psi-iw_X|^2+2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2)+\mathfrak{i}_*=\mathfrak{a}(\mathfrak{d}|_{p_-X_*})-\mathfrak{a}(\mathfrak{d}|_{p_+X_*}), \tag{8-17}$$

with $\mathfrak{i}_*$ in the case of $\mathcal{I}$ and $\mathcal{I}_{\mathrm{tor}1}$ denoting a term with absolute value no greater than $c_0c\mathrm{r}\big(\int_{s^{-1}([-L-3,L+3])}(|F_{\mathbb{A}}|^2)\big)^{1/2}+c_0r^{1+c_0/c}$. In the case of $\mathcal{I}_{\mathrm{tor}0}$, the absolute value of $\mathfrak{i}_*$ is no greater than $c_0cL_*\mathrm{r}$. This bound on $|\mathfrak{i}_*|$ in the case of $\mathcal{I}$ and $\mathcal{I}_{\mathrm{tor}0}$ is a direct consequence of the bounds on the norms of the Riemannian curvature tensor and w_X, the size of L, the volume of the s–inverse image of intervals, and the bound $|\psi|^2\le 2c$ from Step 1. In the case of $\mathcal{I}_{\mathrm{tor}0}$, the bound for $|\mathfrak{i}_*|$ is a consequence of the fact that $dw_X = 0$ on the integration domain, this being the assumption made by item (c) of the fourth bullet of (3-15). By way of an explanation, $\mathfrak{i}_*$ in this case can be written as sum of three terms, these denoted by $\mathfrak{i}_\mathfrak{g}$, $\mathfrak{i}_\mu$ and $\mathfrak{i}_K$. The term that is denoted by $\mathfrak{i}_\mathfrak{g}$ gives the contribution of the scalar curvature term in the Weitzenböck formula for $\mathcal{D}_\mathbb{A}^-\mathcal{D}_\mathbb{A}^+$. As such, it is bounded by the integral of $c_0\mathrm{r}|\psi|^2$ over the $|s| \in [L, L_* - 4]$ part of X_{tor}. The bound $|\psi|^2 \le c_0 c$ leads to a bound on $|\mathfrak{i}_\mathfrak{g}|$ by $c_0 c\mathrm{r}L_*$.

The term that is denoted by $\mathfrak{i}_\mu$ comes by writing $\bigl|F_\mathbb{A}^+ - \mathrm{r}\bigl(\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr) + i\mathfrak{w}_\mu^+\bigr|^2$ as the sum of $\bigl|F_\mathbb{A}^+ - \mathrm{r}\bigl(\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr)\bigr|^2$ with terms that involve w_μ^+. One of these terms has the inner product between $F_\mathbb{A}^+$ and $\mathfrak{w}_\mu^+$. Stokes' theorem identifies the integral of the latter with the contributions to the boundary terms on the right-hand side of (8-17) from the $\mathfrak{e}_\mu$ part of the functional $\mathfrak{a}$. The other $\mathfrak{w}_\mu^+$–terms are bounded by the integral over X_* of $c_0\bigl(\mathrm{r}\bigl||\psi|^2 - \bigl|\tfrac{i}{2}w_X\bigr|\bigr||\mathfrak{w}_\mu^+| + |\mathfrak{w}_\mu^+|^2\bigr)$. This understood, the bounds on $|\psi|^2$ and $|w_X|$ lead to a bound on $|\mathfrak{i}_\mu|$ by $c_0 c\mathrm{r}L_*$.

What follows explains how the term $\mathfrak{i}_K$ in $\mathfrak{i}_*$ arises. The $dw_X = 0$ assumption is used to derive a suitable bound on $|i_K|$. As noted above, the derivation starts by writing $\bigl|F_\mathbb{A}^+ - \mathrm{r}\bigl(\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr) + iw_\mu^+\bigr|^2$ as $\bigl|F_\mathbb{A}^+ - \mathrm{r}\bigl(\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr)\bigr|^2$ plus terms that involve $\mathfrak{w}_\mu^+$. The norm $|F_\mathbb{A}^+ - \mathrm{r}(\psi^\dagger\tau\psi - w_X)|^2$ is then written as a sum of $|F_\mathbb{A}^+|^2$, $\mathrm{r}^2\bigl|\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr|^2$ and twice the inner product between $F_\mathbb{A}^+$ and $\mathrm{r}\bigl(\psi^\dagger\tau\psi - \tfrac{i}{2}w_X\bigr)$. The integral over X_* of the term with the inner product between $F_\mathbb{A}^+$ and $\mathrm{r}\psi^\dagger\tau\psi$ is canceled by the contribution from the $F_\mathbb{A}^+$–term in the Weitzenböck formula for $\mathcal{D}_\mathbb{A}^-\mathcal{D}_\mathbb{A}^+\psi$. The inner product between $F_\mathbb{A}^+$ and $-\tfrac{i}{2}\mathrm{r}w_X$ is equal to that of $F_\mathbb{A}$ with $-\tfrac{i}{2}\mathrm{r}w_X$ and thus its integral is that of $\mathrm{r}F_\mathbb{A} \wedge w_X$. Stokes' theorem identifies most of the latter with the contributions to the boundary terms on the right-hand side of (8-17) from the rw–term in $\mathfrak{a}$. The term designated by $\mathfrak{i}_K$ is what remains after the application of Stokes' theorem. To say more about $\mathfrak{i}_K$, note that the application here of Stokes' theorem requires writing $\mathbb{A}$ as $A_K + 2A_E + \hat{a}_\mathbb{A}$ with $\hat{a}_\mathbb{A}$ being an $i\mathbb{R}$–valued 1–form on X_*. Stokes' theorem involves only $\hat{a}_A$. The $\mathfrak{i}_K$–term is the integral of $\tfrac{i}{2}\mathrm{r}F_{A_K+2A_E} \wedge w_X$. This understood, the bound $|\mathfrak{i}_K| \le c_0 c\mathrm{r}L_*$ follows from the $|w_X| \le c$ assumption.

There is one other subtle point with regards to the derivation of (8-17) in the case when X_* is the $|s| \le L + 3$ part of X, this being that the application of Stokes' theorem requires a Hermitian connection on the bundle $\det \mathbb{S}^+$ whose curvature has norm bounded by $\mathrm{r}^{c_1/c}$ with c_1 being a constant that is independent of $\mathfrak{d}$, r, c, the metric and w_X. The pullback of this connection from the $s \le -L$ and $s \ge L$ part of X via the embeddings from the second and third bullets should also be the respective Y_- and Y_+

versions of $A_K + 2A_E$. Such a connection can be constructed using the isomorphism between de Rham cohomology and the Čech cohomology that is defined by a cover of the $|s| \leq L + 1$ part of X by Gaussian coordinate charts with the property that the any given number of charts have either empty or convex intersection (see Chapter 8 in [3]). The $\mathrm{r}^{1/c}$–bound on the norm of Riemannian curvature and the $\mathrm{r}^{-1/c}$ lower bound on the injectivity radius can be used to obtain such a cover by sets of radius greater than $\mathrm{r}^{-c_0/c}$. As the connection is constructed from the de Rham isomorphism using a subordinate partition of unity, this lower bound on the minimum chart radius can be used to construct a connection on $\det \mathbb{S}^+$ with an $\mathrm{r}^{c_0/c}$ bound on the norm of its curvature.

Section 8.6 says more about $\mathfrak{i}_*$ when the $(c, r = \mathrm{r})$ version of (3-15) is assumed.

Step 4 Define X_*, $\partial_- X_*$ and $\partial_+ X_*$ as in Step 3. Granted Step 3's bound for the norm of the $\mathfrak{i}_*$–term in (8-17), then (8-15) and (8-17) imply that

$$\mathfrak{a}(\mathfrak{d}|_{\partial_+ X_*}) \leq \mathfrak{a}(\mathfrak{d}|_{\partial_- X_*}) + c_0 c^2 \mathrm{r}^2. \tag{8-18}$$

This inequality with the top identity in (8-16) imply that $\mathfrak{a}(\mathfrak{c}_+) \leq \mathfrak{a}(\mathfrak{d}|_s) \leq \mathfrak{a}(\mathfrak{c}_-) + c_0 c^2 \mathrm{r}^2$ when $s \geq L$; and the identity in the bottom bullet of (8-16) and (8-18) imply the inequalities $\mathfrak{a}(\mathfrak{c}_-) \geq \mathfrak{a}(\mathfrak{d}|_s) \geq \mathfrak{a}(\mathfrak{c}_+) - c_0 c^2 \mathrm{r}^2$ when $s \leq -L$. Given these inequalities, then (8-17) implies that

$$\frac{1}{2} \int_{X_*} (|F_{\mathbb{A}}|^2 + \mathrm{r}^2 |\psi^\dagger \tau \psi - i w_X|^2 + 2\mathrm{r} |\nabla_{\mathbb{A}} \psi|^2) \leq \mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) + c_0 c^2 \mathrm{r}^2. \tag{8-19}$$

This last inequality with the identities in (8-15) and (8-16) imply directly the assertion made by the first bullet of Lemma 8.1 and it implies the second bullet when the length 1 interval is part of $[-L-3, L+3]$ or $[-L_* - 1, -L_* + 5]$ or $[L_* - 5, L_* + 1]$.

Granted what was just said, the second bullet of Lemma 8.1 holds if its assertion is true when the length 1 interval is disjoint from $[-L, L]$, $[-L_*, -L_* + 4]$ and $[L_* - 4, L_*]$. To prove the assertion for these cases, use (8-18) with (8-15) and (8-16) to see that $\mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s'}) < c_0 c^2 \mathrm{r}^2$ if $s > s'$ and if both are in the same component of the complement in $\mathbb{R}$ of any of these three intervals. This fact is exploited for the case $s' = s + 1$ using an integration by parts argument to rewrite the integrand on the left-hand side of the $s' = s + 1$ version of (8-15) so as to have the same form as the integrand on the left-hand side of (8-17). The resulting inequality with the bound $\mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s+1}) < c_0 c^2 \mathrm{r}^2$ leads directly to what is asserted by Lemma 8.1's second bullet. $\square$

8.4 Proof of Lemma 8.2

The proof of Lemma 8.2 has five steps. By way of a look ahead, the arguments depend crucially on the fact that the metric with the 2–form $ds \wedge *w + w$ define a Kähler structure on U_C and on U_0. The proof that follows considers only the special case where both μ_- and μ_+ vanish on the respective Y_- and Y_+ versions of U_γ and $\mathcal{H}_0$. The argument in the general case is little different and so not given.

Step 1 Let V_* denote either U_C or U_0. The fact that the metric with $ds \wedge *w + w$ defines an integrable complex structure on V_* has following consequence: View β as a section of the $(0,2)$–part of $\bigwedge^2 T^* V_* \otimes \mathbb{C}$. Then the rightmost equation in (2-10) can be written on either U_C or U_0 as

$$\bar{\partial}_A \alpha + \bar{\partial}_A^\dagger \beta = 0. \tag{8-20}$$

This last equation implies that β obeys

$$\nabla_A^\dagger \nabla_A \beta + \mathrm{r}(1 + |\alpha|^2 + |\beta|^2)\beta + \mathfrak{r}\beta = 0, \tag{8-21}$$

where $\mathfrak{r}$ is determined solely by the metric. In particular, the absolute value of $\mathfrak{r}$ and its derivatives to any specified order are also bounded by c_0. The equation just written implies that $|\beta|^2$ obeys the differential inequality

$$d^\dagger d|\beta|^2 + \mathrm{r}|\beta|^2 + |\nabla_A \beta|^2 \le 0. \tag{8-22}$$

This last inequality is exploited in a moment with the help of the Green's function for the operator $d^\dagger d + \mathrm{r}$.

Let $x \in V_*$ denote a given point and let $G_x(\cdot)$ denote the Dirichlet Green's function for $d^\dagger d + \mathrm{r}$ with pole at x. Keep in mind for what follows the following fact about $G_x(\cdot)$: it is nonnegative and it obeys

$$G_x(\cdot) \le c_0 \frac{1}{\operatorname{dist}(x,\cdot)^2} e^{-\sqrt{\mathrm{r}}\operatorname{dist}(x,\cdot)}. \tag{8-23}$$

Introduce $\mathrm{D}\colon V_* \to [0, c_0)$ to denote the function that measure the distance to the boundary of V_*. Fix x in the interior of D_*, multiply both sides of (8-22) by $G_x(\cdot)$ and integrate the resulting inequality over V_*. An integration by parts in the left-hand integral using the bound $|\beta|^2 \le c_0 c$ from Lemma 8.1 leads directly to the inequalities

$$\begin{cases} |\beta|^2 \le c_0 c e^{-\sqrt{\mathrm{r}}\,\mathrm{D}}, \\ \int_B G_x |\nabla_A \beta|^2 \le c_0 c (1/\mathrm{D}^2) e^{-\sqrt{\mathrm{r}}\,\mathrm{D}}. \end{cases} \tag{8-24}$$

The second inequality is used in Step 3 to derive bounds on the higher-order derivatives of β.

Step 2 This step constitutes a digression to state some very crude bounds for the norms of $F_{\mathbb{A}}$ and $\nabla_{\mathbb{A}}\psi$ and their covariant derivatives. The following lemma states these bounds:

Lemma 8.4 *There exists $\kappa > \pi$ such that given any $c > \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and assume the $(c, r = \mathrm{r})$ version of the first two bullets of (3-15). Assume in addition that $|w_X| \leq c$ and that the norms of its derivatives to order 10 are bounded by $\mathrm{r}^{1/c}$. Fix respective elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1. Use this data to define the equations in (2-10). Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to (2-10) with $F_{\mathbb{A}}$ and $\mathrm{r}^{1/2}|\nabla_{\mathbb{A}}\psi|$ having L^2–norm less than $c\mathrm{r}$ on the s–inverse image of any length 1 interval in $\mathbb{R}$. Then the norm of $F_{\mathbb{A}}$ and $|\nabla_{\mathbb{A}}\psi|$, and those of their derivatives up through order 4 are bounded everywhere by $\kappa_c \mathrm{r}^{\kappa_c}$.*

Proof This follows using a standard elliptic boot-strapping argument since the equations in (2-10) can be viewed as elliptic equations on any given ball in X for a suitable pair on the $C^\infty(X; S^1)$–orbit of $(\mathbb{A}, \psi)$. Except for one remark, the details of this bootstrapping are completely straightforward and so they will not be presented. The remark concerns the fact that the assumed lower bound for the injectivity radius is needed for the proof so as to invoke various Sobolev embedding theorems using embedding constants that are bounded by powers of r. $\square$

The bounds supplied by Lemma 8.4 are used in the next step.

Step 3 To obtain the asserted bound for the covariant derivative of β, differentiate (8-21) and commute covariant derivatives to obtain an equation for $\nabla_A\beta$ that has the schematic form

$$\begin{aligned}(8\text{-}25)\quad \nabla_A^\dagger\nabla_A(\nabla_A\beta) + \mathrm{r}(1 + |\alpha|^2 + |\beta|^2)\nabla_A\beta \\ + \mathfrak{R}_0(F_A)\nabla_A\beta + \mathfrak{R}_1(\nabla F_A)\beta + \mathrm{r}\mathfrak{R}_2(\nabla_A\psi)\nabla_A\beta + \mathfrak{r}_1\nabla_A\beta = 0,\end{aligned}$$

where $\mathfrak{R}_0$, $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are endomorphisms that are linear functions of their entries and are such that $|\mathfrak{R}_*(b)| \leq c_0|b|$. Meanwhile, $\mathfrak{r}_1$ is such that $|\mathfrak{r}_1| \leq c_0$ also. Take the inner product of both sides of (8-25) with $\nabla_A\beta$ and invoke Lemma 8.4 to see that

$$(8\text{-}26)\qquad d^\dagger d(|\nabla_A\beta|^2) + \mathrm{r}|\nabla_A\beta|^2 + |\nabla_A\nabla_A\beta|^2 \leq c_c\mathrm{r}^{c_c}(|\nabla_A\beta|^2 + |\beta|^2),$$

where c_c here and in what follows denotes a constant that is greater than 1 and depends only on c. The value of c_c can be assumed to increase between consecutive appearances.

Fix a point $x \in V_*$ with distance greater than $c_0 \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the boundary of V_*. Having done so, multiply both sides of (8-26) by G_x and integrate both sides over V_*. Use the second bullet in (8-24) to bound integral on the right-hand side of the resulting inequality by $c_0 e^{-\sqrt{\mathrm{r}}/c_0}$ when $\mathrm{r} \ge c_c$. An integration by parts on the left-hand side using Lemma 8.4 to bound $|\nabla_A \beta|$ on the boundary of V_* and the bound just stated implies that

$$|\nabla \beta_A|^2(x) + \int_B G_x |\nabla_A \nabla_A \beta|^2 \le c_0 e^{-\sqrt{\mathrm{r}}/c_0} \tag{8-27}$$

when $\mathrm{r} \ge c_c$. This gives the desired bound for $|\nabla_A \beta|$.

To obtain the bound for $|\nabla_A \nabla_A \beta|$, differentiate (8-25) twice and take the inner product of both sides with $\nabla_A \nabla_A \beta$ after commuting covariant derivatives. The result is an equation that looks much like (8-26) with $\nabla_A \beta$ replaced by $\nabla_A \nabla_A \beta$ on the left-hand side and with the addition of the term $\mathrm{r}^{c_c} |\nabla_A \nabla_A \beta|^2$ on the right-hand side. Granted that this is the case, then the same Green's function argument that led to (8-27) leads to an analogous bound for $|\nabla_A \nabla_A \beta|^2$.

Step 4 This step and Step 5 addresses the assertions of Lemma 8.2 that concern α. To start, act by $\bar{\partial}_A^\dagger$ on both sides of (8-20), commute covariant derivatives and use the bounds from Lemma 8.2 for $|\beta|$ to see that α obeys an equation that has the form

$$\nabla_A^\dagger \nabla_A \alpha - \mathrm{r}(1 - |\alpha|^2)\alpha = \mathfrak{e}, \tag{8-28}$$

where $|\mathfrak{e}| \le e^{-\sqrt{\mathrm{r}}/c_0}$ when $\mathrm{r} \ge c_c$. This equation implies that $\mathrm{w} = 1 - |\alpha|^2$ obeys a differential inequality of the form

$$d^\dagger d \mathrm{w} + \mathrm{r}\mathrm{w} \ge |\nabla_A \alpha|^2 + \mathrm{r}\mathrm{w}^2 - e^{-\sqrt{\mathrm{r}}/c_0}. \tag{8-29}$$

Use of the Green's function G_x with the fact that $|\mathrm{w}| \le c_0 c$ on the boundary of V_* along the same lines as in Steps 1 and 3 finds $\mathrm{w} \ge e^{-\sqrt{\mathrm{r}}/c_0}$ at distances greater than $c_0 \mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the boundary of V_* when $\mathrm{r} \ge c_c$. This is the $|\alpha|^2$ assertion in the first bullet of Lemma 8.2.

The assertion in the third bullet follows directly from (8-20) given Lemma 8.2's bounds for $|\beta|$ and $|\nabla_A \beta|$. The assertion in the fourth bullet follows directly from (8-29) given that $\mathrm{w}(1 - \mathrm{w}) = |\alpha|^2(1 - |\alpha|^2)$ and that this is greater than $\frac{1}{2}\delta^2$ at points where $|\alpha|^2$ is between δ and $1 - \delta$. The assertions in the fifth bullet about the covariant derivatives of α are proved in Step 5.

Step 5 This step derives the asserted bounds in the fifth bullet for the norms of the covariant derivatives of α. To do this, suppose that $x \in V_*$ is such that $|F_A| \le c\mathrm{r}$ on the ball of radius $c_0\mathrm{r}^{-1/2}$ centered at x. Use τ_{r} in what follows to denote the rescaling map from $\mathbb{C}^2$ to $\mathbb{C}^2$ that is given by the rule $x \mapsto \tau_{\mathrm{r}}(x) = \mathrm{r}^{-1/2}x$. The pullback of (A, ψ) by this map is denoted by $(A_{\mathrm{r}}, \psi_{\mathrm{r}})$. The bound $|F_A| \le c\mathrm{r}$ implies that the absolute value of the curvature of A_{r} is bounded in the radius 1 ball about the origin in $\mathbb{C}^2$ is bounded by c. Meanwhile, the pullback of the equations in (2-10) by this map constitutes a uniformly elliptic system of equations (modulo the action of $C^\infty(\mathbb{C}^2; S^1)$) in the radius 1 ball about the origin in $\mathbb{C}^2$ with coefficients that have r–independent bounds for their absolute values and for those of their derivatives to any a priori chosen order. This understood, the fact that $|\psi_{\mathrm{r}}| \le 2$ in this ball and the aforementioned bound by c for the norm of the curvature of A_{r} imply via standard elliptic bootstrapping arguments that the A_{r}–covariant derivatives of ψ_{r} through order 2 are bounded by $c_0 c$ in the radius c_0^{-1} ball about the origin in $\mathbb{C}^2$. Granted these bounds, use the chain rule of calculus to obtain the bounds asserted by the fifth bullet of Lemma 8.2 for the covariant derivative of α. □

8.5 Proof of Lemma 8.3

Use V_* again to denote either U_C or U_0. The functions $\mathfrak{z}_A$ and $\mathfrak{z}_B$ are both equal to $\mathrm{r}|\beta|^2$ on V_* and so what is asserted by the second bullet of Lemma 8.3 follows from the first bullet of Lemma 8.2. The absolute value of r is bounded by $c_0\mathrm{r}|\alpha||\beta|$ on V_* and so the third bullet of Lemma 8.3 also follows from the first bullet of Lemma 8.2. The bounds in the first bullet of Lemma 8.3 follow from the bound in the fourth bullet and that for $|\alpha|^2$ in the first bullet of Lemma 8.2. If the bounds in first through fourth bullets of Lemma 8.3 hold, then $|F_A|$ is bounded by $c_0\mathrm{r}$ at the points in V_* with distance $\frac{1}{200}\rho_D$ from the boundary of V_*. Granted that this is the case, then the rescaling argument in Step 5 of the proof of Lemma 8.2 can be used to derive the bound given in the fifth bullet of Lemma 8.3.

The upcoming Lemma 8.5 is the critical ingredient for the proof of the fourth bullet of Lemma 8.3. The $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \le \mathrm{r}^{2-1/c}$ assumption in Lemma 8.3 and the final three bullets of (3-15) are needed only to invoke Lemma 8.5.

Lemma 8.5 *There exists $\kappa > 100(1+\rho_D^{-1})$ such that given any $c \ge \kappa$, there exists $\kappa_c > \kappa$ with the following significance: Fix $\mathrm{r} \ge \kappa_c$ and assume that the metric and w_X are $(c, r = \mathrm{r})$–compatible. Fix elements μ_- and μ_+ from the Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by* 1 *and use this data to define the equations in* (2-10)*. Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to the* (r, μ_-) *version of* (2-5) *on Y_- and the* (r, μ_+) *version of* (2-5) *on Y_+ with $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \le \mathrm{r}^{2-1/c}$. Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton* *solution to* (2-10) *with* $s \to -\infty$ *limit equal to* $\mathfrak{c}_-$ *and* $s \to \infty$ *limit equal to* $\mathfrak{c}_+$. *Use* B *to denote a ball of radius* κ^{-2} *in the domain* U_C *or in the domain* U_0 *with center at distance* κ^{-1} *or more from the domain's boundary. Then* $\mathrm{r}\int_B \big|1-|\psi|^2\big| \le \kappa_c \mathrm{r}^{1-1/\kappa_c}$.

Lemma 8.5 is proved in Section 8.6. Granted Lemma 8.5, then the six steps that follow prove the fourth bullet of Lemma 8.3 in the case when μ_- and μ_+ are zero on the Y_- and Y_+ versions of U_γ and $\mathcal{H}_0$. The proof when they are not zero but bounded by $e^{-\mathrm{r}^2}$ is little different and so not given.

Step 1 Let V_* denote either U_C or U_0. Keep in mind that metric on V_* has nonnegative Ricci curvature tensor, that the 2–form $w_X = w$ is covariantly constant on V_*, that both $\mathfrak{w}_\mu = 0$ and that B_{A_K} is covariantly constant on V_*. These facts with the bounds from Lemma 8.2 for $|\beta|$ and $|\nabla_A\beta|$ have the following implication: Let s denote $|\mathcal{E}_A - \mathcal{B}_A|$. Granted that $\mathrm{r} \ge c_c$, then the equations in (2-10) imply that s obeys the differential inequality

$$d^\dagger ds + \mathrm{r}|\alpha|^2 s \le \mathrm{r}|\nabla_A\alpha|^2 + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8-30}$$

at the points in V_* with distance greater than $c_0\mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ from the boundary of V_*. Let w again denote $1-|\alpha|^2$ and let q_0 denote $s - \mathrm{r}\mathrm{w}$. It follows from (8-29) and (8-30) that

$$d^\dagger dq_0 + \mathrm{r}|\alpha|^2 q_0 \le e^{-\sqrt{\mathrm{r}}/c_0} \tag{8-31}$$

at the points in V_* with distance $c_0\mathrm{r}^{-1/2}(\ln \mathrm{r})^2$ or more from V_*'s boundary if $\mathrm{r} \ge c_c$.

Step 2 Fix $\rho_* > 0$ such that $\rho_* < 10^{-8}\rho_D$. Fix $s_0 \in \mathbb{R}$. Let $V' \subset V_*$ denote the set of points in the $(s_0 - 1 - \rho_*, s_0 + 1 + \rho_*)$ part of V_* with distance ρ_* or more from the boundary of V_*, and let $V \subset V'$ denote the set of points in V_* with distance greater than $2\rho_*$ from the boundary of V_*. Thus, each point in V has distance ρ_* or more from the boundary of V'.

Fix a sequence $\{\varsigma_n\}_{n=1,\dots}$ of smooth, nonnegative functions on V' with the following properties: Each function in this series is bounded by 1 and is equal to 1 on V. Second, ς_1 has compact support and for each $n \ge 1$, the function ς_{n+1} has compact support where $\varsigma_n = 1$. Finally, the absolute values of the first and second derivatives of the functions in this series enjoy s_0–independent upper bounds.

Step 3 For each integer $n \ge 1$, set $q_n = \max(\varsigma_n q_{n-1}, 0)$. Use $\mathfrak{q}_0$ to denote the maximum of q_0; and for $n \ge 1$, use $\mathfrak{q}_n$ to denote the maximum of q_n. Note that $\mathfrak{q}_n \le \mathfrak{q}_{n-1}$. It follows from (8-31) that if $\mathrm{r} \ge c_c$, then any given $n \ge 1$ version of q_n

obeys

$$d^\dagger dq_n + \mathrm{r}|\alpha|^2 q_n \le (dd^\dagger \varsigma_n)q_{n-1} + 2\langle d\varsigma_n, dq_{n-1}\rangle + c_0 e^{-\sqrt{\mathrm{r}}/c_0}, \tag{8-32}$$

where $\langle\cdot,\cdot\rangle$ denotes the metric inner product. Fix a constant $z_n \ge 1$ to be determined shortly, and let q_{n*} denote the maximum of 0 and $q_n - \mathrm{r}^{-1} z_n \mathfrak{q}_{n-1}$. The function q_{n*} obeys

$$\begin{aligned} d^\dagger dq_{n*} &+ \mathrm{r}|\alpha|^2 q_{n*} \\ &\le z_n \mathfrak{q}_{n-1}\mathrm{w} + (-z_n \mathfrak{q}_{n-1} + (dd^\dagger \varsigma_n)q_{n-1} + 2\langle d\varsigma_n, dq_{n-1}\rangle) + c_0 e^{-\sqrt{\mathrm{r}}/c_0}. \end{aligned} \tag{8-33}$$

Note also that q_{n*} has compact support in V' since $q_n - \mathrm{r}^{-1} z_n \mathfrak{q}_{n-1} = -\mathrm{r}^{-1} z_n \mathfrak{q}_{n-1}$ on the complement of the support of ς_n.

Step 4 Fix x in the interior of V' and let G_x now denote the Dirichlet Green's function for the operator $d^\dagger d$ on V' with pole at x. The function G_x is nonnegative, $|G_x(\cdot)| \le c_0 \operatorname{dist}(x,\cdot)^{-2}$ and $|dG_x(\cdot)| \le c_0 \operatorname{dist}(x,\cdot)^{-3}$. Multiply both sides of (8-33) by G_x and integrate the two sides of the resulting inequality over V'. Integrate by parts on both sides to remove derivatives from q_{n*} and q_{n-1} to obtain the inequality

$$q_{n*}(x) \le z_n \mathfrak{q}_{n-1} \int_{V'} \left(\frac{1}{\operatorname{dist}(x,\cdot)^2}\mathrm{w}\right) + (-c_0^{-1} z_n + e_n)\mathfrak{q}_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0}, \tag{8-34}$$

where $e_n \le c_0 \sup_{x\in V'}(|d^\dagger d\varsigma_n| + |d\varsigma_n|)$. Granted this bound, a purely n–dependent choice for z_n leads from (8-34) to the inequality

$$q_{n*}(x) \le z_n \mathfrak{q}_{n-1} \int_{V'} \left(\frac{1}{\operatorname{dist}(x,\cdot)^2}\mathrm{w}\right) + e^{-\sqrt{\mathrm{r}}/c_0}; \tag{8-35}$$

Lemma 8.5 is used to exploit this inequality.

Step 5 Fix $\rho > 0$ and break up the integral in (8-35) into the part where $\operatorname{dist}(x,\cdot)$ is greater than ρ and the part where $\operatorname{dist}(x,\cdot)$ is less than ρ. Having done so, appeal to Lemma 8.5 and the first bullet of Lemma 8.2 to see that

$$q_{n*}(x) \le z_n(\rho^{-2}\mathrm{r}^{-1/c_0} + \rho^2)\mathfrak{q}_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8-36}$$

when $\mathrm{r} \ge c_c$. Let c_* denote the value of c_0 that appears in (8-36). Take $\rho = \mathrm{r}^{-1/4c_*}$ in (8-36). The resulting right-hand side is independent of x, and this leads directly to the inequality

$$\mathfrak{q}_n \le z_n \mathrm{r}^{-1/2c_*} \mathfrak{q}_{n-1} + e^{-\sqrt{\mathrm{r}}/c_0} \tag{8-37}$$

when $\mathrm{r} \ge c_c$. As Lemma 8.4 finds $\mathfrak{q}_0 < \mathrm{r}^{c_c}$, what is written in (8-37) implies that an $n = c_c$ version of $\mathfrak{q}_n$ is bounded by r^{-200}.

Step 6 Since $\varsigma_n = 1$ on V, the conclusion from Step 5 implies that

$$|\mathcal{E}_A - \mathcal{B}_A| < \mathrm{r}(1-|\alpha|^2) + \mathrm{r}^{-200} \tag{8-38}$$

at all points in V. Square both sides of (8-38). What with the bounds for $|\mathfrak{z}_A|$ and $|\mathfrak{z}_B|$ from Lemma 8.3's second bullet, the resulting inequality implies that

$$(1-2\sigma)^2\mathrm{r}^2(1-|\alpha|^2) + |\mathfrak{X}|^2 \le \mathrm{r}^2(1-|\alpha|^2) + c_0\mathrm{r}^{-198}, \tag{8-39}$$

and rearranging terms writes this as

$$|\mathfrak{X}|^2 \le 2\mathrm{r}^2\sigma(1-\sigma)(1-|\alpha|^2) + c_0\mathrm{r}^{-198}. \tag{8-40}$$

This gives the bound stated in the fourth bullet of Lemma 8.3. □

8.6 Proof of Lemma 8.5

The proof has six parts. Parts 1 and 2 revisit the formula in (8-15) and Part 3 revisits the formula in (8-17). These steps present the proof in the case when $c_1(\det\mathbb{S})$ is nontorsion on all components of the $|s| > 1$ part of X. But for the two remarks that follow, the proof when $X_{\mathrm{tor}} \neq \varnothing$ differs only cosmetically.

The first remark concerns the formula in (8-17) in the case when X_* is the respective $|s| \in [L, L_* - 4]$ part of X_{tor}, the remark being that the absolute value of $\mathfrak{i}_*$ in this case is bounded by $c_0 c^2 \mathrm{r}\ln\mathrm{r}$. The reason is as follows: As noted subsequent to (8-17), the absolute value of the relevant version of $\mathfrak{i}_*$ is bounded in any event by $c_0 c\mathrm{r}L_*$. Meanwhile, the first bullet of (3-15) bounds L_* by $c\ln\mathrm{r}$.

The second remark concerns (8-17) in the case when X_* is the $|s| \in [L_* - 4, L_*]$ part of X_{tor}, this being that the absolute value of the corresponding version of $\mathfrak{i}_*$ is at most c_0 when r is larger than a purely c–dependent constant. Given item (d) of the fourth bullet of (3-15), the proof that this is so differs only in notation from what is said below in Part 2 to prove the analogous bound for the version of $\mathfrak{i}_*$ that appears in (8-17) when X_* is the $|s| \in [L-4, L]$ part of X.

Part 1 Write $\mathfrak{d} = (\mathbb{A}, \psi)$. When X, the metric and w_X are as described by the first bullet of (3-16), use this pair as instructed in the proof of Lemma 8.1 to define the map (A_*, ψ_*) from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(Y_Z; \mathbb{S})$. When the second bullet of (3-16) is relevant, then (A_*, ψ_*) as defined in the proof of Lemma 8.1 denotes a map from $(-\infty, -1]$ to $\mathrm{Conn}(E|_{Y_-}) \times C^\infty(Y_-; \mathbb{S}|_{Y_-})$ and also a map from $[1, \infty)$ to $\mathrm{Conn}(E|_{Y_+}) \times C^\infty(Y_+; \mathbb{S}|_{Y_+})$.

Set $I_L = [-L, L]$ when X, the metric and w_X are as described by the first bullet of (3-16), and set I_L to be either $[-L, -L+4]$ or $[L-4, L]$ otherwise. Use Y_* to denote the constant $s \in I_L$ slice of X, this being either Y_Z, Y_- or Y_+. Write the metric on $I_L \times Y_*$ as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting an s–dependent metric on Y_*. Define the s–dependent 1–form w_* on Y_* by writing w_X as $ds \wedge *w_* + w_*$ with the Hodge dual defined here by $\mathfrak{g}$. The two equations in (2-10) on the $s \in I_L$ part of X are equivalent to equations for (A_*, ψ_*) that can be written as

$$\begin{cases} \frac{\partial}{\partial s} A_* + \mathfrak{B}_{\mathfrak{d}} = 0, \\ \frac{\partial}{\partial s} \psi_* + D_{A_*} \psi_* = 0, \end{cases} \tag{8-41}$$

with $\mathfrak{B}_{\mathfrak{d}}$ denoting the $s \in I_L$ dependent 1–form on Y_*

$$\mathfrak{B}_{\mathfrak{d}} = B_{\mathbb{A}} - \mathrm{r}(\psi^\dagger \tau \psi - i w_*) + i \mathfrak{w}_\mu^+\Big(\frac{\partial}{\partial s}, \cdot\Big) + \tfrac{1}{2} B_{A_K}. \tag{8-42}$$

By way of notation, D_{A_*} in (8-41) denotes the Dirac operator defined by the metric $\mathfrak{g}$, its Levi-Civita connection and the connection $A_K + 2A_*$ on the $\{s\} \times Y_*$ version of $\det \mathbb{S}$.

Part 2 If X, the metric and w_X are as described by the first bullet of (3-16), then the integration and use of Stokes' theorem that leads to (8-15) can be repeated with the domain of integration being $s^{-1}([-L, L])$ to find that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R} \times Y_Z} \Big(\Big| \frac{\partial}{\partial s} A_* \Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big| \frac{\partial}{\partial s} \psi_* \Big|^2 + |D_A \psi|^2 \Big)\Big) + \mathfrak{i}_\mu \\ = \mathfrak{a}(\mathfrak{d}|_{s=-L}) - \mathfrak{a}(\mathfrak{d}|_{s=L}), \end{aligned} \tag{8-43}$$

where $\mathfrak{i}_\mu = 0$ when $\mathfrak{w}_\mu$ is such that X, the metric, w_X and $\mathfrak{w}_\mu$ define the product metric, and where $|\mathfrak{i}_\mu| \le c_0 \big(\int_{s^{-1}([-L,L])} \big| \frac{\partial}{\partial s} A_* \big|^2 \big)^{1/2}$ in any event. This being the case, the second bullet of Lemma 8.1 implies that $|\mathfrak{i}_\mu| \le c_0 \mathrm{r}$.

Assume now that X, the metric and w_X are as described by the second bullet in (3-16). The derivation of (8-15) and (8-43) can be repeated with the domain of integration being $s^{-1}([-L, -L+4])$ and also $s^{-1}([L-4, L])$ to obtain the identities

$$\begin{aligned} \frac{1}{2} \int_{[-L,-L+4] \times Y_Z} \Big(\Big| \frac{\partial}{\partial s} A_* \Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big| \frac{\partial}{\partial s} \psi_* \Big|^2 + |D_A \psi|^2 \Big)\Big) + \mathfrak{i} \\ = \mathfrak{a}(\mathfrak{d}|_{s=-L}) - \mathfrak{a}(\mathfrak{d}|_{s=-L+4}), \\ \frac{1}{2} \int_{[L-4,L] \times Y_Z} \Big(\Big| \frac{\partial}{\partial s} A_* \Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big| \frac{\partial}{\partial s} \psi_* \Big|^2 + |D_A \psi|^2 \Big)\Big) + \mathfrak{i} \\ = \mathfrak{a}(\mathfrak{d}|_{s=L-4}) - \mathfrak{a}(\mathfrak{d}|_{s=L}), \end{aligned} \tag{8-44}$$

where $\mathfrak{i}$ in this case is such that $|\mathfrak{i}| \le c_0 \mathrm{r}^{2-1/c}$ when $c > c_0$ and $\mathrm{r} > c_c$ with c_c denoting a constant that depends only on c. The paragraphs that follow explain how this bound comes about.

The term denoted by $\mathfrak{i}$ can be written as the sum of three integrals, $\mathfrak{i} = \mathfrak{i}_{\mathfrak{g}} + \mathfrak{i}_w + \mathfrak{i}_\mu$. What is denoted by $\mathfrak{i}_\mu$ appears here for the same reason it appears in (8-43) and it has the analogous bound, $|\mathfrak{i}_\mu| \le c_0 \mathrm{r}$. The integral denoted by $\mathfrak{i}_{\mathfrak{g}}$ accounts for the s–dependence of the metric $\mathfrak{g}$ on Y_* when commuting the operators $\frac{\partial}{\partial s}$ and D_{A_*}. In particular, the integrand that defines $\mathfrak{i}_{\mathfrak{g}}$ is bounded by $c_0 \mathrm{r}\big(\big|\frac{\partial}{\partial s}\mathfrak{g}\big|\,|\psi|\,|\nabla_A\psi| + \big|\mathfrak{R}_{\mathfrak{g}}\big(\frac{\partial}{\partial s}, \cdot\big)\big|, |\psi|^2\big)$ with $\mathfrak{R}_g$ denoting the Riemannian curvature tensor of the metric $ds^2 + \mathfrak{g}$. This understood, (3-15) with Lemma 8.1's bounds for $|\psi|^2$ and the L^2–norm of $|\nabla_A\psi|$ imply that $|\mathfrak{i}_{\mathfrak{g}}| \le c_0 \mathrm{r}^{3/2+1/c}$.

The integral $\mathfrak{i}_w$ arises from the contribution to the integral of $\big|\frac{\partial}{\partial s}A_* + \mathfrak{B}_{\mathfrak{d}}\big|^2$ of the metric inner product of $\frac{\partial}{\partial s}A_*$ with $-i\mathrm{r} * w_*$. The integral of this inner product is written as $\int_{I_L} h(s)ds$ with I_L denoting $[-L, -L+4]$ or $[L-4, L]$ as the case may be, and with $h(s)$ denoting the integral of the 3–form $-i\mathrm{r}\frac{\partial}{\partial s}A_* \wedge w_*$ over $\{s\} \times Y_*$. Only a portion of the integral of $-i\mathrm{r}\frac{\partial}{\partial s}A_* \wedge w_*$ contributes to $\mathfrak{i}_w$. To say more, write A_* as $A_E + \hat{a}_A$ with $\hat{a}_A$ denoting an s–dependent 1–form on Y_*. The integral of the 3–form $-i\mathrm{r}\frac{\partial}{\partial s}A_* \wedge w_*$ over $\{s\} \times Y_*$ is written using $\hat{a}_A$ as

$$-i\mathrm{r}\frac{\partial}{\partial s}\bigg(\int_{\{s\}\times Y_*} \hat{a}_A \wedge w_*\bigg) + i\mathrm{r}\bigg(\int_{\{s\}\times Y_*} \hat{a}_A \wedge \frac{\partial}{\partial s} w_*\bigg). \tag{8-45}$$

The contributions of the function w in (2-7) to the right-hand side of (8-44) are given by the integral over I_L of the leftmost term in (8-45), this being a consequence of the fundamental theorem of calculus. What is denoted by $\mathfrak{i}_w$ is the integral over I_L of the rightmost term in (8-45). A bound for the absolute value of the latter is obtained by using the assumption in item (b) of the fourth bullet of (3-15) to write $\frac{\partial}{\partial s}w_*$ as $d\mathfrak{b}$ with $\mathfrak{b}$ as described by this same part of (3-15). Stokes' theorem equates the rightmost integral in (8-45) with the integral of $i\mathrm{r}\, d\hat{a}_A \wedge \mathfrak{b}$. This being the case, it follows from (3-15) that this second contribution to $\mathfrak{i}_w$ has absolute value less than $c_0\mathrm{r}^{2-1/c}$.

Part 3 Integrate $|F_{\mathbb{A}}^+ - \mathrm{r}(\psi^\dagger\tau\psi - iw_X^+) - i\mathfrak{w}_\mu^+|^2 + \mathrm{r}|D_{\mathbb{A}}\psi|^2$ over $s^{-1}([-L+4, L-4])$. Integrate by parts using the fact this integral is zero to derive an identity that can be written as

$$\frac{1}{2}\int_{s^{-1}([-L+4,L-4])} (|F_{\mathbb{A}}|^2 + \mathrm{r}^2|\psi^\dagger\tau\psi - iw_X^+|^2 + 2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2) + \mathfrak{i}_L = \mathfrak{a}(\mathfrak{d}|_{s=-L+4}) - \mathfrak{a}(\mathfrak{d}|_{s=L-4}) \tag{8-46}$$

with $\mathfrak{i}_L$ such that $|\mathfrak{i}_L| \le c_0 \mathrm{r}^{1+c_0/c}$. The paragraphs that follow in a moment derive the latter bound. By way of comparison, the absolute value of the term $\mathfrak{i}$ in (8-17) has the bound $c_0 c\mathrm{r}\bigl(\int_{s^{-1}([-L+4,L-4])} |F_{\mathbb{A}}|^2\bigr)^{1/2} + c_0 \mathrm{r}^{1+c_0/c}$. The difference can be traced to the assumption that w_X is a closed 2–form on $s^{-1}([-L+4, L-4])$.

The bound on $|\mathfrak{i}_L|$ can be seen by writing $\mathfrak{i}_L$ as a sum of four integrals, these denoted by $\mathfrak{i}_\psi$, $\mathfrak{i}_{cs}$, $\mathfrak{i}_w$ and $\mathfrak{i}_\mu$. The integrand of $\mathfrak{i}_\psi$ is $\frac{1}{4}\mathrm{r}|\psi|^2\mathrm{R}$ with R denoting the scalar curvature of X. By way of an explanation, this term comes from the integration by parts and subsequent commuting of covariant derivatives that rewrites the integral of $\mathrm{r}|D_{\mathbb{A}}\psi|^2$ as an integral over the $s^{-1}(-L+4)$ and $s^{-1}(L-4)$ boundaries of the integration domain plus an integral over $s^{-1}([-L+4, L-4])$ whose integrand is the sum of $\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2$, a curvature term involving $F_{\mathbb{A}}^+$ and the product of $\frac{1}{4}\mathrm{r}|\psi|^2\mathrm{R}$ with R denoting the scalar curvature of the metric on X. The boundary terms account for the rightmost integral in (2-6)'s formula for $\mathfrak{a}$. Use the bounds from the first two bullets of (3-15) with the bound $|\psi|^2 \le c_0 c$ from Lemma 8.1 to see that $|i_\psi| \le c_0 \mathrm{r}^{1+2/c}$ if $\mathrm{r} > c_c$ with c_c again denoting a constant that depends only on c.

The integrals $\mathfrak{i}_{cs}$ and $\mathfrak{i}_w$ involve a chosen Hermitian connection on $\det \mathbb{S}^+$ whose curvature has norm bounded by $c\mathrm{r}^{c_0/c}$ and whose pullback from the $s \le -L+8$ and $s \ge L-8$ part of X via the embeddings from the second and third bullets is the respective Y_- and Y_+ versions of $A_K + 2A_E$. Step 3 of the proof of Lemma 8.1 explains why such connections exist. Let $\mathbb{A}_{\mathbb{S}}$ denote a chosen connection with this property.

The integral $\mathfrak{i}_{cs}$ comes by first writing $|F_{\mathbb{A}}^+|^2$ as $\frac{1}{2}|F_{\mathbb{A}}|^2$ plus the Hodge star of $\frac{1}{2}F_{\mathbb{A}} \wedge F_{\mathbb{A}}$. The latter is rewritten using an integration by parts after writing $\mathbb{A}$ as $\mathbb{A}_{\mathbb{S}} + \hat{a}_A$ with $\hat{a}_A$ being an $i\mathbb{R}$–valued 1–form on X. Writing $\mathbb{A}$ in this way yields

$$\tfrac{1}{2}F_{\mathbb{A}} \wedge F_{\mathbb{A}} = \tfrac{1}{2}d\hat{a}_A \wedge d\hat{a}_A + d\hat{a}_A \wedge F_{\mathbb{A}_{\mathbb{S}}} + \tfrac{1}{2}F_{\mathbb{A}_{\mathbb{S}}} \wedge F_{\mathbb{A}_{\mathbb{S}}}. \tag{8-47}$$

An integration by parts writes the integrals of the first two terms on the right side of (8-47) as boundary integrals, these giving the respective cs contributions to $\mathfrak{a}(\mathfrak{d}|_{s=L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$. The integral of the rightmost term in (8-43) is $\mathfrak{i}_{cs}$. Thus $|\mathfrak{i}_{cs}| \le c_0 \mathrm{r}^{c_0/c}$.

The integral $\mathfrak{i}_w$ is obtained by invoking Stokes' theorem to rewrite the term from the inner product between $F_{\mathbb{A}}^+$ and $\frac{i}{2}\mathrm{r}w_X$ that arises when $\bigl|F_{\mathbb{A}}^+ - \mathrm{r}\bigl(\psi^\dagger\tau\psi - \frac{i}{2}w_X\bigr) + i\mathfrak{w}_\mu^+\bigr|^2$ is written as $|F_{\mathbb{A}}^+|^2 + \mathrm{r}|\psi^\dagger\tau\psi - w_X|^2$ plus remainder terms. One of these remainder terms is twice the inner product of $F_{\mathbb{A}}^+$ with $\frac{i}{2}\mathrm{r}w_X$. The integral of the latter is the integral of the 4–form $-i\mathrm{r}F_{\mathbb{A}}^{+}\wedge w_X$. Write $-i\mathrm{r}F_{\mathbb{A}}\wedge w_X$ as the sum of $-i\mathrm{r}d\hat{a}_A\wedge w_X$ and $-2i\mathrm{r}F_{\mathbb{A}}\wedge w_X$. Because w_X is closed, an integration by parts writes the integral of the first of these as an integral over the boundary of the integration domain. The latter accounts for the respective w contributions to $\mathfrak{a}(\mathfrak{d}|_{s=-L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$. The integral of $-2i\mathrm{r}F_{\mathbb{A}}\wedge w_X^{+}$ is $\mathfrak{i}_w$. This being the case, the bound $|\mathfrak{i}_w|\le c_0c\mathrm{r}^{1+c_0/c}$ follows directly from the (3-15) and what is said in Step 3 of the proof of Lemma 8.1 about $|F_{\mathbb{A}_{\mathbb{S}}}|$.

The integral denoted by $\mathfrak{i}_\mu$ has two contributions. The first accounts for the terms with $\mathfrak{w}_\mu$ that arise in the aforementioned rewriting of $|F_{\mathbb{A}}^{+}-\mathrm{r}(\psi^\dagger\tau\psi-iw_X)+i\mathfrak{w}_\mu^{+}|^2$. It follows from the left-hand equation in (2-10) that the integrand for this part of $\mathfrak{i}_\mu$ is bounded by c_0. The second contribution is proportional to the integral of $d\hat{a}_A\wedge\mathfrak{w}_\mu$; it appears when Stokes' theorem is used to write the respective $\mathfrak{e}_\mu$ parts of $\mathfrak{a}(\mathfrak{d}|_{s=-L+4})$ and $\mathfrak{a}(\mathfrak{d}|_{s=L-4})$ as a term that has norm bounded by c_0 and another whose integrand is proportional to $d\hat{a}_A\wedge\mathfrak{w}_\mu$. The norm of the latter is bounded by $c_0c(|F_{\mathbb{A}}|+c^2)$. Granted this, it follows that $|\mathfrak{i}_\mu|\le c_0c\big(\big(\int_{s^{-1}([-L+4,L-4])}|F_{\mathbb{A}}|^2\big)^{1/2}+c^2\big)$ and this is guaranteed by Lemma 8.1 to be less than $c_0c(\mathrm{r}+c^2)$.

Part 4 If the first bullet of (3-16) holds, the assumption $\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{c}_+)<\mathrm{r}^{2-1/c}$ with (8-16) and (8-43) imply that

$$\begin{aligned}\text{(8-48)}\quad \frac{1}{2}\int_{\mathbb{R}\times Y_Z}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{\mathfrak{d}}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)&\le\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{c}_+)+c_0\mathrm{r}\\&\le c_0\mathrm{r}^{2-1/c}\end{aligned}$$

when $c>c_0$ and r is greater than a constant that depends only on c. If the second bullet of (3-16) holds, the assumption $\mathfrak{a}(\mathfrak{c}_-)-\mathfrak{a}(\mathfrak{c}_+)<\mathrm{r}^{2-1/c}$ with (8-16), (8-44) and (8-46) imply the bounds that follow when $c>c_0$ and r is greater than a constant that depends only on c:

$$\text{(8-49)}\quad\begin{aligned}&\int_{(-\infty,-L+4]\times Y_-}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{\mathfrak{d}}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)\le c_0\mathrm{r}^{2-1/c},\\&\int_{[L-4,\infty)\times Y_+}\Big(\Big|\frac{\partial}{\partial s}A_*\Big|^2+|\mathfrak{B}_{\mathfrak{d}}|^2+2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi_*\Big|^2+|D_A\psi|^2\Big)\Big)\le c_0\mathrm{r}^{2-1/c},\\&\int_{s^{-1}([-L+4,L-4])}(|F_{\mathbb{A}}|^2+\mathrm{r}^2|\psi^\dagger\tau\psi-iw_X^{+}|^2+2\mathrm{r}|\nabla_{\mathbb{A}}\psi|^2)\le c_0\mathrm{r}^{2-1/c}.\end{aligned}$$

Put away for now the bounds in (8-48) and those in the first two bullets of (8-49). Assuming that the second bullet of (3-16) holds, the bound in the third bullet of (8-49) implies the bound

$$\mathrm{r}\int_{s^{-1}([-L+4,L-4])} |\psi^\dagger\tau\psi - i w_X^+| \le c_0 \mathrm{r}^{1-1/c} \tag{8-50}$$

when $c > c_0$ and r is greater than a constant that depends only on c. Let B denote the given ball from Lemma 8.5. Use the second and third bullets of (2-9) and (3-14), the first bullet of Lemma 8.2, and (8-50) to see that

$$\mathrm{r}\int_{B\cap s^{-1}([-L+4,L-4])} \bigl|1-|\alpha|^2\bigr| \le c_0 \mathrm{r}^{1-1/c} \tag{8-51}$$

when r is greater than a purely c–dependent constant.

Part 5 If the first bullet of (3-16) holds, then I denotes in what follows any given length 1 interval in $\mathbb{R}$. If the second bullet of (3-16) holds, then I denotes a length 1 interval in either $(-\infty, -L+4]$ or in $(L-4, \infty)$. In either case, reintroduce the 1–form υ_X from the fifth bullet of (3-16). Take the inner product of both sides of (8-41) with $i\upsilon_X$, then integrate the resulting identity over $s^{-1}(I)$. The left-hand side of the result can be written as a sum of four integrals; and the assertion that this sum is zero can be rewritten as the identity

$$\begin{aligned}\int_I\biggl(\int_{Y_*}\upsilon_X\wedge \mathrm{r}(w_* + *i\psi^\dagger\tau\psi)\biggr)\,ds &= \int_I\biggl(\int_{Y_*}\upsilon_X\wedge i\,d\hat{a}_{\mathbb{A}}\biggr)\,ds + \int_I\biggl(\int_{Y_*}\upsilon_X\wedge *\frac{\partial}{\partial s}A_*\biggr)\,ds \\ &\quad + \int_I\biggl(\int_{Y_*}\upsilon_X\wedge *\Bigl(-\mathfrak{w}_\mu^+\Bigl(\frac{\partial}{\partial s},\cdot\Bigr)+\tfrac{1}{2}iB_{\mathbb{A}_{\mathbb{S}}}\Bigr)\biggr)\,ds.\end{aligned} \tag{8-52}$$

Use what is said by either the first bullet in (3-16) or the second and fifth bullets of (3-15) to bound the absolute value of the rightmost integral in (8-52) by a purely c–dependent constant. Meanwhile, Stokes' theorem finds the middle integral on the right-hand side of (8-52) equal to zero. The absolute value of the leftmost integral on the right-hand side of (8-52) is bounded by $c_0 c$ times the L^2–norm over $s^{-1}(I)$ of $\frac{\partial}{\partial s}A_*$. This being the case, use either (8-48) or the first two bullets in (8-49) to bound the absolute value of the leftmost integral on the right side of (8-52) by $\mathrm{r}^{1-1/(2c)}$ when r is greater than a purely c–dependent constant.

It follows as a consequence of what was just said in the preceding paragraph that the absolute value of the integral on the left-hand side of (8-52) is no greater than $\mathrm{r}^{1-1/(2c)}$ when r is large. The plan for what follows is to rewrite this integral as the sum of two terms, one being the integral of $\mathrm{r}|v_X|\big||w_*|-|\psi|^2\big|$ and the other bounded by r^{1-1/c_c}. This is done in Part 7. Part 6 supplies the necessary tools. A bound of this sort with the second and third bullets of (2-9) and (3-14) plus the first bullet of Lemma 8.2 leads directly to the bound

$$\mathrm{r}\int_{B\cap s^{-1}(I)}\big|1-|\alpha|^2\big|\le c_0\mathrm{r}^{1-1/c} \tag{8-53}$$

when B is any given ball from Lemma 8.5. This bound implies Lemma 8.5's assertion if the first bullet of (3-16) holds. This bound with (8-52) imply Lemma 8.5's bound when the second bullet of (3-16) holds.

Part 6 The two lemmas that are stated in a moment and then proved supply what is needed for Part 7. To set the stage for the first lemma, note that Clifford multiplication by w_X splits $\mathbb{S}^+$ where $w_X\ne 0$ as a direct sum of eigenbundles for the endomorphism given by Clifford multiplication by w_X. Write this direct splitting as $\mathbb{S}^+=E_X\oplus(E_X\otimes K_X^{-1})$ with it understood that the leftmost summand is the $i|w_X|$–eigenspace. The upcoming lemma writes a section ψ of $\mathbb{S}^+$ where $w_X\ne 0$ as $|w_X|^{1/2}\eta$ and it writes η with respect to the direct sum decomposition of $\mathbb{S}^+$ as (α,β). The lemma that follows asserts bounds for $|\alpha|$ and $|\beta|$ that are the analogs of those asserted by the first two bullets of Lemma 7.2.

Lemma 8.6 *There exists $\kappa>100$, and given $c\ge\kappa$, there exists κ_c with the following significance: Fix $\mathrm{r}\ge\kappa_c$ and assume that the metric obey the $(c, r=\mathrm{r})$ version of the constraints in the first three bullets of (3-15) and $|w_X|\le c$, or that the first bullet of (3-16) holds. Fix elements μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1 and use all of this data to define the equations in (2-10). Let $\mathfrak{d}=(\mathbb{A},\psi)$ denote an instanton solution to these equations. Fix $m>1$. Then*

$$|\alpha|^2\le 1+\kappa_c m^3\mathrm{r}^{-1+\kappa/c}\quad\textit{and}\quad|\beta|^2<\kappa m^3\mathrm{r}^{-1+\kappa/c}(1-|\alpha|^2)+\kappa^3m^6\mathrm{r}^{-2+\kappa/c}$$

at the points in X where $|w_X|>m^{-1}$.

Proof The proof is much like that of the first two bullets in Lemma 7.2 with the only salient difference being the r–dependent bounds for the norms of the Riemannian curvature and the covariant derivatives of w_X. The paragraphs that follow briefly explain how this r–dependence is dealt with.

The section $\eta=(\alpha,\beta)$ of $\mathbb{S}^+$ obeys an equation of the form $\mathcal{D}_{\mathbb{A}}\eta+\mathfrak{R}\cdot\eta=0$ with $\mathfrak{R}$ being an endomorphism that is bounded by $c_c m^{-1}\mathrm{r}^{1/c}$ on U_{2m}. The Weitzenböck formula for the operator $(\mathcal{D}_{\mathbb{A}} + \mathfrak{R})^2$ leads to an equation for η that has the schematic form

$$\nabla_{\mathbb{A}}^{\dagger}\nabla_{\mathbb{A}}\eta - \tfrac{1}{2}\,\mathrm{cl}(F_{\mathbb{A}}^{+})\eta + \mathfrak{R}_1 \cdot \nabla_{\mathbb{A}}\eta + \mathfrak{R}_0 \cdot \eta = 0, \tag{8-54}$$

where $|R_1| \le c_c m^{-1}$ and $|R_0| \le c_c m^{-2}$. As in the proof of Lemma 7.2, introduce q to denote the maximum of 0 and $|\eta|^2 - 1$. It follows from (8-54) that q obeys the inequality

$$d^{\dagger}dq + \mathrm{r}m^{-1}q \le c_c m^{-2}\mathrm{r}^{2/c} \tag{8-55}$$

on U_{2m} when $\mathrm{r} \ge c_c$. It follows from Lemma 8.1 that $q \le c_c m$ on the boundary of U_{2m}. This understood, the comparison principle using the Green's function for $d^{\dagger}d + \mathrm{r}m^{-1}$ can be used to see that $q - c_c m^3 \mathrm{r}^{-1+2/c}$ is no greater than $c_c m e^{-\sqrt{\mathrm{r}/(2m)}}$ on U_{2m}. This bound on q implies what is said by Lemma 8.6 about $|\alpha|^2$.

To see about the bound for $|\beta|^2$, project (8-54) to the $E_X \otimes K_X^{-1}$–summand of $\mathbb{S}^+$ to see that $|\beta|^2$ obeys a differential inequality on U_{2m} that has the schematic form

$$d^{\dagger}d|\beta|^2 + \mathrm{r}m^{-1}|\beta|^2 \le -2|\nabla\beta|^2 + c_k \mathrm{r}^{-1+c_0/c}m^3|\nabla_A\alpha|^2 + c_0 m^2 \mathrm{r}^{c_0/c} \tag{8-56}$$

when $\mathrm{r} \ge c_c$. Meanwhile, the projection of (8-54) to the E_X–summand can be used to see that $\mathrm{w} = 1 - |\alpha|^2$ on U_{2m} obeys the following analog of any given $\varepsilon > 0$ version of (7-11):

$$d^{\dagger}d\mathrm{w} + \mathrm{r}m^{-1}\mathrm{w} \ge |\nabla\alpha|^2 - c_0\varepsilon|\nabla\beta|^2 - c_0(1+\varepsilon^{-1})m^2\mathrm{r}^{c_0/c}. \tag{8-57}$$

It follows from (8-56) and (8-57) that there are constants z_1 and z_2 that are both bounded by c_c, and there exists an $\varepsilon > c_c^{-1}$ such that $q := |\beta|^2 - z_1\mathrm{r}^{-1+c_0/c}m^3\mathrm{w} - z_2\mathrm{r}^{-2+c_0/c}m^6$ obeys the equation $d^{\dagger}dq + \mathrm{r}m^{-1}q \le 0$ on U_{2m}. This being the case, a comparison principle argument much like that used in the preceding paragraph bounds q by $c_c m e^{-\sqrt{\mathrm{r}/2m}}$ on U_{2m}. This bound implies Lemma 8.6's assertion about $|\beta|^2$. $\square$

The next lemma supplies an analog for X of Lemma 7.3:

Lemma 8.7 *There exists $\kappa > 100$, and given $c \ge \kappa$, there exists κ_c with the following significance: Fix $\mathrm{r} \ge \kappa_c$ and assume that the metric obeys the $(c, r = \mathrm{r})$ version of the constraints in the first three bullets of (3-15) and $|w_X| \le c$, or that the first bullet of (3-16) holds. Fix elements μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm bounded by 1 and use this data to define the equations in (2-10). Let $\mathfrak{d} = (\mathbb{A}, \psi)$ denote an instanton solution to these equations. Fix $m > 1$. Then $|\psi|^2 \le \kappa_c(m^{-1} + c_c\mathrm{r}^{-1+\kappa/c})$ at points in X where $|w_X| \le m^{-1}$.*

Proof The Weitzenböck formula for $\mathcal{D}^2_{\mathbb{A}}$ was used in Step 1 of the proof of Lemma 8.1 to write the differential inequality $d^\dagger d|\psi| + \mathrm{r}(|\psi|^2 - |w_X| - c_c \mathrm{r}^{-1+1/c})|\psi| \le 0$. The maximum principle precludes a local maximum for $|\psi|^2 - m^{-1} - c_c \mathrm{r}^{-1+1/c}$ on $X - U_m$ and Lemma 8.6 implies that $|\psi|^2 \le 2(m^{-1} + c_c m^2 \mathrm{r}^{-1+c_0/c})$ on the boundary of $X - U_m$. □

Part 7 Fix $m > 1$ for the moment and write $(ds \wedge \upsilon_X)^+$ on U_m as $q_X w_X + \mathfrak{b}_X$ with $\mathfrak{b}_X$ being a self-dual 2–form that obeys $\mathfrak{b}_X \wedge w_X = 0$. Note in this regard that

$$q_X |w_X|^2 = *(ds \wedge \upsilon_X \wedge w_X) \tag{8-58}$$

with the $*$ here denoting the Hodge star that is defined by the metric $ds^2 + \mathfrak{g}$ on $I \times Y_*$. Granted (8-58), it follows either from the first bullet of (3-16) or from the fourth bullet and item (c) of the fifth bullet of (3-15) that

$$q_X |w_X|^2 \ge -c_c \mathrm{r}^{-1/c}. \tag{8-59}$$

Noting that $*(ds \wedge \upsilon_X \wedge w_X)$ is also the $\mathfrak{g}$–Hodge star on Y_* of $\upsilon_X \wedge w_*$, the integrand of the U_m part of the integral on the left-hand side of (8-52) is

$$\mathrm{r} q_X |w_*|^2 (1 - |\alpha|^2 + |\beta|^2) + \mathfrak{r} \quad \text{where } |\mathfrak{r}| \le c_c \mathrm{r} |\mathfrak{b}||w_X||\alpha||\beta|. \tag{8-60}$$

Use the bound in (8-59) and the bounds supplied by Lemma 8.6 to see that the U_m part of the integral on the left side of (8-52) can be written as

$$\mathrm{r} \int_{U_m} |q_X||w_*| \big||w_*| - |\psi|^2\big| + \mathfrak{e} \quad \text{where } |\mathfrak{e}| \le c_c (\mathrm{r}^{1-c_0/c} + m^3 \mathrm{r}^{c_0/c}). \tag{8-61}$$

Meanwhile, it follows from Lemma 8.7 that the contribution to the integral on the left side of (8-52) from $X - U_m$ is no greater than $c_c(\mathrm{r} m^{-1} + m^2 \mathrm{r}^{c_0/c})$. Lemma 8.7 also gives such a bound for the integral of $|q_X||w_*|\big||w_*| - |\psi|^2\big|$ over the part of $I \times Y_*$ in $X - U_m$. Granted these bounds, fix for the moment $\varepsilon > 0$ but with $\varepsilon < c_0 c^{-1}$ and take $m = \mathrm{r}^{\varepsilon/c}$. Use the just stated bounds and (8-61) to see that

$$\int_{I \times Y_*} |q_X||w_*|\big||w_*| - |\psi|^2\big| \le \int_{I \times Y_*} ds \wedge \upsilon_X \wedge \mathrm{r}(w_* + *i\psi^\dagger \tau \psi) + c_c \mathrm{r}^{1-\varepsilon/c}. \tag{8-62}$$

This last bound with what is said at the end of Part 5 implies Lemma 8.5. □

8.7 Proof of Proposition 3.8

Fix a smooth, r–independent metric on X whose pullback via the embeddings from the second and third bullets of (2-8) restricts to the $s < -2$ and $s > 2$ parts of X as the product metric $ds^2 + \mathfrak{g}_*$, where $\mathfrak{g}_*$ denotes the given metric on Y_- and Y_+ as the case may be. Use $\mathfrak{m}_X$ to denote this metric. Use this metric to define the bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ over X. The constructions at the beginning of Section 8.3 can be repeated to view the Y_- and Y_+ versions of $\mathbb{S}$ as the restrictions to the respective $s < -1$ and $s > 1$ parts of X of the $\mathfrak{m}_X$ versions of $\mathbb{S}^+$ and $\mathbb{S}^-$. Use this view of these versions of $\mathbb{S}$ to view the Y_- and Y_+ versions $A_K + 2A_E$ as a Hermitian connection on the restriction of the $\mathfrak{m}_X$ version of the bundle $\det \mathbb{S}^+$ to the $|s| > 1$ part of X. This connection has smooth, r–independent extensions to the whole of X as a Hermitian connection on the $\mathfrak{m}_X$ version of $\det \mathbb{S}^+$. Fix such an extension and denote it by $\mathbb{A}_{\mathbb{S}}$.

Use the $s < -1$ and $s > 1$ isomorphisms between the Y_- and Y_+ versions of $\mathbb{S}$ to view the corresponding versions of ψ_E as a section of the $\mathfrak{m}_X$ version of $\mathbb{S}^+$ over the $|s| > 1$ part of X. Fix a smooth extension of the latter to the whole of X and denote it by $\psi_{\mathbb{S}}$.

The metric $\mathfrak{m}_X$ and the pair $\mathfrak{d}_{\mathbb{S}} = (\mathbb{A}_{\mathbb{S}}, \psi_{\mathbb{S}})$ defines a version of the operator that appears in (2.61) of [37]. This operator defines a map from $C^\infty(X; iT^*X \oplus \mathbb{S}^+)$ to $C^\infty(X; \Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R})$. The latter defines an unbounded, Fredholm operator between the L^2 versions of these spaces, and so it has a corresponding Fredholm index, this denoted in what follows by $\iota_{\mathbb{S}}$.

Fix $c > c_0$ so that Proposition 3.7 can be invoked using Y_- and Y_+. Fix $\mathrm{r} \gg 1$ and pairs μ_- and μ_+ from the respective Y_- and Y_+ versions of Ω with $\mathcal{P}$–norm less than 1, and suppose that $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are the corresponding solutions to the Y_- and Y_+ versions of (2-5). Let $\mathfrak{m}$ denote a metric on X that obeys (2-9) and (3-14). Suppose that $\mathfrak{d} = (\mathbb{A}, \psi)$ is a pair of connection on $\det \mathbb{S}^+$ over X and section over X of $\mathbb{S}^+$ with $s \to -\infty$ limit $\mathfrak{c}_-$ and $s \to \infty$ limit $\mathfrak{c}_+$. This metric $\mathfrak{m}$ and $\mathfrak{d}$ together define a corresponding version of the operator that appears in (2.61) of [37]. If both $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are nondegenerate then this operator has an unbounded, Fredholm extension whose domain and range are the respective spaces of square-integrable sections of $iT^*X \oplus \mathbb{S}^+$ and $i\Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R}$. Assume this to be the case for the moment, and let $\iota_{\mathfrak{d}+}$ denote the corresponding Fredholm index. It follows using the excision theorem for the index (or from what is said in [1]) that $\iota_{\mathbb{S}} = \iota_{\mathfrak{d}+} + \mathfrak{f}_s(\mathfrak{c}_-) - \mathfrak{f}_s(\mathfrak{c}_+)$.

With the preceding understood, write $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$ as

$$\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+) - 2\pi(\mathrm{r} - \pi)(\mathfrak{f}_s(\mathfrak{c}_-) - \mathfrak{f}_s(\mathfrak{c}_+)) \tag{8-63}$$

and then use the formula in the last paragraph to write

$$\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) = \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+) + 2\pi(\mathrm{r} - \pi)(\iota_{\mathfrak{d}+} - \iota_{\mathbb{S}}). \tag{8-64}$$

Since $\iota_{\mathbb{S}}$ is independent of r and c, this last formula proves Proposition 3.8 when both $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are nondegenerate.

If one or neither is nondegenerate, fix $\varepsilon > 0$ and fix $\mathfrak{c}'_-$ in the set $\mathfrak{N}_\varepsilon(\mathfrak{c}_-)$ from Section 7.6 that takes on the supremum in the $\mathfrak{c}_-$ version of (7-37). Fix $\mathfrak{c}'_+$ in $\mathfrak{N}_\varepsilon(\mathfrak{c}_+)$ with the analogous property. With $\mathfrak{c}'_-$ and $\mathfrak{c}'_+$ as just described, choose a pair $\mathfrak{d}'$ of connection on $\det\mathbb{S}^+$ and section of $\mathbb{S}^+$ with $s \to -\infty$ limit $\mathfrak{c}'_-$ and $s \to \infty$ limit $\mathfrak{c}'_+$. The metric $\mathfrak{m}$ with $\mathfrak{d}'$ define an unbounded, but now Fredholm version of the operator from (2.62) in [37] with domain and range being the respective spaces of square-integrable sections of $iT^*X \oplus \mathbb{S}^+$ and $i\Lambda^+ \oplus \mathbb{S}^- \oplus i\mathbb{R}$. Let $\iota_{\mathfrak{d}'}$ denote the Fredholm index of this operator. Define $\iota_{\mathfrak{d}+}$ to be $\iota_{\mathfrak{d}'+}$. Note that this definition does not depend on $\mathfrak{c}'_-$, $\mathfrak{c}'_+$ or $\mathfrak{d}'$.

The arguments that lead to (8-64) can be repeated verbatim to obtain the modified version that has $\mathfrak{c}_-$ replaced by $\mathfrak{c}'_-$ and $\mathfrak{c}_+$ replaced by $\mathfrak{c}'_+$. Keeping this in mind, choose $\mathfrak{c}'_-$ so that $|\mathfrak{a}(\mathfrak{c}'_-) - \mathfrak{a}(\mathfrak{c}_-)| < 1$, and choose $\mathfrak{c}'_+$ so that $|\mathfrak{a}(\mathfrak{c}'_+) - \mathfrak{a}(\mathfrak{c}_+)| < 1$. It follows using (7-37) that $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}'_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-)| < 1$ and $|\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}'_+) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+)| < 1$. The latter bound with the $(\mathfrak{c}'_-, \mathfrak{c}'_+)$ analog of (8-64) implies what is asserted by Proposition 3.8 when the nondegeneracy condition does not hold for one or both of $\mathfrak{c}_-$ and $\mathfrak{c}_+$. □

9 Constructing 2–forms on cobordisms

This section mainly supplies proofs for Propositions 3.9, 3.11, 3.13 and 3.14. The proof of Proposition 3.9 is in Section 9.2, that of Proposition 3.11 is in Section 9.4, that of Proposition 3.13 is in Section 9.5, and Section 9.7 contains the proof of Proposition 3.14. The basic issue in each proof is to construct metrics and closed 2–forms on cobordisms with certain prescribed properties. These constructions occupy most of these subsections. By way of a look ahead, these constructions are, on the whole, quite intricate. Note that there is little by way of the Seiberg–Witten equations in this section.

A proof of Proposition 1.5 is given in Section 9.6, using notions introduced in Section 9.5.

9.1 Met_T metrics on $\{Y_k\}_{k\in\{0,\dots,\mathrm{G}\}}$

The eight parts of this section describe a set of preferred metrics on each $k \in \{0,\dots,\mathrm{G}\}$ version of Y_k. These parts also describe the associated harmonic 2–forms with de Rham cohomology class that of $c_1(\det\mathbb{S})$. Let Y_* denote Y_k for any $k \in \{0,\dots,\mathrm{G}\}$. As the $M_\delta \cup \mathcal{H}_0$ parts of Y_* and Y are canonically isomorphic, notions defined on any of them are defined for others and are denoted by the same notation.

Part 1 This part of the subsection summarizes various properties of Y_* that concern $\mathcal{H}_0$ and the curve $\gamma^{(z_0)}$. Most of what is said below can be found in Section II.1.

The handle $\mathcal{H}_0$ in Y_* has coordinates (u,θ,ϕ) with (θ,ϕ) being the standard spherical coordinates on the 2–sphere and with $u \in [-R-\ln(7\delta_*), R+\ln(7\delta_*)]$. As can be seen in (IV.1-5), the 2–form w and the 1–form $\upsilon_\diamond$ restrict to this handle as

$$w = \sin\theta\, d\theta \wedge d\phi \quad \text{and} \quad \upsilon_\diamond = 2(\chi_+ e^{2(|u|-R)} + \chi_- e^{-2(|u|+R)})\, du, \tag{9-1}$$

where $\chi_+ = \chi\big(-u-\frac{1}{4}R\big)$ and $\chi_- = \chi\big(u-\frac{1}{4}R\big)$. The curve $\gamma^{(z_0)}$ intersects $\mathcal{H}_0$ as the $\theta = 0$ line. Meanwhile, the M_δ part of $\gamma^{(z_0)}$ has a tubular neighborhood with coordinates $(t,(\theta,\phi))$ with $t \in [\delta^2, 3-\delta^2]$, with $\theta \in [0,\theta_*)$ and with ϕ the affine coordinate on $\mathbb{R}/(2\pi\mathbb{Z})$. Here, θ_* is positive, smaller than $\frac{1}{100}\delta_*$ but greater $100\delta^3$. The 2–form w here appears as in (9-1) and $\upsilon_\diamond$ appears as dt. The coordinate transition function identifies t with $e^{-2(R-u)}$ near the index 0 critical point and with $e^{-2(R+u)}$ near the index 3 critical point.

Recall the function f on M that plays a central role in much of [19; 20; 21; 22]. This is described in detail in Section II.1. Recall also the vector field v in [20, page 2876]. Set $\varepsilon_* = \delta_* \sin\big(\frac{1}{2}\theta_*\big)$. The coordinates just described can be used to construct a piecewise smooth embedded 2–sphere in the $f \in [\varepsilon_*^2, 3-\varepsilon_*^2]$ part of M_δ as follows:

(9-2)
- The 2–sphere intersects the complement of the radius-δ_* coordinate balls about the index 0 and 3 critical points of f as the cylinder where $\theta = \frac{1}{2}\theta_*$.
- The 2–sphere intersects the $r \in (\varepsilon_*, \delta_*]$ part of the radius-δ_* coordinate ball centered on the index 0 and index 3 critical points of f as the locus where (r,θ,ϕ) are such that $\cos\theta > 0$ and $r\sin\theta = \delta_* \sin\big(\frac{1}{2}\theta_*\big)$.
- The 2–sphere intersects the $r = \varepsilon_*$ spheres centered about the index 0 and index 3 critical points as the locus where $\cos\theta \le 0$.
- The 2–sphere is tangent to v on the rest of M_δ.

As can be seen, this embedding is smooth except along the following loci: It is C^1 on the $\cos\theta = 0$ circle in the boundary of the respective radius ε_* coordinate balls about the index 0 and index 3 critical points of f. It is only C^0 on the $\theta = \frac{1}{2}\theta_*$ circle in the boundary of the respective radius δ_* coordinate balls about the index 0 and index 3 critical points.

The piecewise smooth embedding just described can be smoothed to any desired accuracy so that the vector field $\frac{\partial}{\partial\phi}$ along the resulting 2–sphere is everywhere tangent, the vector field v along the 2–sphere is tangent everywhere on the $f=\frac{3}{2}$ circle but nowhere else, and so that the restriction of f to this sphere has just two critical points (both nondegenerate), these at the points with $\theta=0$ and $\theta=\pi$ on the boundary of the radius ε_* coordinate balls about its respective index 0 and index 3 critical points.

Part 2 It proves useful for what follows to be somewhat more precise about the smoothing of the surface from (9-2) near the $f=\frac{3}{2}$ circle. To this end, introduce first ρ_* to denote $\frac{3}{2}-\varepsilon_*^2$ and $\rho_{1*}=\rho_*+\sqrt{2}\bigl(1-\cos\bigl(\frac{1}{2}\theta_*\bigr)\bigr)$. Return to the $f\in[\delta^2,3-\delta^2]$ tubular neighborhood of $\gamma^{(z_0)}$ with the coordinates $(t,(\theta,\phi))$ as described above. Replace the coordinate θ on a neighborhood of the $\theta=\frac{1}{2}\theta_*$ locus by the function $\hat{\rho}=\sqrt{2}(1-\cos\theta)^{1/2}$. Fix $\varepsilon_1\in(0,c_0^{-1}\varepsilon_*^2)$ and use the coordinate $\hat{\rho}$ to define the smoothing of the $f\in\bigl(\frac{3}{2}-\varepsilon_1,\frac{3}{2}+\varepsilon_1\bigr)$ part of the surface defined by (9-2) to be the locus where

$$\hat{\rho}=\rho_{1*}-\bigl(\rho_*^2-\bigl(t-\tfrac{3}{2}\bigr)^2\bigr)^{1/2}. \tag{9-3}$$

Note that the vector field v is tangent to the locus defined by (9-3) only along the $t=\frac{3}{2}$ circle, and note that the corresponding lines are tangent from the inside. Introduce by way of notation S to denote a smoothing as just described of the original piecewise smooth embedding given by (9-2). (This is the sphere denoted by S_z in [23], about equation (6.2).)

Part 3 Use (x_1,x_2,x_3) for the Euclidean coordinates on $\mathbb{R}^3$. The function f and the $\mathbb{R}/(2\pi\mathbb{Z})$–valued coordinate function ϕ can be used to embed a neighborhood of S into $\mathbb{R}^3$ as the sphere of radius $\rho_*-\frac{3}{2}\varepsilon_*^2$ about the origin by taking $x_3=f-\frac{3}{2}$ and by setting the pair (x_1,x_2) to equal $((\rho_*^2-x_3^2)^{1/2}\cos\phi,(\rho_*^2-x_3^2)^{1/2}\sin\phi)$. Note in this regard that the values of x_3 on the image of S range from $-\rho_*$ to ρ_* because the values of f on S range from ε_*^2 to $3-\varepsilon_*^2$.

This embedding is extended to a neighborhood of S by exploiting the fact that the $\bigl|f-\frac{3}{2}\bigr|>\frac{1}{2}\varepsilon_1$ part of S has a neighborhood with the following property: Let p denote a point in this neighborhood. Then p sits on an integral curve of v that intersects S, and there is precisely one such intersection point with distance $c_\varepsilon^{-1}\varepsilon_1^3$ or less from p. Here, $c_\varepsilon>1$ is a constant that depends on ε_1. Such a neighborhood exists because v is tangent to S only on the $f=\frac{3}{2}$ circle in S. Let $\mathcal{N}_1$ denote this neighborhood. Given $p\in\mathcal{N}_1$, let $\eta(p)\in S$ denote the unique point on the integral curve of v through p with distance less than $c_\varepsilon^{-1}\varepsilon_1^3$ from p. Associate to p the point in $\mathbb{R}^3$ with the coordinates

$$x_1(p)=x_1(\eta(p)),\quad x_2(p)=x_2(\eta(p)),\quad x_3(p)=f(p)-\tfrac{3}{2}. \tag{9-4}$$

To complete the definition of the embedding, suppose next that p is a point near the $f \in \left(\frac{3}{2} - \varepsilon_1, \frac{3}{2} + \varepsilon_1\right)$ part of S where the coordinates $(t, \hat{\rho}, \phi)$ are defined. Associate to p the point in $\mathbb{R}^3$ with the coordinates

$$\begin{aligned} x_1(p) &= |\hat{\rho}(p) - \rho_{1*}| \cos\phi(p), \\ x_2(p) &= |\hat{\rho}(p) - \rho_{1*}| \sin\phi(p), \\ x_3(p) &= t(p) - \tfrac{3}{2}. \end{aligned} \tag{9-5}$$

Note in particular that if p is also in $\mathcal{N}_1$, then it follows from the definition of the function $\hat{\rho}$ and the definition of ρ_1 that the points given by (9-4) and (9-5) are the same.

What is said at the end of the preceding paragraph has the following implication: the map from $\mathcal{N}_1$ to $\mathbb{R}^3$ and the map described in the preceding paragraph together define a smooth, ϕ–equivariant embedding of a neighborhood of S into $\mathbb{R}^3$ that maps S to the radius ρ_* sphere and maps v to $\frac{\partial}{\partial x_3}$.

Fix $\varepsilon > 0$ so that the region in $\mathbb{R}^3$ with $(x_1^2 + x_2^2 + x_3^2)^{1/2} \in (\rho_* - \varepsilon, \rho_* + \varepsilon)$ is in the image of the embedding of $\mathcal{N}_1$. By way of notation, $\mathcal{N}_\varepsilon$ is used in the subsequent discussion to denote both this region in $\mathbb{R}^3$ and its inverse image in M_δ. It is worth keeping in mind for what follows that the points in the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ with distance *greater* than ρ_* from the origin are in the $\mathcal{H}_0$ component of $Y - S$.

By construction, the 1–form $\upsilon_\diamond$ appears on the $\mathbb{R}^3$ version of $\mathcal{N}_\varepsilon$ as dx_3. Meanwhile, the 2–form w must appear here as $\kappa\, dx_1 \wedge dx_2$ with κ being a strictly positive function of $x_1^2 + x_2^2$. This is because w is closed, it annihilates v and v appears on the $\mathbb{R}^3$ version of $\mathcal{N}_\varepsilon$ as $\frac{\partial}{\partial x_3}$.

Use ρ to denote the function $(x_1^2 + x_2^2)^{1/2}$ on $\mathbb{R}^3$ and introduce the $\mathbb{R}/(2\pi\mathbb{Z})$–valued function ϕ by writing x_1 and x_2 as $\rho\cos\phi$ and $\rho\sin\phi$. The observations from the preceding paragraph, the fact that w is harmonic and the fact that its metric Hodge dual is $\upsilon_\diamond$ have the following implication: the metric from M_δ appears on the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ as

$$\mathfrak{g} = \kappa(h^{-2}\, d\rho^2 + h^2 \rho^2\, d\phi^2) + dx_3^2 \tag{9-6}$$

with h denoting a strictly positive function of ρ^2.

Part 4 This part of the subsection says something of the topological significance of S and Part 3's embedding of S and its neighborhood $\mathcal{N}_\varepsilon$ in $\mathbb{R}^3$. To set the stage, recall that Y_0 was obtained from M by attaching the 1–handle $\mathcal{H}_0$. This was done by first deleting the radius $7\delta_*$ coordinate balls about the index 0 and index 3 critical points of f to obtain a manifold with boundary. The resulting boundary spheres were then glued to the $u = R + \ln(7\delta_*)$ and $u = -R - \ln(7\delta_*)$ boundary spheres of $[-R - \ln(7\delta_*), R + \ln(7\delta_*)] \times S^2$.

The sphere S enters a second description of Y_0 as the connected sum of M with the manifold $S^1 \times S^2$. (See [23, (6.2)].) The connected sum description constructs Y_0 by deleting the respective 3–balls from M and $S^1 \times S^2$ and gluing the resulting two boundary spheres to the boundary spheres of the product of an interval with S^2. Denote this product as $I \times S^2$ with $I \subset \mathbb{R}$ being an interval. As explained below, the surface S can be viewed as a cross-sectional sphere of $I \times S^2$.

To see directly this connected sum depiction of Y_0, first view S and $\mathcal{N}_\varepsilon$ as subsets in $\mathbb{R}^3$. Let $r = (\rho^2 + x_3^2)^{1/2}$ denote the radial coordinate on $\mathbb{R}^3$. The connected sum picture of Y_0 results in an embedding of $I \times S^2$ into $\mathbb{R}^3$ whose image is the $r \in \left[\rho_* - \frac{1}{16}\varepsilon, \rho_* + \frac{1}{16}\varepsilon\right]$ part of $\mathcal{N}_\varepsilon$. This depiction of $I \times S^2$ in Y_0 identifies the $r = \rho_* + \frac{1}{16}\varepsilon$ sphere in $\mathcal{N}_\varepsilon$ with the boundary of the complement of a ball in $S^1 \times S^2$. This missing ball can be identified with the $r < \rho_* + \frac{1}{16}\varepsilon$ part of $\mathbb{R}^3$. Indeed, the Y_0 incarnation of the $r = \rho_* + \frac{1}{16}\varepsilon$ sphere in $\mathbb{R}^3$ splits Y_0 into two components. The component that contains the $r > \rho_* + \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the complement of a ball in $S^1 \times S^2$, and $S^1 \times S^2$ is reconstituted in full when this complement is filled in by adding the $r \le \rho_* + \frac{1}{16}\varepsilon$ part of $\mathbb{R}^3$ to the $r > \rho_* + \frac{1}{16}\varepsilon$ incarnation of $\mathcal{N}_\varepsilon$.

The Y_0 incarnation of the $r = \rho_* - \frac{1}{16}\varepsilon$ sphere in $\mathbb{R}^3$ also separates Y_0 into two components. The component that has the $r < \rho_* - \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the complement of a ball in M. This ball is attached to give back M by viewing the complement of its center point as the $r > \rho_* - \frac{1}{16}\varepsilon_*$ part of $\mathbb{R}^3$. To see this, take a second copy of $\mathbb{R}^3$ and use r' to denote the distance to the origin in the latter. Use (θ', φ') to denote the associated spherical coordinates. The manifold M is obtained by attaching the $r' \le \left(\rho_* - \frac{1}{16}\varepsilon\right)^{-1}$ ball in this second copy of $\mathbb{R}^3$ to the $r = \rho_* - \frac{1}{16}\varepsilon$ sphere in the original copy of $\mathbb{R}^3$ via the identifications $r' = r^{-1}$ and $(\theta' = \pi - \theta, \phi' = \phi)$.

Since S splits Y_0 into two parts, it likewise splits Y_* into two parts. The component of $Y_* - S$ that contains $\gamma^{(z_0)}$ has its canonical identification with the $\gamma^{(z_0)}$ component of $Y_0 - S$. The other component of $Y_* - S$ is obtained from the complementary component of $Y_0 - S$ by attaching the $\mathfrak{p} \in \Lambda$ labeled 1–handles.

Both $Y_* - \mathcal{N}_\varepsilon$ and $Y_0 - \mathcal{N}_\varepsilon$ likewise have two components because $\mathcal{N}_\varepsilon$ is a tubular neighborhood of S. A given $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k is obtained from Y_0 by attaching k 1–handles with attaching regions that are disjoint from the component of $Y_0 - \mathcal{N}_\varepsilon$ that contains $\gamma^{(z_0)}$. This understood, $\mathcal{N}_\varepsilon$ can be viewed as a subset of Y_k and $Y_k - \mathcal{N}_\varepsilon$ also has two components. By way of notation, the component of $Y_* - \mathcal{N}_\varepsilon$ or any given $k \in \{0, \dots, \mathrm{G}\}$ version of $Y_k - \mathcal{N}_\varepsilon$ that contains $\gamma^{(z_0)}$ is denoted in what follows by $\mathcal{Y}_0$ and the other component is denoted by $\mathcal{Y}_M$ ($\mathcal{Y}_M$ has a natural interpretation as a sutured manifold, which is denoted by $M(1)$ in Remark 1.3).

Part 5 This part of the subsection introduces a family of distinguished metrics on the $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k that play central roles in the subsequent discussions. Parts 6 and 8 say more about this set.

This distinguished set of metrics is parametrized by a parameter T which is in all cases greater than 1. With T chosen, the corresponding set of metric is denoted in what follows by Met_T. The metrics from Met_T are constructed in a moment from the set of metrics on $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ that are given by (9-6) on $\mathcal{N}_\varepsilon$. This set of metrics on $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ is denoted by $\mathrm{Met}^{\mathcal{N}}$. Note with regards to (9-6) that its formula depicts a 1–parameter family of metrics with the parameter being the length of the curve $\gamma^{(z_0)}$. The length of $\gamma^{(z_0)}$ plays no role of significance. In any event, the length is assumed to be the same for all metrics in $\mathrm{Met}^{\mathcal{N}}$ whether defined on Y or on a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k.

The criteria for membership in Met_T follow directly: All metrics in Met_T agree on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$; the metric they define on this set is denoted in what follows by $\mathfrak{g}_T$. The metric $\mathfrak{g}_T$ on $\mathcal{Y}_0$ is the metric from (3-6). Meanwhile, the metric $\mathfrak{g}_T$ on $\mathcal{N}_\varepsilon$ is defined in the three steps that follow.

Step 1 Introduce χ_r to denote the function on $\mathbb{R}^3$ given by $\chi(64\varepsilon^{-1}(r - \rho_*) - 1)$. This function equals 1 where $r < \rho_* + \frac{1}{64}\varepsilon$ and equals 0 where $r > \rho_* + \frac{1}{32}\varepsilon$. Fix $T > 1$ and introduce r_T to denote $\bigl(1 - \chi_r + \frac{1}{T}\chi_r\bigr)r$. The r derivative of r_T is strictly positive because that of χ_r is nonpositive. Set $\rho_T = r_T \sin\theta$ and $x_{3T} = r_T \cos\theta$. Noting that $d\rho_T$ and dx_{3T} are linearly independent, the quadratic form

$$\mathrm{K}(\rho_T)(h^{-2}(\rho_T)\, d\rho_T^2 + h^2(\rho_T)\rho_T^2\, d\phi^2) + dx_{3T}^2 \tag{9-7}$$

defines a smooth metric on $\mathbb{R}^3$. The metric $\mathfrak{g}_T$ on the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is given by (9-7).

Step 2 The definition of $\mathfrak{g}_T$ on the $r \in \bigl[\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon\bigr]$ part of $\mathcal{N}_\varepsilon$ requires yet another function of r. This one is defined by the rule $r \mapsto \chi(4\varepsilon^{-1}(r - \rho_*) + 2)$ and it is denoted by χ_{r*}. The function χ_{r*} is equal to 1 where $r < \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 0 where $r > \rho_* - \frac{1}{4}\varepsilon$. Set x_{3T*} to denote the function $\bigl(1 - \chi_{r*} + \frac{1}{T}\chi_{r*}\bigr)x_3$. Introduce by way of notation κ_T and h_T to denote the functions $\kappa(\rho/T)$ and $h(\rho/T)$. Noting that dx_1, dx_2 and dx_{3T} are linearly independent, the quadratic form

$$\frac{1}{T^2}\kappa_T(h_T^{-2}\,d\rho^2 + h_T^2\rho^2\,d\phi^2) + \frac{1}{T^2}dx_{3T*}^2 \tag{9-8}$$

defines a smooth metric on the $r \in \bigl[\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon\bigr]$ part of $\mathcal{N}_\varepsilon$. The latter extends the metric given in (9-7) because $\rho_T = \frac{1}{T}\rho$ and $x_{3T} = \frac{1}{T}x_3$ where $\rho < \rho_* + \frac{1}{64}\varepsilon$.

Step 3 The definition of $\mathfrak{g}_T$ on the $r < \rho_* - \frac{1}{2}$ part of $\mathcal{N}_\varepsilon$ requires one more function of r. This one is denoted by χ_{r**} and it is defined by the rule $r \mapsto \chi(4\varepsilon^{-1}(r - \rho_*) + 3)$. This function is equal to 0 where $r > \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 1 where $r < \rho_* - \frac{3}{4}\varepsilon$. With this function in hand, define the function T_* to be $T(1 - \chi_{r**}) + \chi_{r**}$. The function T_* is equal to T where $r > \rho_* - \frac{1}{2}\varepsilon$ and it is equal to 1 where $r < \rho_* - \frac{3}{4}\varepsilon$. The metric $\mathfrak{g}_T$ is defined on the $r \le \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ to be the quadratic form

$$\frac{1}{T_*^2}\kappa_{T_*}(h_{T_*}^{-2}\,d\rho^2 + h_{T_*}^2\,d\phi^2) + \frac{1}{T_*^4}dx_3^2. \tag{9-9}$$

This definition of $\mathfrak{g}_T$ smoothly extends the metric defined in (9-8). Moreover, the metric $\mathfrak{g}_T$ as just defined is the metric in (9-6) where $r < \rho_* - \frac{3}{4}\varepsilon$.

Part 6 This part of the subsection and Part 8 point out some key properties of the Met_T metrics. This part focuses on the metric $\mathfrak{g}_T$, this being the restriction of each Met_T metric to $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$. As explained in the subsequent two paragraphs, each $T > 1$ version of $\mathfrak{g}_T$ on the complement in $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ of the $r \le \rho_*$ part of $\mathcal{N}_\varepsilon$ can be viewed as the pullback of a T–independent metric on $S^1 \times S^2$ by a T–dependent embedding of the $\gamma^{(z_0)}$ component of $Y_* - S$ or $Y - S$ as the case may be. The embedding is denoted by Φ_T.

To define this T–independent metric on $S^1 \times S^2$, view $S^1 \times S^2$ as in Part 4. By way of a reminder, this view comes with a distinguished ball with a distinguished diffeomorphism onto the $r < \rho_* + \frac{1}{16}\varepsilon$ ball in $\mathbb{R}^3$ centered on the origin. There is in addition, a distinguished identification between the complement of the concentric $r \le \rho_*$ ball in $S^1 \times S^2$ and the union of $\mathcal{Y}_0$ and the $r \ge \rho_*$ part of $\mathcal{N}_\varepsilon$. The latter identifies the metric from Section 1 on $\mathcal{Y}_0$ with a metric on $S^1 \times S^2$ whose restriction to the $r \le \rho_* + \frac{1}{16}\varepsilon$ ball in the distinguished coordinate chart appears as $\kappa(\rho)(h^{-2}(\rho)d\rho^2 + h^2(\rho)\rho^2 d\phi^2) + dx_3^2$. This is the desired T–independent metric on $S^1 \times S^2$. This $S^1 \times S^2$ metric is denoted by $\mathfrak{g}_*$.

Fix $T \geq 1$. The promised embedding of the $\mathcal{Y}_0$ component of $Y_* - S$ into $S^1 \times S^2$ is defined as follows: This embedding agrees with the embedding from the preceding paragraph on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Meanwhile, the promised embedding on the $r \in \left(\rho_*, \rho_* + \frac{1}{16}\varepsilon\right)$ part of $\mathcal{N}_\varepsilon$ maps the latter onto the $r \in \left(T^{-1}\rho_*, \rho_* + \frac{1}{16}\varepsilon\right)$ ball in the distinguished coordinate chart. The map here sends the point with spherical coordinates (r, θ, ϕ) to that with spherical coordinates (r_T, θ, ϕ).

Part 7 This part of the subsection describes a certain closed 2–form on a given $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k with compact support in $\mathcal{Y}_M$ and with the following additional property: the de Rham class of this 2–form annihilates all but the $H_2(M; \mathbb{Z})$–summand in the Mayer–Vietoris direct sum decomposition for $H_2(Y; \mathbb{Z})$ in (IV.1-4) or in the analogous direct sum decomposition for $H_2(Y_k; \mathbb{Z})$. Meanwhile, it acts on the $H_2(M; \mathbb{Z})$–summand as $c_1(\det \mathbb{S})$. A version of this 2–form is also defined on M. In all cases, the 2–form is denoted by p. It is used in the upcoming Lemma 9.1 and in later subsections. The construction of p follows directly.

View M_δ as being a subset of each $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. As such, it sits in the $\mathcal{Y}_M$ part of Y_k. It follows from the description of $H_2(Y; \mathbb{Z})$ in Part 4 of Section II.1C that there exists a finite set of the form Θ whose elements are pairs of the form $(\gamma, \mathrm{z}_\gamma)$, with γ being a loop in a level set of M_δ of the function f on M. Meanwhile, z_γ is an integer. The loops from Θ generate the image in any given $k \in \{0, \dots, \mathrm{G}\}$ version of $H_1(Y_k; \mathbb{Z})/\mathrm{Tors}$ of $H_1(M; \mathbb{Z})/\mathrm{Tors}$ via the Mayer–Vietoris homomorphism for the Y_k analog of the direct sum decomposition in (IV.1-4). Meanwhile, the paired integers are such that $\sum_{\gamma \in \Theta} \mathrm{z}_\gamma \gamma$ represents the image of the Poincaré dual of the restriction of $c_1(\det \mathbb{S})$ to the $H_2(M; \mathbb{Z})$–summand in this same direct sum decomposition. Let $(\gamma, \mathrm{z}_\gamma)$ denote a pair from Θ. The loop γ has a tubular neighborhood in M_δ which is the image via an embedding of $S^1 \times D$, where $D \subset \mathbb{R}^2$ is a small radius disk about the origin and where γ corresponds to the image of $S^1 \times \{0\}$. Use $\mathcal{T}_\gamma$ in what follows to denote a tubular neighborhood of this sort. These are to be chosen so that the pairwise distinct versions have disjoint closure that is disjoint from the boundary of the closure of the M_δ part of $\mathcal{N}_\varepsilon$.

Note that there exists such a tubular neighborhood with an embedding that has the following property: the pullback of df via the embedding is a constant 1–form from the D factor of $S^1 \times D$ and the kernel of the pullback via the embedding of the 2–form w is a constant vector field that is tangent to this D factor. The existence of such an embedding follows from two facts, the first being that γ is in an f–level set. The second fact follows from the definition in the first bullet of (IV.1-3) of w on $\mathcal{T}_\gamma$ as the area form for the f–level sets. An embedding of this sort is used in Part 6 of the upcoming Section 9.7.

Fix a compactly supported 2–form on D whose integral is equal to 1. View this 2–form first as an S^1–independent form on $S^1 \times D$ and then as a 2–form on M and on each $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k with compact support in $\mathcal{T}_\gamma$. Use p_γ to denote the latter incarnation; then set $p = \sum_{(\gamma, z_\gamma) \in \Theta} z_\gamma p_\gamma$. By construction, the de Rham class of p agrees with $c_1(\det \mathbb{S})$ on the $H_2(M; \mathbb{Z})$–summand of the Mayer–Vietoris direct sum decomposition of $H_2(Y; \mathbb{Z})$ in (IV.1-4) or its analog for $H_2(Y_0; \mathbb{Z})$ as the case may be. The de Rham class of p also annihilates the $H_2(\mathcal{H}_0; \mathbb{Z})$–summand in these direct sum decompositions. In the case of $H_2(Y; \mathbb{Z})$, the de Rham class of p also annihilates the $\bigoplus_{\mathfrak{p} \in \Lambda} H_2(\mathcal{H}_{\mathfrak{p}}; \mathbb{Z})$–summand in (IV.1-4).

Part 8 Fix $k \in \{0, \dots, \mathrm{G}\}$. Given $T > 1$ and a metric from Met_T on Y_k, the next lemma uses w_T to denote the associated harmonic 2–form on Y_k whose de Rham cohomology class is that of $c_1(\det \mathbb{S})$.

Lemma 9.1 *There exists $\kappa > 1$ with the following significance: Fix a metric from the Y_k version of Met_T so as to define w_T. Let $\|p\|_2$ denote the metric L^2–norm of p, and let w be the closed 2–form from (3-5). Then the L^2–norm of w_T is at most $\kappa(1 + \|p\|_2)$ and the C^1–norm of $w_T - w$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is at most $\kappa T^{-1/2}$.*

Proof The proof has four steps.

Step 1 The L^2–norm of w_T as defined by the metric from Met_T on Y_k is greater than c_0^{-1} because the integral of w_T over $\mathcal{H}_0$ must be greater than c_0^{-1} so as to have integral 2 on each cross-sectional 2–sphere. As explained directly, the L^2–norm of w_T is also less than $c_0(1 + \|p\|_2)$. The proof that this is so uses the fact that a given harmonic form minimizes the L^2–norm amongst all closed forms in its de Rham cohomology class. To obtain such a form, reintroduce the coordinates (t, z) for U_γ and let B denote a smooth function with compact support centered on the origin in $\mathbb{C}$ and with integral 2. Choose a T–independent version of B so that its incarnation as a function on U_γ has support in $U_\gamma \cap \mathcal{H}_0$. With B chosen, set p_0 to denote $\frac{i}{2} \mathrm{B}\, dz \wedge d\bar{z}$. This is a closed, compactly supported 2–form in $\mathcal{Y}_0$ whose de Rham cohomology class when viewed in either $H^2(Y_k; \mathbb{Z})$ has pairing zero with all but the $H_2(\mathcal{H}_0; \mathbb{Z})$–summand in the Y_k version of (IV.1-4). By construction, the de Rham cohomology class of $p_{\mathbb{S}} = p_0 + p$ is that of $c_1(\det \mathbb{S})$. The metric L^2–norm of $p_{\mathbb{S}}$ is less than $c_0(1 + \|p\|_2)$.

Step 2 Use σ to denote the function on $\gamma^{(z_0)}$'s component of $Y_k - S$ that equals 1 on $\gamma^{(z_0)}$'s component of $Y_k - \mathcal{N}_\varepsilon$ and is given near S by the function on the $r \geq \rho_*$ part of $\mathbb{R}^3$ by the radial function $r \mapsto \chi(2 - 128\varepsilon^{-1}(r - \rho_*))$. The function σ is equal to 1 where $r > \rho_* + \frac{1}{64}\varepsilon$ and it is equal to 0 where $r < \rho_* + \frac{1}{128}\varepsilon$.

Use $\mathfrak{e}_T$ to denote the Φ_T^{-1}–pullback to $S^1 \times S^2$ of the 2–form σw_T. This 2–form is supported on the complement in $S^1 \times S^2$ of the $r < \frac{1}{T}\big(\rho_* + \frac{1}{128}\varepsilon\big)$ part of the distinguished coordinate ball. It follows from what is said in Step 1 that the L^2–norm of $\mathfrak{e}_T$ is bounded from below by c_0^{-1} and bounded from above by c_0.

Use $*$ to denote the $\mathfrak{g}_*$–Hodge dual on $S^1 \times S^2$. Note that $d\,\mathfrak{e}_T$ and $d * \mathfrak{e}_T$ are equal to zero on the complement of the $r \leq \frac{1}{T}\big(\rho_* + \frac{1}{64}\varepsilon\big)$ part of the distinguished coordinate chart. Meanwhile, the norms of both are bounded by $c_0 T |(\Phi_T^{-1}) * w_T|_{\mathfrak{g}_*}$ on this same ball. This observation, the fact that the $\mathfrak{g}_T$ metric is the Φ_T–pullback of $\mathfrak{g}_*$ and the fact that the $\mathfrak{g}_*$–volume of the $r \leq \frac{1}{T}\big(\rho_* + \frac{1}{64}\varepsilon\big)$ coordinate ball is bounded by $c_0 T^{-3}$ implies that the L^1–norm of both $d\,\mathfrak{e}_T$ and $d * \mathfrak{e}_T$ is bounded by $c_0(1 + \|\mathfrak{p}\|_2)T^{-1/2}$.

Step 3 The 2–form w appears in the $r \geq \rho_*$ part of the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ as $\kappa(\rho)\rho d\rho \wedge d\phi$. The latter form extends smoothly to the $r \leq \rho_*$ part of $\mathbb{R}^3$ as a $\mathfrak{g}_*$–harmonic 2–form. It follows as a consequence that w's restriction to $\mathcal{Y}_0$ and to the $r \geq \rho_* + \frac{1}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the pullback by all Φ_T of the $\mathfrak{g}_*$–harmonic 2–form on $S^1 \times S^2$ whose de Rham class has pairing equal to 2 with the generator of $H_2(S^1 \times S^2; \mathbb{Z})$. This corresponding form on $S^1 \times S^2$ is $\frac{1}{2\pi}\sin\theta\, d\theta \wedge d\phi$ and also denoted by w.

Step 4 Introduce the operator $\mathfrak{D}_* = *d + d*$ on $S^1 \times S^2$ and use it to write the 2–form $\mathfrak{e}_T$ as $(1 + \mathfrak{z}_T)w + u_T$ with $\mathfrak{z}_T$ denoting a constant with norm bounded by $c_0 T^{-3/2}$ and with u_T denoting a 2–form which is L^2–orthogonal to w and such that $\mathfrak{D}u_T = \mathfrak{D}\mathfrak{e}_T$. As the Green's function kernel for $\mathfrak{D}$ is smooth on the complement of the diagonal in $\times_2(S^1 \times S^2)$, the fact that $\mathfrak{D}\mathfrak{e}_T$ has support where $r < \frac{1}{T}\big(\rho_* + \frac{1}{64}\varepsilon\big)$ and the $c_0(1 + \|\mathfrak{p}\|_2)T^{-1/2}$ bound on its L^1–norm implies that $|u_T| + |\nabla u_T| \leq c_0(1 + \|\mathfrak{p}\|_2)T^{-1/2}$ on $\mathcal{Y}_0$ and also on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. $\square$

9.2 Proof of Proposition 3.9

The three parts of this subsection prove the assertion made by Proposition 3.9.

Part 1 Let Y_Z denote a given compact, oriented 3–manifold and let z denote a nonzero class in $H^2(Y_Z; \mathbb{Z})/\mathrm{Tors}$. Hodge theory associates to each metric on Y_Z a harmonic 2–form whose de Rham cohomology class is z. Of specific interest in what follows are metrics whose associated harmonic 2–form has transverse zeros. There is a residual set of metrics on Y_Z with this property; see for example [11] for a proof.

Fix $k \in \{0, \dots, \mathrm{G}\}$. Let $\mathfrak{g}_{\mathcal{N}}$ denote a metric in the Y_k version of $\mathrm{Met}^{\mathcal{N}}$. Fix $T > 1$ and use $\mathfrak{g}_{\mathcal{N}}$ to define a metric in Met_T, this denoted by $\mathfrak{g}_1$. Let w_1 denote the associated harmonic 2–form with de Rham cohomology class $c_1(\det \mathbb{S})$. If w_1 has degenerate zeros, fix a second metric, $\mathfrak{g}_2$, on Y_k with the following properties: Let w_2 denote the corresponding $\mathfrak{g}_2$ harmonic 2–form. Then w_2 has nondegenerate zeros, and the $\mathfrak{g}_1$–norms of $w_2 - w_1$ and $\mathfrak{g}_2 - \mathfrak{g}_1$ and those of their $\mathfrak{g}_1$–covariant derivatives to order 100 are less than T^{-1}. If w_1 has nondegenerate zeros, take $\mathfrak{g}_2 = \mathfrak{g}_1$.

Part 2 Write w_2 on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as $w + u_2$. By Lemma 9.1, the 2–form is such that $|u_2| \le c_0 T^{-1/2}$. This 2–form is also exact; but, more to the point, u_2 can be written as dz_2, where z_2 is a 1–form with $|z_2| < c_0 T^{-1/2}$ on the $r \ge \rho_* + \frac{5}{8}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Hold on to z_2 for the moment. Let $\sigma_\perp$ denote the function of r on $\mathcal{N}_\varepsilon$ given by $\sigma_\perp = \chi(8\varepsilon^{-1}(r - \rho_*) - 5)$. This function is equal to 1 where $r < \rho_* + \frac{5}{8}\varepsilon$ and it is equal to 0 where $r > \rho_* + \frac{3}{4}\varepsilon$. Use w_3 to denote the closed 2–form on Y_* that is given by w_2 on $\mathcal{Y}_M$, given by w on $\mathcal{Y}_0$ and given by $w + d(\sigma_\perp z_2)$ on $\mathcal{N}_\varepsilon$. The 2–form w_3 has the same de Rham class as w_2, the same zero locus as it agrees with w_2 where both are zero, and $|w_2 - w_3| \le c_0 T^{-1/2}$.

Use $\upsilon_\diamond$ to denote the $\mathfrak{g}_*$–Hodge dual on $S^1 \times S^2$ of the 2–form $w = \sin\theta\, d\theta \wedge d\phi$. Write the $\mathfrak{g}_2$–Hodge star of w_2 as $\upsilon_\diamond + \mathcal{q}_2$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. As both the $\mathfrak{g}_2$–Hodge star of w_2 and $\upsilon_\diamond$ are exact on $\mathcal{N}_\varepsilon$, it follows that $\mathcal{q}_2 = d\mathcal{o}_2$ on $\mathcal{N}_\varepsilon$. Moreover, such a function $\mathcal{o}_2$ can be found with $|\mathcal{o}_2| \le c_0 T^{-1/2}$ on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. This is so because $|w - w_2| < c_0 T^{-1}$ and $|\mathfrak{g}_2 - \mathfrak{g}_*| < c_0 T^{-1}$ on this part of $\mathcal{N}_\varepsilon$. Fix a version of $\mathcal{o}_2$ that obeys this bound. Let υ_3 denote the closed 1–form on Y_* given by $\upsilon_\diamond$ on $\mathcal{Y}_0$, by the $\mathfrak{g}_2$–Hodge star of w_2 on $\mathcal{Y}_M$ and given on $\mathcal{N}_\varepsilon$ by $\upsilon_\diamond + d(\sigma_\perp \mathcal{o}_2)$. This closed 1–form is such that $w_3 \wedge \upsilon_3 \ge 0$ when $T > c_0$ with equality only at the zeros of w_3.

With $T > c_0$ chosen, the upcoming Lemma 9.2 uses what was just said about w_3 and υ_3 as input to supply a metric on Y_* with the properties in the list that follows. This new metric is denoted by $\mathfrak{g}_{3T}$. The $\mathfrak{g}_{3T}$–Hodge star sends w_3 to υ_3; thus w_3 is $\mathfrak{g}_{3T}$–harmonic. The metric $\mathfrak{g}_{3T}$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is the metric $\mathfrak{g}_*$. The metric $\mathfrak{g}_{3T}$ on the $r \in [\rho_* + \frac{1}{2}\varepsilon, \rho_* + \frac{3}{4}\varepsilon]$ part of $\mathcal{N}_\varepsilon$ can be written as $\mathfrak{g}_2 + \mathfrak{h}$ with $\mathfrak{h}$ and its $\mathfrak{g}_2$–covariant derivatives to order 20 having $\mathfrak{g}_2$–norm less than $c_0 T^{-1}$. Finally, the metrics $\mathfrak{g}_{3T}$ and $\mathfrak{g}_2$ are identical except on the rest of Y.

Any sufficiently large T version of the metric $\mathfrak{g}_{3T}$ meets the requirements of Proposition 3.9's space *Met*. Conversely, each metric in *Met* is a sufficiently large T version of a metric $\mathfrak{g}_{3T}$ that is constructed as described above from some metric in $\mathrm{Met}^{\mathcal{N}}$. The lower bound on T depends on various properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric, these being an upper bound on the norm of the metric's Riemann curvature, the metric volume of $\mathcal{Y}_M$, and a lower bound on the metric's injectivity radius.

Part 3 The existence of the metric $\mathfrak{g}_{3T}$ follows from the first lemma below:

Lemma 9.2 *Let Y_Z denote an oriented 3–manifold and let $\mathfrak{g}$ denote a given Riemannian metric on Y_Z. Use $*$ in what follows to denote the Hodge star defined by $\mathfrak{g}$. Suppose that U and V are open sets in Y_Z with the closure of V being a compact subset of U. Let ω and υ denote respectively a 2–form and a 1–form on Y_Z such that $\omega \wedge \upsilon > 0$ on U and such that $*\omega = \upsilon$ on $Y_Z - V$.*

- *There are smooth metrics on Y_Z which equal $\mathfrak{g}$ on $Y_Z - U$ and have Hodge star sending ω to υ. Moreover, there exists metric of this sort whose volume 3–form is the same as the $\mathfrak{g}$–volume 3–form.*
- *Fix $k \in \{0, 1, \dots\}$ and $\mathrm{D} > 1$. There exists $\kappa > 1$ with the following significance: Suppose that the C^k–norms on U of ω and υ and the Riemann curvature tensor of $\mathfrak{g}$ are less than D. Then Y_Z has a metric that obeys the conclusions of the first bullet and differs from $\mathfrak{g}$ by a tensor whose $\mathfrak{g}$–norm and those of its $\mathfrak{g}$–covariant derivatives to order k are bounded by κ times the C^k–norm of $*\omega - \upsilon$.*

Lemma 9.2 has a generalization that holds for 1–parameter families of data sets. This parametrized version is given below but used in the next subsection.

Lemma 9.3 *Let $\{(\mathfrak{g}_\tau, \omega_\tau, \upsilon_\tau)\}_{\tau \in [0,1]}$ denote a smoothly parametrized family of metrics, 2–forms and 1–forms on Y_Z with $\omega_\tau \wedge \upsilon_\tau > 0$ on U and such that the $\mathfrak{g}_\tau$–Hodge dual of ω_τ is υ_τ on $Y_Z - V$. There is a corresponding smooth, 1–parameter family of metrics such that each $\tau \in [0, 1]$ member obeys the conclusion of the first bullet of Lemma 9.2. Moreover, this new family of metrics can be chosen to obey the properties listed below:*

- *Let $I \subset [0, 1]$ denote an open neighborhood of one or both of the endpoints. Suppose that the conclusions of the first bullet of Lemma 9.2 hold for $(\mathfrak{g}_\tau, \omega_\tau, \upsilon_\tau)$ when $\tau \in I$. There is a neighborhood $I' \subset I$ of the endpoints such each $\tau \in I'$ member of the new family is the corresponding $\mathfrak{g}_\tau$.*

- *Given a nonnegative integer k and $\mathrm{D} \geq 1$, there exists $\kappa > 1$ with the following significance: Suppose that the conditions of the second bullet of Lemma 9.2 are satisfied for each $\tau \in [0,1]$ and that the C^k–norms of the τ–derivatives to order k of $\{(g_\tau, \omega_\tau, \upsilon_\tau)\}_{\tau\in[0,1]}$ are also bounded by D. There is a 1–parameter family of metrics that obeys the preceding bullet and the first and second bullets of Lemma 9.2. In addition, each $\tau \in [0,1]$ member of the family differs from the corresponding metric $\mathfrak{g}_\tau$ by a tensor whose τ–derivatives to order k have C^k–norm bounded by κ times the C^k–norm of the sum of the τ–derivatives to order k of the difference between υ_τ and the $\mathfrak{g}_\tau$–Hodge star of ω_τ.*

Proof of Lemmas 9.2 and 9.3 Let Ω denote $\mathfrak{g}$'s volume 3–form. Write $\omega \wedge \upsilon$ as $\mathfrak{q}\Omega$ with $\mathfrak{q}$ being a nonnegative function on U. Let v denote the vector field on U that is annihilated by ω and has pairing $\mathfrak{q}$ with υ. Let $\mathrm{Ker}(\upsilon) \subset TU$ denote the 2–plane bundle that is annihilated by υ. The 2–form ω is symplectic on $\mathrm{Ker}(\upsilon)$ and so orients $\mathrm{Ker}(\upsilon)$. Choose an ω–compatible almost complex structure on $\mathrm{Ker}(\upsilon)$, denoted by J below. Note in this regard that there are no obstructions to finding such an almost complex structure. This is so because the space of almost complex structures that are compatible with a constant symplectic form on $\mathbb{R}^2$ is contractible. The construction just given yields a new metric with volume 3–form Ω.

With J chosen, a metric on U is defined as follows: The vector field v has norm $\mathfrak{q}^{1/2}$ and is orthogonal to $\mathrm{Ker}(\upsilon)$. The inner product between vectors v and v' in a given fiber of $\mathrm{Ker}(\upsilon)$ is $\mathfrak{q}^{-1/2}\omega(v, Jv')$. A metric of this sort has $*\omega = \upsilon$ and is such that both ω and υ have norm $\mathfrak{q}^{1/2}$. Moreover, any metric with these two properties is of the form just described. In particular, any two differ only with respect to the choice of the almost complex structure on the $\mathrm{Ker}(\upsilon)$.

Let J_1 denote a chosen, ω–compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_U$ and let $\mathfrak{g}_1$ denote the corresponding metric. The metric $\mathfrak{g}$ on $U - V$ is by necessity of the sort just described, thus it differs from $\mathfrak{g}_1$ only on $\mathrm{Ker}(\upsilon)$. In particular, the metric $\mathfrak{g}$ on $\mathrm{Ker}(\upsilon)$ is given by $\mathfrak{q}^{-1/2}\omega(v, J_{\mathfrak{g}}v')$ with $J_{\mathfrak{g}}$ being an ω–compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_{U-V}$. As noted above, if point $p \in U$, then the space of $\omega|_p$–compatible almost complex structures on $\mathrm{Ker}(\upsilon)|_p$ is contractible. This understood, there are no obstructions to choosing an ω–compatible almost complex structure on $\mathrm{Ker}(\upsilon)|_U$ that agrees with $J_{\mathfrak{g}}$ near $Y_Z - U$ and agrees with J_1 on V. Let J_2 denote an almost complex structure of this sort. The metric defined as instructed above by J_2 has the properties that are asserted by the first bullet of Lemma 9.2.

The assertions of the second bullet of Lemma 9.2 and those of Lemma 9.3 are proved by taking care with the choice of J_2 and its $\tau \in [0,1]$ counterparts. As the details are straightforward and rather tedious, they are omitted. □

9.3 Met_T metrics on cobordisms

Lemma 9.1 has an analog given below that concerns self-dual forms on cobordisms. The cobordism manifold is denoted below by X and it is assumed to be of the sort that is described in Section 3.3 with its constant s slices where $s < -1$ and $s > 1$ given as follows: Either one is Y and the other is Y_{G}, or one is some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or one is Y_0 and the other is $M \sqcup S^1 \times S^2$. The case when both are Y or both some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k is also allowed, but only the case where both are Y_{G} are needed in what is to come. The topology of X is further constrained by the requirement that s have 1 critical point when it is not diffeomorphic to a product with $\mathbb{R}$. If one of these slices is Y and the other Y_{G}, or if both are Y or both Y_k for $k \in \{1, \dots, \mathrm{G}\}$, then s has no critical points and the cobordism manifold X is $\mathbb{R} \times Y$ or $\mathbb{R} \times Y_k$ as the case may be, with the projection to $\mathbb{R}$ given by the function s.

One more constraint on X is needed. By way of background, what is said in Part 4 of Section 9.1 identifies $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ as a subset of Y and Y_k, and also $S^1 \times S^2$. This extra constraints uses $\mathcal{Y}_{0\varepsilon}$ to denote the union of $\mathcal{Y}_0$ and the $r > \rho_* + \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Here is the extra constraint:

(9-10) There is a distinguished embedding of $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ into X with the following property: the respective $s < 0$ and $s > 0$ slices of the image of this embedding, when written using the diffeomorphisms from the second and third bullets of (2-8), appear as the incarnation of $\mathcal{Y}_{0\varepsilon}$ in either Y, Y_k or $S^1 \times S^2$ as the case may be.

The metric for X is assumed to obey a constraint that requires membership in an analog for X of the various $T > 1$ versions of the space Met_T. The definition of this X version of Met_T requires the a priori selection of metrics $\mathfrak{g}_-$ and $\mathfrak{g}_+$ from the respective Y_- and Y_+ versions of Met_T with it understood that Met_T in the case of $M \sqcup (S^1 \times S^2)$ is the space consisting of the metric $\mathfrak{g}_*$ on $S^1 \times S^2$ and a metric on M of the following sort: If $c_1(\det \mathbb{S})$ is torsion on M, then any metric on M is allowed. If this class is not torsion, then the metric's associated harmonic 2–form with de Rham cohomology class $c_1(\det(\mathbb{S}|_M))$ has nondegenerate zeros. Meanwhile, Met_T for any given $k \in \{1, \dots, \mathrm{G}\}$ version of $Y_k \sqcup (S^1 \times S^2)$ consists of a Met_T metric for Y_{k-1} and any metric for $S^1 \times S^2$. Reintroduce from Part 5 of Section 9.1 the metric $\mathfrak{g}_T$ on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$. Of immediate interest in what follows is $\mathfrak{g}_T$'s restriction to $\mathcal{Y}_{0\varepsilon}$. By way of a reminder, $\mathfrak{g}_T$ on $\mathcal{Y}_{0\varepsilon}$ is the metric $\mathfrak{g}_*$ on $\mathcal{Y}_0$ and it is the metric in (9-7) on the $r > \rho_* + \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

The analog of Met_T for X consists of the space of metrics with the following three properties:

(9-11)
- The metric obeys the $L = 100$ version of (2-9).
- The metric pulls back via the embedding in (9-10) as the metric $ds^2 + \mathfrak{g}_T$.
- The metric pulls back from the $s \le -104$ part of X via the embedding in the second bullet of (3-10) as $ds^2 + \mathfrak{g}_-$, and it pulls back from the $s > 104$ part of X via the embedding from the third bullet of (2-8) as $ds^2 + \mathfrak{g}_+$.

This analog for X of Met_T is denoted in what follows by Met_T also, its dependence on $\mathfrak{g}_-$ and $\mathfrak{g}_+$ being implicit.

Lemma 9.4 given in a moment supplies the promised analog to Lemma 9.1. To set the notation, suppose that a metric on X has been specified and that p_X is a differential form on X. The lemma uses $\langle p_X \rangle_2$ to denote the L^2–norm of p_X over the $|s| < 104$ part of X. Lemma 9.4 uses w_- and w_+ to denote the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2–forms with de Rham cohomology class that of $c_1(\det \mathbb{S})$; and it uses the embeddings from the second and third bullets of (2-8) to view w_- and w_+ as 2–forms on the $s \le -1$ and $s > 1$ parts of X.

Lemma 9.4 *Let X denote a cobordism manifold of the sort described above. Given metrics $\mathfrak{g}_-$ and $\mathfrak{g}_+$ in the respective Y_- and Y_+ versions of Met_T, there exists $\kappa > 1$ with the following significance: Fix $T > 1$, and fix a Riemannian metric on X from the corresponding set Met_T. There is a self-dual, harmonic 2–form on X whose pullback to the constant s slices of X converges as $s \to -\infty$ to w_- and as $s \to \infty$ to w_+. Let p_X denote a closed 2–form on X that equals w_- where $s < -102$, that equals w_+ where $s > 102$, and with de Rham cohomology class that of $c_1(\det \mathbb{S})$.*

- *The L^2–norm of this harmonic self-dual 2–form on the s–inverse image of any length 1 interval in $\mathbb{R}$ is bounded by $\kappa\langle p_X \rangle_2$.*
- *The pullback of this harmonic self-dual 2–form to the constant $s > 1$ and $s < -1$ slices differs in the C^1–topology from w_- and w_+ by at most $\kappa\langle p_X \rangle_2 e^{-|s|/z}$ with $z \ge 1$ depending on the corresponding limit metric.*

- *The pullback of this harmonic self-dual 2–form to $\mathbb{R}\times\mathcal{Y}_{0\varepsilon}$ via the embedding from* (9-10) *differs from $ds\wedge\upsilon_\diamond + w$ by a 2–form whose C^1–norm on $\mathbb{R}\times\mathcal{Y}_0$ and on the $r>\rho_*+\frac{1}{2}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$ is less than $\kappa\langle p_X\rangle_2 T^{-1/2}$.*

Proof The existence of a closed, self-dual harmonic 2–form with the desired $s\to-\infty$ and $s\to\infty$ limits follows from the index theorem in [1]. This 2–form is denoted in what follows by ω. Given the first bullet, then the assertion in the second bullet follows from the eigenfunction expansion that is depicted below in (9-13). As explained next, the third bullet also follows from the second bullet.

To prove the third bullet, fix $s_0\subset\mathbb{R}$ and introduce σ_0 to denote the function on $\mathbb{R}$ given by the rule $s\mapsto\chi(|s-s_0|-1)$. This function equals 1 where $|s-s_0|$ is less than 1 and it equals zero where $|s-s_0|$ is greater than 2. Let σ denote the function from Step 2 of the proof of Lemma 9.1 and let Φ_T denote the embedding from Part 5 of Section 9.1. View the Φ_T^{-1} pullback of $\sigma_0\sigma\omega$ as a 2–form on $\mathbb{R}\times(S^1\times S^2)$ with support where $|s-s_0|<2$. The assumed L^2–bound for ω with a Green's function argument much like that used in Step 4 of the proof of Lemma 9.1 can be used to derive the pointwise bound that is asserted by Lemma 9.4. The derivation differs little from that in Step 4 of the proof of Lemma 9.1 save for the fact that the Green's function in question is that for the elliptic operator

$$\mathcal{D}\colon C^\infty(\mathbb{R}\times(S^1\times S^2);\Lambda^+\oplus\mathbb{R})\to C^\infty\big(\mathbb{R}\times(S^1\times S^2);T^*(\mathbb{R}\times(S^1\times S^2))\big) \tag{9-12}$$

given by the formula $\mathcal{D}=*_X d_X+d_X$, where d_X denotes the 4–dimensional exterior derivative $ds\wedge\frac{\partial}{\partial s}(\cdot)+d$ and where $*_X$ denotes the Hodge star for the metric $ds^2+\mathfrak{g}_*$.

The lemma's first bullet is proved in the four steps that follow.

Step 1 Let ω denote the relevant closed, self-dual harmonic form. Fix an integer $n\in\{106,107,\dots\}$ and introduce by way of notation $I_n\subset\mathbb{R}$ to denote a closed interval of length $2n$ whose endpoints have distance 106 or more from the origin. Let C denote the space of closed 2–forms on the domain $s^{-1}(I_n)$ that agree with ω on some neighborhood of the s–inverse images of the boundary points of I_n. The 2–form ω is the minimizer in C of the functional that is defined by the rule $\mathfrak{w}\mapsto\int_{s^{-1}(I_n)}|\mathfrak{w}^+|^2$.

Step 2 Use the embedding from the second bullet of (2-8) to write the $s\ge 100$ part of X as $[100,\infty)\times Y_+$ and likewise write the $s<-100$ part of X as $(-\infty,-100]\times Y_-$. Let Y_* for the moment denote either Y_+ or Y_-. Let $*$ denote either the $\mathfrak{g}_-$ or $\mathfrak{g}_+$ version of the Hodge star on Y_*. The corresponding operator $d*$ defines an unbounded, self-adjoint operator on the space of closed 2–forms on Y_*. Let Ξ^- denote an L^2–

orthonormal basis of eigenvectors of $d*$ on the space of closed 2–forms with negative eigenvalue and let Ξ^+ denote an L^2–orthonormal basis of eigenvectors of $d*$ with positive eigenvalue. The eigenvalue of $d*$ on a given eigenvector, a, is denoted by λ_{a}.

The 2–form ω on $(-\infty,-1]\times Y_-$ and on $[1,\infty)\times Y_+$ can be written as

$$\text{(9-13)}\quad \begin{cases} \omega = ds\wedge *w_- + w_- + \sum_{\mathrm{a}\in\Xi^+} \mathrm{z}_{\mathrm{a}} e^{\lambda_{\mathrm{a}}(s+1)}(ds\wedge *\mathrm{a}+\mathrm{a}) & \text{where } s\le -104,\\ \omega = ds\wedge *w_+ + w_+ + \sum_{\mathrm{a}\in\Xi^-} \mathrm{z}_{\mathrm{a}} e^{\lambda_{\mathrm{a}}(s-1)}(ds\wedge *\mathrm{a}+\mathrm{a}) & \text{where } s\ge 104.\end{cases}$$

What is denoted by $\mathrm{z}_{(\cdot)}$ in (9-13) is a real number. Keep in mind for what follows that any given version of $e^{\lambda_{\mathrm{a}}s}(ds\wedge *\mathrm{a}+\mathrm{a})$ is the exterior derivative on its domain of definition of the 1–form $\mathfrak{q}_{\mathrm{a}} = \lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}s} *\mathrm{a}$.

Step 3 Fix $m>1$. Let a denote an eigenvector in the Y_- version of Ξ^+. Introduce σ_{a} to denote the function on $\mathbb{R}$ given by the rule $s\mapsto\sigma_{\mathrm{a}}(s)=1-\chi(-m^{-1}\lambda_{\mathrm{a}}(s+102)-1)$. This function equals 0 where $s>-102-m\lambda_{\mathrm{a}}^{-1}$ and it equals 1 where $s<-102-2m\lambda_{\mathrm{a}}^{-1}$. If a is in the Y_+ version of Ξ^-, then σ_{a} is given by the rule

$$s\mapsto\sigma_{\mathrm{a}}(s)=1-\chi(-m^{-1}\lambda_{\mathrm{a}}(s-102)-1).$$

This version of σ_{a} is 0 where $s<102+m|\lambda_{\mathrm{a}}|^{-1}$ and it is 1 where $|s|>102+2m|\lambda_{\mathrm{a}}|^{-1}$. Meanwhile, use χ_* to denote the function $\chi(102-|s|)$. This function is 1 where $|s|>102$ and 0 where $|s|<101$.

Use p_X and these functions to define the 2–form $\mathfrak{w}$ on X by the rule

$$\text{(9-14)}\quad \mathfrak{w} = \chi_* ds\wedge *p_X + p_X + \sum_{\mathrm{a}\in\Xi^+} \mathrm{z}_{\mathrm{a}}\, d(\sigma_{\mathrm{a}}\lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}(s+102)} *\mathrm{a}) + \sum_{\mathrm{a}\in\Xi^-} \mathrm{z}_{\mathrm{a}}\, d(\sigma_{\mathrm{a}}\lambda_{\mathrm{a}}^{-1}e^{\lambda_{\mathrm{a}}(s-102)} *\mathrm{a}).$$

This is a closed 2–form whose de Rham cohomology class is the same as ω. Let E denote the smallest of the numbers from the set $\{\lambda_{\mathrm{a}} \mid \mathrm{a}\in\Xi^+\}\cup\{|\lambda_{\mathrm{a}}| \mid \mathrm{a}\in\Xi^-\}$ with it understood that Ξ^+ refers to the Y_- version and Ξ^- refers to the Y_+ version. The 2–form $\mathfrak{w}$ equals ω where $|s|\ge 1+2m\mathrm{E}^{-1}$.

Step 4 The square of the L^2–norm of $\mathfrak{w}^+$ over the $|s|\le 102+2m\mathrm{E}^{-1}$ part of X is no greater than

$$\text{(9-15)}\quad \int_{s^{-1}([-102,102])} |p_X|^2 + c_0 m^{-2}e^{-2m} \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_{\mathrm{a}}|^{-1}|\mathrm{z}_{\mathrm{a}}|^2 + 4m\mathrm{E}^{-1}(\|w_-\|_2^2+\|w_+\|_2^2) + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_{\mathrm{a}}|^{-1}|\mathrm{z}_{\mathrm{a}}|^2(e^{-2m}-e^{-4|\lambda_{\mathrm{a}}|m/\mathrm{E}}).$$

Meanwhile, the integral of ω over this same part of X is equal to

$$\begin{aligned}(9\text{-}16)\quad &\int_{s^{-1}([-102,102])} |\omega|^2 \\ &\quad + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_\mathrm{a}|^{-1}|\mathrm{Z_a}|^2(1-e^{-2m}) + 4m\mathrm{E}^{-1}(\|w_-\|_2^2 + \|w_+\|_2^2) \\ &\qquad + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_\mathrm{a}|^{-1}|\mathrm{Z_a}|^2(e^{-2m} - e^{-4|\lambda_\mathrm{a}|m/\mathrm{E}}).\end{aligned}$$

As noted in Step 1, the expression in (9-16) cannot be greater than what is written in (9-15). This being the case, the $m > c_0$ versions of (9-15) and (9-16) imply the bound

$$(9\text{-}17)\qquad \int_{s^{-1}([-102,102])} |\omega|^2 + \sum_{\mathrm{a}\in\Xi^+\cup\Xi^-} |\lambda_\mathrm{a}|^{-1}|\mathrm{Z_a}|^2 \le c_0(1 + \langle p_X\rangle_2^2).$$

This last bound has the following corollary: let $I \subset \mathbb{R}$ denote any interval of length 1. Then $\int_{s^{-1}(I)} |\omega|^2 \le c_0(1 + \langle p_X\rangle_2^2)$. □

9.4 Proof of Proposition 3.11

To explain the first bullet, identify a neighborhood of the critical point of the function s with a ball about the origin in $\mathbb{R}^4$ using coordinates (y_1, y_2, y_3, y_4) and write s in terms of these coordinates as $s = y_4^2 - y_1^2 - y_2^2 - y_3^2$ when the constant $s < -1$ slices of X are Y_0 and the constant $s > 1$ slices are $M \sqcup (S^1 \times S^2)$. With the ends reversed, the function s appears as $s = -y_4^2 + y_1^2 + y_2^2 + y_3^2$. The embeddings given in the second and third bullets of (2-8) are defined using a pseudogradient vector field for s. This pseudogradient vector field in the $Y_- = Y_0$ and $Y_+ = M \sqcup (S^1 \times S^2)$ case can be chosen so as to have the following properties: The inverse image of the descending 3–ball from the critical point via the embedding given by the second bullet of (2-8) appears as the locus $(-\infty, 0) \times S$ with S being the 2–sphere that is described in Part 4 of Section 9.1. Meanwhile, the inverse image via the embedding given by the third bullet of (2-8) of one of the ascending arcs from this critical point intersects the $(0,\infty) \times (S^1 \times S^2)$ component of $(0,\infty) \times (M \sqcup (S^1 \times S^2))$ as the locus $(0,\infty) \times p_*$ with $p_* \in S^1 \times S^2$ being the $r = 0$ point in the ball that is described in the third paragraph of Part 4 in Section 9.1. The other ascending arc intersects the $(0,\infty) \times M$ component as the $r' = 0$ point in the ball that is described in the fourth paragraph of Section 9.1. There is a completely analogous picture of X when Y_0 is the constant $s > 0$ slice of X and $S^1 \times S^2$ is the constant $s < 0$ slice.

What is said above about the descending and ascending submanifolds from the critical point has the following consequence: the pseudogradient vector field that defines the embeddings from the second and third bullets of (2-8) can be chosen so that (3-11) are obeyed and likewise the condition in (9-10). These properties are assumed in what follows. The fact that S carries no homology implies that the fourth bullet of (2-8) holds for X.

Parts 1–10 of this subsection construct large L versions of the form w_X and the metric that are used in Part 11 to satisfy the requirements of the second bullet of Proposition 3.11. These constructions require the choice of parameters $T \gg 1$, $L_0 \gg 1$ and $L_1 > L_0 + 1$. Granted these large choices, Parts 1–10 construct a closed 2–form denoted by ω_{T*} and a metric denoted by $\mathfrak{m}_{T*}$ that makes ω_{T*} self-dual. Any $L > L_1 + 20$ version of ω_{T*} can serve for Proposition 3.11's desired 2–form w_X and the corresponding version of $\mathfrak{m}_{T*}$ can serve for the desired metric.

Proposition 3.11 requires as input a metric on $M \sqcup (S^1 \times S^2)$ and asserts that such a metric determines a certain subset of the set *Met* on Y_0. To say more about this subset, recall from Part 2 of Section 9.2 that each metric in *Met* is determined in part by a metric from Section 9.1's set $\mathrm{Met}^{\mathcal{N}}$ and a large choice for a number denoted by T. A metric of this sort was denoted by $\mathfrak{g}_{3T}$ in Section 9.2. As noted at the end of Part 2 of Section 9.2, a lower bound on T is determined by certain properties of the metric from $\mathrm{Met}^{\mathcal{N}}$. A metric of this sort is in Proposition 3.11's subset if and only if T is greater than a new lower bound that is determined by the aforementioned properties of the $\mathrm{Met}^{\mathcal{N}}$ metric. Suffice it to say for the purposes of the proof that this new lower bound is defined implicitly by the constructions in the subsequent eleven parts of this subsection.

The upcoming Parts 1–10 are written so as to simultaneously supply a metric and a closed, self-dual 2–form for Section 9.5's proof of Proposition 3.13 and Section 9.7's proof of Proposition 3.14. This is done by considering a cobordism space X as described in the previous section whose limit manifolds Y_- and Y_+ are as follows: Either one is Y_0 and the other is $M \sqcup (S^1 \times S^2)$, or one is some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, or both are Y_{G}. Although not needed for what follows, the constructions in Parts 1–10 can be done when both limit manifolds are Y or both are some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k.

Part 1 When Y_- or Y_+ is not $M \sqcup (S^1 \times S^2)$, choose metrics, $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$ in the respective Y_- and Y_+ versions of $\mathrm{Met}^{\mathcal{N}}$ as the case may be. In the case when one of

Y_- or Y_+ is some $k \in \{1, \dots, \mathrm{G}\}$ version of Y_k and the other is $Y_{k-1} \sqcup (S^1 \times S^2)$, what is denoted by $\mathrm{Met}^{\mathcal{N}}$ allows any metric for the $S^1 \times S^2$ component. Fix a $T \gg 1$; in particular so that Lemma 9.1 can be invoked for the metric in Met_T defined using $\mathfrak{g}_{1-}$ in the case of Y_- and $\mathfrak{g}_{1+}$ in the case of Y_+. Use $\mathfrak{g}_{1-}$ to choose a metric $\mathfrak{g}_2$ as directed in Part 2 of Section 9.2 on Y_-. Then set $\mathfrak{g}_- = \mathfrak{g}_2$. Meanwhile, use $\mathfrak{g}_2$ to construct a version of the metric $\mathfrak{g}_{3T}$ and denote it by $\mathfrak{g}_{-T}$. Do the same using $\mathfrak{g}_{1+}$; denote the chosen $\mathfrak{g}_2$ metric on Y_+ by $\mathfrak{g}_+$ and use $\mathfrak{g}_{+T}$ to denote the resulting $\mathfrak{g}_{3T}$ metric. If either of Y_- or Y_+ is $M \sqcup (S^1 \times S^2)$, take the metric of the sort described in Part 1 of Section 3.5 for M and the metric $\mathfrak{g}_*$ on $S^1 \times S^2$. Denote the resulting metric on $M \sqcup (S^1 \times S^2)$ as $\mathfrak{g}_-$ in the Y_- case and $\mathfrak{g}_+$ in the Y_+ case. With $T \geq 1$ chosen, this same metric is also denoted at times by $\mathfrak{g}_{-T}$ and $\mathfrak{g}_{+T}$ as the case may be.

By way of notation, the constant c_0 in what follows depends implicitly on the various properties of the metrics $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$. In particular, c_0 depends on an upper bound for the norm of the metric's curvature, upper and lower bounds on the metric's volume and a lower bound on the injectivity radius.

Let $\mathfrak{m}$ denote a chosen metric in the $\mathfrak{g}_-$ and $\mathfrak{g}_+$ version of Met_T on X. Certain constraints on $\mathfrak{m}$ are imposed later in this subsection. Note that some of the latter impose constraints on $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$.

Part 2 Use w_- and w_+ to denote the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2–forms on Y_- and Y_+ with de Rham cohomology class that of $c_1(\det \mathbb{S})$. Fix for the moment a closed 2–form p_X on X as described in Lemma 9.4. Use ω to denote the self-dual 2–form on X given by Lemma 9.4 for the case when the metric on X is $\mathfrak{m}$. The distinguished embedding from (9-11) pulls ω back to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ as a 2–form that can be written as

$$\omega = ds \wedge \upsilon_\diamond + w + ds \wedge \frac{\partial}{\partial s} q + dq, \tag{9-18}$$

with q being an s–dependent 1–form on $\mathcal{Y}_{0\varepsilon}$. Lemma 9.4 says that the C^1–norms of $\frac{\partial}{\partial s} q$ and dq on $\mathbb{R} \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathbb{R} \times \mathcal{N}_\varepsilon$ are less than $c_0 \langle p_X \rangle_2 T^{-1/2}$.

An s– and T–independent open cover of $\mathcal{Y}_{0\varepsilon}$ by balls of radius $c_0^{-1}\varepsilon$ can be used to write q on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{17}{32}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as $q_0 + d\mathfrak{k}$ with q_0 obeying $|q_0| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$ and $\left|\frac{\partial}{\partial s} q_0\right| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$. Meanwhile, $\mathfrak{k}$ is a smooth function with $|d(\mathfrak{k})| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$. Both q_0 and $\mathfrak{k}$ can be constructed so as to depend smoothly on s. It follows as a consequence of the bound $\left|d\left(\frac{\partial}{\partial s}\mathfrak{k}\right)\right| \leq c_0 \langle p_X \rangle_2 T^{-1/2}$

that an s–dependent constant can be added to $\mathfrak{k}$ so that the resulting function, $\mathfrak{k}_0$, depends smoothly on s and obeys $\left|\frac{\partial}{\partial s}\mathfrak{k}_0\right| \le c_0\langle p_X\rangle_2 T^{-1/2}$.

Reintroduce $\sigma_\perp$ from Part 2 of Section 9.2. The 2–form $w + d(\sigma_\perp \mathfrak{q}_0)$ is equal to w on $\mathbb{R}\times\mathcal{Y}_0$ on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$. Meanwhile, it is equal to $w + dq$ on the $r < \rho_* + \frac{5}{8}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$. Moreover, the norm of the difference between this 2–form and w on the $r > \rho_* + \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is bounded by $c_0\langle p_X\rangle_2 T^{-1/2}$, this being a consequence of the bounds in the preceding paragraph for $\mathfrak{q}_0$.

Of interest in what follows is the 2–form on $\mathbb{R}\times\mathcal{Y}_{0\varepsilon}$ given by

$$ds\wedge\mathfrak{b} + w + d(\sigma_\perp\mathfrak{q}_0) \quad \text{with } \mathfrak{b} = \upsilon_\diamond + \sigma_\perp\frac{\partial}{\partial s}\mathfrak{q}_0 + d\Big(\sigma_\perp\frac{\partial}{\partial s}\mathfrak{k}_0\Big). \tag{9-19}$$

This is a closed 2–form on $\mathbb{R}\times\mathcal{Y}_{0\varepsilon}$ which is $ds\wedge *w + w$ on $\mathbb{R}\times\mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$. The bounds given above on the norms of $\mathfrak{k}_0$, its s–derivative, and on the norms of $\mathfrak{q}_0$, $d\mathfrak{q}_0$ and $\frac{\partial}{\partial s}\mathfrak{q}_0$ imply the following: there exists $c_\diamond > 1$ such that each $s\in\mathbb{R}$ version of the 3–form $\mathfrak{b}\wedge(w + d(\sigma\mathfrak{q}_0))$ is strictly positive on $\mathcal{Y}_{0\varepsilon}$ if

$$\langle p_X\rangle_2 T^{-1/2} \le c_\diamond^{-1}. \tag{9-20}$$

Assume in what follows that this bound holds. Granted (9-20), then Lemma 9.3 supplies a smooth, s–dependent metric on $\mathcal{Y}_{0\varepsilon}$ with the properties listed below; the notation uses $\mathfrak{g}_X$ to denote the metric at any given $s\in\mathbb{R}$:

(9-21)
- The Hodge star of $\mathfrak{g}_X$ sends $w + d(\sigma\mathfrak{q}_0)$ to $\mathfrak{b}$.
- The metric $\mathfrak{g}_X$ is $\mathfrak{g}_*$ on $\mathbb{R}\times\mathcal{Y}_0$ and on the $\mathrm{r} > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$.
- The metric $\mathfrak{g}_X$ is the metric in (9-7) on the $\mathrm{r} < \rho_* + \frac{5}{8}\varepsilon$ part of $\mathbb{R}\times\mathcal{N}_\varepsilon$.
- Given $k\in\{1,2,\dots\}$, there exists $c_k > 1$ such that the $s < -104$ and $s > 104$ versions of $\mathfrak{g}_X$ and their derivatives to order $k \ge 1$ differ by at most $c_k e^{-|s|/c_0}$ from the metric $\mathfrak{g}_{-T}$ on Y_- incarnation of $\mathcal{N}_\varepsilon$ or $\mathfrak{g}_{+T}$ on the Y_+ incarnation as the case may be.

By way of an explanation for the fourth bullet, this follows from (9-19) and the third bullet of Lemma 9.4 given the following fact: the derivatives to order k of any given coclosed eigenvector of $*d$ on Y_- or Y_+ with L^2–norm 1 is bounded by a polynomial function of the norm of the eigenvalue with coefficients that are determined solely by the given metric.

Part 3 Let $\mathfrak{m}_T$ denote the metric on X that is equal to $\mathfrak{m}$ on the complement of the image of (9-10)'s embedding and whose pullback to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ via this embedding is the metric $ds^2 + \mathfrak{g}_X$. This is a smooth metric on X whose pullback by the embeddings from the second and third bullets of (2-8) converge as $s \to -\infty$ to the metric $ds^2 + \mathfrak{g}_{-T}$ and converge as $s \to \infty$ to the metric $ds^2 + \mathfrak{g}_{+T}$. These pullbacks are also independent of s for $|s| > 104$ at points of the form (s, p) if p is in either $\mathcal{Y}_0$, the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ or $\mathcal{Y}_M$.

Let ω_T denote the closed 2–form on X given by ω on the complement of the image of (9-10)'s embedding and whose pullback to $\mathbb{R} \times \mathcal{Y}_{0\varepsilon}$ via this embedding is the 2–form in (9-19). The 2–form ω_T is closed. This 2–form is also self-dual when self-duality is defined by the metric $\mathfrak{m}_T$, this being a consequence of the first bullet in (9-21). Let w_{-T} and w_{+T} denote the $\mathfrak{g}_{-T}$ and $\mathfrak{g}_{+T}$ harmonic 2–forms with de Rham cohomology class that of $c_1(\det \mathbb{S})$. Use $*$ in what follows to denote either the $\mathfrak{g}_{-T}$– or $\mathfrak{g}_{+T}$–Hodge dual. The pullbacks of ω_T via the embedding from the second bullet of (2-8) differs from $ds \wedge *w_{-T} + w_{-T}$ in the C^1–topology by at most $c_T e^{-|s|/c_T}$ with $c_T > 1$ being a constant. The pullback via the embedding from the third bullet of (2-8) differs from $ds \wedge *w_{+T} + w_{+T}$ in the C^1–topology by at most c_T. By way of an explanation, these bounds follow from the second and third bullet of Lemma 9.4. Keep in mind that ω_T obeys the second and third bullets of (2-8).

Neither ω_T nor $\mathfrak{m}_T$ is likely to be s–independent where $|s|$ is sufficiently large. This is a defect that is remedied in Parts 4–7 below.

Part 4 Both w_{-T} and w_{+T} have nondegenerate zeros on the components of Y_- and Y_+ where they are not identically zero, these being the components where $c_1(\det \mathbb{S})$ is not torsion. Let $Y_* \subset Y_-$ denote such a component and let $p \in Y_*$ denote a zero of w_{-T}. Let $B \subset Y_*$ denote a small radius ball centered on p with the following properties: the point p is the only zero of w_{-T} in the closure of B; and B is disjoint from $\mathcal{Y}_0$ and from the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Since w_{-T} vanishes transversely at p, there exists $L_0 > 1$ such that each $s < -L_0$ version of $w_{-T} + dq$ vanishes transversely in the closure of B at a single point. Let p_s denote this point. Note in particular that $\operatorname{dist}(p, p_s) \le c_- e^{-|s|/c_-}$. Granted that $\operatorname{dist}(p, p_s) \ll 1$ for $s \ll -1$, there exists $s_0 > 1$ such that $\operatorname{dist}(p, p_s)$ is less than $\frac{1}{8}$ times the radius of B when $s \le -s_0$. This being the case, there exists $L_0 > s_0$, $c_- > 1$ and a family of diffeomorphisms of Y_* parametrized by $(-\infty, -L_0]$ with the properties in the list that follows. The list uses Ψ_s to denote the diffeomorphism labeled by a given $s \in (-\infty, -L_0]$.

(9-22) • If $s > -L_0 - 1$, then Ψ_s is the identity map.

- Every $s \in (-\infty, -L_0]$ version of Ψ_s is the identity where $\operatorname{dist}(\cdot, p) > 2\operatorname{dist}(p, p_s)$.
- Every $s \in (-\infty, -L_0]$ version of Ψ_s differs from the identity in the C^{10}–topology by at most $e^{-|s|/c}$.
- $\Psi_s(p) = p_s$ when $s < -L_0 - 2$.

This family of diffeomorphisms defines a diffeomorphism of X which is the identity on the $s > -L_0 - 1$ part of X, and on the image in X of $(-\infty, -L_0] \times (Y_- - Y_*)$ via the diffeomorphism in the second bullet of (2-8). This diffeomorphism is defined on the image of $(-\infty, -L_0] \times Y_*$ via the second bullet of (2-8) by that of $(-\infty, -L_0] \times Y_*$ that sends a given point (s, q) to $(s, \Psi_s(q))$. Use Ψ_p to denote this diffeomorphism of X. Various versions of this diffeomorphism are defined by the zeros of w_{-T} on the components of Y_- where $c_1(\det \mathbb{S})$ is not torsion. These diffeomorphisms pairwise commute. Use Ψ to denote their composition.

Introduce $\mathfrak{m}_{T0}$ to denote $\Psi^*\mathfrak{m}_T$ and ω_{T0} to denote $\Psi^*\omega_T$. The 2–form ω_{T0} is closed and it is self-dual if the notion of self-duality is defined using $\mathfrak{m}_{T0}$. The form ω_{T0} can be written on $(-\infty, -L_0] \times Y_*$ as $ds \wedge (*w_{-T} + n) + (w_{-T} + m)$, where n and m have C^1–norm less than $c_- e^{-|s|/c_-}$ and both vanish on $(-\infty, -1] \times \mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $(-\infty, -1] \times \mathcal{N}_\varepsilon$. By way of notation, c_- denotes here and in what follows a constant that is greater than 1. Its value can increase between successive appearances. Note that the fact that ω_{T0} is closed requires that dn equals $\frac{\partial}{\partial s} m$.

The pullback of $w_{-T} + m$ to each constant s slice of $(-\infty, -1] \times Y_*$ defines the same cohomology class as w_{-T}. This implies in particular that $m = du$ with u being an s–dependent 1–form on Y_*. Any s–dependent, closed 1–form can be added to u without changing du, and this fact is used to choose u so that the conditions that follow hold:

(9-23)
- The 1–form u is zero on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{3}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$.
- The C^2–norm of u is less than $c_- e^{-|s|/c_-}$.
- Let p denote a zero of w_{-T} in $(-\infty, -L_0] \times Y_*$. Then $|u|$ and the norm of u's covariant derivative along $\frac{\partial}{\partial s}$ at any $s \in (-\infty, -L_0] \times Y_*$ is bounded by $c_- \operatorname{dist}(\cdot, p)^2 e^{-|s|/c_-}$.

To explain how the third bullet can be satisfied, let p again denote a zero of w_{-T}. Use the metric $\mathfrak{g}_{-T}$ to construct a Gaussian coordinate chart centered at p so as to identify B with a small radius ball in $\mathbb{R}^3$. The corresponding coordinate map to $\mathbb{R}^3$ is denoted by x or (x_1, x_2, x_3). Write the 2–form ω_{T0} as $ds \wedge (*w_{-T} + \mathfrak{n}) + (w_{-T} + \mathfrak{m})$. The 2–form $\mathfrak{m}$ appears in these coordinates as

(9-24) $$\mathfrak{m} = \tfrac{1}{2}\mathfrak{o}^{ij} x^i \varepsilon^{jnm} dx^n \, dx^m + \cdots,$$

where the summation convention over repeated indices is used. The unwritten terms in (9-24) are $\mathcal{O}(|x|^2)$. What is denoted by $\{\varepsilon^{jnm}\}_{1\le j,n,m\le 3}$ is antisymmetric with respect to interchanging indices and so defined by the rule $\varepsilon^{123} = 1$. Meanwhile, $\{\mathfrak{o}^{ij}\}_{i,j=1,2,3}$ are the entries of a traceless, s–dependent matrix whose norm and that of its s–derivative are at most $c_{-}e^{-|s|/c_1}$. The matrix is traceless because $\mathfrak{m}$ is closed. The fact that this matrix $\mathfrak{o}$ is traceless implies that $\mathfrak{m}$ on B can be written as du_B with $u_B = \frac{1}{6}\mathfrak{o}^{ij} x^i x^n \varepsilon^{jnm} dx^m + \cdots$, where the unwritten terms are $\mathcal{O}(|x|^3)$. Since $u - u_B = dp$ on B, it follows that u can be modified with no change near the boundary of B so that $u = u_B$ on a small radius ball in B centered at p.

Part 5 Fix $L_1 > L_0 + 1$ and let χ_1 denote the function on $\mathbb{R}$ given by $\chi(-L_1 - s)$. This function equals zero where $s < -L_1 - 1$ and it equals 1 when $s > -L_1$. Use χ_1' to denote the derivative of χ_1. The function χ_1 and the 2–form ω_{T0} are used next to define the 2–form on $(-\infty, -L_0] \times Y_*$ to be denoted by ω_{T1}. This 2–form is ω_{T0} on the $s > -L_1$ part of $(-\infty, -L_0] \times Y_*$, and it is given where $s \le -L_1$ by the formula that follows for its pullback via the embedding from (2-8)'s second bullet:

(9-25) $$\omega_{T1} = ds \wedge (*w_{-T} + \chi_1 \mathfrak{n} + \chi_1' u) + w_{-T} + \chi_1 \, du.$$

The 2–form ω_{T1} is a closed 2–form on $(-\infty, -L_0] \times Y_*$. The remainder of this part of the subsection and Part 6 describe a metric on the $s \in (-\infty, -L_0] \times Y_*$ that makes ω_{T1} self-dual. This new metric is equal to $\mathfrak{m}_{T0}$ where $s \ge -L_1 + 1$ and it is equal to $ds^2 + \mathfrak{g}_{-T}$ where $s < -L_1 - 2$. This new metric is denoted below by $\mathfrak{m}_{T1}$. The five steps that follow describe the metric $\mathfrak{m}_{T1}$ at points in $(-\infty, -L_0] \times Y_*$ that project to Y_* near the zero locus of w_{-T}.

Step 1 The 2–form w_{-T} and the 1–form $*w_{-T}$ on B can be written using the Gaussian coordinates (x_1, x_2, x_3) on B as

(9-26) $$w_{-T} = \tfrac{1}{2}\mathrm{A}^{ij} x^i \varepsilon^{jnm} dx^n \, dx^m + \cdots \quad \text{and} \quad *\, w_{-T} = \mathrm{A}^{ij} x^i \, dx^j + \cdots$$

with summations over repeated indices implicit. The various $i, j \in \{1, 2, 3\}$ versions of A^{ij} in (9-26) are the entries of an invertible matrix, this denoted by A. The unwritten terms in (9-26) vanish to order $|x|^2$. The fact that w_{-T} is closed implies that A is traceless and the fact that $*w_{-T}$ is self-dual implies that A is symmetric.

The unwritten terms in (9-26) are incorporated using the notation whereby w_{-T} and $*w_{-T}$ on $(-\infty, -L_0] \times B$ are written as

$$ds \wedge (f_1 \hat{e}^1 + f_2 \hat{e}^2 + f_3 \hat{e}^3) + f_1 \hat{e}^2 \wedge \hat{e}^3 + f_2 \hat{e}^3 \wedge \hat{e}^1 + f_3 \hat{e}^1 \wedge \hat{e}^2, \tag{9-27}$$

where $\{\hat{e}^k\}_{1\le k\le 3}$ denotes a $\mathfrak{g}_{-T}$–orthonormal set for T^*B with $\hat{e}^k = dx^k + \mathcal{O}(|x|^2)$ for $1 \le k \le 3$ and where $\{f_k\}_{1\le k\le 3}$ are functions with $f_k = \sum_{1\le i\le 3} \mathrm{A}^{ik} x^i + \mathcal{O}(|x|^2)$ for $1 \le k \le 3$. Note in particular that these are such that $df_1 \wedge df_2 \wedge df_3 > \frac{1}{2} \det \mathrm{A}$ on a concentric ball in B centered at the origin. This ball is denoted by B'. It is assumed in what follows that L_0 is chosen so that $\omega_{T0} = \omega_T$ on the complement of a concentric ball in B' with radius one-fourth that of B'. In particular, it is assumed that (9-22)'s diffeomorphism Ψ_s is the identity for all s on a neighborhood in B of $B - B'$.

Step 2 The Ψ–pullback of $\{ds, \hat{e}^1, \hat{e}^2, \hat{e}^3\}$ is $\mathfrak{m}_T$–orthonormal. The Ψ–pullback of ds is ds. Meanwhile, Ψ can be chosen so that

$$\Psi^* \hat{e}^k = \hat{e}^k + \sum_{1\le k\le 3} \mathfrak{p}^k ds + \sum_{1\le j\le 3} \mathfrak{p}^{kj} \hat{e}^j, \tag{9-28}$$

where $\sum_{1\le k\le 3} |\mathfrak{p}^k| \le c_- e^{-|s|/c_-}$ and $\sum_{1\le k,j\le 3} |\mathfrak{p}^{kj}| \le c_- |x| e^{-|s|/c_-}$ when $s < -L_0 - 1$. This is done by defining (9-22)'s diffeomorphism Ψ_s using the Gaussian coordinates in (9-25) by the rule $x \mapsto \Psi_s(x) = x + p_s$ at points (s, x) with $|x| < \frac{3}{2}|p_s|$ and $s < -L_0 - 1$. Use $\{\hat{e}_s^k\}_{1\le k\le 3}$ to denote $\{\Psi^* \hat{e}^k\}_{1\le k\le 3}$. Granted this notation, the 2–form ω'_{T0} near p can be written as

$$\omega'_{T0} = ds \wedge (f_{s_1} \hat{e}_s^1 + f_{s_2} \hat{e}_s^2 + f_{s_3} \hat{e}_s^3) + f_{s_1} \hat{e}_s^2 \wedge \hat{e}_s^3 + f_{s_2} \hat{e}_s^3 \wedge \hat{e}_s^1 + f_{s_3} \hat{e}_s^1 \wedge \hat{e}_s^2, \tag{9-29}$$

where $\{f_{s_k} = \Psi_s^* f_k\}_{1\le k\le 3}$.

Step 3 Introduce $\{e_{s_\chi}^k = \hat{e}^k + \chi_0 \sum_{1\le j\le 3} \mathfrak{p}^{kj} \hat{e}^j\}_{1\le k\le 3}$. Use this s–dependent basis to write the (9-25)'s 2–form $w_{-T} + \chi_1\, du$ on B' as

$$w_{-T} + \chi_1\, du = f_{s_\chi 1} e_{s_\chi}^2 \wedge e_{s_\chi}^3 + f_{s_\chi 2} e_{s_\chi}^3 \wedge e_{s_\chi}^1 + f_{s_\chi 3} e_{s_\chi}^1 \wedge e_{s_\chi}^2, \tag{9-30}$$

where $\{f_{s_\chi k}\}_{1\le k\le 3}$ are smoothly varying functions of s and the coordinate x with the property that $f_{s_\chi(\cdot)} = f_{(\cdot)}$ when $s < -L_1 - 1$ and $f_{s_\chi(\cdot)} = f_{s(\cdot)}$ when $s > -L_1$. This depiction can be derived from the fact that $\{f_k\}_{1\le k\le 3}$ generate $C^\infty(B')$. Note that $f_{s_\chi k} = f_k + \cdots$ with the unwritten terms such that their norms are bounded by $c_- e^{-|s|/c_-}|x|$ and such that their first derivatives have norms bounded by $c_- e^{-|s|/c_-}$. This implies in particular that the functions $\{f_{s_\chi k}\}_{1\le k\le 3}$ also generate $C^\infty(B')$ and that $df_{s_\chi 1} \wedge df_{s_\chi 2} \wedge df_{s_\chi 3} > \det \mathrm{A}$ on B' when $L_0 > c_-$.

The 1–form $*w_{-T} + \chi_0 n + \chi_0' u$ can be written schematically on $(-\infty, -L_0] \times B'$ using the basis $\{e^k_{s_\chi}\}_{1\le k\le 3}$ as

$$*w_{-T} + \chi_0 n + \chi_0' u = \sum_{1\le k,i\le 3} f_{s_\chi k} \mathfrak{c}_{ki} e^i_{s_\chi}, \tag{9-31}$$

with $\{\mathfrak{c}_{ki}\}_{1\le i,k\le 3}$ denoting a matrix of smooth functions of s and the coordinate x. Given that the functions $\{f_{s_\chi k}\}_{1\le k\le 3}$ also generate $C^\infty(B')$, such a depiction follows because both n and u vanish at p. Keep in mind for what follows that the matrix with coefficients $\{\mathfrak{c}_{ki}\}_{1\le i,k\le 3}$ differs from the identity matrix by at most $c_- e^{-|s|/c_-}$.

Step 4 A particular set of three smooth functions of $s \in (-\infty, -L_0]$ and the coordinate x is specified in a moment. Let $\{\mathfrak{q}^k\}_{1\le k\le 3}$ denote any given set of such functions. Use this set to define 1–forms $\{\hat{e}^k_{s_\chi}\}_{1\le k\le 3}$ on $(-\infty, -L_0] \times B'$ by the rule

$$\hat{e}^k_{s_\chi} = e^k_{s_\chi} - \mathfrak{q}^k\, ds. \tag{9-32}$$

Given the formulas in (9-31) and (9-32), it follows that ω_{T1} on $(-\infty, -L_0] \times B'$ can be written using $\{\hat{e}^k_{s_\chi}\}_{1\le k\le 3}$ as

$$ds \wedge (f_{s_\chi k}(\mathfrak{c}_{ki} + \varepsilon^{kni}\mathfrak{q}^n))\hat{e}^i_{s_\chi} + \tfrac{1}{2} f_{s_\chi k}\varepsilon^{knm}\hat{e}^n_{s_\chi} \wedge \hat{e}^m_{s_\chi}. \tag{9-33}$$

This equation uses the summation convention over repeated indices.

Step 5 The set $\{\mathfrak{q}^k\}_{1\le k\le 3}$ is introduced for the following reason: there is a unique choice for $\{\mathfrak{q}^k\}_{1\le k\le 3}$ that makes the matrix with entries $\{\mathfrak{c}_{ki} + \varepsilon^{kni}\mathfrak{q}^n\}_{1\le i,k\le 3}$ a symmetric matrix, this being $\{\mathfrak{q}^k = \frac{1}{2}\varepsilon^{kin}\mathfrak{c}_{ni}\}$. This choice is used in what follows. With this choice understood, a metric is defined on $(-\infty, -L_0] \times B'$ by the following rules:

(9-34)
- ds has norm 1 and it is orthogonal to $\{\hat{e}^{ks_\chi}\}_{1\le k\le 3}$.
- Given $(i,k) \in \{1,2,3\}$, then the inner product between $\hat{e}^k_{s_\chi}$ and $\hat{e}^i_{s_\chi}$ is $\mathfrak{c}_{ki} + \varepsilon^{kni}\mathfrak{q}^n$.

The inner product defined by the second bullet is positive definite if $L_0 > c_0 c_-$ because of the aforementioned fact that the matrix defined by $\{\mathfrak{c}_{ki}\}_{1\le i,k\le 3}$ differs by at most $c_- e^{-|s|/c_-}$ from the identity matrix.

The metric just defined is the metric $\mathfrak{m}_{T0}$ when $s > -L_1$, and it is the metric $ds^2 + \mathfrak{g}_{-T}$ when $s < -L_1 - 1$. Moreover, the 2–form ω_{T1} is self-dual on $(-\infty, -L_0] \times B'$ when self-duality is defined by this metric. Denote this metric by $\mathfrak{m}_{T1p}$.

Let $B'' \subset B'$ denote the concentric ball whose radius is one-half that of B'. The desired metric $\mathfrak{m}_{T1}$ is defined to equal $\mathfrak{m}_{T1p}$ on $(-\infty, -L_0] \times B''$.

Part 6 Use U to denote the union of the various versions of the ball B''. The two steps that follow directly describe the metric $\mathfrak{m}_{T1}$ on $(-\infty, -L_0] \times (Y_* - U)$.

Step 1 This step describes a metric on $(-\infty, -L_0] \times (Y_* - U)$ to be denoted by $\mathfrak{m}_{T1\diamond}$. The metrics $\mathfrak{m}_{T1}$ and $\mathfrak{m}_{T1\diamond}$ agree on the product of $(-\infty, -L_0]$ with the complement in Y_* of the union of the various versions of the ball B'. The definition of this metric $\mathfrak{m}_{T1\diamond}$ assumes that $L_0 > c_\diamond$ with $c_\diamond$ such that $\omega = w_{-T} + \chi_1\, du$ and $\upsilon = *w_{-T} + \chi_1 n + \chi_1' u$ from (9-25) obey $\upsilon \wedge \omega > 1/c_\diamond$ on $(-\infty, -L_0] \times (Y_* - U)$. The existence of $c_\diamond$ follows from (9-23). Let p denote a zero of w_{-T} and let $B_\diamond \subset B'$ denote the concentric ball whose radius is three quarters that of B'. Use V to denote the union of the various versions of $B_\diamond$. Invoke Lemma 9.3 on $(-\infty, -L_0] \times (Y_* - U)$ using ω and υ to obtain a smooth family of metrics on $Y_* - V$ parametrized by $(-\infty, -L_0]$ with the properties listed in the upcoming (9-35). The notation uses $\mathfrak{g}_\diamond$ to denote any given $s \in (-\infty, -L_0]$ member of the family. To explain more of the notation, note first that pullbacks of $\mathfrak{m}$ and Part 4's metric $\mathfrak{m}_{T0}$ via the embedding from the second bullet of (2-8) agree on $(-\infty, -L_0] \times (Y_* - U)$. In particular, the pullback of $\mathfrak{m}_{T0}$ to this part of $(-\infty, -L_0] \times Y_*$ can be written as $ds^2 + \mathfrak{g}_X$ with $\mathfrak{g}_X$ denoting here a smooth, s–dependent metric on $Y_* - U$. This metric $\mathfrak{g}_X$ is the metric $\mathfrak{g}_{-T}$ on $\mathcal{Y}_M - U$ and it is the metric from (9-19) on $\mathcal{Y}_{0\varepsilon}$.

(9-35)
- Each $s \in (-\infty, -L_1 - 1]$ version of $\mathfrak{g}_\diamond$ is $\mathfrak{g}_{-T}$ and each $s \in [-L_1, -L_0]$ version is the corresponding version of $\mathfrak{g}_X$.
- The $\mathfrak{g}_X$–Hodge dual of the 2–form $w_{-T} + \chi_1\, du$ on $Y_* - V$ is the 1–form $*w_{-T} + \chi_1 n + \chi_1' u$.

The metric $\mathfrak{m}_{T1\diamond}$ on $(-\infty, -L_0] \times (Y_* - U)$ is defined to be $ds^2 + \mathfrak{g}_\diamond$. It follows directly from the second bullet in (9-35) that the 2–form ω_{T1} is self-dual on $(-\infty, -L_0] \times (Y_* - V)$ when the notion of self-duality is defined using the metric $\mathfrak{m}_{T1\diamond}$.

Step 2 Let p denote a zero of w_{-T}. The metrics $\mathfrak{m}_{T1\diamond}$ and $\mathfrak{m}_{T1p}$ are both metrics on $(-\infty, -L_0] \times (B' - B_\diamond)$. The 2–form ω_{T1} is self-dual on $(-\infty, -L_0] \times (B' - B_\diamond)$ when the latter notion is defined by either metric. Use $z_\diamond$ and z_p to denote the respective $\mathfrak{m}_{T1\diamond}$– and $\mathfrak{m}_{T1p}$–norms of ω_{T1}. Since $\omega_{T1} \wedge \omega_{T1} > c_-^{-1}$ here, there are ω_{T1}–compatible almost complex structures for $(-\infty, -L_0] \times (B' - B_\diamond)$, these denoted by $J_\diamond$ and J_p, such that

$$\mathfrak{m}_{T1\diamond} = z_\diamond^{-1} \omega_{T1}(\cdot\,, J_\diamond(\cdot)) \quad \text{and} \quad \mathfrak{m}_{T1p} = z_p^{-1} \omega_{T1}(\cdot\,, J_p(\cdot)). \tag{9-36}$$

As the space of ω_{T1}–compatible almost complex structures on $(-\infty,-L_0]\times(B'-B_\diamond)$ is contractible, there exists an almost complex structure with two properties, the first of which is as follows: The almost complex structure is J_p at points with $B'-B_\diamond$ component in a neighborhood of the boundary of the closure of $B_\diamond$, and it is $J_\diamond$ at points with $B'-B_\diamond$ component in the B' part of a neighborhood of the boundary of the closure of B' in B. To state the second property, keep in mind that $J_\diamond = J_p$ in some neighborhood of $(-\infty,-L_1-1]\times(B'-B_\diamond)$ and also in some neighborhood of $[-L_1,-L_0]\times(B'-B_\diamond)$. What follows is the second property: the new almost complex structure is $J_\diamond$ and thus J_p in a slightly smaller neighborhood of $(-\infty,-L_1-1]\times(B'-B_\diamond)$ and $[-L_1,-L_0]\times(B'-B_\diamond)$. Use J_* to denote an almost complex structure of the sort just described.

Fix a smooth, strictly positive function on $(-\infty,-L_1-1]\times(B'-B_\diamond)$ that is equal to $z_\diamond$ where $J_* = J_\diamond$ and equal to z_p where $J_* = J_p$. Denote this function by z_*. Use J_* and z_* to define the metric $\mathfrak{m}_{T1}$ on $(-\infty,-L_1-1]\times(B'-B_\diamond)$ by the rule $\mathfrak{m}_{T1} = z_*^{-1}\omega_{T1}(\cdot\,, J_*(\cdot))$. This metric smoothly extends the metrics defined in Step 1 and in Step 5 of Part 5 and it has all of the desired properties.

Part 7 Let $Y_*\subset Y_-$ now denote a component where w_{-T} is identically zero, thus a component where $c_1(\det\mathbb{S})$ is torsion. Suppose that $L>1$ has been chosen. Let ω_{T0} now denote the pullback of ω_T to $(-\infty,-L]\times Y_*$ via the embedding from the second bullet of (2-8). It follows from Lemma 9.4 that the C^1–norm of ω_{T0} is bounded by $c_0\langle p_X\rangle_2 e^{-|s|/c_0}$. The 2–form ω_{T0} is exact on $(-\infty,-L]\times Y_*$, it can be written as $ds\wedge\frac{\partial}{\partial s}u + du$ with d denoting here the exterior derivative along the constant s slices of $(-\infty,-L]\times Y_*$ and with u denoting a smooth, s–dependent 1–form on Y_* with $|u|$, $|du|$ and $\left|\frac{\partial}{\partial s}u\right|$ bounded by $c_0\langle p_X\rangle_2 e^{-|s|/c_0}$.

With the preceding understood, fix $L_{\mathrm{tor}} > L+4$ and let χ_* denote the function on $\mathbb{R}$ defined by the rule $s\mapsto\chi(-L_{\mathrm{tor}}+3-s)$. This function equals 1 where $s>-L_{\mathrm{tor}}+3$ and it equals 0 where $s<-L_{\mathrm{tor}}+2$. Use χ_* to define a self-dual form on $(-\infty,-L]\times Y_*$ by the following rules: This form is equal to ω_{T0} on $[-\infty,-L_{\mathrm{tor}}+4,-L]\times Y_*$, it is identically 0 on $[-\infty,-L_{\mathrm{tor}}]\times Y_*$ and it is equal to $\chi_*\big(ds\wedge\frac{\partial}{\partial s}u + du\big)$ on $[-L_{\mathrm{tor}},-L_{\mathrm{tor}}+4]\times Y_*$. Denote this 2–form by ω_{T1}.

The 2–form ω_{T1} can be written as $ds\wedge *w_* + w_*$ with $w_* = d(\chi_* u)$ with it understood again that d here denotes the exterior derivative along Y_*. Let χ_*' denote the derivative of the function $s\mapsto\chi_*(s)$. The 2–form w_* on $[-L_{\mathrm{tor}},-L_{\mathrm{tor}}+4]\times Y_*$ can be written as $d\mathfrak{b}$ with $\mathfrak{b} = \chi_*'u + \chi_* u$. Note in particular that $|\mathfrak{b}|\le c_0 c\langle p_X\rangle_2 e^{-|s|/c_0}$.

Fix $c > 1$. The bound just given for $|\mathfrak{b}|$ leads to the following conclusion: Fix $\mathrm{r} > 1$. Then $|\mathfrak{b}| \leq \mathrm{r}^{-10}$ if $L_{\mathrm{tor}} > c_0\big(|\ln(\langle p_X\rangle_2)| + \ln \mathrm{r}\big)$.

Part 8 Define the 2–form ω_{T*} on the $s \leq 0$ part of X as follows: This 2–form is equal to ω_T where $s \in [-L, 0]$. Meanwhile, its pullback to each component of the $s < -1$ part of X via the embedding from the second bullet of (2-8) is the corresponding version of the 2–form ω_{T1}. Modulo notation, what is said in Parts 4–7 can be repeated for the $s > 0$ part of X to extend the definition of ω_{T*} and the metric $\mathfrak{m}_{T*}$ to the whole of X. The form ω_{T*} is self-dual if the latter notion is defined by $\mathfrak{m}_{T*}$. This construction has the following additional property: Suppose that p_X obeys (9-20). Fix $c > c_0$. If $r > 1$ has been chosen to be greater than a purely c–dependent constant, then the $(L = c, L_{\mathrm{tor}} = c \ln \mathrm{r})$ version of $\mathfrak{m}_{T*}$ and ω_{T*} obey the constraints given by (2-9), (2-12), (3-13), (3-14) and the (c, r) version of (3-15). Here, the closed 1–form υ_X can be chosen so that it is s–independent and $\upsilon_X = *w_{\pm T}$ over constant s slices of X where $|s| > L - 4$. The bounds in items (4)(b), (4)(d) and (5)(c) of (3-15) follow from the bounds on u in (9-23) and those for $\mathfrak{b}$ in Part 7 above.

Part 9 The happy conclusions of Part 8 are contingent on the existence of a closed 2–form, p_X, on X with the following properties: the de Rham cohomology class of p_X is $c_1(\det \mathbb{S})$, it equals w_- where $s < -102$, it equals w_+ where $s > 102$, and it obeys the bound in (9-20).

The subsequent four steps in this part of the subsection construct p_X on various parts of X. These constructions are used in Part 11 and they are also used in the proofs of Propositions 3.13 and 3.14.

Step 1 This step first states and then proves a lemma that supplies a crucial tool for what is to come.

Lemma 9.5 *Let U denote a 3–manifold and let $V \subset U$ denote an open set with compact closure in U. Given the data set consisting of U, V and a Riemannian metric on U, there exists $\kappa > 1$ with the following significance: Let u denote a closed, exact 2–form on U. There is a 1–form on U, this denoted by q, with $\int_V |q|^2 \leq \kappa \int_U |u|^2$ and such that $dq = u$.*

To set the notation used below, the L^2–norm of a function or differential form over a given set $W \subset U$ is denoted by $\|\cdot\|_W$.

Proof The set V has a finite cover by Gaussian coordinate balls with centers in U with the property that the mutual intersection of balls from this cover is either empty or convex. This cover can also be chosen so that each ball has the same radius and such that no ball intersects more than c_0 others. The minimal number of balls in such a cover, their common radius and the combinatorial properties of the mutual intersections are determined a priori by U, V and the metric. Let $\mathfrak{U}$ denote such a cover and let σ denote the radius of its constituent balls.

Let $B \in \mathfrak{U}$. The fact that B is convex can be used to write u on B as $u = dq_B$ with $\|q_B\|_B \le c_0\sigma\|u\|_B$. Let B and B' denote two sets from $\mathfrak{U}$. Then $dq_B - dq_{B'} = 0$ on their intersection, and so $q_B - q_{B'} = d\mathfrak{k}_{BB'}$ with $\mathfrak{k}_{BB'}$ being a function on $B' \cap B$. It follows that $\|d\mathfrak{k}_{BB'}\|_{B'\cap B} \le c_0\sigma(\|u\|_B + \|u\|_{B'})$. Changing $\mathfrak{k}_{BB'}$ by a constant if needed produces a version with $\|\mathfrak{k}_{BB'}\|_{B\cap B'} \le c_0\sigma\|d\mathfrak{k}_{BB'}\|_{B'\cap B}$ and thus $\|d\mathfrak{k}_{BB'}\|_{B'\cap B} \le c_0\sigma^2(\|u\|_B + \|u\|_{B'})$.

Now suppose that B, B' and B'' are from $\mathfrak{U}$ with a point in common. Let $c_{BB'B''}$ denote $\mathfrak{k}_{BB'} + \mathfrak{k}_{B'B''} + \mathfrak{k}_{B''B}$. This $c_{BB'B''}$ is constant and the collection of such numbers is a Čech cohomology cocycle whose cohomology class gives the class of u via the de Rham isomorphism. It follows that this cocycle is zero, and so $c_{BB'B''} = c_{BB'} + c_{B'B''} + c_{B''B}$ with each term being constant. Noting that $|c_{BB'B''}| \le c_0\sigma^{-1}(\|u\|_B + \|u\|_{B'} + \|u\|_{B''})$, it follows that $|c_{BB'}| \le c_*\sigma^{-1}\sup_{B''\in U: B''\cap B'\cap B\neq\varnothing}(\|u\|_B + \|u\|_{B'} + \|u\|_{B''})$ with $c_* \ge 1$ determined a priori by the combinatorics of the cover $\mathfrak{U}$.

Let $\{\chi_B\}_{B\in\mathfrak{U}}$ denote a partition of unity subordinate to the cover $\mathfrak{U}$. Note that these functions can be chosen so that $|d\chi_B| \le c_0\sigma^{-1}$. Define now a 1–form q on B by the rule $q|_B = q_B + d\big(\sum_{B'}\chi_{B'}(\mathfrak{k}_{BB'} - c_{BB'})\big)$. This defines a smooth 1–form on V with $dq = u$ and with $\|q\|_V \le c_*\sigma\|u\|_U$.

Step 2 The lemma that is stated and then proved in this step makes the first application of Lemma 9.5.

Lemma 9.6 *There exists $\kappa > 0$ with the following significance: Fix $k \in \{0,\dots,\mathrm{G}\}$ and then $T > 1$ so as to define* Met_T *on Y_k. Let $\mathfrak{g}$ denote a* Met_T *metric on Y_k and let $w_{\mathfrak{g}}$ denote the corresponding harmonic 2–form whose de Rham cohomology class is that of $c_1(\det\mathbb{S})$. The 2–form $w_{\mathfrak{g}}$ on the $r \in \big[\rho_* - \frac{1}{16}\varepsilon, \rho_* + \frac{1}{128}\varepsilon\big]$ part of $\mathcal{N}_\varepsilon$ can be written as dq with q being a 1–form whose L^2–norm on this part of $\mathcal{N}_\varepsilon$ is bounded by κ/T times that of $w_{\mathfrak{g}}$.*

Proof The metric on the $r \in \big[\rho_* - \frac{1}{8}\varepsilon, \rho_* + \frac{1}{64}\varepsilon\big]$ part of $\mathcal{N}_\varepsilon$ is the metric given by (9-7) with $\rho_T = \rho/T$ and with $x_{3T} = x_3/T$. The functions K and h are smooth around $\rho = 0$ with $h(0)$ and $\mathrm{K}(0) = 1$. It follows as a consequence that the metric in the region of interest when written using ρ_T and x_T is uniformly close for $T > c_0$ to the Euclidean metric on the part of the radius $\bigl(\rho_* + \frac{1}{64}\varepsilon\bigr)/T$ ball about the origin in $\mathbb{R}^3$ that lies outside the concentric ball of radius $\bigl(\rho_* - \frac{1}{8}\varepsilon\bigr)/T$. Take this to be the region U for Lemma 9.5 and take V to be the part of this same ball where the radius is between $\bigl(\rho_* - \frac{1}{16}\varepsilon\bigr)/T$ and $\bigl(\rho_* + \frac{1}{128}\varepsilon\bigr)/T$. A cover $\mathfrak{U}$ can be found as in the proof of Lemma 9.5 with a T–independent bound on the number of sets, a T–independent combinatorial structure to the intersections between them, and a common radius for the balls, c_0. This can be done because the T–dependence is just given by scaling the coordinates. Granted all of this, then the claim by the lemma follows by appeal to Lemma 9.5. □

Step 3 This step supplies a part of what will be p_X on the $s \in [-102, -101]$ part of X when Y_- is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, and on the $s \in [100, 102]$ part of X when Y_+ is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. The constructions that follow use the embeddings from the second and third bullets of (2-8) to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0) \times Y_-$ and as $(0, \infty) \times Y_+$.

Let $\chi_{\diamond 1}$ denote the function on $\mathbb{R}$ given by the rule $\chi(|s| - 101)$. Denote its derivative by $\chi'_{\diamond 1}$. This function is equal to 0 where $|s| \geq 102$ and it is equal to 1 where $|s| \leq 101$. Use χ to construct a smooth function on $\mathcal{N}_\varepsilon$ that equals 0 where $|r - \rho_*| > \frac{1}{128}\varepsilon$ and equals 1 where $|r - \rho_*| < \frac{1}{256}\varepsilon$. Construct this function of r so that its derivative is bounded by c_0. Use σ_1 to denote this new function of r.

If Y_- is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, let q_{1-} denote the $w_{\mathfrak{g}} = w_-$ version of q that is given by Lemma 9.6. Define $p_{\mathcal{N}1}$ where $s \in [-102, -101]$ to be

$$p_{\mathcal{N}1} = -ds \wedge \chi'_{\diamond 1}\sigma_1 q_{1-} + w_- - \chi_{\diamond 1} d(\sigma_1 q_{1-}). \tag{9-37}$$

This is a closed form with de Rham cohomology class that of $c_1(\det \mathbb{S})$ and it equals w_- where $s \leq -102$. Of particular note is the fact that $p_{\mathcal{N}1} = 0$ on the $|r - \rho_*| < \frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$ where $s > -101$ and that it equals w_- on the complement of the $|r - \rho_*| < \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$. It follows from Lemma 9.6 that the L^2–norm of $p_{\mathcal{N}1}$ at any given $s \in [-102, -101]$ is bounded by c_0 times that of w_-.

If Y_+ is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, then very much the same formula defines an $s \in [101, 102]$ analog to $p_{\mathcal{N}1}$. The latter is obtained by using Lemma 9.6 with $w_{\mathfrak{g}} = w_+$. Lemma 9.6 supplies a 1–form q_{1+} with $dq_{1+} = w_+$ on the $|r - \rho_*| < \frac{1}{128}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Use w_+ and q_{1+} in (9-37) in lieu of w_- and q_- to define $p_{\mathcal{N}1}$ where $s \in [101, 102]$.

Step 4 This step extends the definition of $p_{\mathcal{N}1}$ to the $s \in [-101, -100]$ part of X when Y_- is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, and to the $s \in [100, 101]$ part of X when Y_+ is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. The embeddings from the second and third bullets of (2-8) are again used to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0) \times Y_-$ and as $(0, \infty) \times Y_+$.

The extension of $p_{\mathcal{N}1}$ uses the function $\chi_{\diamond 2}$ on $\mathbb{R}$ that is given by $\chi(|s| - 100)$. The latter function is 0 where $|s| \geq 101$ and it is equal to 1 where $|s| \leq 100$. The derivative of $\chi_{\diamond 2}$ is denoted by $\chi'_{\diamond 2}$. Reintroduce the closed 2–form p_0 from Step 1 of the proof of Lemma 9.1. By way of a reminder, this 2–form has compact support on $\mathcal{Y}_0$, and it has integral 2 over each cross sectional 2–sphere in $\mathcal{H}_0$.

Suppose that Y_- is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k. The extension of $p_{\mathcal{N}1}$ will equal $p_{\mathcal{N}1}$ on the complement in Y_- of the union of $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Lemma 9.5 is used in a moment to obtain a 1–form to be denoted by q_{2-} with the following properties: the 1–form q_{2-} has compact support on $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$, its exterior derivative is $w_{\mathfrak{g}} = w_- - p_0 + d(\sigma_1 q_{1-})$, and its L^2–norm is bounded by c_0 times that of w_-. Granted such a 1–form, the extension of $p_{\mathcal{N}1}$ is given by

$$p_{\mathcal{N}2} = -ds \wedge \chi'_{\diamond 2} q_{2-} + w_- - d(\sigma_1 q_{1-}) + \chi_{\diamond 2}\, dq_{2-}. \tag{9-38}$$

This is a closed 2–form that equals $p_{\mathcal{N}1}$ where $s \leq -101$ and for all $s \in [-101, -100]$ on the complement of $\mathcal{Y}_0$ and the $r \geq \rho_* + \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. This 2–form for $s \geq -100$ is equal to p_0 on $\mathcal{Y}_0$ and the $r \geq \rho_*$ part of $\mathcal{N}_\varepsilon$.

The application of Lemma 9.5 takes $U = V = S^1 \times S^2$. The diffeomorphism Φ_T in Part 6 of Section 9.1 is used to view $p_0 - (w_- - d(\sigma_1 q_{1-}))$ as a smooth 2–form on $S^1 \times S^2$, and viewed as such, Lemma 9.5 is applied using this 2–form for $w_{\mathfrak{g}}$. Lemma 9.5 then finds a 1–form, q, on $S^1 \times S^2$ with $dq = p_0 - (w_- - d(\sigma_1 q_{1-}))$ and with L^2–norm bounded by c_0 times the L^2–norm of w_- on Y_-. The next two paragraphs explain how to obtain q_{2-} from q.

View $p_0 - (w_- - d(\sigma_1 q_{1-}))$ as a 2–form on $S^1 \times S^2$ as done in the preceding paragraph. As explained in Part 4 of Section 9.1, the coordinates (ρ, ϕ, x_3) for $\mathcal{N}_\varepsilon$ can be viewed where $r \leq \rho_* + \frac{1}{16}\varepsilon$ as coordinates for a ball of this same radius in $S^1 \times S^2$. The 2–form $p_0 - (w_- - d(\sigma_1 q_{1-}))$ vanishes on the concentric ball of radius $\bigl(\rho_* + \frac{1}{256}\varepsilon\bigr)/T$. It follows as a consequence that q can be written as $d\mathfrak{k}$ with $\mathfrak{k}$ being a smooth function on this ball. Since the L^2–norm of $d\mathfrak{k}$ on this ball is bounded by c_0 times the L^2–norm of w_- over Y_-, it follows that k can be modified by adding a constant if necessary so that its L^2–norm over this ball is bounded by c_0/T times the L^2–norm of w_- over Y_-.

Use χ to construct a smooth function of the radial coordinate on this ball with compact support that equals 1 on the concentric ball of radius $\big(\rho_* + \frac{1}{512}\varepsilon\big)/T$. In particular, such a function can be constructed so that the absolute value of its derivative is bounded by $c_0 T$. Let σ_2 denote such a function and define q_* to be $q - d(\sigma k)$. This 1–form has the same properties as q but it is zero on the complement of the image of the embedding Φ_T from Part 6 of Section 9.1. The desired 1–form q_{2-} is $\Phi_T^* q_*$.

If Y_+ is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, then there is an analogous construction that defines $p_{\mathcal{N}2}$ on the $s \in [100, 101]$ part of X. The formula for the latter is given by replacing w_-, q_{1-} and q_{2+} by w_+, q_{1+} and a 1–form, q_{2+}, that is defined by the rules given in the preceding paragraph with w_+ and q_{1+} used in lieu of w_- and q_{1-}.

Part 10 Constructions in Part 11 and in the proof of Proposition 3.13 require a particular choice for the metric $\mathfrak{m}$ on certain parts of X. The constraint given in a moment holds on the $s \in [-100, -96]$ part of X when Y_- is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k, and it holds on the $s \in [96, 100]$ part of X when Y_+ is a $k \in \{0, \dots, \mathrm{G}\}$ version of Y_k.

The statement of the constraint uses the embeddings from the second and third bullets of (2-8) to view the $s < 0$ and $s > 0$ part of X as $(-\infty, 0] \times Y_-$ and as $(0, \infty) \times Y_+$. Viewed this way, the constraint on the metric $\mathfrak{m}$ involves only the $r \in \big[\rho_* - \frac{15}{16}\varepsilon, \rho_*\big)$ parts of $[-100, -96] \times \mathcal{N}_\varepsilon$ and $[96, 100] \times \mathcal{N}_\varepsilon$. To define $\mathfrak{m}$ on these parts of X, construct a smooth, nondecreasing function on $\mathbb{R}$ to be denoted by $T_\diamond$: This function equals T where $|s| \geq 99$ and it equals 1 where $|s| \leq 98$. The ubiquitous function χ can be used to define this function $T_\diamond$. Reintroduce the metric $\mathfrak{g}_T$ on $\mathcal{N}_\varepsilon$ that is defined in Part 5 of Section 9.1. The assignment $s \mapsto \mathfrak{g}_{T_\diamond}$ defines a 1–parameter family of metrics on $\mathcal{N}_\varepsilon$ with parameter space either $[-100, -96]$ or $[96, 100]$ as the case may be. The $|s| = 100$ end member of this family is $\mathfrak{g}_T$ and the $|s| = 96$ member is the metric in (9-6).

Use χ to construct a smooth function of the coordinate r on $\mathcal{N}_\varepsilon$ that is equal to 1 where $r < \rho_* - \frac{1}{1024}\varepsilon$ and equal to 0 where $r > \rho_* - \frac{1}{2048}\varepsilon$. Use σ_* to denote this function.

The metric $\mathfrak{m}$ is constrained by requiring that its pullback to $[-100, -96] \times \mathcal{N}_\varepsilon$ via the embedding from the second bullet of (2-8) or to $[96, 100] \times \mathcal{N}_\varepsilon$ via the embedding from the third bullet of (2-8) to be the metric

$$ds^2+\sigma_*\mathfrak{g}_{T_\diamond}+(1-\sigma_*)\mathfrak{g}_T. \tag{9-39}$$

Note in particular that this metric smoothly extends $ds^2+\mathfrak{g}_T$ near $|s|=100$ and it smoothly extends $ds^2+\mathfrak{g}_T$ from the $r\le\rho_*-\frac{15}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ for all s in the relevant interval.

An important observation is given in a moment about the versions of the L^2–norm of $w_--d(\sigma_1 q_{1-})$ on the $r\le\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Keep in mind in what follows that this 2–form is zero on the $r>\rho_*-\frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Given $s\in[-100,-96]$, the notation uses $\|w_--d(\sigma_1 q_{1-})\|_s$ to denote the version of the L^2–norm of $w_--d(\sigma_1 q_{1-})$ on the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. There is the analogous definition for $\|w_+-d(\sigma_1 q_{1+})\|_s$ when $s\in[96,100]$. Here is the key observation:

(9-40)
- Each $s\in[-100,-96]$ version of $\|w_--d(\sigma_1 q_{1-})\|_s$ is bounded by c_0 times the L^2–norm of w_- on Y_-.
- Each $s\in[96,100]$ version of $\|w_+-d(\sigma_1 q_{1+})\|_s$ is bounded by c_0 times the L^2–norm of w_+ on Y_+.

To see about (9-40), write any $s\in[-100,-96]$ or $s\in[96,100]$ version of $\mathfrak{g}_{T_\diamond}$ at any given point in the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$ as

$$\mathfrak{g}_{T_\diamond}=\lambda_1\hat{e}^1\otimes\hat{e}^1+\lambda_2\hat{e}^2\otimes\hat{e}^2+\lambda_3\hat{e}^3\otimes\hat{e}^3 \tag{9-41}$$

with λ_1, λ_2 and λ_3 being positive numbers and with $\{\hat{e}^k\}_{k=1,2,3}$ being a $\mathfrak{g}_T$–orthonormal frame. It follows from (9-7)–(9-9) that each λ_k can be written as $(T/T_\diamond)^2 e_k$, where the numbers e_1, e_2 and e_3 are such that $c_0^{-1}\le e_1,e_2\le c_0$ and $c_0^{-1}\le e_3\le c_0(T/T_\diamond)^2$. It follows from this that the volume form of the metric is less than $c_0(T/T_\diamond)^4$ times that of $\mathfrak{g}_T$. It also follows from this that the square of the $\mathfrak{g}_{T_\diamond}$–norm of $w_--d(\sigma_1 q_{1-})$ is less than $c_0(T/T_\diamond)^4$ times the square of its $\mathfrak{g}_T$–norm. These last two observations imply that the integrand whose integral gives $\|w_--d(\sigma_1 q_{1-})\|_s^2$ is no greater than c_0 times the integrand whose integral computes the square of the $\mathfrak{g}_T$ version of the L^2–norm of $w_--d(\sigma_1 q_{1-})$. This last fact implies directly the first bullet of (9-40). But for replacing $-$ subscripts with $+$ subscripts, the same argument proves the second bullet of (9-40).

Part 11 This part of the subsection completes the proof of Proposition 3.11. According to Part 8, it is sufficient to find the closed 2–form p_X with certain special properties. This is done given two more constraints on $\mathfrak{m}$. The first constraint affects $\mathfrak{m}$ only on the

$|s| \in [96, 100]$ part of X. The statement of this uses the embeddings from the second and third bullets of (2-8) to view the $s < 0$ and $s > 0$ parts of X as $(-\infty, 0] \times Y_-$ and as $(0, \infty) \times Y_+$:

(9-42) The metric $\mathfrak{m}$ on $[-100, -96] \times \mathcal{Y}_M$ is the product metric $ds^2 + \mathfrak{g}_-$ when $Y_- = Y_0$; and when $Y_+ = Y_0$, the metric $\mathfrak{m}$ on $[96, 100] \times \mathcal{Y}_M$ is the product metric $ds^2 + \mathfrak{g}_+$.

To state the second constraint, reintroduce from Part 7 of Section 9.1 the set Θ and the associated collection $\{\mathcal{T}_\gamma\}_{(\gamma, Z_\gamma) \in \Theta}$ of subsets of M_δ. The following observation views them as subsets of Y_0 and M:

There exists an embedding of $\mathbb{R} \times (\bigcup_{(\gamma, Z_\gamma) \in \Theta} \mathcal{T}_\gamma)$ into X with the following two properties:

(9-43)
- The function s on X pulls back via the embedding to its namesake on the $\mathbb{R}$ factor of $\mathbb{R} \times \left(\bigcup_{(\gamma, Z_\gamma) \in \Theta} \mathcal{T}_\gamma\right)$.
- The composition of this embedding of the $|s| > 1$ part of $\mathbb{R} \times \left(\bigcup_{(\gamma, Z_\gamma) \in \Theta} \mathcal{T}_\gamma\right)$ with the inverse of the embeddings from the second and third bullets of (2-8) is the tautological inclusion map.

The existence of such an embedding is implied by what is said in the first paragraph of this section about the ascending and descending manifolds from the critical point of s. The second constraint uses $\mathfrak{m}_-$ and $\mathfrak{m}_+$ to denote the metrics $ds^2 + \mathfrak{g}_-$ and $ds^2 + \mathfrak{g}_+$ on the product $\mathbb{R} \times \left(\bigcup_{(\gamma, Z_\gamma) \in \Theta} \mathcal{T}_\gamma\right)$.

(9-44) There exists a T–independent constant, $c_* > 1$, with the following significance: The pullback of $\mathfrak{m}$ via the embedding in (9-43) obeys $c_*^{-1} \mathfrak{m}_- \le \mathfrak{m} \le c_* \mathfrak{m}_-$ and $c_*^{-1} \mathfrak{m}_+ \le \mathfrak{m} \le c_* \mathfrak{m}_+$.

Granted these constraints, the three steps that follow construct p_X when $Y_- = Y_0$. The construction when $Y_+ = Y_0$ is not given as it has the identical description but for changes of $-$ to $+$ in various places.

Step 1 Define p_X on the $s \in [-102, -101]$ part of X to be $p_{\mathcal{N}1}$ and define p_X on the $s \in [-101, -100]$ part of X to be $p_{\mathcal{N}2}$. The rest of this step extends the definition of p_X to the $s \in [-100, -98]$ part of X. To this end, use the embedding from the second bullet of (2-8) to view this part of X as $[-100, -98] \times Y_0$.

The 2–form $p_{\mathcal{N}2}$ near $s = -100$ is the s–independent 2–form on Y_0 given by p_0 on $\mathcal{Y}_0$ and $w_- - d(\sigma_1 q_{1-})$ on the rest of Y_0. This understood, p_X is extended to the $s \in [-100, -98]$ part of X as this s–independent 2–form on Y_0.

Write the metric $\mathfrak{m}$ appearing on $[-100,-98]\times Y_0$ as $ds^2+\mathfrak{g}$ with $\mathfrak{g}$ denoting an s–dependent metric on Y_0. The constraint in (9-42) asserts that $\mathfrak{g}=\mathfrak{g}_-$ on $\mathcal{Y}_M$. Meanwhile, $\mathfrak{g}$ is Part 9's metric on the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. It therefore follows from (9-40) that the L^2–norm of p_X on Y as defined by any $s\in[-100,-98]$ version of $\mathfrak{g}$ is bounded by c_0.

Step 2 This step extends the definition of p_X to the $s\in[-98,-96]$ part of X. To do this, view the $s\in[-98,-96]$ part of X as $[-98,-96]\times Y_0$ as in Step 1. Keep in mind for what follows that the metric $\mathfrak{m}$ here has the form $ds^2+\mathfrak{g}_M$ with $\mathfrak{g}_M$ being an s–independent metric on Y_0. Note in particular that $\mathfrak{g}_M=\mathfrak{g}_-$ on $\mathcal{Y}_M$ and it is the metric that is depicted in (9-6) on the $r\le\rho_*-\frac{1}{1024}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Lemma 9.5 is invoked in a moment to construct a 1–form on the union of $\mathcal{Y}_M$ and the $r<\rho_*$ part of $\mathcal{N}_\varepsilon$, with the three properties listed in a moment. The list of properties denotes the 1–form by q_{3-} and it reintroduces the 2–form p from Part 7 of Section 9.1. Here are the three properties: The 1–form q_{3-} obeys $dq_{3-}=p-w_-+d(\sigma_1q_{1-})$, it vanishes on the $\mathrm{r}\ge\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$, and its L^2–norm as defined by the $\mathfrak{g}_M$ is bounded by c_0 times the L^2–norm of w_- on Y.

Let $\chi_{\diamond 3}$ denote the function on $\mathbb{R}$ given by $\chi(|s|-97)$. The function $\chi_{\diamond 3}$ equals 0 where $|s|\ge 98$ and it equals 1 where $|s|\le 97$. Introduce $\chi'_{\diamond 3}$ to denote its derivative. The 2–form p_X on the $s\in[-98,-96]$ part of X is p_0 on $\mathcal{Y}_0$ and it is given on the rest of Y_0 by

$$ds\wedge\chi'_{\diamond 3}q_{3-}+w_--d(\sigma_1q_{1-})+\chi_{\diamond 3}\,dq_{3-}. \tag{9-45}$$

Of particular note is that the $\mathfrak{m}$ version of the L^2–norm of the 2–form p_X on $[-98,-96]\times Y_0$ is bounded by c_0. What follows is a key point to keep in mind for Step 3: the 2–form p_X on $[-97,-96]\times Y_0$ is the 2–form p_0+p from Y_0.

Lemma 9.5 is invoked using for the set U the union of $\mathcal{Y}_M$ and the $r<\rho_*-\frac{1}{1024}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Lemma 9.5's set V is the union of $\mathcal{Y}_M$ and the $r<\rho_*-\frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$. The 2–form $w_{\mathfrak{g}}$ is $p-w_-+d(\sigma_1q_{1-})$. Note that this 2–form is zero on the $r>\rho_*-\frac{1}{256}\varepsilon$ part of U. The metric used for the lemma is the metric $\mathfrak{g}_M$. It follows from (9-40) and (9-42) that the L^2–norm of $p-w_-+d(\sigma_1q_{1-})$ as defined by $\mathfrak{g}_M$ is bounded by c_0. As neither U, V nor $\mathfrak{g}_M$ depend on T, Lemma 9.5 finds a 1–form q on U with $dq=p-w_-+d(\sigma_1q_{1-})$ whose L^2–norm on V is bounded by c_0. To obtain q_{3-} from q, note that q on the $r>\rho_*-\frac{1}{256}\varepsilon$ part of $\mathcal{N}_\varepsilon$ is given by $d\mathfrak{k}$ with $\mathfrak{k}$ denoting a smooth function. Changing $\mathfrak{k}$ by a constant if necessary supplies a version whose L^2–norm is bounded by c_0 times that of $d\mathfrak{k}$; thus by c_0. Take such a version. Meanwhile, use χ to construct a smooth function of r on $\mathcal{N}_\varepsilon$ that equals 0 where $r \geq \rho_* - \frac{1}{512}\varepsilon$, equals 1 where r $\leq \rho_* - \frac{1}{256}\varepsilon$ and whose derivative has norm bounded by c_0. Denote this function by σ_3 and set $q_{3-} = q - d(\sigma_3 \mathfrak{k})$.

Step 3 This step extends the definition of p_X to the $s \in [-96, 102]$ part of X. To this end, consider first the definition of p_X on the $s \in [-96, 100]$ part of X. As p_0 is supported in the image of the embedding from (9-10) and as the 2–form p is supported in the image of the embedding from (9-43), these embeddings can be used to view $p_0 + p$ as a 2–form on the $s \in [-96, 100]$ part of X. View them in this light and define p_X on this same part of X to be $p_0 + p$. The constraint in (9-44) has the following implication: the L^2–norm of p_X on the $s \in [-96, 100]$ part of X is bounded by c_0.

The definition of p_X on the $s \in [100, 102]$ part of X views this part of X via the embedding from the third bullet of (2-8) as $[100, 102] \times (M \sqcup (S^1 \times S^2))$. The 2–form p_0 on $S^1 \times S^2$ can be written as $w + dq_0$ with q_0 being a smooth 1–form. Likewise, the 2–form p on M can be written as $w_+|_M + dq_M$ with q_M denoting a smooth 1–form. Set $q_{4+} = q_0 + q_M$. Let $\chi_{\diamond 4}$ denote the function on $\mathbb{R}$ given by $\chi(s - 100)$. This function $\chi_{\diamond 4}$ is equal to 1 where $s < 100$ and it is equal to 0 where $s > 101$. Use $\chi'_{\diamond 4}$ to denote its derivative.

Define p_X on the $s \in [100, 102]$ part of X to be the 2–form

$$ds \wedge \chi'_{\diamond 4} q_{4+} + p_0 + p - \chi_{\diamond 4}\, dq_{4+}. \tag{9-46}$$

This form is closed, and it extends p_X as a 2–form that equals w_+ where $s > 101$. Of particular note is that the L^2–norm of p_X on the $s \in [100, 102]$ part of X is bounded by c_0. □

9.5 Proof of Proposition 3.13

The proof of this proposition has two parts. Of the two possible cases, only that where $Y_- = Y_k$ and $Y_+ = Y_{k-1} \sqcup (S^1 \times S^2)$ is discussed as the case when the roles are switched is proved with the same argument but for changing the direction of various inequalities and signs that involve s.

Part 1 of what follows proves the first bullet of Proposition 3.13. Part 2 of this subsection addressses the assertion in the second bullet and, in doing so, defines implicitly the required subset $\mathrm{Met}(Y_k)$. To make the definition only slightly less implicit, return momentarily to what is said about *Met* just prior to Part 1 of Section 9.4. By way of a reminder, each metric in *Met* is determined in part by a metric from the Y_0 version of Section 9.1's set $\mathrm{Met}^{\mathcal{N}}$ and a sufficiently large choice for a number denoted by T. A lower bound on T is determined by certain properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric. This said, a metric from *Met* is in Proposition 3.13's subset $\mathrm{Met}(Y_k)$ if and only if the chosen value for T is larger than a new lower bound. This new lower bound is determined in part by the same properties of the chosen $\mathrm{Met}^{\mathcal{N}}$ metric that determine the $\mathrm{Met}(Y_0)$ lower bound. The chosen metrics on the $S^1 \times S^2$ components also determine in part the lower bound for T. By the way, no generality is lost by taking the metrics on these components to be the product of the standard Euclidean S^1 and the standard round metric on S^2. In any event, this new lower bound is determined implicitly by the constructions in Part 2.

Part 1 This part discusses the first bullet of the proposition. The notation used below is that used to describe Y and its geometry in [19; 20; 21; 22]. In particular, the manifold Y and its 2–form w are described in Section II.1. A summary of the salient features can be found in Section IV.1.1. The notation used below is the same as that used in Sections II.1 and IV.1.1.

To set the stage, label the G pairs in the set Λ as $\{\mathfrak{p}_1, \dots, \mathfrak{p}_{\mathrm{G}}\}$. A $k \in \{1, \dots, \mathrm{G}\}$ version of the manifold Y_k is obtained from Y_0 by attaching k 1–handles, these being the handles from the set $\{\mathcal{H}_{\mathfrak{p}}\}_{\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}}$. Thus, Y_k is obtained from Y_{k-1} by attaching just the handle $\mathcal{H}_{\mathfrak{p}_k}$. By way of a short review, Y is obtained from Y_0 by a surgery that attaches G 1–handles to $Y_0 - \mathcal{H}_0$. The attaching region of each handle are disjoint coordinate balls centered around a pair of points in $Y_0 - \mathcal{H}_0$. The set of such pairs is denoted by Λ. The 1–handle that corresponds to a given pair $\mathfrak{p} \in \Lambda$ is denoted by $\mathcal{H}_{\mathfrak{p}}$. The geometry of Y_k near $\mathcal{H}_{\mathfrak{p}_k}$ is as follows: The handle $\mathcal{H}_{\mathfrak{p}_k}$ is diffeomorphic to $[-R - 7\ln\delta_*, R + 7\ln\delta_*] \times S^2$ given by the preferred coordinates $(u, (\theta, \phi))$ with u denoting the Euclidean coordinate for interval factor and with (θ, ϕ) denoting spherical coordinates on the constant u cross-sectional spheres of $\mathcal{H}_{\mathfrak{p}_k}$. The handle is attached to Y_{k-1} using the identifications given in (3-3) with it understood that $(r_+, (\theta_+, \phi_+))$ and $(r_-, (\theta_-, \phi_-))$ are certain preferred spherical coordinates for respective balls about the two points that form the pair $\mathfrak{p}_k$.

The definition of X requires choosing a properly embedded arc in the $\mathcal{Y}_M$ part of Y_{k-1}. The arc has one endpoint at one of the points in $\mathfrak{p}_k$ and the other endpoint at the other. This arc intersects a neighborhood of the boundary of the radius $7\delta_*$ coordinate ball centered at the points from $\mathfrak{p}_k$ as a ray from the origin when viewed using the coordinate system that is specified in Section II.1A. Part 7 of Section 9.1 introduces a finite set of pairs Θ in M_δ with one partner in each pair being an embedded loop in M_δ. Part 7 of Section 9.1 associates each such loop a small radius tubular neighborhood, this being $\mathcal{T}_\gamma$ when γ is the loop in question. The arc must be chosen so as to lie in the complement of the closure of all such tubular neighborhoods. The arc can and should be chosen to intersect the $f = \frac{3}{2}$ Heegaard surface in M_δ transversely in a single point. Denote this arc by $\lambda_{\mathfrak{p}_k}$.

Let $S_{\mathfrak{p}_k} \subset \mathcal{Y}_M$ denote an embedded 2–sphere boundary of neighborhood of the arc $\lambda_{\mathfrak{p}_k}$ with each point having distance between 2δ and 4δ from the arc. The neighborhood in question and $S = S_{\mathfrak{p}_k}$ should be disjoint from the closures of the tubular neighborhoods of the loops from Θ. The sphere S appears in Y_k as an embedded 2–sphere that separates Y_k into two components. One of these contains $\mathcal{H}_{\mathfrak{p}_k}$ and is diffeomorphic to the complement in $S^1 \times S^2$ of an embedded ball.

The following is a consequence of what is said above about the descending and ascending submanifolds from the critical points of s: the pseudogradient vector field that defines the embeddings from the second and third bullets of (2-8) can be chosen so that (3-11) are obeyed and likewise (3-12) and the conditions in (9-10) and (9-43). These properties are assumed in what follows. The condition for the first Chern class is satisfied if and only it has zero pairing with the cross-sectional 2–spheres in each $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_{k-1}\}$ version of the Y_{k-1} version of $\mathcal{H}_\mathfrak{p}$ and annihilates the generator of $H_2(S^1 \times S^2; \mathbb{Z})$.

Part 2 Proposition 3.13 requires as input a metric from a certain subset of a set of metrics on Y_{k-1} that is denoted by $\mathrm{Met}(Y_{k-1})$ and a metric from a set of metrics on Y_k, this denoted by $\mathrm{Met}(Y_k)$. These subsets of metrics are in the respective Y_{k-1} and Y_k versions of *Met*. They are defined roughly as follows: Let Y_* for the moment denote either Y_{k-1} or Y_k. Each metric in the Y_* version of *Met* is determined in part by a metric from the corresponding version of Met_N, this defined in Section 9.1. The second input for the definition is a large choice for the parameter T. A metric in *Met* of this sort is denoted in Section 9.2 by $\mathfrak{g}_{3T}$. A Y_* metric $\mathfrak{g}_{3T}$ is in $\mathrm{Met}(Y_*)$ if T is greater than a certain lower bound that is determined implicitly by the chosen Met_N metric. As in the case of Proposition 3.13's implicit definition of $\mathrm{Met}(Y_0)$, this lower bound is determined implicitly by the requirements of subsequent constructions. In any event, it is determined by certain curvature norms, injectivity radius lower bounds and volume.

The construction of a suitable metric on X starts by choosing metrics $\mathfrak{g}_{1-}$ and $\mathfrak{g}_{1+}$ from the respective Y_- and Y_+ versions of Met_N. This done, use what is said in Parts

1–10 of Section 9.4 to define a metric $\mathfrak{m}_{T*}$ and self-dual 2–form ω_{T*} on X. It then follows from what is said in Part 8 and at the start of Part 9 of Section 9.4 that the pair $\mathfrak{m}_{T*}$ and ω_{T*} satisfy the requirements of Proposition 3.13 if there exists a suitable closed 2–form p_X on X with the following properties: The de Rham cohomology class of p_X is that of $c_1(\det \mathbb{S})$. In addition, p_X must equal w_- where $s < -102$ and w_+ where $s > 102$ with w_- and w_+ being the respective $\mathfrak{g}_-$ and $\mathfrak{g}_+$ harmonic 2–forms on Y_- and Y_+ with de Rham cohomology class that of $c_1(\det \mathbb{S})$.

The construction of p_X in this case differs in only one respect from the construction described in Parts 9–11 of Section 9.4, this involving Step 3 in Part 11 of Section 9.4. To say more about this difference, require as in Part 11 of Section 9.4 that the metric $\mathfrak{m}$ obey (9-42). Require in addition that (9-43) is obeyed; as noted in Part 1 above, such a requirement can be met. With (9-43) understood, the metric $\mathfrak{m}$ is chosen so as to obey the constraints in (9-44). Proceed with the constructions in Steps 1 and 2 of Part 11 in Section 9.4. Step 3 in Part 11 of Section 9.4 is replaced with the following Step $3'$:

Step 3′ Define p_X on the $s \in [-96, 96]$ part of X by viewing $p_0 + p$ as a 2–form on this part of X via the embeddings in (9-10) and (9-43). The constraint in (9-44) implies that such a definition yields a version of p with L^2–norm bounded by c_0 on the $s \in [-96, 96]$ part of X. Extend p_X to the $[96, 102]$ part of X by copying almost verbatim what is done in Steps 1 and 2 with the direction of s reversed and with the metric $\mathfrak{g}_+$ in (9-42) used in lieu of $\mathfrak{g}_-$.

9.6 Proof of Proposition 1.5

This subsection provides a proof of Proposition 1.5 and thus completes the proof of Theorem 1.4.

Explicit formulas for the differentials and $\boldsymbol{A}_\dagger(Y)$–actions on the chain complex used to define ech° are given in Theorem 1.1 of [21]. These formulas were also written in terms of a factorization of $\boldsymbol{A}_\dagger(Y)$ into a tensor product $\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$, which is however *different* from that in (1-8), the factorization used in the statement of the proposition. The difference originates from a different choice of splitting for $H_1(Y;\mathbb{Z})/\mathrm{Tors}$ from that in (1-4).

In [21], an "M–adapted 1–cycle basis" is assigned to $H_1(Y;\mathbb{Z})/\mathrm{Tors}$, whose basis elements are represented by "M–adapted 1–cycles" in Y. Each "M–adapted 1–cycle" is of one of the following three types:

- $\hat{\imath}^{(z)}$ for every $z \in ¥ - z_0$;

- $\gamma^{(z_0)}$; and
- $\hat{\iota}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Lambda$.

Decompose $H_1(Y;\mathbb{Z})/\mathrm{Tors}$ accordingly into

$$H_1(Y;\mathbb{Z})/\mathrm{Tors} \simeq H_1(M;\mathbb{Z})/\mathrm{Tors} \oplus H_1(S^1 \times S^2;\mathbb{Z}) \oplus \bigoplus_{\mathfrak{p}\in\Lambda} H_1((S^1 \times S^2)_{\mathfrak{p}};\mathbb{Z}) \tag{9-47}$$

with the first, second, and third summand generated by the ordered sets $\{[\hat{\iota}^{(z)}]\}_{z \in ¥ - z_0}$, $\{[\gamma^{(z_0)}]\}$ and $\{[\hat{\iota}_{\mathfrak{p}}]\}_{\mathfrak{p}\in\Lambda}$, respectively.

On the other hand, in Section 1.1 we split $H_1(Y;\mathbb{Z})/\mathrm{Tors}$ differently using a connected-sum decomposition of Y. Namely, by combining (1-2) and (1-4) we get another splitting

$$\begin{aligned} H_1(Y;\mathbb{Z})/\mathrm{Tors} & \\ &\simeq H_1(\underline{M};\mathbb{Z})/\mathrm{Tors} \oplus \bigoplus_{\mathfrak{p}\in\Lambda} H_1((S^1 \times S^2)_{\mathfrak{p}};\mathbb{Z}) \\ &\simeq H_1(M;\mathbb{Z})/\mathrm{Tors} \oplus H_1(S^1 \times S^2;\mathbb{Z}) \oplus \bigoplus_{\mathfrak{p}\in\Lambda} H_1((S^1 \times S^2)_{\mathfrak{p}};\mathbb{Z}). \end{aligned} \tag{9-48}$$

Note that the preceding splitting (9-48) depends on the relative homology class of the chosen arcs γ_z and $\lambda_{\mathfrak{p}}$. The summands from this splitting are generated by elements in $H_1(Y;\mathbb{Z})/\mathrm{Tors}$ represented by the following sets of 1–cycles in Y:

- For the first summand $H_1(M;\mathbb{Z})/\mathrm{Tors}$, $b_1(M)$ 1–cycles from the M–summand of the connected sum decomposition $Y \simeq M \#_{\mathrm{G}+1} (S^1 \times S^2)$, so that as 1–cycles in M they avoid all the arcs γ_z and $\lambda_{\mathfrak{p}}$ and their homology classes together form an (arbitrary) basis for $H_1(M;\mathbb{Z})/\mathrm{Tors}$. For example, the set of 1–cycles $\{[\hat{\iota}^{(z)}]\}_{z \in ¥ - z_0}$ is a possible choice.
- For the second summand $H_1(S^1 \times S^2;\mathbb{Z})$, the cycle coming from the 1–cycle $\underline{\gamma}$ in the $\underline{M}$–summand of the connected sum $Y \simeq \underline{M} \#_{\mathrm{G}} (S^1 \times S^2)$. (This cycle in Y was called $\gamma^{(z_0)}$ in [21].)
- For each $H_1((S^1 \times S^2)_{\mathfrak{p}};\mathbb{Z})$–summand in $\bigoplus_{\mathfrak{p}\in\Lambda} H_1((S^1 \times S^2)_{\mathfrak{p}};\mathbb{Z})$, a 1–cycle $\underline{\lambda}_{\mathfrak{p}} \subset Y$ constructed from the arc $\lambda_{\mathfrak{p}} \subset M$ in Part 1, in a way parallel to the construction of $\gamma^{(z_0)}$ or $\underline{\gamma}$ from $\gamma_z \subset M$.

Here is a more precise description of the cycles $\underline{\lambda}_{\mathfrak{p}}$. For $\mathfrak{p} = \mathfrak{p}_k$ with $k = 1, \dots, \mathrm{G}$, let $\underline{\lambda}_{\mathfrak{p}_k}$ be a 1–cycle in Y_k characterized by the properties listed below. Recall the sphere

$S_{\mathfrak{p}_k}$ from Part 1. Let $\mathcal{N}_{k,\varepsilon}$ denote the version of $\mathcal{N}_\varepsilon$ from Part 4 of Section 9.1 when the sphere S therein is set to be $S_{\mathfrak{p}_k}$. Then

- on $Y_k - \mathcal{N}_{k,\varepsilon}$, $\underline{\lambda}_{\mathfrak{p}_k}$ agrees with $\lambda_{\mathfrak{p}_k} \cup \hat{\imath}_{\mathfrak{p}_k}$;
- on $\mathcal{N}_{k,\varepsilon}$, $\underline{\lambda}_{\mathfrak{p}_k}$ is transverse to the spheres $\{\rho\} \times S^2$ for all $\rho \in I$ under the identification $\mathcal{N}_{k,\varepsilon} \simeq I \times S^2$ in Part 4 of Section 9.1.

Recall that $Y_{k+1}, \dots, Y_{\mathrm{G}} = Y$ are constructed from Y_k by iteratively connected summing with $S^1 \times S^2$, and thus they all contain a 1–cycle inherited from the $\underline{\lambda}_{\mathfrak{p}_k} \subset Y_k$ described above. We use the same notation $\underline{\lambda}_{\mathfrak{p}_k}$ for all such cycles in $Y_{k+1}, \dots, Y_{\mathrm{G}} = Y$.

With the above understood, the splitting (9-47) adopted in [21] is related to the splitting (9-48) used in this article's Section 1.1 via a transformation matrix of the block form

$$\begin{bmatrix} \mathbb{X} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ \mathbb{Y} & \vec{1} & \mathrm{Id} \end{bmatrix}, \tag{9-49}$$

where $\vec{1}$ denotes a row vector of all entries 1, $\mathbb{X}$ is an automorphism of $H_1(M;\mathbb{Z})/\mathrm{Tors}$, and $\mathbb{Y}$ depends on the relative homology classes of $\lambda_{\mathfrak{p}}$'s. One may choose the arcs $\lambda_{\mathfrak{p}}$ so that the entry $\mathbb{Y}$ vanishes. *Such a choice of the* $\lambda_{\mathfrak{p}}$ *is adopted in this article.*

Use $(C^\circ_{\mathrm{ech}}, \partial_{\mathrm{ech}})$ to denote the underlying chain complex of ech°, and let $(\mathrm{CF}^\circ, \partial_{\mathrm{HF}})$ be the Heegaard Floer complex. In [21]'s notation, the chain module C^∞_{ech} is generated by the set $\hat{\mathcal{Z}}_{\mathrm{ech},M}$, which is a $\mathbb{Z}$–bundle over the set $\mathcal{Z}_{\mathrm{ech},M}$. The latter is written in [21, (1-10)] as a product of $\mathcal{Z}_{\mathrm{HF}}$, the generating set for the Heegaard Floer chain module $\widehat{\mathrm{CF}}$, and for each $\mathfrak{p} \in \Lambda$, a copy of $\mathbb{Z} \times \mathrm{O}$. This can be used to write the ech chain module C°_{ech} as a tensor product of CF° and, for each $\mathfrak{p} \in \Lambda$, a polynomial algebra $C_{\mathfrak{p}} = \mathbb{Z}[\tau_{\mathfrak{p}}, \tau_{\mathfrak{p}}^{-1}, y_{\mathfrak{p}}^+, y_{\mathfrak{p}}^-]$. Here, $\tau_{\mathfrak{p}}$ is an even variable and corresponds to the generator $1 \in \mathbb{Z}$ of the first factor in $\mathbb{Z} \times \mathrm{O}$, and $y_{\mathfrak{p}}^+$ and $y_{\mathfrak{p}}^-$ are odd variables such that the polynomials 1, $y_{\mathfrak{p}}^+$, $y_{\mathfrak{p}}^-$ and $y_{\mathfrak{p}}^+ y_{\mathfrak{p}}^- = -y_{\mathfrak{p}}^- y_{\mathfrak{p}}^+$ correspond respectively to the elements 0, 1, -1 and $\{1, -1\}$ of O in [21]'s notation.

Recall that a U–map on the ech–chain complex C°_{ech}, and for each M–adapted 1–cycle $\hat{\gamma}$, a map we shall denote by $\mathfrak{t}_{\hat{\gamma}}$, were defined in [20, Appendix; 21, Section 1]. Together they define the $\boldsymbol{A}_\dagger(Y)$–action on ech°.

Stated in the language of this article, Theorem 1.1 of [21] asserts the following, with respect to the aforementioned decomposition of the chain module

$$C^\circ_{\mathrm{ech}} \simeq \mathrm{CF}^\circ \otimes \bigotimes_{\mathfrak{p} \in \Lambda} C_{\mathfrak{p}} : \tag{9-50}$$

- $(C^\circ_{\mathrm{ech}}, \partial_{\mathrm{ech}})$ is the product complex of the Heegaard Floer chain complex $(\mathrm{CF}^\circ, \partial_{\mathrm{HF}})$ and for each $\mathfrak{p} \in \Lambda$, the chain complex
$$(C_{\mathfrak{p}}, \partial_{\mathfrak{p}}) := (\mathbb{Z}[\tau_{\mathfrak{p}}, \tau_{\mathfrak{p}}^{-1}, y_{\mathfrak{p}}^+, y_{\mathfrak{p}}^-], (1+\tau_{\mathfrak{p}})(\partial_{y_{\mathfrak{p}}^+} + \tau_{\mathfrak{p}}^{-1}\partial_{y_{\mathfrak{p}}^-})),$$
where $\tau_{\mathfrak{p}}$ has degree 0 and $y_{\mathfrak{p}}^+$ and $y_{\mathfrak{p}}^-$ both have degree 1. Note that the homology $H(C_{\mathfrak{p}}, \partial_{\mathfrak{p}})$ has two generators, one of degree 0 and the other of degree 1, and they are respectively represented by the elements 1 and $y_{\mathfrak{p}}^+ - \tau_{\mathfrak{p}} y_{\mathfrak{p}}^-$ in the polynomial algebra $\mathbb{Z}[\tau_{\mathfrak{p}}, \tau_{\mathfrak{p}}^{-1}, y_{\mathfrak{p}}^+, y_{\mathfrak{p}}^-]$.
- The U–map on C°_{ech} acts only on the CF° factor, namely $U_{\mathrm{ech}} = U_{\mathrm{HF}} \otimes \bigotimes_{\mathfrak{p}\in\Lambda} \mathrm{Id}$, and the map U_{HF} on CF° induces the U–action on HF°.
- The $\mathfrak{t}_{\gamma^{(z_0)}}$–action on C°_{ech} likewise has the form $\mathfrak{t}_{\mathrm{HF}}^{(z_0)} \otimes \bigotimes_{\mathfrak{p}\in\Lambda} \mathrm{Id}$ under the decomposition (9-50).
- The $\mathfrak{t}_{\hat{\iota}^{(z)}}$–action on C°_{ech} has the form $\mathfrak{t}_{\mathrm{HF}}^{(z)} \otimes \bigotimes_{\mathfrak{p}\in\Lambda} \mathrm{Id}$ under the decomposition (9-50), and the map $\mathfrak{t}_{\mathrm{HF}}^{(z)}$ induces the action of $[\hat{\iota}^{(z)}] \in H_1(Y;\mathbb{Z})/\mathrm{Tors}$ on HF°.
- For $\mathfrak{p} = \mathfrak{p}_k$ with $k = 1, \dots, \mathrm{G}$, the $\mathfrak{t}_{\hat{\iota}_{\mathfrak{p}}}$–action on C°_{ech} is the tensor product of $\partial_{y_{\mathfrak{p}}^+}$ on the $C_{\mathfrak{p}}$ factor, and Id on all other factors of C°_{ech}. Note that with $\partial_{y_{\mathfrak{p}}^+}$ identified as the generator of the algebra $H_{-*}(S^1) \simeq \mathbb{Z}[\partial_{y_{\mathfrak{p}}^+}] \simeq \bigwedge^* H_1((S^1\times S^2)_{\mathfrak{p}};\mathbb{Z})$, the homology $H(C_{\mathfrak{p}}, \partial_{\mathfrak{p}})$ is identified with the module $H_*(S^1)$ with the standard $H_{-*}(S^1)$–action.

View HF° as a module over $\boldsymbol{A}_\dagger(M) \otimes \bigwedge^* H_1(S^1\times S^2;\mathbb{Z})$ with the $\bigwedge^* H_1(S^1\times S^2;\mathbb{Z})$ factor part of the action generated by the induced map from $\mathfrak{t}_{\mathrm{HF}}^{(z_0)}$ in the fourth bullet above. It follows that there is an isomorphism between ech° and $\mathrm{HF}^\circ \boxtimes H_*(S^1)^{\boxtimes \mathrm{G}}$ as modules over
$$(\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}.$$
Here, the $(\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$–module structure on ech° comes from the decomposition (9-47) to identify $\boldsymbol{A}_\dagger(Y)$ with
$$\begin{aligned}\boldsymbol{A}_\dagger(Y) &\simeq_{i_{\mathrm{cycle}}} \big(\boldsymbol{A}_\dagger(M) \otimes {\textstyle\bigwedge^*} H_1(S^1\times S^2;\mathbb{Z})\big) \otimes \bigotimes_{\mathfrak{p}\in\Lambda} {\textstyle\bigwedge^*} H_1((S^1\times S^2)_{\mathfrak{p}};\mathbb{Z}) \\ &\simeq (\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}},\end{aligned}$$
which is isomorphic to the external tensor product $\mathrm{HF}^\circ \boxtimes H_*(S^1)^{\boxtimes \mathrm{G}}$ as modules over $(\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$. The two factorizations of $\boldsymbol{A}_\dagger(Y)$, i_{sum} in (1-8) and, above, i_{cycle}, are related via (9-49) (where $\mathbb{Y} = 0$). According to Theorem 1.1(2),

the middle $H_{-*}(S^1)$ factor in (1-8)'s factorization of $\boldsymbol{A}_\dagger(Y)$ acts trivially on $H^\circ(Y)$. Recalling from [22] that $H^\circ(Y)$ and ech° are canonically isomorphic as $\mathbb{A}_\dagger(Y)$–modules, this means that the $\boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$–action on ech° in the statement of Proposition 1.5 is the same as the one arising from composing the inclusion

$$\boldsymbol{A}_\dagger(M) \otimes 1 \otimes H_{-*}(S^1)^{\otimes \mathrm{G}} \hookrightarrow \boldsymbol{A}_\dagger(M) \otimes H_{-*}(S^1) \otimes H_{-*}(S^1)^{\otimes \mathrm{G}}$$

with i_{cycle}. The assertion of the proposition then follows from the isomorphism $\mathrm{ech}^\circ \simeq \mathrm{HF}^\circ \boxtimes H_*(S^1)^{\boxtimes \mathrm{G}}$ described above. □

9.7 Proof of Proposition 3.14

The construction of the cobordism manifold X, its metric and self-dual 2–form has nine parts.

Part 1 This part sets some of the notation for the construction in the subsequent parts of the subsection of the desired metric on X and the 2–form w_X. Fix a metric on Y of the sort that is described in Part 2 of Section 3.5 and denote the latter by $\mathfrak{g}_Y$. The 2–form w on Y has $\mathfrak{g}_Y$–norm equal to 1 and its Hodge dual is the 1–form $\hat{a}$ that is described in Section II.3A; see also (IV.1-6). The constant L for use in (2-9) is specified at the end of the proof. Assume until then that $L > 100$ has been chosen.

The description of the metric for X and the 2–form w_X on the $s \in [-L, -L+8]$ part of X requires the formula for w on a given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ from (IV.1-3),

$$\text{(9-51)} \quad w = 6x \cos\theta \sin\theta \, d\theta \, du - \sqrt{6} f' \cos\theta \sin^2\theta \, du \, d\phi + \sqrt{6} f (1 - 3\cos^2\theta) \sin\theta \, d\theta \, d\phi.$$

The notation here uses x and f to denote a pair of nonnegative functions on $\mathcal{H}_\mathfrak{p}$, these given in (IV.1-2), with f' denoting the derivative of f. Both x and f are invariant under the reflection $u \mapsto -u$. The function x has compact support and is a nonzero constant where $|u| < 2$. This constant is denoted by x_0. The function f on the $|u| < 4$ part of $\mathcal{H}_\mathfrak{p}$ is given by the rule $u \mapsto f(u) = x_0 + 4e^{-2R} \cosh(2u)$.

The 1–form $\upsilon_\diamond$ given in (IV.1-5) plays a central role in what follows. This 1–form on the $|u| < 4$ part of $\mathcal{H}_\mathfrak{p}$ can be written as

$$\text{(9-52)} \quad \upsilon_\diamond = 4e^{-2R} \cosh(2u)(1 - 3\cos^2\theta) \, du + 12e^{-2R} \sinh(2u) \cos\theta \sin\theta \, d\theta.$$

The 1–form $\upsilon_\diamond$ is a closed form on Y, and its zero locus are the loci in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ where both u and the function $1 - 3\cos^2\theta$ are zero. Note also that

$*w = \upsilon_\diamond$ on the complement in Y of the $|u| \geq R + \ln\delta - 9$ parts of each $\mathfrak{p} \in \Lambda$ handle $\mathcal{H}_\mathfrak{p}$. A second point to note is that $*(w \wedge \upsilon_\diamond) \geq c_0^{-1}|\upsilon_\diamond|^2$ on the whole of Y.

Part 2 Let $*$ denote for the moment the Hodge star of the metric $\mathfrak{g}_Y$ on Y. The desired metric for X must pull back to $(-\infty, -L] \times Y$ via the embedding from the second bullet of (2-8) as the metric $ds^2 + \mathfrak{g}_Y$. Meanwhile, the corresponding pullback of w_X must equal $ds \wedge *w + w$. This 2–form is self-dual but it is not closed; this is because $d * w \neq 0$ on the $|u| \leq R + \ln\delta - 9$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$. This last fact follows from the formula in (IV.1-6).

The rest of this part of the subsection describes w_X for $s \in [-L, -L+3]$. The metric on this part of X still pulls back as $ds^2 + \mathfrak{g}_Y$ via the second bullet of (2-8).

Let $\chi_{\diamond 1}$ denote the function on $\mathbb{R}$ given by the rule $s \mapsto \chi(-s - L + 2)$. This function is equal to 0 where $s < -L + 1$ and it is equal to 1 where $s > -L + 2$. The derivative of $\chi_{\diamond 1}$ is denoted in subsequent equations by $\chi'_{\diamond 1}$. Fix $m > 1$ and introduce χ_m to denote the function of the coordinate s given by the rule $s \mapsto \chi(m|u| - 1)$. This function equals 0 where $|u| > 2m^{-1}$ and it equals 1 where $|u| < m^{-1}$. By way of a look ahead, m will be set equal to $r^{1/c_0 c}$ when the time comes to verify the requirements of Proposition 3.13.

Use w_1 to denote the s–dependent 2–form on Y that is equal to w on the $M_\delta \cup \mathcal{H}_0$ part of Y, and equal to the following 2–form below on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$:

$$\begin{aligned} w_1 = d(x(1 - \chi_{\diamond 1}\chi_m)(1 - 3\cos^2\theta)\,du) - \sqrt{6} f' \cos\theta \sin^2\theta \,du\,d\phi \\ + \sqrt{6} f\,(1 - 3\cos^2\theta) \sin\theta \,d\theta\,d\phi. \end{aligned} \tag{9-53}$$

Note that $|w_1| \leq c_0$. Meanwhile, $\frac{\partial}{\partial s} w_1 = d\mathfrak{b}$ with $\mathfrak{b} = -x\chi'_{\diamond 1}\chi_m(1 - 3\cos^2\theta)\,du$. As $\chi_m = 0$ where $|u| > 2m^{-1}$, the L^2–norm of $\mathfrak{b}$ on $[-L, -L+3] \times Y$ is no greater than $c_0 m^{-1}$. The appearance of $\chi_{\diamond 1}$ in the definition guarantees that $w_1 = w$ where $s \leq -L$. Note that w_1 is a closed 2–form on Y for each s. A key point to note is that the zero set of the $s > -L + 1$ version of w_1 consists of two circles in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$, these being the circles where u and $1 - 3\cos^2\theta$ are both zero.

The desired 2–form w_X pulls back to $[-L, -L+3] \times Y$ via the embedding from the second bullet of (2-8) as $ds \wedge *w_1 + w_1$.

Part 3 What follows directly describes the desired metric and the 2–form w_X on the $s \in [-L+3, -L+4]$ part of X. To this end, let $\chi_{\diamond 2}$ denote the function on $\mathbb{R}$ that is given by the rule $s \mapsto \chi(s + L - 3)$. This function is equal to 1 where $s \leq -L + 3$

and it is equal to 0 where $s \geq -L+4$. A smooth metric on Y will be constructed in a moment, whose Hodge star sends the $s \geq -L+3$ versions of w_1 to $\upsilon_\diamond$, thus making w_1 harmonic. Let $\mathfrak{g}_1$ denote this metric. Use $\mathfrak{g}$ to denote the s–dependent metric $\chi_{\diamond 2}\mathfrak{g}_Y + (1-\chi_{\diamond 2})\mathfrak{g}_1$ and let $*$ now denote its Hodge dual. The metric on X pulls back $[-L+3, -L+4] \times Y$ via the embedding from the second bullet of (2-8) as $ds^2 + \mathfrak{g}$. The pullback of w_X to $[-L+3, -L+4] \times Y$ is the 2–form $ds \wedge *w_1 + w_1$. This 2–form is self-dual when s is near $-L+4$. The two steps that follow construct the metric $\mathfrak{g}_1$.

Step 1 The 2–form w_1 is equal to w on the $M_\delta \cup \mathcal{H}_0$ part of Y and its $\mathfrak{g}_Y$–Hodge star here is $\upsilon_\diamond$. This understood, the metric $\mathfrak{g}_1$ on $M_\delta \cup \mathcal{H}_0$ is set equal to $\mathfrak{g}_Y$. To define $\mathfrak{g}_1$ on a given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$, note first that the function $\chi_{\diamond 1}$ in (9-53) is equal to 1 when $s \in [-L+3, -L+4]$. This implies that w_1 is s–independent when $s \in [-L+3, -L+4]$. More to the point, it also implies that the $s \in [-L+3, -L+4]$ version of w_1 shares the same zero locus with the closed 1–form $\upsilon_\diamond$, this being the circles in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ where u and $1-3\cos^2\theta$ are both zero. Meanwhile, $w_1 \wedge \upsilon_\diamond > 0$ on the complement of their common zero locus. This last observation can be used with Lemma 9.2 to construct the desired metric $\mathfrak{g}_1$ on any part of the complement in $\mathcal{H}_\mathfrak{p}$ of the $u=0$ and $1-3\cos^2\theta = 0$ locus as a smooth extension of the metric $\mathfrak{g}_Y$ from $M_\delta \cup \mathcal{H}_0$.

Step 2 Let $\mathcal{T} \subset \mathcal{H}_\mathfrak{p}$ denote the $|u| < m^{-1}$ part of $\mathcal{H}_\mathfrak{p}$. The function χ_m in (9-36) is equal to 1 on $\mathcal{T}$ and $f = x_0 + 4e^{-2R}\cosh(2u)$ on $\mathcal{T}$. This being the case, it follows from (9-51) and (9-52) that the metric on $\mathcal{T}$ with volume 3–form $\Omega = \sin\theta\, du\, d\theta\, d\phi$ and Hodge star defined by the rules

$$
\begin{aligned}
*\sin\theta\, d\theta\, d\phi &= \frac{1}{\sqrt{6}} \frac{4e^{-2R}\cosh(2u)}{x_0 + 4\mathrm{r}^{-2R}\cosh(2u)}\, du, \\
*\sin\theta\, d\phi\, du &= \tfrac{3}{2\sqrt{2}}\, d\theta, \\
*du\, d\theta &= \tfrac{3}{2\sqrt{2}} \sin\theta\, d\phi
\end{aligned}
\tag{9-54}
$$

sends w_1 to $\upsilon_\diamond$. Note that a suitable change of coordinates near the $\theta = 0$ and $\theta = \pi$ loci can be used to prove that the metric defined by (9-54) is smooth on the whole of $\mathcal{T}$.

As noted previously, Lemma 9.2 can be used to extend the metric defined in (9-54) to the whole of $\mathcal{H}_\mathfrak{p}$ so as to agree with $\mathfrak{g}_Y$ on $\mathcal{H}_\mathfrak{p} \cap M_\delta$. This must be done with some care so as to obtain an $m = \mathrm{r}^{1/c_0 c}$ extension that can be used to satisfy the second item of (3-15). With this goal in mind, note that Lemma 9.2 can be used to find an extension with the following three properties:

(9-55)
- The norm of the Riemannian curvature tensor and those of its covariant derivatives to order 20 are bounded by c_0.
- The injectivity radius is bounded from below by c_0^{-1}.
- The metric volume of Y is at most c_0.

The first bullet of Lemma 9.2 gives metrics that obey the third bullet of (9-55) and the second bullet of Lemma 9.2 supplies metrics that obey all three bullets.

Part 4 The desired metric for X and the 2–form w_X on the $s \in \left[-L+4, -\frac{3}{4}L+2\right]$ portion of X are described below. This is done by specifying their pullbacks via the embedding from the second bullet of (2-8) to $\left[-L+4, -\frac{3}{4}L+2\right] \times Y$. In this part, we use $\chi_{\diamond 2}$ to denote the function on $\mathbb{R}$ given by the rule $s \mapsto \chi\left(\frac{4}{L-20}(s+L-5)\right)$. This function is equal to 1 where $s < -L+5$ and it is equal to zero where $s > -\frac{3}{4}L$. Use $\chi'_{\diamond 2}$ to denote the derivative of $\chi_{\diamond 2}$. Note in particular that $|\chi'_{\diamond 2}| \le c_0 L^{-1}$.

Let w_2 denote the s–dependent 2–form on Y given by w_1 for $s < -L+4$, given by w on $M_\delta \cup \mathcal{H}_0$, and given on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ for $s \ge -L+4$ by

$$w_2 = \chi_{\diamond 2}\, d(x(1-\chi_m)(1-3\cos^2\theta)\, du) - \sqrt{6} f' \cos\theta \sin^2\theta\, du\, d\phi \\ + \sqrt{6} f (1-3\cos^2\theta) \sin\theta\, d\theta\, d\phi. \tag{9-56}$$

The 2–form w_2 is a closed 2–form on Y for each s, it has the same zero locus as w_1 and it has the property that $w_2 \wedge \upsilon_\diamond = w_1 \wedge \upsilon_\diamond$.

An s–dependent metric on Y is described in a moment for the cases when $L > c_0$. This metric is denoted by $\mathfrak{g}$. Let $*$ denote the corresponding Hodge dual. By way of a look ahead, $\mathfrak{g}$ is chosen so that $d * w_2 = \frac{\partial}{\partial s} w_2$. The pullback of the desired metric on X to $\left[-L+4, -\frac{3}{4}L+2\right] \times Y$ via the embedding from the second bullet of (2-8) is the quadratic form $ds^2 + \mathfrak{g}$, and the corresponding pullback of w_X is $ds \wedge *w_2 + w_2$. Note in particular that w_X is self-dual and closed if self-duality is defined by the metric $ds^2 + \mathfrak{g}$.

The metric $\mathfrak{g}_1$ from Part 3 is s–independent and so it is defined where $s > -L+4$. This understood, the metric $\mathfrak{g}$ is set equal to $\mathfrak{g}_1$ where $s < -L+5$. It is also set equal to $\mathfrak{g}_1$ for all $s \in \left[-L+4, -\frac{3}{4}L+2\right]$ on $M_\delta \cup \mathcal{H}_0$. This is to say that it equals $\mathfrak{g}_Y$ for all such s on $M_\delta \cup \mathcal{H}_0$. The metric $\mathfrak{g}$ is chosen where $s \ge -L+5$ on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ so that its Hodge star on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ acts on w_2 as

$$*w_2 = \chi'_{\diamond 2} x (1-\chi_m)(1-3\cos^2\theta)\, du + \upsilon_\diamond. \tag{9-57}$$

As will be explained directly, if $L > c_0$, there are metrics of the sort just described that obey the $c_0 = 1$ version of (9-55) where $s > -\frac{3}{4}L + 1$.

To see about these requirements, consider first constructing a metric of the desired sort where $s > -\frac{3}{4}L$. The metric that is defined by (9-54) with volume form $\sin\theta\, du\, d\theta\, d\phi$ satisfies the requirements where $|u| < 2$. Since $w_2 \wedge \upsilon_\diamond > 0$ on the rest of $\mathcal{H}_\mathfrak{p}$ and the $\mathfrak{g}_Y$–Hodge star of w_2 is $\upsilon_\diamond$ on $M_\delta \cup \mathcal{H}_0$, Lemma 9.2 finds an extension of the latter metric from the $|u| < 1$ part of each $\mathcal{H}_\mathfrak{p}$ that has the desired properties. Use $\mathfrak{g}_2$ to denote this s–independent metric.

Consider next the story where $s < -\frac{3}{4}L + 1$. The metric on any given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ that is defined by (9-54) with volume form $\sin\theta\, du\, d\theta\, d\phi$ has Hodge star sending w_2 to $\upsilon_\diamond$ where $|u| < m^{-1}$. Let υ denote the 1–form on the right-hand side of (9-57). The 3–form $\upsilon \wedge w_2$ can be written where $|u| \ge \frac{1}{2}m^{-1}$ as $\mathfrak{q}\upsilon_\diamond \wedge w_2$ and it follows from the fact that $|\chi'_{\diamond 2}| < c_0 L^{-1}$ that $\mathfrak{q} > c_0^{-1} - c_0 L^{-1}$. Thus, $\upsilon \wedge w_2 > 0$ where $|u| > \frac{1}{2}m^{-1}$. Given this positivity and given what was said in the preceding paragraphs, Lemmas 9.2 and 9.3 can be used to construct an s–dependent metric where $s < -\frac{3}{4}L + 1$ that equals $\mathfrak{g}_2$ where $s > -\frac{3}{4}L + \frac{1}{2}$, that equals $\mathfrak{g}_1$ where $s < -L + 5$ and equals $\mathfrak{g}_Y$ on $M_\delta \cup \mathcal{H}_0$.

Part 5 This part and Part 6 construct the desired metric for X and the 2–form w_X where $s \in \left[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2\right]$. By way of a look ahead, the metric pulls back from this part of X via the embedding from the second bullet of (2-8) as $ds^2 + \mathfrak{g}_3$ with $\mathfrak{g}_3$ being an s–dependent metric on Y that equals the metric $\mathfrak{g}_*$ for all s on the set $\mathcal{Y}_{0\varepsilon}$ from (9-10).

The metric $\mathfrak{g}_3$ is independent of s on the whole of Y when $s \in \left[-\frac{1}{2}L + 1, -\frac{1}{2}L + 2\right]$. This s–independent version of $\mathfrak{g}_3$ is in a large T version of the space Met_T that is defined in Part 5 of Section 9.1. For the purposes to come, the choice of T requires choosing $L > c_T$ with c_T denoting here and in what follows a constant that depends on T and is greater than $c_0 T^2$ in any event. The value of c_T may increase between appearances.

Use $*$ now to denote the $\mathfrak{g}_3$–Hodge star on Y. The 2–form w_X pulls back via the embedding from the second bullet of (2-8) to $\left[-\frac{3}{4}L+1, -\frac{1}{2}L+2\right] \times Y$ as $ds \wedge * w_3 + w_3$, with w_3 denoting an s–dependent, closed 2–form on Y. The 2–form w_3 is also independent of s where $s \in \left[-\frac{1}{2}L + 1, -\frac{1}{2}L + 2\right]$ and it is independent of s on $\mathcal{Y}_{0\varepsilon}$ for all s. With regards to the motivation for what follows below and in Part 6, keep in mind that $ds\wedge *w_3 + w_3$ is closed if and only if both $dw_3 = 0$ and $d(*w_3) = \frac{\partial}{\partial s} w_3$ for all s.

This part of the subsection assumes that $c_1(\det \mathbb{S})$ annihilates the $H_2(M;\mathbb{Z})$–summand of the direct sum decomposition for $H_2(Y;\mathbb{Z})$ given in (IV.1-4). This assumption makes for a simpler construction. Even so, much of what is done here is used again for Part 6's construction for the general case.

The construction that follows has six steps. Note that some of these steps use notation from Section 9.1.

Step 1 Let $\chi_{\diamond 3}$ denote the function of s given by $\chi\big(\frac{3}{L-8}(s+\frac{3}{4}L-2)\big)$. This function equals 1 for $s < -\frac{3}{4}L + 2$ and it equals 0 for $s \geq -\frac{1}{2}L$. Reintroduce the notation from Section 9.1 and let χ_r denote the function on $\mathbb{R}^3$ given by $\chi(64\varepsilon_*^{-1}(r-\rho_*)-1)$. This function equals 1 where $r < \rho_* + \frac{1}{64}\varepsilon$ and it equals 0 where $r > \rho_* + \frac{1}{32}\varepsilon$. Let $T \geq 1$ and use χ_r with $\chi_{\diamond 3}$ to define the s–dependent function on $\mathbb{R}^3$ given by

$$r_{s_T} = \chi_{\diamond 3} r + (1-\chi_{\diamond 3})\Big(1-\chi_r+\frac{1}{T}\chi_r\Big)r. \tag{9-58}$$

Note in particular that $\frac{\partial}{\partial s} r_{s_T} > 0$ because χ_r is a nonincreasing function of r. Use ρ_{s_T} and $x_{s_T 3}$ to denote the respective s–dependent functions on $\mathbb{R}^3$ given by $r_{s_T}\sin\theta$ and $r_{s_T}\cos\theta$.

Define the s–dependent 2–form w_3 on Y by setting $w_3 = w_2$ for $s \leq -\frac{3}{4}L+2$ and setting it equal to w on the $\mathcal{Y}_0$ component of $Y - \mathcal{N}_\varepsilon$. The 2–form w_3 is defined on $\mathcal{N}_\varepsilon$ by specifying it on the $\mathbb{R}^3$ incarnation of $\mathcal{N}_\varepsilon$ to be $\kappa(\rho_{s_T})\rho_{s_T}\, d\rho_{s_T}\, d\phi$. The definition of w_3 on the rest of Y uses τ to denote the function of s given by $(\chi_{\diamond 3}+(1-\chi_{\diamond 3})/T)^2$. The latter function equals 1 where $s < -\frac{3}{4}L+2$ and it is equal to $\frac{1}{T^2}$ where $s > -\frac{1}{2}L$. The 2–form w_3 is defined on $\mathcal{Y}_M \cap M_\delta$ to be τw_2, and it is defined on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ by the upcoming (9-59). This upcoming definition uses χ_Δ to denote the function of u and θ given by $\chi(|u|^2-1)\chi(4(1-3\cos^2\theta)-1)$. The function χ_Δ is equal to 1 where both $|u|<1$ and $|1-3\cos^2\theta| < \frac{1}{4}$, and it is equal to 0 where either $|u|>2$ or $|1-3\cos^2\theta| > \frac{1}{2}$. Note in particular that the support of χ_Δ consists of two open sets. These are mirror images under the involution $\theta\mapsto\pi-\theta$, with one being a neighborhood of the $u=0$ and $\cos\theta = \frac{1}{\sqrt{3}}$ circle with $0<\theta<\frac{\pi}{2}$ on its closure. Define

$$w_3 = -\sqrt{6}\tau\, d(f\cos\theta\sin^2\theta\, d\phi - (x_0+4e^{-2R})\operatorname{sign}(\cos\theta)\chi_\Delta\, d\phi) \tag{9-59}$$

on $\mathcal{H}_\mathfrak{p}$ for $s > \frac{3}{4}L+2$.

By way of comparison, the 2–form w_2 on $\mathcal{H}_{\mathfrak{p}}$ can be written as $\sqrt{6}\,d(f\cos\theta\sin^2\theta\,d\phi)$. What is written in (9-59) adds a 2–form with support on $\mathcal{H}_{\mathfrak{p}}$ to τw_2.

The 2–form w_3 on Y is closed for each s. Moreover, it defines the s–independent de Rham cohomology class $c_1(\det\mathbb{S})$ because the latter class is assumed to annihilate the $H_2(M;\mathbb{Z})$–summand in (IV.1-4).

Step 2 The s–dependent metric $\mathfrak{g}_3$ is defined when $s\in\bigl[-\frac{3}{4}L+1,-\frac{1}{2}L+2\bigr]$ with the help of a certain s–dependent 1–form, $\mathfrak{b}$. The 1–form $\mathfrak{b}$ should obey $d\mathfrak{b}=\frac{\partial}{\partial s}w_3$. There are four additional constraints on $\mathfrak{b}$. The first is that $\mathfrak{b}$ should vanish on $\mathcal{Y}_0$ and on the part of $\mathcal{N}_\varepsilon$ where $r>\rho_*+\frac{1}{16}\varepsilon$. The second constraint specifies $\mathfrak{b}$ on the $|u|<4$ part of $\mathcal{H}_{\mathfrak{p}}$:

$$\mathfrak{b}=-\sqrt{6}\tau'(f\cos\theta\sin^2\theta-(x_0+4e^{-2R})\operatorname{sign}(\cos\theta)\chi_\Delta)\,d\phi, \tag{9-60}$$

where τ' denotes $\frac{\partial}{\partial s}\tau$. The third constraint asks that $\mathfrak{b}$'s norm at $s\in\bigl[-\frac{3}{4}L+1,-\frac{1}{2}L+2\bigr]$ when measured by the metric $\mathfrak{g}_Y$ obeys $|\mathfrak{b}|_{\mathfrak{g}_-}\le c_T L^{-1}$. The fourth constraint requires the following: Fix $k\in\{0,\dots,20\}$. Then the $\mathfrak{g}_Y$–covariant derivatives up to order 20 of $\bigl(\frac{\partial}{\partial s}\bigr)^k\mathfrak{b}$ are bounded by $c_T L^{-k-1}$.

To see about satisfying these constraints, note first that $\mathfrak{b}$ can be chosen to vanish on $\mathcal{Y}_0$ and on the $r>\rho_*+\frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$ because w_3 is constant on these parts of Y, and because the first cohomology of the $r\in\bigl[\rho_*+\frac{1}{32}\varepsilon,\rho_*+\frac{1}{64}\varepsilon\bigr]$ part of $\mathcal{N}_\varepsilon$ is zero. The c_0L^{-1} bound on $|\chi'_{\diamond 3}|$ implies that $\mathfrak{b}$ can be chosen to vanish on $\mathcal{Y}_0$ and so that its norm elsewhere when measured by the metric $\mathfrak{g}_Y$ is bounded by c_0L^{-1}. A 1–form of this sort can be chosen so that the $\mathfrak{g}_Y$–norms of its derivatives also have the required norm bound. Let $\mathfrak{b}_*$ denote such a choice, and let $\mathfrak{b}_\Lambda$ denote the 1–form on any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ given by (9-51). Their difference, $\mathfrak{b}_*-\mathfrak{b}_\Lambda$, is a closed 1–form on $\mathcal{H}_{\mathfrak{p}}$. As $H^1(\mathcal{H}_{\mathfrak{p}}\cap M_\delta;\mathbb{R})=0$, this difference can be written as $d\mathfrak{k}$ with $\mathfrak{k}$ denoting a function on $\mathcal{H}_{\mathfrak{p}}$. The function $\mathfrak{k}$ can be taken so that $|\mathfrak{k}|\le c_0L^{-1}$ since the $\mathfrak{g}_Y$–norms of both $\mathfrak{b}_*$ and $\mathfrak{b}_\Lambda$ obey a similar c_0L^{-1} bound. Granted this bound on $\mathfrak{k}$, then $\mathfrak{b}=\mathfrak{b}_*-d(\chi(|u|-4)\mathfrak{k})$ has all of the requisite properties.

Step 3 The definition of the upcoming Steps 4 and 6 use observations made below about w_3 and $\mathfrak{b}$ on the $|u|\le 4$ part of each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. The first series of observations concern w_3. To start, note that the zero locus of the 2–form in (9-60) is the same as that of $\upsilon_\diamond$, this being the locus where both $u=0$ and $1-3\cos^2\theta=0$. The reason being that f' and χ_Δ have the same sign where $\chi_\Delta\ne 0$, and likewise the functions $1-3\cos^2\theta$ and $\operatorname{sign}(\cos\theta)\chi_\Delta$ have the same sign where $\chi_\Delta\ne 0$. In fact, these comments about the derivatives of χ_Δ imply that w_3 on $\mathcal{H}_\mathfrak{p}$ can be written schematically as

$$w_3 = -(1+\mathrm{A}_1)\tau\sqrt{6}f'\cos\theta\sin^2\theta\,du\,d\phi + (1+\mathrm{A}_2)\tau\sqrt{6}f(1-3\cos^2\theta)\sin\theta\,d\theta\,d\phi, \tag{9-61}$$

where A_1 and A_2 are smooth, nonnegative functions of u and θ that equal zero where both $|u|<1$ and $|1-3\cos^2\theta|<\frac{1}{4}$ and where either $|u|>2$ or $|1-3\cos^2\theta|>\frac{1}{2}$. Given that w_2 on $\mathcal{H}_\mathfrak{p}$ is $-\sqrt{6}\,d(f\cos\theta\sin^2\theta\,d\phi)$, these last remarks imply that

$$w_3\wedge\upsilon_\diamond \geq \tau w_2\wedge\upsilon_\diamond \quad \text{on } \mathcal{H}_\mathfrak{p} \tag{9-62}$$

with the inequality being a strict one only where $d\chi_\Delta \neq 0$.

The next series of remarks concern the 1–form $\mathfrak{b}$ on the $|u|\leq 4$ part of $\mathcal{H}_\mathfrak{p}$, the first point of note being that $f(u)\cos\theta\sin^2\theta$ is equal to $(x_0+4e^{-2R})\frac{2}{3\sqrt{3}}\,\mathrm{sign}(\cos\theta)$ on the zero locus of $\upsilon_\diamond$. It follows as a consequence that $\mathfrak{b}$ can be written as

$$\mathfrak{b} = -\mathrm{B}_1\tau'f'\cos\theta\sin^2\theta\,d\phi + \mathrm{B}_2\tau'f(1-3\cos^2\theta)\sin\theta\,d\phi, \tag{9-63}$$

where B_1 and B_2 are smooth functions of u and θ.

Step 4 The metric $\mathfrak{g}_3$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ is defined to be the metric from Part 5 for $s<-\frac{3}{4}L+2$. The metric $\mathfrak{g}_3$ on $\mathcal{H}_\mathfrak{p}$ at other values of s is defined in part so that its Hodge star obeys

$$*w_3 = \tau\upsilon_\diamond + \mathfrak{b}. \tag{9-64}$$

There is one other constraint. To explain it, note first that the metric $\mathfrak{g}_2$ does not depend on s when $s\in\left[-\frac{3}{4}L+1, -\frac{3}{4}L+2\right]$. Use $\mathfrak{g}_{2+}$ to denote this s–independent metric. Look at (9-45) to see that the $s>-\frac{1}{2}L+1$ version of w_3 on the $|u|>4$ part of each $\mathcal{H}_\mathfrak{p}$ is $\frac{1}{T^2}w_2$. Since $\mathfrak{b}$ is zero when $s>-\frac{1}{2}L+1$, the constraint in (9-64) is satisfied by taking the Hodge star to be that defined by $\mathfrak{g}_{2+}$. This understood, the final constraint is as follows:

(9-65) The metric $\mathfrak{g}_3$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ when $s>-\frac{1}{2}L+1$ must be both s–independent and T–independent, and it must equal $\mathfrak{g}_{2+}$ where $|u|>4$.

As explained in what follows, an s–dependent metric with all of these requisite properties exists if L is greater than a T–dependent constant.

Consider first the existence of a metric with the desired properties where $|u|<1$ and $|1-3\cos^2\theta|<\frac{1}{4}$, this being a neighborhood of the common zero locus of w_3 and $\upsilon_\diamond$.

The metric $\mathfrak{g}$ is defined on this part of $\mathcal{H}_{\mathfrak{p}}$ by its volume 3–form $\Omega = \sin\theta\, du\, d\theta\, d\phi$ and the Hodge duals

$$\begin{aligned} *\sin\theta\, d\theta\, d\phi &= \frac{1}{\sqrt{6}} \frac{4e^{-2R}\cosh(2u)}{x_0 + 4e^{-2R}\cosh(2u)}\, du + \tau^{-1}\tau' \mathrm{B}_2 \sin\theta\, d\phi, \\ *\sin\theta\, d\phi\, du &= \tfrac{\sqrt{3}}{2\sqrt{2}}\, d\theta - \tfrac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_1 \sin\theta\, d\phi, \\ *du\, d\theta &= \tfrac{\sqrt{3}}{2\sqrt{2}} \sin\theta\, d\phi + \tfrac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_2\, du - \tfrac{1}{\sqrt{6}}\tau^{-1}\tau'\mathrm{B}_1\, d\theta. \end{aligned} \tag{9-66}$$

These formulas for the Hodge dual define a symmetric, bilinear form on the cotangent bundle of this part of $\mathcal{H}_{\mathfrak{p}}$. This bilinear form is positive definite if $\tau^{-1}|\tau'| < c_0^{-1}$, which is guaranteed if $T^2 L^{-1} < c_0^{-1}$ since $\tau^{-1} < T^2$ and $|\tau'| < c_0 L^{-1}$.

To see about defining $\mathfrak{g}_3$ on the rest of $\mathcal{H}_{\mathfrak{p}}$, use the fact that $|\mathfrak{b}| \le c_0 L^{-1}$ to draw the following conclusion: If $L > c_0 T^2$, then $w_3 \wedge (\tau \upsilon_\diamond + \mathfrak{b}) > 0$ on the complement in Y of the $|u| < \frac{1}{2}$ and $|1 - 3\cos^2\theta| < \frac{1}{8}$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. This being the case, then Lemma 9.3 can be used directly to obtain a family of metrics on $\mathcal{H}_{\mathfrak{p}}$ parametrized by the set $\left[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2\right]$ so as to obey (9-64) and (9-65). Use $\mathfrak{g}_{3\Lambda}$ to denote this family of metrics on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$.

Step 5 The 1–form $\upsilon_\diamond$ is used here to construct another closed, s–dependent 1–form that plays a central role in the upcoming definition of the $s \in \left[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2\right]$ versions of $\mathfrak{g}_3$ on $M_\delta \cup \mathcal{H}_0$. This new 1–form is denoted by $\upsilon_{\diamond 3}$ and its definition is given in the subsequent paragraph.

The 1–form $\upsilon_{\diamond 3}$ on $\mathcal{Y}_0$ is $\upsilon_\diamond$ and it is defined on the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$ to be $dx_{s_T 3}$ with the latter defined in Step 1. Since $\upsilon_\diamond = dx_3$ on $\mathcal{N}_\varepsilon$, it follows from the definition of $x_{s_T 3}$ that $\upsilon_{\diamond 3}$ as defined so far is a 1–form on the union of $\mathcal{Y}_0$ and the $r > \rho_* - \frac{1}{4}\varepsilon$ part of $\mathcal{N}_\varepsilon$. The definition of $\upsilon_{\diamond 3}$ on the $r \in \left[\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon\right]$ part of $\mathcal{N}_\varepsilon$ requires the reintroduction of the function χ_{r*} from Step 2 in Part 5 of Section 9.1. This function is used here to define $x_{s_T 3*} = \bigl(\chi_{\diamond 3} + (1 - \chi_{\diamond 3})\bigl(1 - \chi_{r*} + \frac{1}{T}\chi_{\mathrm{r}*}\bigr)\bigr)x_3$. Define $\upsilon_{\diamond 3}$ on the $r \in \left[\rho_* - \frac{1}{2}\varepsilon, \rho_* - \frac{1}{4}\varepsilon\right]$ part of $\mathcal{N}_\varepsilon$ to be $\tau^{1/2} dx_{s_T 3*}$. It follows from the definitions of $x_{s_T 3}$ and $x_{s_T 3*}$ that the definition just given defines a smooth 1–form on the union of $\mathcal{Y}_0$ with the $r > \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. As the latter's restriction near the $r = \rho_* - \frac{1}{2}\varepsilon$ is $\tau\, dx_3$, a smooth 1–form on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ is defined by setting $\upsilon_{\diamond 3} = \tau\, dx_3$ on the $r \le \rho_* - \frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$. Noting that $\tau\, dx_3 = \tau\upsilon_\diamond$, defining $\upsilon_{\diamond 3}$ on $\mathcal{Y}_M$ to be $\tau\upsilon_\diamond$ defines a smooth, closed 1–form on Y.

The 1–form $\upsilon_{\diamond 3}$ has the four properties that are listed below.

Property 1 *The* 1*–form* $\upsilon_{\diamond 3}$ *is equal to* $\upsilon_\diamond$ *where* $s \in \left[-\frac{3}{4}L+1, -\frac{3}{4}L+2\right]$.

This follows because $\chi_{\diamond 3} = 1$ at these values of s.

Property 2 *The zero locus of each* $s \in \left[-\frac{3}{4}L+1, -\frac{1}{2}L+2\right]$ *version of* $\upsilon_{\diamond 3}$ *is identical to that of* $\upsilon_\diamond$.

This is because $\upsilon_{\diamond 3}$ has no zeros on $\mathcal{Y}_0 \cup \mathcal{N}_\varepsilon$ and it is equal to $\tau\upsilon_\diamond$ on $\mathcal{Y}_M$.

Property 3 *Each* $s \in \left[-\frac{3}{4}L+1, -\frac{1}{2}L+2\right]$ *version of* $w_3 \wedge \upsilon_{\diamond 3}$ *is positive on the complement of the common zero locus of* w_3 *and* $\upsilon_{\diamond 3}$.

This property follows directly from the definitions on $Y - \left(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\right)$ and from (9-61) on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$.

To set the stage for the fourth property, note that w_3 and $\upsilon_{\diamond 3}$ do not depend on s when $s \in \left[-\frac{3}{4}L+1, -\frac{3}{4}L+2\right]$. Use w_{3+} and $\upsilon_{\diamond 3+}$ to denote these s–independent differential forms. To continute the stage setting, let $\mathfrak{g}_{3\Lambda_+}$ denote the s–independent metric on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$ given by the $s \in \left[-\frac{3}{4}L+1, -\frac{3}{4}L+2\right]$ version of Part 5's metric $\mathfrak{g}_{3\Lambda}$. What with (9-55), this metric on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$ with $\mathfrak{g}_{2+}$ on $Y - \left(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\right)$ define a smooth, s– and T–independent metric on Y. Denote the latter by $\mathfrak{g}_\diamond$. The restriction of $\mathfrak{g}_\diamond$ to $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ is in the space $\mathrm{Met}^{\mathcal{N}}$ from Part 5 of Section 9.1. This understood, let $\mathfrak{g}_{\diamond T}$ denote the Met_T metric that is constructed in Part 5 of Section 9.1 from T and $\mathcal{Y}_M \cup \mathcal{N}_\varepsilon$ part of $\mathfrak{g}_\diamond$.

Property 4 *The* $\mathfrak{g}_{\diamond T}$*–Hodge star of* w_{3+} *is* $\upsilon_{\diamond 3+}$.

The definitions in Part 5 of Section 9.1 with those given above for w_{3+} and $\upsilon_{\diamond 3+}$ imply this on $Y - \left(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\right)$ and (9-64)–(9-65) imply this on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$.

Step 6 This step completes the definition of $\mathfrak{g}_3$ on Y so as to satisfy five constraints, the first being that $*w_3 = \upsilon_{\diamond 3} + \mathfrak{b}$ at each $s \in \left[-\frac{3}{4}L+1, -\frac{1}{2}L+2\right]$. The second constraint asks that the $s \in \left[-\frac{3}{4}L+1, -\frac{3}{4}L+2\right]$ versions are independent of s; and the third asks that the $s \in \left[-\frac{1}{2}L+1, -\frac{1}{2}L+2\right]$ versions are also independent of s and that this s–independent metric is $\mathfrak{g}_{\diamond T}$. The fourth constraint asks that $\mathfrak{g}_3 = \mathfrak{g}_{3\Lambda}$ on the $|u| < R + \ln\delta$ part of each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$. The fifth and final constraint asks that $\mathfrak{g}_3 = \mathfrak{g}_*$ on $\mathcal{Y}_0$ and on the $r > \rho_* + \frac{1}{16}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Use Property 3 and what is said in Step 4 with the bound $|\mathfrak{b}|_{\mathfrak{g}_-} < c_0 L^{-1}$ to see that $w_3 \wedge (\upsilon_{\diamond 3} + \mathfrak{b}) > 0$ on the complement in Y of the common zeros of w_3 and $\upsilon_{\diamond 3}$ if

$L \geq c_T$. Given this bound, Lemma 9.3 with the input from Step 4 and Property 4 of Step 5 find a metric with all of the desired properties. Take such a metric for $\mathfrak{g}_3$. Note for future reference that the s–independent, $s > -\frac{1}{2}L + 1$ version of $\mathfrak{g}_3$ is equal to $\mathfrak{g}_Y$ on $\mathcal{Y}_M \cap M_\delta$.

Part 6 This part of the subsection puts no constraints on the restriction of $c_1(\det \mathbb{S})$ to the $H_2(M;\mathbb{Z})$–summand in $H_2(Y;\mathbb{Z})$. The s–dependent metric $\mathfrak{g}_3$ and the 2–form w_3 in this case are identical to their namesakes in Part 5 on $Y - \big(\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma\big)$. The three steps that follow define $\mathfrak{g}_3$ and w_3 on $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$.

Step 1 Reintroduce from Part 7 of Section 9.1 the closed 2–form p on Y. By way of a reminder, the de Rham class of p has pairing 0 with the $H_2(\mathcal{H}_0;\mathbb{Z})\oplus\big(\bigoplus_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\big)$–summand in (IV.1-4)'s decomposition of $H_2(Y;\mathbb{Z})$ and its pairing with the $H_2(M;\mathbb{Z})$–summand is the same as that of $c_1(\det \mathbb{S})$. Since p's support lies in $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$ and thus in $\mathcal{Y}_M - \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\big)$, setting w_3 on $\mathcal{Y}_M \cup \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}\big)$ to be $w_3 = \tau w_2 + (1-\tau)p$ defines a closed 2–form on Y for each $s \in \big[-\frac{3}{4}L + 1, -\frac{1}{2}L + 2\big]$ with de Rham cohomology class $c_1(\det \mathbb{S})$.

The metric $\mathfrak{g}_3$ is defined on $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$ so that its Hodge star maps w_3 to $\tau \upsilon_\diamond + \mathfrak{b}$ with $\mathfrak{b}$ denoting a certain 1–form with $d\mathfrak{b} = \frac{\partial}{\partial s} w_3$. As done previously, Lemma 9.3 will be used to construct a metric with this property that meets all of the other requirements.

Step 2 The definition of $\mathfrak{g}_3$ and $\mathfrak{b}$ on $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$ requires what is said here about the w_2 and p in the support of p. To start, reintroduce from Part 7 of Section 9.1 the set Θ and write p as $\sum_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathrm{z}_\gamma p_\gamma$ with each $(\gamma,\mathrm{z}_\gamma)$ version of p_γ being a closed 2–form with support in the tubular neighborhood $\mathcal{T}_\gamma$ that is described in Part 7 of Section 9.1. Part 7 of Section 9.1 describes a diffeomorphism from $S^1 \times D$ to $\mathcal{T}_\gamma$ with D denoting a small radius disk about the origin in $\mathbb{R}^2$. The diffeomorphism identifies γ with $S^1 \times \{0\}$ and it has two important properties that concern the 2–form w on Y and the function f from Section II.1. As noted in Part 7 of Section 9.1, the 1–form df pulls back via the embedding of $S^1 \times D$ as a constant 1–form on the D factor and the kernel of the pullback via the embedding of the 2–form w is a constant vector field that is tangent to this D factor. These last properties are exploited in the next paragraph.

As can be seen in (IV.1-5), the 1–form $\upsilon_\diamond$ on $\mathcal{T}_\gamma$ is df. Meanwhile, the 2–form w_2 on $\mathcal{T}_\gamma$ is still the original 2–form w on Y as described in (IV.1-3). This understood, what was said above about df and the kernel of w imply that $S^1 \times D$ has coordinates

$(t,(x,y))$ with t denoting an affine coordinate for the S^1 factor and (x,y) coordinates for D with the following two properties: The 1–form $\upsilon_\diamond$ pulls back as dx and the 2–form w_2 pulls back as $\mathrm{H}_\gamma(y,t)\,dy\,dt$ with H_γ denoting a positive function. Granted these coordinates, the 2–form p_γ has the form $\mathfrak{h}_\gamma(x,y)\,dx\,dy$ with $\mathfrak{h}_\gamma$ denoting a function with compact support in a small radius disk about the origin in the (x,y)–plane and with total integral equal to 1.

Step 3 An almost verbatim repeat of what is said in Step 2 of Part 6 supplies a version of the 1–form $\mathfrak{b}$ which obeys the four properties listed in the first paragraph of Step 2 in Part 6 with it understood that w_3 is now defined as in Step 1.

It follows as a consequence of what is said in Step 2 that

$$(\tau w_2 + (1-\tau)p) \wedge \upsilon_\diamond = \tau w_2 \wedge \upsilon_2; \tag{9-67}$$

thus, the $\mathfrak{g}_Y$–norm of $(\tau w_2+(1-\tau)p)\wedge(\tau\upsilon_\diamond+\mathfrak{b})$ is no less than $\tau^2(c_0^{-1}-c_T T^2 L^{-1})$. This being the case, Lemma 9.3 supplies an s–dependent metric on Y with all of the desired properties if L is larger than a purely T–dependent constant.

Let $\mathfrak{g}_{3+}$ denote the s–independent metric on Y given by the $s\in\left[-\frac{1}{2}L+1,-\frac{1}{2}L+2\right]$ versions of $\mathfrak{g}_3$. This is the metric $\mathfrak{g}_Y$ on $(\mathcal{Y}_M\cap M_\delta)-\left(\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta}\mathcal{T}_\gamma\right)$. It proves necessary for what follows to take some care with regards to the choice of $\mathfrak{g}_{3+}$ on $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta}\mathcal{T}_\gamma$. In particular, Lemmas 9.2 and 9.3 will construct a version of $\mathfrak{g}_3$ with $\mathfrak{g}_{3+}$ on each $\mathcal{T}_\gamma$ by $\mathfrak{g}_Y$–volume 3–form $\mathrm{H}_\gamma\,dx\,dy\,dt$ and the Hodge star rules

$$\begin{aligned}
*dx\,dy &= \mathrm{A}_\gamma\,dt-\mathrm{A}_\gamma\tau^{-1}(1-\tau)\,\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma\,dx+\mathrm{B}_\gamma\,dy,\\
*dy\,dt &= \mathrm{H}_\gamma^{-1}(1+\tau^{-2}\mathrm{H}_\gamma^{-1}\mathrm{A}_\gamma(1-\tau)^2\mathrm{Z}_\gamma\mathfrak{h}_\gamma)\,dx-\mathrm{A}_\gamma\tau^{-1}(1-\tau)\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma\,dt,\\
*dt\,dx &= dy+\mathrm{B}_\gamma\,dt,
\end{aligned}\tag{9-68}$$

with A_γ being a positive function and with τ equal to $\frac{1}{T^2}$. The function A_γ is constrained for the moment only to the extent that $\mathrm{A}_\gamma<c_0^{-1}\tau^2$ on the support of $\mathrm{Z}_\gamma\mathfrak{h}_\gamma$ and that A_γ is independent of T on the complement in $\mathcal{T}_\gamma$ of a T–independent open set that contains the support of $\mathfrak{h}_\gamma$ and has compact closure in $\mathcal{T}_\gamma$. This set is denoted by $\mathcal{T}'_\gamma$. This upper bound on A_γ is needed so that (9-68) defines a positive definite metric. As for B_γ, it is zero on $\mathcal{T}_\gamma$ and it is independent of T elsewhere.

Part 7 This part of the subsection defines the desired metric on X and 2–form w_X on the $s\in\left[-\frac{1}{2}L+1,-\frac{1}{2}L+5\right]$ part of X. As done previously, these are defined by their pullbacks via the embedding from the second bullet of (2-8). The pullback of the metric will have the form $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ denoting an s–dependent metric on Y. Meanwhile, the pullback of w_X will have the form $ds \wedge *w_4 + w_4$, with w_4 denoting a closed, s–dependent 2–form on Y and with $*$ denoting the Hodge $*$ defined by $\mathfrak{g}$. The de Rham cohomology class of w_4 at each s is $c_1(\det \mathbb{S})$.

The metric $\mathfrak{g}$ is independent of s for $s \in \left[-\frac{1}{2}L + 1, -\frac{1}{2}L + 2\right]$ and the 2–form w_4 is independent of s for $s \in \left[-\frac{1}{2}L + 1, -\frac{1}{2}L + 3\right]$. Both the metric and w_4 are independent of s when $s \in \left[-\frac{1}{2}L + 4, -\frac{1}{2}L + 5\right]$. Moreover, the restriction of both to $Y - \left(\bigcup_{\mathfrak{p} \in \Lambda} \mathcal{H}_{\mathfrak{p}}\right)$ are independent of s for all values of s. The salient difference between the $s \le -\frac{1}{2}L + 3$ version of w_4 and the $s \ge -\frac{1}{2}L + 4$ version is that the latter has nondegenerate zeros and the former does not.

The construction of $\mathfrak{g}$ and w_4 has two steps.

Step 1 Let $\mathfrak{g}_{3+}$ denote the $-\frac{1}{2}L + 2$ version of the metric that is supplied in Parts 5 and 6, and let w_{3+} denote the $s = -\frac{1}{2}L + 2$ version of w_3. The 2–form w_{3+} is $\mathfrak{g}_{3+}$–harmonic but it does not vanish transversely. By way of a reminder, the zero locus of w_{3+} consists of the two circles in each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ where both $u = 0$ and $1 - 3\cos^2\theta = 0$. Note in this regard that w_{3+} on $\mathcal{H}_{\mathfrak{p}}$ is the 2–form

$$\sqrt{6}T^{-2}(-f' \cos\theta \sin^2\theta \, du \, d\phi + f(1 - 3\cos^2\theta) \sin\theta \, d\theta \, d\phi). \tag{9-69}$$

The construction of w_4 starts by introducing $\chi_{\diamond 4}$ to denote the function on $\mathbb{R}$ given by $s \mapsto \chi(s + L - 3)$. This function is equal to 1 where $s < -\frac{1}{2}L + 3$ and it is equal to 0 where $s > -\frac{1}{2}L + 3$. The derivative of $\chi_{\diamond 3}$ is denoted by $\chi'_{\diamond 3}$. Use χ_* to denote the function of u given by the rule $u \mapsto \chi(|u| - 1)$. This function is equal to 1 where $|u| \le 1$ and it is equal to 0 where $|u| > 2$. One last function is needed for what follows, this denoted by χ_θ. It is a function on $[0, \pi]$ with values in $[0, 2]$ which has the following two properties: It is zero near the endpoints, and has two local minima at the two values of θ where $1 - 3\cos^2\theta = 0$. Moreover, χ_θ should appear on a neighborhood of these minima as $1 + (1 - 3\cos^2\theta)^2$. Take χ_θ so that $\chi_\theta(\theta) = \chi_\theta(\pi - \theta)$.

Fix $z > 1$ and define the 2–form w_z by

$$\begin{aligned} w_z = -\Big(\sqrt{6}f' \cos\theta \sin^2\theta + z^{-1} \cos\phi \, \chi_{\diamond 4} \, \chi_* \sin\theta \, \frac{\partial}{\partial\theta}\chi_\theta\Big) \, du \, d\phi \\ + \sqrt{6}f(1 - 3\cos^2\theta) \sin\theta \, d\theta \, d\phi \\ - z^{-1} \sin\phi \, \chi_{\diamond 4}\chi_* \frac{\partial}{\partial\theta}\Big(\sin\theta \, \frac{\partial}{\partial\theta}\chi_\theta\Big) \, du \, d\theta. \end{aligned} \tag{9-70}$$

This is a closed 2–form for all s that equals w_{3+} for $s \le -\frac{1}{2}L+3$ and for all s where $|u|>2$. This 2–form is independent of s when $s \ge -\frac{1}{2}L+4$. Moreover, if $z>c_0$, then the s–independent version of w_z defined where $s \ge -\frac{1}{2}L+4$ has a nondegenerate zero locus, this being the four points where $\sin\phi = 0$, $1-3\cos^2\theta = 0$ and $u=0$.

The desired 2–form w_4 is defined to be w_{3+} on $Y - \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}\big)$ and it is defined on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ to be a $z>c_0$ version of $T^{-2}w_z$.

Step 2 This step defines the metric $\mathfrak{g}$. This is done by first constructing $\mathfrak{g}$ near the zero locus of w_4 in each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ and then extending the result to the rest of Y with the help of Lemma 9.3.

Fix $z>c_0$ so that w_z as defined in (9-70) has nondegenerate zeros. The 2–form w_z can be written as $d\mathfrak{b}_z$, where $\mathfrak{b}_z$ is given by

$$\text{(9-71)}\quad \frac{\sqrt{3}}{2\sqrt{2}} z^{-1}\cos\phi\, \chi_{\diamond 4}(\chi_*'\chi_\theta\, du + \chi_*\chi_\theta'\, d\theta) - \frac{\sqrt{3}}{2\sqrt{2}} z^{-1}\sin\phi\, \chi_{\diamond 4}\chi_*\chi_\theta\, d\phi \\ + z^{-1}\sin\phi\, \chi_{\diamond 4}'\,\chi_*\sin\theta\,\frac{\partial}{\partial\theta}\chi_\theta\, du.$$

Granted this formula, then $\upsilon + \mathfrak{b}_z$ has the same zero locus as w_z if $z>c_0$, and it also vanishes transversely. Moreover, $w_z\wedge(\upsilon_\diamond + \mathfrak{b}_z)$ can be written as $\mathrm{Q}\sin\theta\, du\, d\theta\, d\phi$ and a calculation finds that $\mathrm{Q}\ge 0$ with equality only on the joint zero locus of w_z and $\upsilon_\diamond + \mathfrak{b}_z$. In fact, the calculation finds $\mathrm{Q} \ge c_0^{-1}\big(|u|^2 + (1-3\cos^2\theta)^2 + z^{-2}\sin^2\phi\sin^2\theta\big)$ if $z>c_0$.

With z large and w_4 defined by (9-70) on $\mathcal{H}_{\mathfrak{p}}$, the metric $\mathfrak{g}$ is defined near the zeros of (9-71) so that its Hodge star sends w_z to $\upsilon_\diamond + \mathfrak{b}_z$. The definition requires the introduction of yet another function of s, this denoted by $\chi_{\diamond\diamond 4}$ and defined by the rule whereby $\chi_{\diamond\diamond 4}(s) = \chi\big(s+\frac{1}{2}L-2\big)$. This function equals 1 where $s< -\frac{1}{2}L+2$ and it equals 0 where $s> -\frac{1}{2}L+3$. The desired metric $\mathfrak{g}$ is defined by taking its volume form to be $\sin\theta\, du\, d\theta\, d\phi$ and its Hodge star to act as follows:

$$\begin{aligned} *\sin\theta\, d\theta\, d\phi &= \tfrac{1}{\sqrt{6}}(4e^{-2R}\cosh(2u) + 12z^{-1}\sin\phi\,\chi_{\diamond 4}'\cos\theta\sin^2\theta)\, du,\\ \text{(9-72)}\quad *\sin\theta\, d\phi\, du &= \frac{\sqrt{3}}{2\sqrt{2}}\, d\theta,\\ *du\, d\theta &= \frac{\sqrt{3}}{2\sqrt{2}}\Big(\chi_{\diamond\diamond 4}\sin\theta + (1-\chi_{\diamond\diamond 4})\chi_\theta\Big(\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial}{\partial\theta}\chi_\theta\Big)\Big)^{-1}\Big)\, d\phi. \end{aligned}$$

By way of a parenthetical remark, the metric $\mathfrak{g}_{3+}$ near the zeros of w_z is defined by the same volume form but with Hodge star rule given by (9-54). The appearance of $\chi_{\diamond\diamond 4}$ in the third line of (9-72) guarantees that $\mathfrak{g} = \mathfrak{g}_{3+}$ where $s \le -\frac{1}{2}L$.

As noted previously, $w_z \wedge (\upsilon_\diamond + \mathfrak{b}_z) > 0$ on the complement of the common zero locus of w_z and $(\upsilon_\diamond + \mathfrak{b}_z)$. Having constructed $\mathfrak{g}$ on a neighborhood of this locus with the desired properties, Lemma 9.3 provides an extension to the whole of Y which is independent of s where $s < -\frac{1}{2}L+2$, where $s > -\frac{1}{2}L+4$ and on $Y - \bigl(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}\bigr)$. This extension is such that the 2–form $ds \wedge * w_4 + w_4$ is self-dual on $\bigl[-\frac{1}{2}L+1, -\frac{1}{2}L+5\bigr] \times Y$ when self-duality is defined by the metric $ds^2 + \mathfrak{g}$.

Part 8 This part of the subsection supplies the input for the definition in Part 9 of the desired metric and the 2–form w_X on the $s \in \bigl[-\frac{1}{2}L+4, L\bigr]$ part of X. The discussion in this section refers to an auxiliary copy of the space X, this denoted by X_*. The manifold X_* is the same as X, but its metric is not a metric of the sort that is described in Parts 1–7. The eight steps that follow construct a metric on X_* and a corresponding self-dual 2–form with certain desirable properties.

Step 1 Fix a metric in the Y_{G} version of $\mathrm{Met}^{\mathcal{N}}$. The latter with a sufficiently large choice for T determines metrics in the set $\mathrm{Met}(Y_{\mathrm{G}})$. This understood, choose T large enough that this is the case and that two additional requirements are met, the first being that Part 7's metric $\mathfrak{g}$ and 2–form w_4 can be constructed for any choice of $L > c_T$ with c_T denoting a constant that is greater than 1 and depends only on T. The second requirement is given in Step 2.

Let $\mathfrak{g}_-$ and w_- denote the respective $s \in \bigl[-\frac{1}{2}L+4, -\frac{1}{2}L+5\bigr]$ versions of $\mathfrak{g}$ and w_4, these being independent of s. The metric $\mathfrak{g}_-$ is in the Y version of the space Met_T, so it can be used for the metric $\mathfrak{g}_1$ in Part 1 of Section 9.2, and since w_- has nondegenerate zeros, it can also be used for the metric $\mathfrak{g}_2$ in Part 1 of Section 9.2. This part of Section 9.2 uses w_2 to denote the $\mathfrak{g}_2$–harmonic 2–form with de Rham cohomology class that of $c_1(\det \mathbb{S})$. This 2–form w_2 is w_-. The 2–form w_- is equal to w on $\mathcal{Y}_0$ and on the $r \geq \rho_* + \frac{5}{8}\varepsilon$ part of $\mathcal{N}_\varepsilon$ and so it follows that w_- is also the 2–form that is denoted by w_3 in Part 2 of Section 9.2. This fact implies that the metric $\mathfrak{g}_-$ is also a version of what Part 2 of Section 9.2 denotes as $\mathfrak{g}_{3T}$. Parts 1–10 of Section 9.4 will be invoked in the upcoming steps using X_* and the $\mathfrak{g}_-$ version of $\mathfrak{g}_{3T}$. These parts of Section 9.4 denote the latter version of $\mathfrak{g}_{3T}$ by $\mathfrak{g}_{-T}$. What Parts 1–10 of Section 9.4 denote as w_{-T} in this case is the 2–form w_-.

Step 2 Let $\mathfrak{g}_\diamond$ denote the given metric from $\mathrm{Met}(Y_{\mathrm{G}})$. By way of a reminder, the metric $\mathfrak{g}_\diamond$ is determined in part by Step 1's chosen metric from the Y_{G} version of $\mathrm{Met}^{\mathcal{N}}$ and T.

As explained in Part 1 of Section 9.2, a metric denoted by $\mathfrak{g}_2$ determines various versions of the metric $\mathfrak{g}_{3T}$, and $\mathfrak{g}_\diamond$ can be any one of these $\mathfrak{g}_{3T}$ metrics. Set $\mathfrak{g}_+$ to be the version of $\mathfrak{g}_2$ that is used to construct $\mathfrak{g}_\diamond$ and set $\mathfrak{g}_{+T}$ to denote $\mathfrak{g}_\diamond$. What follows is the second requirement for T: It should be large enough that the $Y_- = Y$ and $Y_+ = Y_{\mathrm{G}}$ versions of the constructions in Parts 1–10 from Section 9.4 can be invoked using X_* and the metrics $\mathfrak{g}_-$ on Y_- and $\mathfrak{g}_+$ on Y_+.

The constructions in Parts 1–8 of Section 9.4 require a closed 2–form on X_*, this denoted by p_X, whose de Rham cohomology class is $c_1(\det \mathbb{S})$ and which has the following additional properties: it equals w_- where $s < -102$, it equals w_+ where $s > 102$ and it obeys the bound in (9-20). Given such a 2–form, Parts 1–8 of Section 9.4 supply $L_1 \gg 1$, a metric on X_* and a 2–form on X_* with the properties listed below; the metric and 2–form are denoted in the list and subsequently by $\mathfrak{m}_{T*}$ and ω_{T*}:

(9-73)
- The metric $\mathfrak{m}_{T*}$ obeys (2-9) and (3-14) when the version of L in the latter is greater than $L_1 + 20$.
- The pullback of $\mathfrak{m}_{T*}$ from the $s < -L_1 - 1$ part of X via the embedding from the second bullet of (2-8) is $ds^2 + \mathfrak{g}_-$ and the pullback of $\mathfrak{m}_{T*}$ from the $s > L_1 + 1$ part of X_* by the embedding from the third bullet of (2-8) is $ds^2 + \mathfrak{g}_+$.
- The 2–form ω_{T*} is self-dual when self-duality is defined by $\mathfrak{m}_{T*}$. In addition, the pullback of ω_{T*} to any constant $s > 1$ slice of X_* is closed.
- The pullback of ω_{T*} from the $s < -L_1 - 1$ part of X_* by the embedding from the second bullet of (2-8) is $ds \wedge *w_- + w_-$ with $*$ denoting here the $\mathfrak{g}_-$–Hodge star.
- The pullback of ω_{T*} from the $s > L_1 + 1$ part of X_* via the embedding from the third bullet of (2-8) is $ds \wedge *w_+ + w_+$ with $*$ now denoting the $\mathfrak{g}_+$–Hodge star and with w_+ denoting the $\mathfrak{g}_+$–harmonic 2–form with de Rham cohomology class $c_1(\det \mathbb{S})$.
- The 2–form ω_{T*} obeys the constraint in (3-13).
- The norm of ω_{T*} and those of its $\mathfrak{m}_{T*}$–covariant derivatives to order 10 are less than c_0.

When comparing the notation in (9-73) with the notation in Parts 1–10 of Section 9.4, keep in mind that this case has $\mathfrak{g}_{-T} = \mathfrak{g}_-$ and $w_{-T} = w_-$, and $\mathfrak{g}_{+T} = \mathfrak{g}_\diamond$ and $w_{+T} = w_\diamond$.

The remaining steps construct a version of p_X with the required properties.

Step 3 The construction of p_X requires the three constraints on $\mathfrak{m}_{T*}$ that are described here and a fourth constraint that is described in Step 4. The first constraint is that imposed in Part 10 of Section 9.4.

The remaining constraints and that in Step 4 refer to the subset $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma \subset M_\delta$, this viewed as a subset of Y and also as a subset of Y_{G}. The second constraint uses the embeddings from the first and second bullets of (2-8) to view the $s<0$ and $s>0$ parts of X_* as $(-\infty,0]\times Y$ and as $(0,\infty)\times Y_{\mathrm{G}}$. This constraint is the analog of that given in (9-42):

(9-74) The metric $\mathfrak{m}_{T*}$ on $[-100,-96]\times\mathcal{Y}_M$ is the product metric $ds^2+\mathfrak{g}_Y$. The metric $\mathfrak{m}_{T*}$ on $[96,100]\times\mathcal{Y}_M$ is the product metric $ds^2+\mathfrak{g}_+$.

By way of background for the third constraint, note that (9-43) holds for X_*, this being a consequence of what is said in Part 1 about the ascending and descending manifolds from the critical points of s. The third constraint refers to this embedding; it also uses $\mathfrak{m}_Y$ and $\mathfrak{m}_+$ to denote the metrics $ds^2+\mathfrak{g}_Y$ and $ds^2+\mathfrak{g}_+$ on $\mathbb{R}\times\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma$:

(9-75) There exists a T–independent constant, $c_*>1$, with the following significance: the pullback of $\mathfrak{m}_{T*}$ from the $s>-94$ part of X_* via the embedding in (9-43) obeys $c_*^{-1}\mathfrak{m}_Y \le \mathfrak{m} \le c_*\mathfrak{m}_Y$ and $c_*^{-1}\mathfrak{m}_+ \le \mathfrak{m} \le c_*\mathfrak{m}_+$.

This third constraint is the analog of the constraint in (9-44).

Step 4 This step describes the fourth constraint on $\mathfrak{m}_{T*}$. This constraint on $\mathfrak{m}_{T*}$ specifies its pullback to $[-96,-94]\times\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma$ via the embedding from the second bullet of (2-8). The constraint asks that this pullback have the form $ds^2+\mathfrak{g}$ with $\mathfrak{g}$ denoting a certain s–dependent metric on $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma$. The upcoming description of $\mathfrak{g}$ refers to the depiction in (9-68) of $\mathfrak{g}_-$ on $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma$; and it refers to an analogous depiction of the metric $\mathfrak{g}_Y$ on $\bigcup_{(\gamma,\mathrm{Z}_\gamma)\in\Theta} \mathcal{T}_\gamma$. The metric $\mathfrak{g}_Y$ on each $\mathcal{T}_\gamma$ has the same form as (9-68) but with $\mathfrak{h}_\gamma=0$ and with different versions of A_γ and B_γ. The $\mathfrak{g}_Y$ versions of these functions are denoted by A_{Y_γ} and B_{Y_γ}. Note that $\mathrm{A}_{Y_\gamma}\ge c_0^{-1}$.

The specification of $\mathfrak{g}$ uses two functions on $\mathbb{R}$, the first being the function $\chi^{\mathcal{T}}_{\diamond 1}$ given by $\chi(s+96)$. This function equals 1 where $s<-96$ and it equals 0 where $s\ge -95$. The second function is denoted by $\chi^{\mathcal{T}}_{\diamond 2}$, it is given by $\chi(s+95)$. The latter is equal to 1 where $s<-95$ and it is equal to 0 where $s>-94$.

The metric $\mathfrak{g}$ on $\mathcal{T}_\gamma$ is defined by its volume form, this being $\mathrm{H}_\gamma\, dx\, dy\, dt$, and by the following Hodge star rules:

$$(9\text{-}76)\quad \begin{aligned} *dx\,dy &= (\chi^{\mathcal{T}}_{\diamond 2}\mathrm{A}_\gamma + (1-\chi^{\mathcal{T}}_{\diamond 2})\mathrm{A}_{\mathrm{Y}_\gamma})\,dt - \chi^{\mathcal{T}}_{\diamond 1}\mathrm{A}_\gamma\tau^{-1}(1-\tau)\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma\,dx + \mathrm{B}_\gamma\,dy,\\ *dy\,dt &= \mathrm{H}_\gamma^{-1}(1+\chi^{\mathcal{T}}_{\diamond 1}\tau^{-2}\mathrm{H}_\gamma^{-1}\mathrm{A}_\gamma(1-\tau)^2\mathrm{Z}_\gamma\mathfrak{h}_\gamma)\,dx\\ &\qquad\qquad - \chi^{\mathcal{T}}_{\diamond 1}\mathrm{A}_\gamma\tau^{-1}(1-\tau)\mathrm{H}_\gamma^{-1}\mathrm{Z}_\gamma\mathfrak{h}_\gamma\,dt,\\ *dt\,dx &= dy + \mathrm{B}_\gamma\,dt. \end{aligned}$$

Important points to note are that $\mathfrak{g}$ is independent of T and s on a neighborhood of $s=-94$, that $\mathfrak{g}=\mathfrak{g}_-$ on a neighborhood of $s=-96$ and that $\mathfrak{g}=\mathfrak{g}_-$ for all s on the complement of $\mathcal{T}'_\gamma$.

Step 5 This step describes p_X and says more about the metric $\mathfrak{m}_{T*}$. The 2–form p_X and the metric $\mathfrak{m}_{T*}$ on the $s\in[-102,-98]$ part of X_* are described by the analog of Step 1 in Part 11 of Section 9.4 that has Y replacing Y_{G}. By way of a summary, p_X is defined on the $s\in[-102,-101]$ part of X to be the 2–form $p_{\mathcal{N}1}$ that is described in the Y version of Step 3 from Part 9 of Section 9.4. The 2–form p_X is defined on the $s\in[-101,-100]$ part of X to be the Y version of the 2–form $p_{\mathcal{N}2}$ that is described in Step 4 from Part 9 of Section 9.4. The definition of p_X on the $s\in[-100,-98]$ part of X is made by specifying its pullback via the embedding from the second bullet of (2-8). This pullback is the s–independent 2–form that equals p_0 on $\mathcal{Y}_0$ and $w_- - d(\sigma_1 q_{1-})$ on the rest of Y. The metric $\mathfrak{m}_{T*}$ on this part of X pulls back via the embedding from the second bullet of (2-8) as $ds^2+\mathfrak{g}$ with $\mathfrak{g}$ denoting the metric given by $\mathfrak{g}_-$ on $\mathcal{Y}_M$, the metric in (9-39) on $[-100,-98]\times\mathcal{N}_\varepsilon$ and the metric $\mathfrak{g}_*$ on $[-100,-98]\times\mathcal{Y}_0$. Note in this regard that $\mathfrak{m}_{T*}$ is in any event described by (9-10).

Step 6 This step describes p_X and the metric on the $s\in[-98,-96]$ part of X. But for one significant difference, the description of p_X here is similar to the description of its namesake given in Step 2 from Part 11 in Section 9.4. Both p_X and the metric on this part of X are described by their pullbacks via the embedding from the second bullet of (2-8). The metric pulls back as $ds^2+\mathfrak{g}$ with $\mathfrak{g}$ given by $\mathfrak{g}_*$ on $\mathcal{Y}_{0\varepsilon}$ and by the metric in (9-39) on $\mathcal{N}_\varepsilon$. The metric $\mathfrak{g}$ on $\mathcal{Y}_M$ is the metric $\mathfrak{g}_-$.

As in the Step 2 from Part 11 of Section 9.4, a 1–form to be denoted by q_{3-} is constructed with the following properties: it obeys $dq_{3-}=p-w_-+d(\sigma_1 q_{1-})$, it vanishes on the $\mathrm{r}\ge\rho_*-\frac{1}{2}\varepsilon$ part of $\mathcal{N}_\varepsilon$ and its L^2–norm is bounded by c_0. Reintroduce $\chi_{\diamond 3}$ to denote the function on $\mathbb{R}$ given by $\chi(|s|-97)$ and use $\chi'_{\diamond 3}$ to denote its derivative. The 2–form p_X on $[-98,-96]\times Y$ is p_0 on $\mathcal{Y}_0$ and given on the rest of Y by the formula in (9-45). Note that p_X is p_0+p near $\{-96\}\times Y$, and that its L^2–norm on this part of X is bounded by c_0.

To start the description of q_{3-}, let γ denote a loop from a pair in the set Θ. The 2–form w_- on $\mathcal{T}_\gamma$ is given by $\tau w + (1-\tau)\mathrm{z}_\gamma p_\gamma$ and so it can be written as

$$p_\gamma + \tau(\mathrm{Q}_\gamma\, dt - \mathrm{z}_\gamma \mathfrak{q}_\gamma\, dx), \tag{9-77}$$

where Q_γ is a function of y and t whose y–derivative is H_γ, and where $\mathfrak{q}_\gamma$ is a function of x and y whose y–derivative is $\mathfrak{h}_\gamma$. Meanwhile, $\tau = \frac{1}{T^2}$. Let q_γ denote $\tau(\mathrm{Q}_\gamma\, dt - \mathrm{z}_\gamma \mathfrak{q}_\gamma\, dx)$. Use (9-68) to see that the $q_\gamma \wedge * q_\gamma$ can be written as $\sigma \mathrm{H}_\gamma\, dx\, dy\, dt$ with $|\sigma| \le c_0 \tau^2 \mathrm{A}_\gamma^{-1}$. Now, A_γ is constrained to be positive and less than $c_0^{-1}\tau^2$, and these constraints are met if A_γ is chosen greater than $c_0^{-2}\tau^2$. Take A_γ so that this is the case, and then the L^2–norm (and pointwise norm) of q_γ is bounded by c_0.

The 2–form $w_- - d(\sigma q_{1-})$ is exact on $\mathcal{Y}_M - \bigl(\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}'_\gamma\bigr)$ and on the $r \le \rho_*$ part of $\mathcal{N}_\varepsilon$. This being the case, it can be written as dq_* on this part of Y. More to the point, Lemma 9.5 can be used as in the last paragraph of Step 2 from Part 11 in Section 9.4 to obtain a version of q_* that is zero where $\mathrm{r} \ge \rho_* - \frac{1}{512}\varepsilon$ and has L^2–norm bounded by c_0 on $\mathcal{Y}_M - \bigl(\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}'_\gamma\bigr)$ and on the $\mathrm{r} \le \rho_* - \frac{1}{512}\varepsilon$ part of $\mathcal{N}_\varepsilon$.

Let γ again denote a loop from a pair in Θ. The difference $q_* - q_\gamma$ on $\mathcal{T}_\gamma - \mathcal{T}'_\gamma$ is exact. This being the case, it follows from the Mayer–Vietoris exact sequence and from the fact that the various loops from Θ freely generate $H_1(M_\delta;\mathbb{R})$ that there is a closed 1–form, $\mathfrak{k}$, on $\mathcal{Y}_M$ with the following three properties: First, $q_* - q_\gamma = \mathfrak{k}$ on each $(\gamma,\mathrm{z}_\gamma) \in \Theta$ version of $\mathcal{T}_\gamma - \mathcal{T}'_\gamma$. Second, $\mathfrak{k} = 0$ near $\mathcal{N}_\varepsilon$ and on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$. Finally, the L^2–norm of $\mathfrak{k}$ is bounded by c_0. This understood, the sought after 1–form q_{3-} is defined to be q_γ on each $(\gamma,\mathrm{z}_\gamma) \in \Theta$ version of $\mathcal{T}_\gamma$ and to be $q_* - \mathfrak{k}$ on $\mathcal{Y}_M - \bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$.

Step 7 This step describes p_X and the metric on the $s \in [-96,-94]$ part of X. The story with p_X is simple: it is the 2–form $p_0 + p$. The metric on X is described by its pullback to $[-96,-94] \times Y$ via the embedding from the second bullet of (2-8). In particular, it pulls back as $ds^2 + \mathfrak{g}$ with $\mathfrak{g}$ being an s–dependent metric on Y. The s–dependence involves only $\mathfrak{g}$'s restriction to $\bigcup_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathcal{T}_\gamma$ where it is given in Step 4. The metric $\mathfrak{g}$ is independent of s on the rest of Y. As explained in the next paragraph, this metric on X is such that the L^2–norm of p_X on the $[-96,-94]$ part of X is bounded by c_0, a T–independent constant.

The aforementioned L^2–norm bound holds for p_0. To see about p, write it as $\sum_{(\gamma,\mathrm{z}_\gamma)\in\Theta} \mathrm{z}_\gamma p_\gamma$. A given version of p_γ has support in $\mathcal{T}_\gamma$, where the metric is given by (9-76). Fix $s \in [-96,-94]$ and since $p_\gamma = h_\gamma\, dx\, dy$, the first bullet of (9-76) can be used to write $p_\gamma \wedge *p_\gamma$ as $\mathrm{P}|\mathfrak{h}_\gamma|^2\mathrm{H}_\gamma\, dx\, dy\, dt$ with $\mathrm{P} = (\chi^{\mathcal{T}}_{\diamond 2}\mathrm{A}_\gamma + (1-\chi^{\mathcal{T}}_{\diamond 2})\mathrm{A}_{Y_\gamma})^2$. Since $\mathrm{P} < c_0$, so the L^2–norm of p_γ at any $s \in [-96, -94]$ slice of $[-96, -94] \times Y$ is bounded by c_0.

Step 8 This last step describes p_X and the metric on the $s \in [-94, 102]$ part of X. The description of p_X starts where $s \in [96, 102]$. The 2–form p_X here is described by the $Y_+ = Y_{\mathrm{G}}$ version of the 2–form that is defined in Steps 1 and 2 from Part 11 of Section 9.4. The $s \in [96, 100]$ part of the constraint in (9-74) and the constraint from Part 10 of Section 9.4 are needed to repeat Steps 1 and 2 from Part 11 in the case at hand. These steps define a version of p_X whose L^2–norm on the $s \in [96, 102]$ part of X is bounded by c_0 times the L^2–norm of w_+ on Y_{G}. This version of p_X is w_+ near the $s = 102$ slice of X_* and it is the 2–form $p_0 + p$ near the $s = 96$ slice. The 2–form p_X is set equal to $p_0 + p$ on the $s \in [-94, 96]$ part of X. Its L^2–norm on the $s \in [-94, 96]$ part of X is bounded by c_0, this being a consequence of (9-75).

Part 9 Taking up where Part 8 left off, this last part of the subsection defines the desired metric on X and w_X on the $s \in \bigl[-\frac{1}{2}L + 4, L\bigr]$ part of X. To this end, fix T large and then $L_1 \geq c_T$ so as to use the constructions in Part 8 of the metric $\mathfrak{m}_{T*}$ and ω_{T*}. With L_1 chosen, assume that $L > 4L_1$. The metric $\mathfrak{m}_{T*}$ where $s \in \bigl[-\frac{1}{2}L + 4, -\frac{1}{2}L + 5\bigr]$ is the same as the $s \in \bigl[-\frac{1}{2}L + 4, -\frac{1}{2}L + 5\bigr]$ version of the metric from Part 8, and ω_{T*} on this same part of X is the $s \in \bigl[-\frac{1}{2}L + 4, -\frac{1}{2}L + 5\bigr]$ version of Part 8's 2–form w_X. This understood, the desired metric for X is taken to be $\mathfrak{m}_{T*}$ where $s \geq -\frac{1}{2}L + 4$, and the 2–form w_X is taken to be ω_{T*} on this same part of X. Here, υ_X is set to be the s–independent 1–form $\upsilon_\diamond$, and the bounds in items (4)(b) and (5)(c) of (3-15) are verified by choosing the parameter m to be sufficiently small, as directed in Part 2 above. □

References

[1] **M F Atiyah**, **V K Patodi**, **I M Singer**, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69 MR

[2] **J Bloom**, **T Mrowka**, **P Ozsváth**, *A Künneth formula for monopole Floer homology*, in preparation

[3] **R Bott**, **L W Tu**, *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Springer (1982) MR

[4] **H Cartan**, *La transgression dans un groupe de Lie et dans un espace fibré principal*, from "Colloque de topologie (espaces fibrés)", Georges Thone, Liège (1951) 57–71 MR

[5] **V Colin**, **P Ghiggini**, **K Honda**, **M Hutchings**, *Sutures and contact homology, I*, Geom. Topol. 15 (2011) 1749–1842 MR

[6] **S K Donaldson**, *The Seiberg–Witten equations and* 4*–manifold topology*, Bull. Amer. Math. Soc. 33 (1996) 45–70 MR

[7] **S K Donaldson**, *Floer homology groups in Yang–Mills theory*, Cambridge Tracts in Mathematics 147, Cambridge Univ. Press (2002) MR

[8] **S K Donaldson**, **P B Kronheimer**, *The geometry of four-manifolds*, Oxford Univ. Press (1990) MR

[9] **K Fukaya**, *Floer homology of connected sum of homology* 3*–spheres*, Topology 35 (1996) 89–136 MR

[10] **M Goresky**, **R Kottwitz**, **R MacPherson**, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998) 25–83 MR

[11] **K Honda**, *Transversality theorems for harmonic forms*, Rocky Mountain J. Math. 34 (2004) 629–664 MR

[12] **M Hutchings**, **C H Taubes**, *Gluing pseudoholomorphic curves along branched covered cylinders, II*, J. Symplectic Geom. 7 (2009) 29–133 MR

[13] **J D S Jones**, *Cyclic homology and equivariant homology*, Invent. Math. 87 (1987) 403–423 MR

[14] **A Juhász**, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. 6 (2006) 1429–1457 MR

[15] **J L Koszul**, *Sur un type d'algèbres différentielles en rapport avec la transgression*, from "Colloque de topologie (espaces fibrés)", Georges Thone, Liège (1951) 73–81 MR

[16] **P B Kronheimer**, **C Manolescu**, *Periodic Floer pro-spectra from the Seiberg–Witten equations*, preprint (2002) arXiv

[17] **P Kronheimer**, **T Mrowka**, *Monopoles and three-manifolds*, New Mathematical Monographs 10, Cambridge Univ Press (2007) MR

[18] **P Kronheimer**, **T Mrowka**, *Knots, sutures, and excision*, J. Differential Geom. 84 (2010) 301–364 MR

[19] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, I: Heegaard Floer homology and Seiberg–Witten Floer homology*, Geom. Topol. 24 (2020) 2829–2854

[20] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, II: Reeb orbits and holomorphic curves for the ech/Heegaard–Floer correspondence*, Geom. Topol. 24 (2020) 2855–3012

[21] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, III: Holomorphic curves and the differential for the ech/Heegaard Floer correspondence*, Geom. Topol. 24 (2020) 3013–3218

[22] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, HF = HM*, IV: The Seiberg–Witten Floer homology and ech correspondence*, Geom. Topol. 24 (2020) 3219–3469

[23] **Y-J Lee**, *Heegaard splittings and Seiberg–Witten monopoles*, from "Geometry and topology of manifolds" (H U Boden, I Hambleton, A J Nicas, B D Park, editors), Fields Inst. Commun. 47, Amer. Math. Soc., Providence, RI (2005) 173–202 MR

[24] **Y-J Lee**, *Reidemeister torsion in Floer–Novikov theory and counting pseudoholomorphic tori, I*, J. Symplectic Geom. 3 (2005) 221–311 MR

[25] **Y-J Lee**, **C H Taubes**, *Periodic Floer homology and Seiberg–Witten–Floer cohomology*, J. Symplectic Geom. 10 (2012) 81–164 MR

[26] **K Lefèvre-Hasegawa**, *Sur les A_∞–catègories*, PhD thesis, Université Paris 7 (2003) arXiv

[27] **C Manolescu**, *Seiberg–Witten–Floer stable homotopy type of three-manifolds with* $b_1 = 0$, Geom. Topol. 7 (2003) 889–932 MR

[28] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR

[29] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR

[30] **C H Taubes**, SW ⇒ Gr*: from the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996) 845–918 MR

[31] **C H Taubes**, *Seiberg Witten and Gromov invariants for symplectic* 4*–manifolds*, First International Press Lecture Series 2, International, Somerville, MA (2000) MR

[32] **C H Taubes**, *Asymptotic spectral flow for Dirac operators*, Comm. Anal. Geom. 15 (2007) 569–587 MR

[33] **C H Taubes**, *The Seiberg–Witten equations and the Weinstein conjecture*, Geom. Topol. 11 (2007) 2117–2202 MR

[34] **C H Taubes**, *The Seiberg–Witten equations and the Weinstein conjecture, II: More closed integral curves of the Reeb vector field*, Geom. Topol. 13 (2009) 1337–1417 MR

[35] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, I*, Geom. Topol. 14 (2010) 2497–2581 MR

[36] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, II*, Geom. Topol. 14 (2010) 2583–2720 MR

[37] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, III*, Geom. Topol. 14 (2010) 2721–2817 MR

[38] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, IV*, Geom. Topol. 14 (2010) 2819–2960 MR

[39] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, V*, Geom. Topol. 14 (2010) 2961–3000 MR

[40] **D Toledo**, **Y L L Tong**, *Duality and intersection theory in complex manifolds, I*, Math. Ann. 237 (1978) 41–77 MR

Department of Mathematics, University at Buffalo
Buffalo, NY, United States

Institute of Mathematical Sciences, The Chinese University of Hong Kong
Shatin, NT, Hong Kong

Department of Mathematics, Harvard University
Cambridge, MA, United States

kutluhan@buffalo.edu, yjlee@math.cuhk.edu.hk,
chtaubes@math.harvard.edu

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