MTH–636 Spring 2014
Lecture Notes

1. January 27–January 31

Smooth Manifolds

Definition. Let M be a topological manifold, i.e. a topological space that is Hausdorff, second countable, and locally Euclidean. A smooth *atlas* on M is a family $\{(U_\alpha, \varphi_\alpha)\}$ of coordinate neighborhoods (or charts) on M where

(1) $\{U_\alpha\}$ is an open cover of M, $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$ are homeomorphisms onto their images,
(2) If $U_\alpha \cap U_\beta \neq \emptyset$, then the *transition map* $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism.

Two smooth atlases are said to be *compatible* if their union is also a smooth atlas. A maximal smooth atlas, i.e. the union of all smooth atlases compatible with a given smooth atlas, is called a *smooth structure* on M.

A topological manifold M together with a smooth structure on it is called a smooth manifold. Note that an open subset of a smooth manifold is also a smooth manifold.

Definition. A smooth manifold M is *orientable* if it admits an atlas $\{(U_\alpha, \varphi_\alpha)\}$ with the property that the Jacobian of $\varphi_\beta \circ \varphi_\alpha^{-1}$ is positive whenever $U_\alpha \cap U_\beta \neq \emptyset$. An atlas with this property is called (*coherently*) *oriented*, and the choice of a maximal atlas with this property is called an *orientation*.

Definition. A function $f : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U}$ of a smooth manifold M is called *smooth* if for any coordinate neighborhood (U, φ) the function $f \circ \varphi^{-1} : \varphi(\mathcal{U} \cap U) \to \mathbb{R}$ is smooth.

Definition. Let M and N be two smooth manifolds. A map $F : M \to N$ is called *smooth* if for any point $p \in M$, any coordinate neighborhood (U, φ) around p, and any coordinate neighborhood (V, ϕ) around $F(p)$ with $F(U) \subset V$, the map $\phi \circ F \circ \varphi^{-1} : \varphi(U) \to \phi(V)$ is smooth.

Definition. The *rank* of a smooth map $F : M \to N$ at $p \in M$ is the rank of the Jacobian matrix of $\phi \circ F \circ \varphi^{-1}$ at $\varphi(p)$ for any coordinate neighborhoods (U, φ) around p and (V, ϕ) around $F(p)$ with $F(U) \subset V$.

Remark. Around any point $p \in M$ one can find a coordinate neighborhood (U, φ) such that $\varphi(U)$ is a ball centered at the origin, and φ maps p onto the origin. Furthermore, if $dim(M) = m$, $dim(N) = n$, and $rank(F) = k$, then there exist such coordinate neighborhoods (U, φ) around p and (W, ϕ) around $F(p)$ with $F(U) \subset W$ satisfying

$$\phi \circ F \circ \varphi^{-1}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

See [2, III-Remark 4.2, II-Theorem 7.1] for a proof.

Definition. A smooth map $F : M \to N$ is called an *immersion* if $rank(F) = dim(M)$ at any point in M. It is called a *submersion* if $rank(F) = dim(N)$ at any point in M.

Remark. Note that the image of an injective immersion $F : M \to N$ admits two topologies: one that is induced by the topology of M, making the map $F : M \to F(M)$ a homeomorphism, and the other is the subspace topology induced by N. These two topologies on $F(M)$ need not be homeomorphic.

Example 1. Figure-eight: Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth monotone increasing function with the property that $f(0) = \pi$, $\lim_{t\to-\infty} f(t) = 0$, and $\lim_{t\to\infty} f(t) = 2\pi$. Then the map $F : \mathbb{R} \to \mathbb{R}^2$ defined by

$$F(t) := (2\cos(f(t) - \frac{\pi}{2}), \sin 2(f(t) - \frac{\pi}{2}))$$

is an injective immersion, but it is not a homeomorphism onto its image since the pre-image of any sufficiently small open neighborhood of the origin is disconnected.

Definition. A smooth map $F : M \to N$ is called an *embedding* if it is an injective immersion that induces a homeomorphism between M and $F(M)$ with the subspace topology inherited from N. The image of an embedding $F : M \to N$ is called a (*regular*) *submanifold* of N.

Tangent and Cotangent Spaces

Let M be a smooth m-dimensional manifold, and $p \in M$ be a point. Then consider the set of all smooth parametrized curves $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$. Two such curves γ_1 and γ_2, are deemed equivalent if for any coordinate neighborhood (U, φ) around p the derivatives of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ at 0 agree. The tangent space to M at p can be identified with the set $\Gamma(p)$ of equivalence classes of smooth parametrized curves in M through p.

Alternatively, consider the set $C^\infty(p)$ of smooth functions defined in some open neighborhood of p. There is an equivalence relation on $C^\infty(p)$ which declares two functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ equivalent if they agree on a smaller open neighborhood $W \subset U \cap V$ of p. An equivalence class of such functions is called a *germ* of smooth functions at p, and these equivalence classes form an algebra

$$\mathfrak{G}(p) := C^\infty(p)/\sim = \{[f] | f \in C^\infty(p)\},$$

with addition, scalar multiplication, and multiplication operations inherited from $C^\infty(p)$. With the preceding understood, the tangent space to M at p is defined as follows.

Definition. A *derivation* at p is a linear map $X_p : \mathfrak{G}(p) \to \mathbb{R}$ satisfying the Leibniz Rule:

$$X_p([f] \cdot [g]) = (X_p[f])g(p) + f(p)(X_p[g]).$$

Derivations at p form a vector space: for any two derivations X_p and Y_p at p, and $c \in \mathbb{R}$,

(1) $(X_p + Y_p)[f] = X_p[f] + Y_p[f]$,
(2) $(cX_p)[f] = cX_p[f]$.

The tangent space to M at p, denoted T_pM, is the vector space of derivations at p.

Example 2. Consider the Euclidean space $\mathbb{R}^m$ with the standard smooth structure. Let $x^1, \ldots, x^m$ denote the Euclidean coordinates. Then the tangent space to $\mathbb{R}^m$ at any point $p = (p^1, \ldots, p^m)$ is the space of vectors based at p, i.e. $T_p\mathbb{R}^m = \{\vec{px} \mid x \in \mathbb{R}^n\}$. Any tangent vector $\vec{px} = \langle a^1, \ldots, a^m \rangle_p$ defines a unique derivation X_p at p by

$$X_p[f] = \sum_{i=1}^{m} a^i \frac{\partial f}{\partial x^i}(p)$$

In fact, any derivation D_p at p is obtained this way. In order to see this, first note that any smooth function defined in a neighborhood of p can be restricted to a ball centered at p so that

$$\begin{aligned} f(x) &= f(p) + \int_0^1 \frac{df}{dt}(p + t(x-p))dt \\ &= f(p) + \sum_{i=1}^{m} (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x-p))dt \end{aligned}$$

at any point x in the ball. Then, for any smooth function f defined in a neighborhood of p

$$D_p[f] = \sum_{i=1}^{m} D_p[x^i] \frac{\partial f}{\partial x^i}(p).$$

As a result, the tangent vector $\langle D_p[x^1], \ldots, D_p[x^m] \rangle_p$ corresponds to the derivation D_p via the above association. In particular, the vector space of derivations at p has dimension m, and a canonical basis is $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$.

Next, we have the following theorem:

Theorem 1.1. *Let $F : M \to N$ be a smooth map. Then for any $p \in M$ the map*

$$F^*{}_p : \mathfrak{G}(F(p)) \to \mathfrak{G}(p)$$

defined by $F^*{}_p([f]) = [f \circ F]$ is an algebra homomorphism that induces a homomorphism between the vector spaces

$$dF_p : T_pM \to T_{F(p)}N$$

defined by $dF_p(X_p)[f] = X_p[F^*{}_p([f])]$. Furthermore,

1. *Both $id_M{}^*{}_p$ and $d(id_M)_p$ are the identity homomorphisms,*
2. *If $F : M \to N'$ and $G : N' \to N$ are two smooth maps, and $p \in M$, then,*

$$(G \circ F)^*{}_p = F^*{}_p \circ G^*{}_{F(p)} \qquad \text{and} \qquad d(G \circ F)_p = dG_{F(p)} \circ dF_p.$$

Proof. We will prove that the map dF_p is a homomorphism of vector spaces, and it behaves well under compositions. The statements about $F^*{}_p$ are left as an exercise. First, for any $X_p \in T_pM$, we prove that $dF_p(X_p) \in T_{F(p)}N$. For this, we need to check that $dF_p(X_p)$ is a derivation at $F(p)$:

- Linearity: For any $X_p \in T_pM$, $f, g \in C^\infty(p)$, and $a, b \in \mathbb{R}$,

$$\begin{aligned} dF_p(X_p)(a[f]+b[g]) &= dF_p(X_p)([af+bg]) \\ &= X_p([(af+bg)\circ F]) \\ &= X_p([a(f\circ F)+b(g\circ F)]) \\ &= aX_p[f\circ F]+bX_p[g\circ F] \\ &= adF_p(X_p)[f]+bdF_p(X_p)[g]. \end{aligned}$$

- Leibniz rule: For any $X_p \in T_pM$, and $f, g \in C^\infty(F(p))$,

$$\begin{aligned} dF_p(X_p)([f]\cdot[g]) = dF_p(X_p)[fg] &= X_p[fg\circ F] = X_p([(f\circ F)\cdot(g\circ F)]) \\ &= X_p([f\circ F]\cdot[g\circ F]) \\ &= (X_p[f\circ F])\, g(F(p)) + f(F(p))\, X_p[g\circ F] \\ &= dF_p(X_p)[f]\, g(F(p)) + f(F(p))\, dF_p(X_p)[g]. \end{aligned}$$

Next, we show that the map dF_p is linear. For $X_p, Y_p \in T_pM$, $a, b \in \mathbb{R}$, and $f \in C^\infty(F(p))$,

$$\begin{aligned} dF_p(aX_p+bY_p)[f] &= (aX_p+bY_p)[f\circ F] \\ &= aX_p[f\circ F]+bY_p[f\circ F] \\ &= a\,dF_p(X_p)[f]+b\,dF_p(Y_p)[f]. \end{aligned}$$

Therefore, dF_p is a linear map between the tangent spaces T_pM and $T_{F(p)}N$.

Finally, we show that if $F: M \to N'$ and $G: N' \to N$ are two smooth maps, and $p \in M$, then $d(G\circ F)_p = dG_{F(p)} \circ dF_p$: For any $X_p \in T_pM$, and $f \in C^\infty(G(F(p)))$,

$$\begin{aligned} d(G\circ F)_p(X_p)[f] &= X_p[f\circ G\circ F] \\ &= dF_p(X_p)[f\circ G] \\ &= dG_{F(p)}(dF_p(X_p))[f] \\ &= (dG_{F(p)}\circ dF_p)(X_p)[f]. \end{aligned}$$

□

Using Example 2 and applying Theorem 1.1 to a coordinate map, we deduce that the dimension of the tangent space to a smooth manifold M at any point $p \in M$ is equal to the topological dimension of M. Moreover, given a coordinate neighborhood (U, φ) around p with coordinate functions $(x^1, \ldots, x^m)$, denote, without ambiguity, the tangent vector $d\varphi^{-1}{}_{\varphi(p)}(\frac{\partial}{\partial x^i})$ by $\frac{\partial}{\partial x^i}|_p$.

Let $F: M \to N$ be a smooth function, and $p \in M$. Choose coordinate neighborhoods (U, φ) around p with coordinate functions $(x^1, \ldots, x^m)$ and (V, ϕ) around $F(p)$ with coordinate functions $(y^1, \ldots, y^n)$ so that $F(U) \subset V$. Then the map $\phi\circ F\circ\varphi^{-1}$ appears as $\phi\circ F\circ\varphi^{-1}(x^1, \ldots, x^m) = (F^1(x^1, \ldots, x^m), \ldots, F^n(x^1, \ldots, x^m))$. Now, we claim that

$$dF_p(\frac{\partial}{\partial x^i}|_p) = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i}(\varphi(p))\frac{\partial}{\partial x'^j}|_p.$$

To prove this, it suffices to compute $dF_p(\frac{\partial}{\partial x^i}|_p)[x'^j]$. In this regard, note that

$$\begin{aligned} dF_p(\frac{\partial}{\partial x^i}|_p)[x'^j] &= dF_p(d(\varphi^{-1})_{\varphi(p)}(\frac{\partial}{\partial x^i}))[x'^j] \\ &= (dF_p \circ d(\varphi^{-1})_{\varphi(p)})(\frac{\partial}{\partial x^i})[x'^j] \\ &= \frac{\partial}{\partial x^i}[x'^j \circ F \circ d(\varphi^{-1})] \\ &= \frac{\partial}{\partial x^i}[F^j] = \frac{\partial F^j}{\partial x^i}(\varphi(p)). \end{aligned}$$

In particular, if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two coordinate neighborhoods around p with coordinate functions $(x_\alpha^1, \ldots, x_\alpha^m)$ and $(x_\beta^1, \ldots, x_\beta^m)$, respectively, then

$$\frac{\partial}{\partial x_\alpha^j}|_p = \sum_{i=1}^{m} \underbrace{\frac{\partial x_\beta^i}{\partial x_\alpha^j}(\varphi_\alpha(p))}_{\text{Jacobian of } \varphi_\beta \circ \varphi_\alpha^{-1}} \frac{\partial}{\partial x_\beta^i}|_p.$$

Returning to the first description of the tangent space via smooth parametrized curves, note that an equivalence class $[\gamma]$ of smooth parametrized curves in a smooth manifold M through a point $p \in M$ yields a unique tangent vector to M at p by

$$X_p = \sum_{i=1}^{m} \frac{d(x^i \circ \gamma)}{dt}(0)\frac{\partial}{\partial x^i}|_p.$$

Conversely, any tangent tangent vector $X_p = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}|_p \in T_pM$ defines a line $L(t) := \varphi(p) + t(a^1, \ldots, a^m)$ where $t \in (-\epsilon, \epsilon)$ for $\epsilon > 0$ sufficiently small. The image of $L(t)$ under φ^{-1} yields a smooth parametrized curve γ in M through p for which $\frac{d(x^i \circ \gamma)}{dt}(0) = a^i$ for each $i = 1, \ldots, m$.

Definition. Let $X_p \in T_pM$ and $f \in C^\infty(p)$. Then $X_p[f]$ is called the *directional derivative of f at p.* If γ is a smooth parametrized curve in M through p that represents X_p in the above sense, then $X_p[f] = \frac{d(f \circ \gamma)}{dt}(0)$. Given a coordinate neighborhood (U, φ) around p with coordinate functions $x^1, \ldots, x^m$, in which $X_p = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}|_p$, we have

$$X_p[f] = \sum_{i=1}^{m} a^i \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)).$$

An ordered basis of T_pM defines an orientation on T_pM. Two ordered bases define the same orientation if the change of basis matrix has positive determinant. Having said that, M is orientable if and only if it is possible to choose an orientation at any point in M so that every point $p \in M$ has a coordinate neighborhood (U, φ) with $d\varphi_p$ mapping the orientation of T_pM to the same orientation of $\mathbb{R}^m$.

The tangent bundle TM of M is the disjoint union $\bigsqcup_{p\in M} T_pM$. There exists a projection map

$$\pi : TM \to M$$

sending X_p to p. The tangent bundle is endowed with a topology defined as follows: let $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas for M. Then one can define $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$ and

$$\tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \varphi(U_\alpha) \times \mathbb{R}^m$$

where $\tilde{\varphi}_\alpha(X_p) = (\varphi_\alpha(p), \langle X_p[x^1], \ldots, X_p[x^m]\rangle)$. The topology on TM is generated by a basis consisting of pre-images of all open subsets of $\varphi(U_\alpha) \times \mathbb{R}^m$ endowed with the product topology.

Exercise. Show that the family $\{(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)\}$ is a smooth atlas for TM, and hence the tangent bundle is a smooth manifold of dimension twice the dimension of M. Furthermore, π is a smooth map.

The pairs $(\tilde{U}_\alpha, (\varphi_\alpha^{-1} \times id) \circ \tilde{\varphi}_\alpha)$ are *local trivializations* for TM. The manifold M is called *parallelizable* if there exists a global trivialization.

Example 3. The unit circle: Consider S^1 as embedded in $\mathbb{R}^2$, and defined by the equation $x^2 + y^2 = 1$. The embedding allows us to see the tangent space to S^1 at a point (x, y) as a subspace of $T_{(x,y)}\mathbb{R}^2$ spanned by $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. As a result, tangent bundle of S^1 can be identified with $S^1 \times \mathbb{R}$ via the map sending $\lambda(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})|_{(x,y)}$ to $((x, y), \lambda)$. Note that S^1 is parallelizable.

2. February 3–February 7

Definition. A *section* of the tangent bundle is a map $s : M \to TM$ such that $s \circ \pi = id_M$. It is *smooth* if s is a smooth map.

A section of the tangent bundle TM is called a *vector field* on M. Given a coordinate neighborhood (U, φ) on M with coordinate functions $x^1, \ldots, x^m$, X can be locally described by

$$X : p \mapsto (p, \langle X^1(p), \ldots, X^m(p)\rangle),$$

where $X_p = \sum_{i=1}^m X^i(p)\frac{\partial}{\partial x^i}|_p$ for any $p \in U$. Alternatively, a vector field X on M can be regarded as a linear operator from $C^\infty(M)$ to itself satisfying the Leibniz rule. Then a vector field X on M is smooth if and only if the function Xf defined by

$$Xf(p) = X_p[f]$$

is smooth for any $f \in C^\infty(M)$. The space of smooth vector fields on M, denoted $\mathfrak{X}(M)$, is a real vector space.

The cotangent space of M at p is defined to be the vector space dual of the tangent space to M at p. To be more explicit, let $\mathfrak{F}(p)$ be the subspace of $\mathfrak{G}(p)$ defined by

$$\mathfrak{F}(p) := \{[f] \mid \frac{d(f\circ\gamma)}{dt}(0) = 0 \; \forall[\gamma] \in \Gamma(p)\}.$$

Then the cotangent space T_p^*M is defined to be $\mathfrak{G}(p)/\mathfrak{F}(p)$. The cotangent vector corresponding to the germ of a function f is denoted df_p and called the *differential of f at p*. Given a function $f \in C^\infty(p)$, we can regard df_p as a linear map $df_p : T_pM \to \mathbb{R}$ defined by

$$df_p(X_p) := X_p[f].$$

In particular, given a coordinate neighborhood (U, φ) around p with coordinate functions $(x^1, \ldots, x^m)$, the corresponding cotangent vectors are denoted $\{dx_p^1, \ldots, dx_p^m\}$, and they form a canonical basis for T_p^*M dual to the canonical basis $\{\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p\}$ for T_pM. More precisely, for any $f \in C^\infty(p)$

$$df_p = \sum_{i=1}^m \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p))dx_p^i.$$

Theorem 2.1. *Let $F : M \to N$ be a smooth map. Then for any $p \in M$ the map*

$$F^*{}_p : T^*_{F(p)}N \to T_p^*M$$

defined by $F^*{}_p(df_{F(p)}) = d(f\circ F)_p$ is a homomorphism.

Proof. This map is induced by the algebra homomorphism defined in Theorem 1.1. We only need to show that $F^*{}_p$ maps $\mathfrak{F}(F(p))$ into $\mathfrak{F}(p)$. In this regard, note that if $f \in C^\infty(F(p))$ such that $[f] \in \mathfrak{F}(F(p))$, then $[f\circ F] \in \mathfrak{F}(p)$ since for any smooth parametrized curve γ in M through p, $F\circ\gamma$ is a smooth parametrized curve in N through $F(p)$, and

$$\frac{d((f\circ F)\circ\gamma)}{dt}(0) = \frac{d(f\circ(F\circ\gamma))}{dt}(0) = 0$$

for all $[\gamma] \in \Gamma(p)$. This completes the proof. □

As a result, given two coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) around p with respective coordinate functions $(x_\alpha^1, \ldots, x_\alpha^m)$ and $(x_\beta^1, \ldots, x_\beta^m)$, we have

$$dx^j_{\beta p} = \sum_{i=1}^m \underbrace{\frac{\partial x_\beta^j}{\partial x_\alpha^i}(\varphi_\alpha(p))}_{\text{Jacobian of } \varphi_\beta\circ\varphi_\alpha^{-1}} dx^i_{\alpha p}.$$

The cotangent bundle T^*M of M is the disjoint union $\bigsqcup_{p\in M} T_p^*M$. There exists a projection map

$$\pi : T^*M \to M$$

sending df_p to p, and a natural bijection between the tangent and cotangent bundles of M such that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "\simeq"] \arrow[d, "\pi"'] & T^*M \arrow[d, "\pi"] \\ M \arrow[r, "id"] & M \end{tikzcd}$$

The cotangent bundle of M is endowed with a topology to make this bijection a homeomorphism, and it admits a smooth structure when M is smooth: a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ for M yields a smooth atlas $\{(\tilde{U}_\alpha, \tilde{\varphi}_\alpha)\}$ for T^*M where $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$ and

$$\tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$$

is defined by $\tilde{\varphi}_\alpha(df_p) = (\varphi_\alpha(p), \langle df_p(\frac{\partial}{\partial x^1}), \ldots, df_p(\frac{\partial}{\partial x^m})\rangle)$.

Sections of the cotangent bundle are called *covector fields* on M. A covector field ω is smooth if $\omega(X) \in C^\infty(M)$ for any $X \in \mathfrak{X}(M)$.

Example 4. The differential of a function $f \in C^\infty(M)$ defined by

$$df(X)(p) := X_p[f]$$

for any $X \in \mathfrak{X}(M)$, is a smooth covector field on M.

A smooth map $F : M \to N$ yields smooth maps $dF : TM \to TN$ and $F^* : T^*N \to T^*M$ defined pointwise as in Theorems 1.1 and 2.1, respectively. Although dF does not map vector fields on M to vector fields on N, F^* maps covector fields on N to covector fields on M.

Tensors and Tensor Fields

We begin with a review of some concepts from linear algebra.

Definition. Let V be an m-dimensional vector space over a field $\mathcal{K}$. A *tensor* T of type (r, s) on V is a multi-linear map

$$T : \mathrm{V}^{\times r} \times \mathrm{V}^{*\times s} := \underbrace{\mathrm{V} \times \cdots \times \mathrm{V}}_{r} \times \underbrace{\mathrm{V}^* \times \cdots \times \mathrm{V}^*}_{s} \to \mathcal{K}.$$

Here, r denotes the *covariant order*, and s denotes the *contravariant order* of the tensor.

Definition. Let W be an n-dimensional vector space over a field $\mathcal{K}$. The *tensor product* $\mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$ is defined to be the vector space over $\mathcal{K}$ that is the quotient of the vector space freely generated over $\mathcal{K}$ by $\mathrm{V} \times \mathrm{W}$ by the subspace spanned by

- $(v_1, w) + (v_2, w) - (v_1 + v_2, w)$,
- $(v, w_1) + (v, w_2) - (v, w_1 + w_2)$,
- $c(v, w) - (cv, w)$,
- $c(v, w) - (v, cw)$,

where $v, v_1, v_2 \in \mathrm{V}$, $w, w_1, w_2 \in \mathrm{W}$, and $c \in \mathcal{K}$. For any $v \in \mathrm{V}$ and $w \in \mathrm{W}$, the equivalence class of the pair (v, w) is denoted by $v \otimes w$.

Given bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ for V and W, respectively, the set $\{v_i \otimes w_j\}$ yields a basis for $\mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$. To see this, it suffices to show that for any $v \in \mathrm{V}$ and $w \in \mathrm{W}$, $v \otimes w$ can be uniquely written as a linear combination of vectors from this set. In this regard, first note that the set $\{v_i \otimes w_j\}$ is linearly independent. Then write

$$v = \sum_{i=1}^{m} a_i v_i, \qquad w = \sum_{j=1}^{n} b_j w_j$$

where $a_i, b_j \in \mathcal{K}$ to see that

$$v \otimes w = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j v_i \otimes w_j.$$

In particular, the dimension of $\mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$ is equal to $m \cdot n$.

Note also that $\mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$ has the property that any bilinear map $\alpha : \mathrm{V} \times \mathrm{W} \to \mathrm{Z}$ into a vector space Z factors through $\mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$. More precisely, there exists a unique linear map $\beta : \mathrm{V} \otimes_{\mathcal{K}} \mathrm{W} \to \mathrm{Z}$ such that the diagram

$$\begin{tikzcd} \mathrm{V} \times \mathrm{W} \arrow[r, "h"] \arrow[dr, "\alpha"'] & \mathrm{V} \otimes_{\mathcal{K}} \mathrm{W} \arrow[d, "\beta"] \\ & \mathrm{Z} \end{tikzcd}$$

commutes, where $h : \mathrm{V} \times \mathrm{W} \to \mathrm{V} \otimes_{\mathcal{K}} \mathrm{W}$ is defined by $h(v, w) = v \otimes w$. This is the universal property that defines the tensor product up to isomorphism.

Theorem 2.2. *Let* V *be an* n*-dimensional vector space over a field* $\mathcal{K}$*. The space* ${T^r}_s(\mathrm{V})$ *of type* (r, s)*-tensors on* V *is a vector space equipped with addition and scalar multiplication operations defined by*

- $(T + S)(v_1, \ldots, v_r, v_1'^*, \ldots, v_s'^*) := T(v_1, \ldots, v_r, v_1'^*, \ldots, v_s'^*) + S(v_1, \ldots, v_r, v_1'^*, \ldots, v_s'^*)$,
- $(cT)(v_1, \ldots, v_r, v_1'^*, \ldots, v_s'^*) := cT(v_1, \ldots, v_r, v_1'^*, \ldots, v_s'^*)$.

Furthermore, ${T^r}_s(\mathrm{V})$ *is isomorphic to the tensor product*

$$\mathrm{V}^{*\otimes r} \otimes_{\mathcal{K}} \mathrm{V}^{\otimes s} := \underbrace{\mathrm{V}^* \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} \mathrm{V}^*}_{r} \otimes_{\mathcal{K}} \underbrace{\mathrm{V} \otimes_{\mathcal{K}} \cdots \otimes_{\mathcal{K}} \mathrm{V}}_{s}.$$

In particular, it has dimension m^{r+s}*.*

Proof. The fact that ${T^r}_s(\mathrm{V})$ is a vector space is easy to check, and it is left as an exercise. In order to prove the latter claim, fix a basis $\{e_1, \ldots, e_m\}$ for V and consider the dual basis $\{e^1, \ldots, e^m\}$ for V^* defined by $e^i(e_j) = {\delta^i}_j$. Then define a homomorphism

$$\Phi : {T^r}_s(\mathrm{V}) \to \mathrm{V}^{*\otimes r} \otimes_{\mathcal{K}} \mathrm{V}^{\otimes s},$$

by

$$\Phi(T)=\sum_{i_k,j_k\in\{1,\dots,m\}} T(e_{i_1},\dots,e_{i_r},e^{j_1},\dots,e^{j_s})e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}.$$

Note that the map Φ is injective since $\Phi(T)=0$ implies that $T(e_{i_1},\dots,e_{i_r},e^{j_1},\dots,e^{j_s})=0$ for any $i_1,\dots,i_r,j_1,\dots,j_s\in\{1,\dots,m\}$, which in turn implies that $T=0$ by multi-linearity. Meanwhile, by a generalization of the above discussion for tensor products, the set

$$\{e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}\}$$

is a basis for $\mathrm{V}^{*\otimes r}\otimes_{\mathcal{K}}\mathrm{V}^{\otimes s}$, and for each basis element $e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}$ there exists a canonically defined multi-linear map sending a vector $(v_1,\dots,v_r,v_1'^*,\dots,v_s'^*)$ to

$$e^{i_1}(v_1)\cdots e^{i_r}(v_r)\cdot v_1'^*(e_{j_1})\cdots v_s'^*(e_{j_s}).$$

The map Φ sends the latter to the basis element $e^{i_1}\otimes\cdots\otimes e^{i_r}\otimes e_{j_1}\otimes\cdots\otimes e_{j_s}$. Therefore, Φ is surjective. □

In light of Theorem 2.2, we shall regard tensors as both multi-linear functionals on $\mathrm{V}^{\times r}\times\mathrm{V}^{*\times s}$ and elements of $\mathrm{V}^{*\otimes r}\otimes_{\mathcal{K}}\mathrm{V}^{\otimes s}$.

Let $\{e_1,\dots,e_m\}$ be a basis for V. Pick another basis $\{e_1',\dots,e_m'\}$, and let $\mathrm{B}^i{}_j$ be the change of base matrix, that is, $e_j'=\sum_{i=1}^m \mathrm{B}^i{}_j e_i$. Let $\mathrm{A}^i{}_j$ be the inverse of the matrix $\mathrm{B}^i{}_j$. Then

$$\begin{aligned}&e'^{i_1}\otimes\cdots\otimes e'^{i_r}\otimes e'_{j_1}\otimes\cdots\otimes e'_{j_s}\\ &\quad=\sum_{k_1,\dots,k_r,l_1,\dots,l_s\in\{1,\dots,n\}}\mathrm{A}^{k_1}{}_{i_1}\cdots\mathrm{A}^{k_r}{}_{i_r}\cdot\mathrm{B}^{l_1}{}_{j_1}\cdots\mathrm{B}^{l_s}{}_{j_q}e^{k_1}\otimes\cdots\otimes e^{k_s}\otimes e_{l_1}\otimes\cdots\otimes e_{l_s}\end{aligned}$$

Definition. The product of two tensors one of type (r_1,s_1) and the other of type (r_2,s_2) is a tensor of type (r_1+r_2,s_1+s_2). More precisely, if $T\in T^{r_1}{}_{s_1}(\mathrm{V})$ and $S\in T^{r_2}{}_{s_2}(\mathrm{V})$, then $T\otimes S\in T^{r_1+r_2}{}_{s_1+s_2}(\mathrm{V})$ is defined by

$$\begin{aligned}&(T\otimes S)(v_1,\dots,v_{r_1},v_{r_1+1},\dots,v_{r_1+r_2},v_1'^*,\dots,v_{s_1}'^*,v_{s_1+1}'^*,\dots,v_{s_1+s_2}'^*)\\ &\quad:=T(v_1,\dots,v_{r_1},v_1'^*,\dots,v_{s_1}'^*)\cdot S(v_{r_1+1},\dots,v_{r_1+r_2},v_{s_1+1}'^*,\dots,v_{s_1+s_2}'^*).\end{aligned}$$

Exercise. Show that the tensor product operation is bilinear, associative, and it is distributive over addition.

Denote by $T(\mathrm{V})$ the direct sum

$$\bigoplus_{rs\geq 0} T^r{}_s(\mathrm{V}).$$

An element of $T(\mathrm{V})$ is a finite formal sum. The infinite dimensional vector space $T(\mathrm{V})$ together with tensor product is an associative algebra, called the *tensor algebra*.

Definition. The *symmetrizing* and *alternating* maps on ${T^r}_0(\mathrm{V})$ are defined respectively as follows:

$$
\begin{aligned}
(\mathcal{S}T)(v_1,\ldots,v_r) &:= \frac{1}{r!}\sum_{\sigma\in \mathrm{S}_r} T(v_{\sigma(1)},\ldots,v_{\sigma(r)}),\\
(\mathcal{A}T)(v_1,\ldots,v_r) &:= \frac{1}{r!}\sum_{\sigma\in \mathrm{S}_r} sgn(\sigma)T(v_{\sigma(1)},\ldots,v_{\sigma(r)}),
\end{aligned}
$$

where S_r is the symmetric group on a r element set, and $sgn(\sigma)$ is the sign of $\sigma\in\mathrm{S}_r$.

Definition. A covariant tensor $T\in {T^r}_0(\mathrm{V})$ is called *symmetric* if

$$T(v_1,\ldots,v_r)=T(v_{\sigma(1)},\ldots,v_{\sigma(r)}).$$

for any $v_1,\ldots,v_r\in\mathrm{V}$ and $\sigma\in\mathrm{S}_r$. It is called *alternating* if

$$T(v_1,\ldots,v_r)=sgn(\sigma)T(v_{\sigma(1)},\ldots,v_{\sigma(r)}).$$

for any $v_1,\ldots,v_r\in\mathrm{V}$ and $\sigma\in\mathrm{S}_r$.

Remark. It follows from the definitions that $T\in {T^r}_0(\mathrm{V})$ is symmetric (resp. alternating) if and only if $\mathcal{S}T=T$ (resp. $\mathcal{A}T=T$).

The set $\Lambda^r(\mathrm{V})$ of alternating tensors form a subspace of ${T^r}_0(\mathrm{V})$

Definition. Given two alternating tensors $T\in\Lambda^{r_1}(\mathrm{V})$ and $S\in\Lambda^{r_2}(\mathrm{V})$, we define the *exterior* (or *wedge*) product of T and S by

$$T\wedge S:=\frac{(r_1+r_2)!}{r_1!\cdot r_2!}\mathcal{A}(T\otimes S).$$

Together with the exterior product, the direct sum

$$\Lambda(\mathrm{V}):=\bigoplus_{r\geq 0}\Lambda^r(\mathrm{V})$$

is an associative algebra, called the *exterior algebra.*

Exercise. To show associativity, note that if $T\in\Lambda^{r_1}(\mathrm{V})$, $S\in\Lambda^{r_2}(\mathrm{V})$, and $R\in\Lambda^{r_3}(\mathrm{V})$, then

$$T\wedge S\wedge R=\frac{(r_1+r_2+r_3)!}{r_1!\cdot r_2!\cdot r_3!}\mathcal{A}(T\otimes S\otimes R).$$

Theorem 2.3. *Given two alternating tensors $T\in\Lambda^{r_1}(\mathrm{V})$ and $S\in\Lambda^{r_2}(\mathrm{V})$, we have*

$$T\wedge S=(-1)^{r_1r_2}S\wedge T.$$

Proof. Consider the permutation τ sending $k\in\{1,\ldots,r_1+r_2\}$ to $k+r_2$ (mod r_1+r_2). This permutation can be written as r_2 power of the r_1+r_2-cycles sending $k\in\{1,\ldots,r_1+r_2\}$ to $k+1$ (mod r_1+r_2). Therefore, $sgn(\sigma)=(-1)^{r_1r_2+{r_2}^2-r_2}$. By definition,

$$T\wedge S(v_1,\ldots,v_{r_1+r_2})=(-1)^{r_1r_2}T\wedge S(v_{\tau(1)},\ldots,v_{\tau(r_1+r_2)}).$$

Meanwhile,

$$
\begin{aligned}
&T \wedge S(v_{\tau(1)}, \ldots, v_{\tau(r_1+r_2)}) \\
&\quad = \frac{1}{r_1! \cdot r_2!} \sum_{\sigma \in \mathrm{S}_{r_1+r_2}} sgn(\sigma) T(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(r_1)}) \cdot S(v_{\sigma\tau(r_1+1)}, \ldots, v_{\sigma\tau(r_1+r_2)}) \\
&\quad = \frac{1}{r_1! \cdot r_2!} \sum_{\sigma \in \mathrm{S}_{r_1+r_2}} sgn(\sigma) T(v_{\sigma(r_1+1)}, \ldots, v_{\sigma(r_1+r_2)}) \cdot S(v_{\sigma(1)}, \ldots, v_{\sigma(r_1)}) \\
&\quad = S \wedge T(v_1, \ldots, v_{r_1+r_2}).
\end{aligned}
$$

This completes the proof. □

One immediate outcome of this is that for $T \in \Lambda^r(\mathrm{V})$ where r is odd, $T \wedge T = 0$. As a result, $\Lambda^r(\mathrm{V})$ is empty for $r > m$.

Theorem 2.4. *Given a basis $\{e_1, \ldots, e_m\}$ for the vector space* V*, the set $\{e^{i_1} \wedge \cdots \wedge e^{i_r}\}$ where $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$ such that $i_1 < \cdots < i_r$ is a basis for $\Lambda^r(\mathrm{V})$. In particular, $\Lambda^r(\mathrm{V})$ has dimension*

$$\frac{m!}{r!(m-r)!}.$$

Proof. Since the set $\{e^{i_1} \otimes \cdots \otimes e^{i_r}\}$ where $i_1, \ldots, i_r \in \{1, \ldots, m\}$ is a basis for $T^r(\mathrm{V})$, and $\mathcal{A}(T^r(\mathrm{V})) = \Lambda^r(\mathrm{V})$, we conclude that $\{\mathcal{A}(e^{i_1} \otimes \cdots \otimes e^{i_r}) = e^{i_1} \wedge \cdots \wedge e^{i_r}\}$ spans $\Lambda^r(\mathrm{V})$. Next, we show that the set $\{e^{i_1} \wedge \cdots \wedge e^{i_r}\}$ is linearly independent. In order to see this, first note that $e^1 \wedge \cdots \wedge e^m \neq 0$. The latter follows from

$$
\begin{aligned}
e^1 \wedge \cdots \wedge e^m(e_1, \ldots, e_m) &= m!\, \mathcal{A}(e^1 \otimes \cdots \otimes e^m)(e_1, \ldots, e_m) \\
&= \sum_{\sigma \in \mathrm{S}_m} sgn(\sigma) e^1 \otimes \cdots \otimes e^m(e_{\sigma(1)}, \ldots, e_{\sigma(m)}) \\
&= 1.
\end{aligned}
$$

Now, we show that the set $\{e^{i_1} \wedge \cdots \wedge e^{i_r}\}$ is linearly dependent. Let

$$\sum_{1 \leq i_1 < \cdots < i_r \leq m} c_{i_1 \cdots i_r} e^{i_1} \wedge \cdots \wedge e^{i_r} = 0$$

but not all coefficients $c_{i_1 \cdots i_r} \in \mathcal{K}$ are zero. Let $\{j_1, \ldots, j_r\} \subset \{1, \ldots, m\}$ such that $c_{j_1 \cdots j_r} \neq 0$, and $\{k_1, \ldots, k_{m-r}\} = \{1, \ldots, m\} \smallsetminus \{j_1, \ldots, j_r\}$. Then

$$\pm c_{j_1 \cdots j_r} = \Big[\sum_{1 \leq i_1 < \cdots < i_r \leq m} c_{i_1 \cdots i_r} e^{i_1} \wedge \cdots \wedge e^{i_r} \Big] \wedge e^{k_1} \wedge \cdots \wedge e^{k_{m-r}}(e_1, \ldots, e_m) = 0,$$

which is a contradiction. Hence, there is no such linear combination. □

Exercise. Show that a collection of vectors $\{v_1, \ldots, v_r\} \subset \mathrm{V}$ is linearly dependent if and only if $v_1^* \wedge \cdots \wedge v_r^* = 0$.

Corollary 2.5. *For any $r > n$, the vector space $\Lambda^r(\mathrm{V})$ is trivial. Hence,*

$$\Lambda(\mathrm{V}) = \bigoplus_{r=0}^{m} \Lambda^r(\mathrm{V}),$$

and it has dimension 2^m.

Proof. This follows from Theorems 2.3 and 2.4. □

Let $H : \mathrm{V} \to \mathrm{W}$ be a linear map. Then for any $r > 0$, H induces a linear map $H^* : \Lambda^r(\mathrm{W}) \to \Lambda^r(\mathrm{V})$ defined by

$$(H^*T)(v_1, \ldots, v_r) := T(H(v_1), \ldots, H(v_r)),$$

where $v_1, \ldots, v_r \in \mathrm{V}$.

Theorem 2.6. *Given a linear map $H : V \to W$, $H^* : \Lambda(\mathrm{W}) \to \Lambda(\mathrm{V})$ commutes with the exterior product, i.e. for any $T \in \Lambda^{r_1}(\mathrm{W})$ and $S \in \Lambda^{r_2}(\mathrm{W})$*

$$H^*(T \wedge S) = H^*T \wedge H^*S.$$

Proof. Let $v_1, \ldots, v_r, v_{r_1+1}, \ldots, v_{r_1+r_2} \in \mathrm{V}$. Then

$$\begin{aligned}
&H^*(T \wedge S)(v_1, \ldots, v_{r_1}, v_{r_1+1}, \ldots, v_{r_1+r_2}) \\
&\quad = (T \wedge S)(H(v_1), \ldots, H(v_{r_1}), H(v_{r_1+1}), \ldots, H(v_{r_1+r_2})) \\
&\quad = \frac{1}{(r_1+r_2)!} \sum_{\sigma \in \mathrm{S}_{r_1+r_2}} T(H(v_{\sigma(1)}), \ldots, H(v_{\sigma(r_1)})) \cdot S(H(v_{\sigma(r_1+1)}), \ldots, H(v_{\sigma(r_1+r_2)})) \\
&\quad = \frac{1}{(r_1+r_2)!} \sum_{\sigma \in \mathrm{S}_{r_1+r_2}} (H^*T)(v_{\sigma(1)}, \ldots, v_{\sigma(r_1)}) \cdot (H^*S)(v_{\sigma(r_1+1)}, \ldots, v_{\sigma(r_1+r_2)}) \\
&\quad = (H^*T \wedge H^*S)(v_1, \ldots, v_{r_1}, v_{r_1+1}, \ldots, v_{r_1+r_2}).
\end{aligned}$$

□

Definition. Given $v \in \mathrm{V}$ and $T \in \Lambda^r(\mathrm{V})$, we define the *contraction* of T by v as the tensor $\iota_v T \in \Lambda^{r-1}(\mathrm{V})$ by

$$\iota_v T(v_1, \ldots, v_{r-1}) := T(v, v_1, \ldots, v_{r-1}).$$

Definition. For any $r, s > 0$, the *contraction operator*

$$c^i{}_j : T^r{}_s(\mathrm{V}) \to T^{r-1}{}_{s-1}(\mathrm{V})$$

is defined by

$$c^i{}_j(v_1^* \otimes \cdots \otimes v_r^* \otimes v_1' \otimes \cdots \otimes v_s') := v_i^*(v_j')v_1^* \otimes \cdots \otimes \hat{v}_i^* \otimes \cdots \otimes v_r^* \otimes v_1' \otimes \cdots \otimes \hat{v}_j' \otimes \cdots \otimes v_s',$$

where $v_1', \ldots, v_s' \in \mathrm{V}$ and $v_1^*, \ldots, v_r^* \in \mathrm{V}^*$ arbitrary.

Lemma 2.7. *Let $v \in \mathrm{V}$, $T \in \Lambda^{r_1}(\mathrm{V})$, and $S \in \Lambda^{r_2}(\mathrm{V})$. Then*

$$\iota_v(T \wedge S) = (\iota_v T) \wedge S + (-1)^{r_1} T \wedge (\iota_v S).$$

Homework-1
Due 2/20/14

Please work these problems out on your own, and if you consult a book or any other reference, please include it in your solution. Feel free to email me or visit me in my office if you need clarification.

(1) Show that the tangent bundle of a smooth manifold M is a smooth orientable manifold.

(2) Similar to Example 2, give a description of the tangent bundle of the 3-sphere explicit enough to deduce that S^3 is parallelizable. (Hint: Consider S^3 as embedded in $\mathbb{R}^4$ with coordinates $x^1, \dots, x^4$, and note that the restriction of the vector field

$$x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^3}$$

to S^3 is nowhere vanishing.)

In fact, all closed orientable 3-manifolds are parallelizable. Specific to the case of spheres, both S^1 and S^3 admit the structure of a Lie group; therefore, they are parallelizable. Although these are the only spheres with a Lie group structure, there is another sphere, namely S^7, which is also parallelizable. Bott and Milnor, and independently Kervaire, proved that S^1, S^3, and S^7 are the only parallelizable spheres, a result which requires sophisticated tools from algebraic topology.

(3) Let M be a smooth m-dimensional manifold, $p \in M$, and $f_1, \dots, f_k \in C^\infty(p)$ where $k \le m$. Show that one can find a coordinate neighborhood (U, φ) around p for which $f_1, \dots, f_k$ are some of the coordinate functions if and only if $df_{1p}, \dots, df_{kp}$ are linearly independent in $T_p^*(M)$.

(4) Let V be a $2n$-dimensional vector space over $\mathbb{R}$, and $\Omega \in \Lambda^2(\mathrm{V})$ be such that $\underbrace{\Omega \wedge \cdots \wedge \Omega}_{n-times} \neq 0$.

(a) Show that for any non-zero $v \in \mathrm{V}$ there exists $w \in \mathrm{V}$ such that $\Omega(v, w) = 1$.

(b) Show that V^* admits a basis $\{e^1, \dots, e^n, f^1, \dots, f^n\}$ such that

$$\Omega = \sum_{i=1}^{n} e^i \wedge f^i.$$

(Hint: Use part (a), and the definition that for a subspace $\mathrm{W} \subset \mathrm{V}$

$$\mathrm{W}^\Omega := \{v \in \mathrm{V} \mid \Omega(v, w) = 0 \ \forall w \in \mathrm{W}\}.)$$

An alternating tensor such as Ω is called a symplectic form on V.

(5) Prove Lemma 2.7.

3. February 10–February 14

Tensor Bundles and Tensor Fields

Let M be smooth m-dimensional manifold, and $p \in M$. Denote by ${T^r}_s(p)$ the vector space of type (r,s) tensors on T_pM. Then,

$$ {T^r}_s(M) := \bigsqcup_{p \in M} {T^r}_s(p) $$

is called the (r,s)-*tensor bundle* on M. The topology on ${T^r}_s(M)$ is described similarly to the topology on the tangent/cotangent bundle of M. To be more precise, let $\pi : {T^r}_s(M) \to M$ be the projection map, and $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas on M. For each coordinate neighborhood $(U_\alpha, \varphi_\alpha)$ with coordinate functions $x_\alpha^1, \dots, x_\alpha^m$, define local trivializations

$$ \tilde{U}_\alpha := \pi^{-1}(U_\alpha), $$
$$ \Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times [\underbrace{(\mathbb{R}^m)^{*\otimes r} \otimes (\mathbb{R}^m)^{\otimes s}}_{\cong \mathbb{R}^{m^{r+s}}}], $$

by

$$ \begin{aligned} &\Psi_\alpha\Big(\sum_{i_1,\dots,i_r,j_1,\dots,j_s \in \{1,\dots,m\}} {c^{j_1\cdots j_s}}_{i_1\cdots i_r} d(x_\alpha^{i_1})_p \otimes \cdots \otimes d(x_\alpha^{i_r})_p \otimes \frac{\partial}{\partial x_\alpha^{j_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial x_\alpha^{j_s}}|_p \Big) \\ &= \Big(p, \sum_{i_1,\dots,i_r,j_1,\dots,j_s \in \{1,\dots,m\}} {c^{j_1\cdots j_s}}_{i_1\cdots i_r} e^{i_1} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes \cdots \otimes e_{j_s}\Big), \end{aligned} $$

where $\{e_1, \dots, e_m\}$ is the standard basis for $\mathbb{R}^m$ and $\{e^1, \dots, e^m\}$ is its dual. Then the topology of ${T^r}_s(M)$ is generated by pre-images under $\tilde{\phi}_\alpha$ of open subsets of $U_\alpha \times \mathbb{R}^{m^{r+s}}$, endowed with the product topology. With this topology and the atlas $\{(\tilde{U}_\alpha, (\varphi_\alpha \times id) \circ \Psi_\alpha)\}$, the tensor bundle ${T^k}_l(M)$ becomes a smooth manifold of dimension $m + m^{r+s}$, and π is a smooth map. In particular, the (0,1)-tensor bundle is the tangent bundle of M, while the (1,0)-tensor bundle is the cotangent bundle of M. Similarly, we define the *degree-k exterior bundle* of M as

$$ \Lambda^k(M) := \bigsqcup_{p \in M} \Lambda^k(T_p^*M), $$

endowed with the topology induced from ${T^k}_0(M)$, and the projection $\pi : \Lambda^k(M) \to M$.

Note that whenever $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$ \Psi_\alpha \circ \Psi_\beta^{-1}(p, T) = (p, g_{\alpha\beta}(p)T), $$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(m^{r+s}, \mathbb{R})$ is smooth such that

- $g_{\alpha\alpha}(p) = id$ for any $p \in U_\alpha$,
- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha}(p) = id$ for any $p \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

The smooth maps $g_{\alpha\beta}$ are called the *transition functions* for the tensor bundle ${T^r}_s(M)$. The data provided by the transition functions is enough to construct the tensor bundle.

Definition. A smooth (r,s)-*tensor field* on M is a smooth section of $\pi : T^r{}_s(M) \to M$. A smooth section of the exterior bundle $\pi : \Lambda^k(M) \to M$ is called a *degree-k differential form*, or simply a *k-form*. Tensor product, exterior product, and contraction operations can be carried over to smooth tensor fields via pointwise definitions.

In a given coordinate neighborhood $(U_\alpha, \varphi_\alpha)$ with coordinate functions $x_\alpha^1, \dots, x_\alpha^m$, a (r,s)-tensor field T can be written as

$$T|_{U_\alpha} = \sum_{i_1,\dots,i_r,j_1,\dots,j_l \in \{1,\dots,m\}} T^{j_1\cdots j_s}{}_{i_1\cdots i_r} dx_\alpha^{i_1} \otimes \cdots \otimes dx_\alpha^{i_r} \otimes \frac{\partial}{\partial x_\alpha^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_\alpha^{j_s}},$$

where $T^{j_1\cdots j_s}{}_{i_1\cdots i_r} \in C^\infty(U_\alpha)$. If (U_β, φ_β) is another coordinate neighborhood with coordinate functions $x_\beta^1, \dots, x_\beta^m$ such that $U_\alpha \cap U_\beta \neq \emptyset$, and

$$T|_{U_\beta} = \sum_{k_1,\dots,k_r,\ell_1,\dots,\ell_s \in \{1,\dots,m\}} T^{\ell_1\cdots \ell_s}{}_{k_1\cdots k_r} dx_\beta^{k_1} \otimes \cdots \otimes dx_\beta^{k_r} \otimes \frac{\partial}{\partial x_\beta^{\ell_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_\beta^{\ell_s}},$$

where $T^{\ell_1\cdots \ell_s}{}_{k_1\cdots k_r} \in C^\infty(U_\beta)$, then

$$T^{\ell_1\cdots \ell_s}{}_{k_1\cdots k_r} = \sum_{i_1,\dots,i_r,j_1,\dots,j_s \in \{1,\dots,m\}} (T^{j_1\cdots j_s}{}_{i_1\cdots i_r} \circ \varphi_\alpha^{-1} \circ \varphi_\beta) \cdot \frac{\partial x_\alpha^{i_1}}{\partial x_\beta^{k_1}} \cdots \frac{\partial x_\alpha^{i_r}}{\partial x_\beta^{k_r}} \cdot \frac{\partial x_\beta^{\ell_1}}{\partial x_\alpha^{j_1}} \cdots \frac{\partial x_\beta^{\ell_s}}{\partial x_\alpha^{j_s}}.$$

Given a smooth vector field X on M, we can differentiate smooth tensor fields along X, generalizing the notion of directional derivative of a smooth function along a smooth vector field:

Theorem 3.1. *Let $X \in \mathfrak{X}(M)$, and $p \in M$. There exists an open neighborhood U of p, $\epsilon > 0$, and a unique smooth map*

$$\rho : U \times (-\epsilon, \epsilon) \to M,$$

such that for any $q \in U$, and $t \in (-\epsilon, \epsilon)$,

$$d\rho(\frac{d}{dt}|_{(q,t)}) = X_{\rho(q,t)},$$

with $\rho(q, 0) = q$.

Proof. This is a local statement, and the proof is an application of the standard existence and uniqueness result on initial value problems for ordinary differential equations. For more details, see [2, IV-Theorem 4.2]. □

Definition. Given a smooth vector field X on M and a point $p \in M$, the smooth parametrized curve $\rho(p, \cdot) : (-\epsilon, \epsilon) \to M$ is called an *integral curve* of X through p. The collection of smooth maps $\{\rho(\cdot, t)\}$ form a *local one-parameter group of diffeomorphisms*. (In particular, $\rho(\cdot, -t) = \rho(\cdot, t)^{-1}$.)

Now let X and Y be smooth vector fields on M, and ω be a k-form on M.

Definition. The *Lie derivatives* of f, Y, and ω along X are respectively defined by

$$\begin{aligned}(\mathcal{L}_X Y)_p &:= \left.\frac{d}{dt}\right|_{t=0} (d\rho(\cdot,-t)(Y))_p,\\ (\mathcal{L}_X \omega)_p &:= \left.\frac{d}{dt}\right|_{t=0} (\rho(\cdot,t)^*\omega)_p.\end{aligned}$$

Note that if $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$, then

$$(\mathcal{L}_X f)(p) = \left.\frac{d}{dt}\right|_{t=0} [(\rho(\cdot,t)^* f)(p)] = \left.\frac{d}{dt}\right|_{t=0} [f \circ \rho(p,t)] \,| = Xf(p).$$

More generally, we can define the Lie derivative of a smooth tensor field along a smooth vector field on M in such a way that it is linear, and it satisfies the Leibniz rule with respect to tensor product, i.e. if T and S are smooth tensor fields on M, then

$$\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S).$$

For this, define the Lie derivative of a smooth tensor field T by a smooth vector field X by

$$(\mathcal{L}_X T)_p := \left.\frac{d}{dt}\right|_{t=0} (\rho(\cdot,t)^* T)_p,$$

where it is understood that $\rho(\cdot,t)^*$ acts on the covariant part of the tensor field as $\rho(\cdot,t)^*$, and it acts on the contravariant part of the tensor field as $\rho(\cdot,-t)_*$.

Theorem 3.2. *The contraction operator commutes with the Lie derivative, i.e. if T is a smooth tensor field on M, and $X \in \mathfrak{X}(M)$, then*

$$\mathcal{L}_X({c^i}_j T) = {c^i}_j(\mathcal{L}_X T).$$

Proof. Let $p \in M$. Then

$$\begin{aligned}(\mathcal{L}_X({c^i}_j T))_p &= \left.\frac{d}{dt}\right|_{t=0} [\rho(\cdot,t)^*({C^i}_j T)]_p\\ &= \left.\frac{d}{dt}\right|_{t=0} [{c^i}_j(\rho(\cdot,t)^* T)]_p\\ &= {c^i}_j \left.\frac{d}{dt}\right|_{t=0} [\rho(\cdot,t)^* T]_p\\ &= ({c^i}_j \mathcal{L}_X T)_p\end{aligned}$$

□

Lemma 3.3. *Let $X, Y \in \mathfrak{X}(M)$, and ω be a differential form. Then*

$$\mathcal{L}_X(\iota_Y \omega) - \iota_Y \mathcal{L}_X \omega = \iota_{\mathcal{L}_X Y}\omega.$$

Proof. Consider the tensor $\omega \otimes Y$. Note that $\iota_Y \omega = {c^1}_1[\omega \otimes Y]$, and hence

$$\mathcal{L}_X(\iota_Y \omega) = \mathcal{L}_X({c^1}_1[\omega \otimes Y]) = {c^1}_1 \mathcal{L}_X(\omega \otimes Y),$$

by Lemma 3.2. Meanwhile, by Leibniz rule,

$$c^1{}_1\mathcal{L}_X(\omega \otimes Y) = c^1{}_1[\mathcal{L}_X(\omega) \otimes Y + \omega \otimes \mathcal{L}_X Y] = \iota_Y \mathcal{L}_X \omega + \iota_{\mathcal{L}_X Y}\omega.$$

This completes the proof. □

Lemma 3.4. *Let T be a smooth (r,s)-tensor field on M, and $X \in \mathfrak{X}(M)$. Then for any $X_1, \dots, X_r \in \mathfrak{X}(M)$ and any 1-forms $\omega_1, \dots, \omega_s$ on M*

$$\begin{aligned} \mathcal{L}_X[T(X_1,\dots,X_r,\omega_1,\dots,\omega_s)] &= (\mathcal{L}_X T)(X_1,\dots,X_r,\omega_1,\dots,\omega_s) \\ &\quad + \sum_{i=1}^{r} T(X_1,\dots,\mathcal{L}_X X_i,\dots,X_r,\omega_1,\dots,\omega_s) \\ &\quad + \sum_{j=1}^{s} T(X_1,\dots,X_r,\omega_1,\dots,\mathcal{L}_X\omega_j,\dots,\omega_s). \end{aligned}$$

Proof. This follows by applying Theorem 3.2, together with the Leibniz rule, to

$$\underbrace{c^1{}_1 \cdots c^1{}_1}_{(r+s)-times} T \otimes \omega_1 \otimes \cdots \otimes \omega_s \otimes X_1 \otimes \cdots \otimes X_r.$$

□

Definition. Let X and Y be smooth vector fields on M. Define their *Lie bracket* $[X,Y]$ pointwise by

$$[X,Y]_p[f] = X_p[Yf] - Y_p[Xf],$$

for any $f \in C^\infty(p)$.

Theorem 3.5. *For any $X, Y \in \mathfrak{X}(M)$, we have*

$$\mathcal{L}_X Y = [X,Y].$$

Proof. Let $p \in M$ and $f \in C^\infty(p)$. Then

$$\begin{aligned} (\mathcal{L}_X Y)_p[f] &= \left.\frac{d}{dt}\right|_{t=0} (d\rho(\cdot,-t)(Y))f(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\rho(\cdot,t)^* Y(f \circ \rho(\cdot,-t)))(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\rho(\cdot,t)^* Yf)(p) + Y_p[\left.\frac{d}{dt}\right|_{t=0} f \circ \rho(\cdot,-t)] \\ &= \left.\frac{d}{dt}\right|_{t=0} Yf(\rho(p,t)) + Y_p[-Xf] \\ &= X_p[Yf] - Y_p[Xf]. \end{aligned}$$

□

Next, we show that the space of smooth vector fields on M together with the Lie bracket has the structure of a Lie algebra:

Theorem 3.6. *Let $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Then*

(1) $[X, Y] = -[Y, X]$,
(2) $[X + Y, Z] = [X, Z] + [Y, Z]$,
(3) $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$,
(4) ***(Jacobi's identity)*** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Proof. The proof simply uses the definition of Lie bracket. To prove (4), let $f \in C^\infty(M)$ and write

$$[X, [Y, Z]]f = XY(Zf) - XZ(Yf) - YZ(Xf) + ZY(Xf).$$

After cyclically permuting, we get similar equations for $[Y, [Z, X]]f$ and $[Z, [X, Y]]f$. Add the three equations side-by-side and observe the cancellations. □

Proposition 3.7. *Let $X, Y \in \mathfrak{X}(M)$, and ω be a k-form on M. Then*

$$\mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega) = \mathcal{L}_{[X,Y]}\omega.$$

Proof. The proof is by induction on k. In this regard, note first that the above equation is satisfied for any $f \in C^\infty(M)$, namely,

$$\mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f) = X(Yf) - Y(Xf) = [X, Y]f = \mathcal{L}_{[X,Y]}\omega.$$

Suppose that the equation holds for all n-forms where $n < k$. Let $\omega \in \Omega^k(M)$, and $Z \in \mathfrak{X}(M)$ be arbitrary. Then, by assumption,

$$\mathcal{L}_{[X,Y]}(\iota_Z\omega) = \mathcal{L}_X(\mathcal{L}_Y\iota_Z\omega) - \mathcal{L}_Y(\mathcal{L}_X\iota_Z\omega).$$

Meanwhile,

$$\mathcal{L}_{[X,Y]}(\iota_Z\omega) = \iota_Z\mathcal{L}_{[X,Y]}\omega + \iota_{[[X,Y],Z]}\omega,$$

and

$$\begin{aligned} \mathcal{L}_X(\mathcal{L}_Y\iota_Z\omega) &= \mathcal{L}_X(\iota_Z\mathcal{L}_Y\omega + \iota_{Y,Z}\omega) \\ &= \iota_Z\mathcal{L}_X(\mathcal{L}_Y\omega) + \iota_{[X,Z]}\mathcal{L}_Y\omega + \iota_{[Y,Z]}\mathcal{L}_X\omega + \iota_{[X,[Y,Z]]}\omega, \end{aligned}$$

by Lemma 3.3. Similarly,

$$\begin{aligned} \mathcal{L}_y(\mathcal{L}_X\iota_Z\omega) &= \mathcal{L}_Y(\iota_Z\mathcal{L}_X\omega + \iota_{X,Z}\omega) \\ &= \iota_Z\mathcal{L}_Y(\mathcal{L}_X\omega) + \iota_{[Y,Z]}\mathcal{L}_X\omega + \iota_{[X,Z]}\mathcal{L}_Y\omega + \iota_{[Y,[X,Z]]}\omega. \end{aligned}$$

Therefore,

$$\mathcal{L}_X(\mathcal{L}_Y\iota_Z\omega) - \mathcal{L}_Y(\mathcal{L}_X\iota_Z\omega) = \iota_Z[\mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega)] + \iota_{[X,[Y,Z]]}\omega + \iota_{[Y,[Z,X]]}\omega.$$

Since by Jacobi's identity,

$$\iota_{[[X,Y],Z]}\omega = \iota_{[X,[Y,Z]]}\omega + \iota_{[Y,[Z,X]]}\omega,$$

we have

$$\iota_Z[\mathcal{L}_X(\mathcal{L}_Y\omega) - \mathcal{L}_Y(\mathcal{L}_X\omega)] = \iota_Z\mathcal{L}_{[X,Y]}\omega$$

for any $Z \in \mathfrak{X}(M)$. This completes the proof. $\square$

Proposition 3.8. *Let $X, Y_1, \ldots, Y_k \in \mathfrak{X}(M)$, and ω be a k-form on M. Then*

$$(\mathcal{L}_X\omega)(Y_1, \ldots, Y_k) = \mathcal{L}_X[\omega(Y_1, \ldots, Y_k)] - \omega([X, Y_1], Y_2, \ldots, Y_k) - \cdots - \omega(Y_1, \ldots, Y_{k-1}, [X, Y_k]).$$

Proof. This follows from Lemma 3.4. $\square$

Next we focus on the exterior bundle. Denote by $\Omega^k(M)$ the vector space of all k-forms on M. Then the exterior product endows

$$\Omega(M) := \bigoplus_{k=0}^{m} \Omega^k(M),$$

with the structure of an algebra, called the *exterior algebra*. In fact, $\Omega(M)$ is a *graded* algebra where the grading is the degree of differential forms. Furthermore, if ω_1 and ω_2 are differential forms on M, and $X \in \mathfrak{X}(M)$, then

$$\mathcal{L}_X(\omega_1 \wedge \omega_2) = (\mathcal{L}_X\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X\omega_2.$$

Hence, Lie derivative along a smooth vector field on M is a derivation on $\Omega(M)$. This follows from the fact that Lie derivative satisfies the Leibniz rule with respect to tensor product, and it commutes with the alternating operator.

Theorem 3.9. *Let M be a smooth m-dimensional manifold. Then there exists a unique real linear map $d : \Omega(M) \to \Omega(M)$ such that*

(1) *If $f \in C^\infty(M) = \Omega^0(M)$, then df is the differential of f.*
(2) *If $\omega_1 \in \Omega^k(M)$ and $\omega_2 \in \Omega^\ell(M)$, then*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

(3) $d \circ d = 0$.

This linear operator is called the **exterior derivative**.

Proof. We prove the theorem by showing that exterior derivative is a local operator. In other words, for any k-form ω, $d\omega$ is uniquely defined by its restriction on coordinate neighborhoods.

Step 1: Let $\{(U_\alpha, \varphi_\alpha)\}$ be a maximal smooth atlas on M, and fix a coordinate neighborhood $(U_\alpha, \varphi_\alpha)$ with coordinate functions $x_\alpha^1, \ldots, x_\alpha^m$. The restriction of any k-form ω on M to U_α can be written as

$$\omega|_{U_\alpha} = \sum_{1 \le i_1 < \cdots < i_k \le m} f_{i_1 \cdots i_k} dx_\alpha^{i_1} \wedge \cdots \wedge dx_\alpha^{i_k},$$

where $f_{i_1 \cdots i_k} \in C^\infty(U_\alpha)$. Define

$$d_{U_\alpha}\, \omega|_{U_\alpha} := \sum_{1 \le i_1 < \cdots < i_k \le m} df_{i_1 \cdots i_k} \wedge dx_\alpha^{i_1} \wedge \cdots \wedge dx_\alpha^{i_k}.$$

where

$$df_{i_1\cdots i_k} = \sum_{j=1}^{m} \frac{\partial f_{i_1\cdots i_k}}{\partial x_\alpha^j} dx_\alpha^j.$$

From this, properties (1) and (3) follow immediately. In order to prove property (2), let

$$\begin{aligned} \omega_1|_{U_\alpha} &= \sum_{1\le i_1<\cdots<i_k\le m} f^1_{i_1\cdots i_k} dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}, \\ \omega_2|_{U_\alpha} &= \sum_{1\le j_1<\cdots<j_l\le m} f^2_{j_1\cdots j_l} dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l}. \end{aligned}$$

Then

$$\begin{aligned}
& d_{U_\alpha}\ (\omega_1\wedge\omega_2)|_{U_\alpha} \\
= \ & d_{U_\alpha}\left(\Big(\sum_{1\le i_1<\cdots<i_k\le m} f^1_{i_1\cdots i_k} dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\Big)\wedge\Big(\sum_{1\le j_1<\cdots<j_l\le m} f^2_{j_1\cdots j_l} dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l}\Big)\right) \\
= \ & d_{U_\alpha}\left(\sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} f^1_{i_1\cdots i_k} f^2_{j_1\cdots j_l} dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l}\right) \\
= \ & \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} d(f^1_{i_1\cdots i_k} f^2_{j_1\cdots j_l}) dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l} \\
= \ & \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} (f^2_{j_1\cdots j_l} df^1_{i_1\cdots i_k} + f^1_{j_1\cdots j_l} df^2_{i_1\cdots i_k}) dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l} \\
= \ & \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} f^2_{j_1\cdots j_l} df^1_{i_1\cdots i_k}\wedge dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l} \\
& + \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} f^1_{i_1\cdots i_k} df^2_{j_1\cdots j_l}\wedge dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l} \\
= \ & \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} (df^1_{i_1\cdots i_k}\wedge dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k})\wedge(f^2_{j_1\cdots j_l}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l}) \\
& + \sum_{\substack{1\le i_1<\cdots<i_k\le m\\ 1\le j_1<\cdots<j_l\le m}} (-1)^k (f^1_{i_1\cdots i_k} dx_\alpha^{i_1}\wedge\cdots\wedge dx_\alpha^{i_k})\wedge(df^2_{j_1\cdots j_l}\wedge dx_\alpha^{j_1}\wedge\cdots\wedge dx_\alpha^{j_l}) \\
= \ & d_{U_\alpha}(\omega_1|_{U_\alpha})\wedge\omega_2|_{U_\alpha} + (-1)^k\ \omega_1|_{U_\alpha}\wedge d_{U_\alpha}(\omega_2|_{U_\alpha}).
\end{aligned}$$

Step 2: Now, if $d_M : \Omega(M)\to\Omega(M)$ is a linear operator with the properties listed in the theorem, then we want to show that $(d_M\omega)|_{U_\alpha} = d_{U_\alpha}\omega|_{U_\alpha}$ for any U_α. Having fixed $p\in U_\alpha$, there exists a compact subset $K\subset U_\alpha$ and an open subset $V\subset \mathring{K}$ containing p, and smooth functions $h, g\in C^\infty(M)$ such that $h(q)=1$ at any $q\in\mathring{K}$ and $h(q)=0$ for any $q\notin U_\alpha$, while $g(p)=1$ and $g(q)=0$ for any $q\notin V$. Denote by $\tilde\omega$ the k-form $h\omega$ on M.

Note that $\tilde{\omega}|_V = \omega|_V$. Write

$$\omega|_{U_\alpha} = \sum_{1\le i_1<\cdots<i_k\le m} f_{i_1\cdots i_k} dx_\alpha^{i_1} \wedge \cdots \wedge dx_\alpha^{i_k},$$

and then

$$\tilde{\omega} = \sum_{1\le i_1<\cdots<i_k\le m} h f_{i_1\cdots i_k} dx_\alpha^{i_1} \wedge \cdots \wedge dx_\alpha^{i_k}.$$

Since $g(\omega - \tilde{\omega}) = 0$ and d_M is a linear operator, $d_M(g\omega - g\tilde{\omega}) = 0$. But then

$$dg \wedge (\omega - \tilde{\omega}) + g\, d_M(\omega - \tilde{\omega}),$$

by properties of the operator d_M. Since dg can only be non-zero in V, where $\tilde{\omega} = \omega$, we have

$$g\, d_M\omega = g\, d_M\tilde{\omega}.$$

Meanwhile,

$$\begin{aligned}(g\, d_M\tilde{\omega})_p &= \sum_{1\le i_1<\cdots<i_k\le m} df_{i_1\cdots i_k p} \wedge dx_{\alpha p}^{i_1} \wedge \cdots \wedge dx_{\alpha p}^{i_k} \\ &= (d_{U_\alpha}\omega|_{U_\alpha})_p.\end{aligned}$$

Since the above equality is true for any $p \in U_\alpha$, we have the desired result.

Finally, the uniqueness assertion follows from repeating the above argument to deduce that for two coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) such that $U = U_\alpha \cap U_\beta \neq \emptyset$, we have

$$(d_{U_\alpha}\omega|_{U_\alpha})|_U = d_U\omega|_U = (d_{U_\beta}\omega|_{U_\beta})|_U.$$

□

4. February 17–February 21

Proposition 4.1. *For $\omega \in \Omega^{k-1}(M)$, and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, we have*

$$\begin{aligned}(d\omega)(X_1, \ldots, X_k) &= \sum_{i=1}^{k} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned}$$

Proof. It suffices to check this formula locally. In this regard, let (U, φ) be a coordinate neighborhood with coordinate functions $x^1, \ldots, x^m$. Then any $X \in \mathfrak{X}(U_\alpha)$ can be written as a $C^\infty(U)$-linear combination of the smooth vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$. Meanwhile,

$$\omega|_{U_\alpha} = \sum_{1\le i_1<\cdots<i_{k-1}\le m} f_{i_1\cdots i_{k-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

where $f_{i_1\cdots i_{k-1}} \in C^\infty(U)$. Since differential forms are multi-linear on $\mathfrak{X}(M)$ regarded as a $C^\infty(M)$-module, it is enough to consider the case where $X_1, \ldots, X_k \in \mathfrak{X}(U)$ are chosen among $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$. Fix $\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}$. Then consider

$$\begin{aligned}(d\omega|_{U_\alpha})(\frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_k}}) &= \sum_{1\le i_1<\cdots<i_k\le m} df_{i_1\cdots i_{k-1}}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}}(\frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_k}})\\ &= \sum_{1\le i_1<\cdots<i_{k-1}\le m}(-1)^{s+1}df_{i_1\cdots i_{k-1}}(\frac{\partial}{\partial x^{j_s}})\cdot\delta^{i_1}_{j_1}\cdots\delta^{i_{s-1}}_{j_{s-1}}\cdot\delta^{i_s}_{j_{s+1}}\cdots\delta^{i_{k-1}}_{j_k}\\ &= \sum_{s=1}^k(-1)^{s+1}\frac{\partial f_{j_1\cdots j_{s-1}j_{s+1}\cdots j_k}}{\partial x^{j_s}}.\end{aligned}$$

Meanwhile, the right-hand side of the formula is equal to

$$\sum_{s=1}^k(-1)^{s+1}\frac{\partial}{\partial x^{j_s}}\omega(\frac{\partial}{\partial x^{j_1}},\ldots,\frac{\partial}{\partial x^{j_{s-1}}},\frac{\partial}{\partial x^{j_{s+1}}}\ldots,\frac{\partial}{\partial x^{j_k}})=\sum_{s=1}^k(-1)^{s+1}\frac{\partial f_{j_1\cdots j_{s-1}j_{s+1}\cdots j_k}}{\partial x^{j_s}},$$

since $[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}]=0$ for all $i,j\in\{1,\ldots,m\}$. This completes the proof. □

Corollary 4.2. *For any $\omega\in\Omega^1(M)$ and $X,Y\in\mathfrak{X}(M)$,*

$$d\omega(X,Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y]).$$

Theorem 4.3. *Let $F:M\to N$ be a smooth map, and $\omega\in\Omega^k(N)$. Then*

$$F^*d_N\omega=d_MF^*\omega.$$

Proof. It suffices to check this property locally. In this regard, let (U,φ) and (V,ϕ) be coordinate neighborhoods on M with coordinate functions $x^1,\ldots,x^m$ and $y^1,\ldots,y^n$, respectively, such that $F(U)\subset V$. Now consider a k-form ω on N whose restriction to V can be written as

$$\omega|_V=\sum_{1\le i_1<\cdots<i_k\le n}f_{i_1\cdots i_k}dy^{i_1}\wedge\cdots\wedge dy^{i_k},$$

where $f_{i_1\cdots i_k}\in C^\infty(V)$. Then

$$d_N\omega|_V=\sum_{1\le i_1<\cdots<i_k\le n}df_{i_1\cdots i_k}\wedge dy^{i_1}\wedge\cdots\wedge dy^{i_k},$$

and

$$\begin{aligned}F^*(d_N\omega|_V) &= \sum_{1\le i_1<\cdots<i_k\le n}(F^*df_{i_1\cdots i_k})\wedge(F^*dy^{i_1})\wedge\cdots\wedge(F^*dy^{i_k})\\ &= \sum_{1\le i_1<\cdots<i_k\le n}d(f_{i_1\cdots i_k}\circ F)\wedge d(y^{i_1}\circ F)\wedge\cdots\wedge d(y^{i_k}\circ F)\\ &= d_M\sum_{1\le i_1<\cdots<i_k\le n}(f_{i_1\cdots i_k}\circ F)\wedge d(y^{i_1}\circ F)\wedge\cdots\wedge d(y^{i_k}\circ F)\\ &= d_MF^*\sum_{1\le i_1<\cdots<i_k\le n}f_{i_1\cdots i_k}dy^{i_1}\wedge\cdots\wedge dy^{i_k}=d_M(F^*\omega)|_U.\end{aligned}$$

□

Theorem 4.4 (Cartan Formula)**.** *Let $\omega \in \Omega^k(M)$, and $X \in \mathfrak{X}(M)$. Then*

$$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega.$$

Proof. To prove the claim, let $X_1, \ldots, X_k \in \mathfrak{X}(M)$. Then by Proposition 4.1

$$\begin{aligned}(d\iota_X\omega)(X_1,\ldots,X_k) &= \sum_{i=1}^k (-1)^{i+1} X_i(\omega(X, X_1, \ldots, \hat{X}_i, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega(X, [X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned}$$

Meanwhile,

$$\begin{aligned}(\iota_X d\omega)(X_1,\ldots,X_k) &= d\omega(X, X_1, \ldots, X_k) \\ &= X\omega(X_1,\ldots,X_k) + \sum_{j=1}^k (-1)^j \omega([X, X_j], X_1, \ldots, \hat{X}_j, \ldots, X_k) \\ &\quad + \sum_{i=1}^k (-1)^i X_i(\omega(X, X_1, \ldots, \hat{X}_i, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned}$$

Adding the right-hand sides of the equations for $d\iota_X\omega$ and $\iota_X d\omega$, we see that

$$(d\iota_X\omega + \iota_X d\omega)(X_1,\ldots,X_k) = X\omega(X_1,\ldots,X_k) + \sum_{j=1}^k (-1)^j \omega([X, X_j], X_1, \ldots, \hat{X}_j, \ldots, X_k),$$

which is precisely $(\mathcal{L}_X\omega)(X_1,\ldots,X_k)$ by Lemma 3.4. □

Corollary 4.5. *Lie derivative commutes with the exterior derivative.*

Proof. This follows at once from Theorem 4.4. □

Definition. A smooth *k-dimensional distribution* Δ on M is a smooth assignment of a k-dimensional subspace $\Delta_p \subset T_pM$ to every point $p \in M$ such that it is locally spanned by k linearly independent smooth vector fields $X_1, \ldots, X_k$. A smooth k-dimensional distribution Δ on M is called *involutive* if it is closed under Lie bracket. It is called *completely integrable* if every point $p \in M$ has a coordinate neighborhood (U, φ) with coordinate functions $x^1, \ldots, x^m$ such that $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\}$ is a local basis for Δ. Given a smooth k-dimensional distribution Δ on M, an *integral submanifold* N_p through a point $p \in M$ is a smooth manifold such that $T_qN_p = \Delta_q$ at every point $q \in N_p$.

Remark. Given a completely integrable smooth k-dimensional distribution on M, integral submanifolds through every point in M exist.

Theorem 4.6 (Frobenius Theorem)**.** *A smooth k-dimensional distribution Δ on M is completely integrable if and only if it is involutive.*

Proof. See [2, IV-Theorem 8.3] or [4, Theorem 4.4] for a proof. □

Alternatively, we may regard a smooth k-dimensional distribution locally as the kernel of $m-k$ linearly independent 1-forms on M. Then we can rephrase the condition for a smooth k-dimensional distribution to be involutive as follows:

Theorem 4.7. *Let Δ be a smooth k-dimensional distribution on M. Then Δ is involutive if and only if every point $p \in M$ has an open neighborhood U on which there exist $m-k$ linearly independent 1-forms $\eta^{k+1}, \dots, \eta^m$ vanishing on Δ and satisfying*

$$d\eta^j = \sum_{i=k+1}^{m} \theta_i^j \wedge \eta^i,$$

for some 1-forms θ_i^j on U.

Proof. Given $p \in M$, there exists an open neighborhood U of p such that a local basis $X_1, \dots, X_k$ for Δ can be completed to a local basis $X_1, \dots, X_k, \dots, X_n$ of TM over U. Let $\eta^1, \dots, \eta^k, \dots, \eta^m$ be the dual basis of covector field on U, and write

$$[X_i, X_j] = \sum_{s=1}^{m} c^s{}_{ij} X_s,$$

where $c^s{}_{ij} \in C^\infty(U)$. Then Δ is involutive if and only if $c^s{}_{ij} = 0$ for $1 \le i < j \le k$ and $k < s \le m$. By Corollary 4.2

$$d\eta^r(X_i, X_j) = -\eta^r([X_i, X_j]) = -\eta_r(\sum_{s=1}^{m} c^s{}_{ij} X_s) = -c^r{}_{ij},$$

for any $1 \le i < j \le m$ and $1 \le r \le m$. Meanwhile,

$$d\eta^r = \sum_{1 \le s < t \le m} \kappa^r{}_{st}\, \eta^s \wedge \eta^t,$$

for any $1 \le r \le m$. Hence

$$d\eta^r(X_i, X_j) = \sum_{1 \le s < t \le m} \kappa^r{}_{st}\, \eta^s \wedge \eta^t (X_i, X_j) = \kappa^r{}_{ij},$$

for any $1 \le i < j \le m$ and $1 \le r \le m$, and hence, $\kappa^r{}_{ij} = -c^r{}_{ij}$. Therefore, Δ is involutive if and only if for any $k < r \le m$,

$$d\eta^r = \sum_{\substack{k<t \\ 1 \le s < t}} \kappa^r{}_{st}\, \eta^s \wedge \eta^t.$$

Taking

$$\theta_t^r = \sum_{1 \le s < t} \kappa^r{}_{st}\, \eta^s,$$

for each $k < r, t \le m$ completes the proof. □

Integration on Smooth Manifolds

Recall that every topological manifold is paracompact, namely, every open cover of the manifold admits a locally finite refinement. This is used to prove that every smooth manifold admits a smooth partition of unity:

Definition. A smooth *partition of unity* on a smooth manifold M is a collection $\{f_\lambda\}$ of smooth functions on M such that

(1) $f_\lambda \geq 0$ on M,
(2) $\{supp(f_\lambda) := \overline{\{p \in M \mid f_\lambda(p) \neq 0\}}\}$ is a locally finite cover of M,
(3) $\sum_\lambda f_\lambda(p) = 1$ for any $p \in M$.

Given an open cover $\{U_\alpha\}$ of M, a smooth partition of unity $\{f_\lambda\}$ is said to be *subordinate* to this cover if for every f_λ there exists U_α such that $supp(f_\lambda) \subset U_\alpha$.

Theorem 4.8. *Every open cover of M admits a smooth partition of unity subordinate to it.*

Proof. We can find a countable basis $\{U_\lambda\}$ for the topology of M consisting of relatively compact coordinate neighborhoods. We may further assume that this basis is a locally finite cover of M, by paracompactness, and that every U_λ contains an open set V_λ such that $\overline{V}_\lambda \subset U_\lambda$ and $\{V_\lambda\}$ is an open cover of M. Now there exist smooth functions $h_\lambda \in C^\infty(M)$ such that $0 \leq h_\lambda \leq 1$ and

$$h_\lambda(p) = \begin{cases} 1 & \text{if } p \in V_\lambda \\ 0 & \text{if } p \notin U_\lambda. \end{cases}$$

By local finiteness,

$$\sum_\lambda h_\lambda(p)$$

is a finite sum, and hence defines a smooth function h on M. Moreover, $h \geq 1$ since for every point $p \in M$ there exists U_λ such that $p \in U_\lambda$ and $h_\lambda(p) = 1$. Then the functions

$$f_\lambda := \frac{h_\lambda}{h},$$

form a partition of unity subordinate to $\{U_\lambda\}$, and therefore to any open cover of M. $\square$

With the help of the above theorem, we give a new and more practical criteria for a smooth manifold to be orientable.

Theorem 4.9. *Let M be a smooth m-dimensional manifold. Then M is orientable if and only if M admits a nowhere vanishing m-form.*

Proof. Suppose that M admits a nowhere vanishing m-form Ω. Given a smooth maximal atlas on M, and a coordinate neighborhood (U, φ) with coordinate functions $x_\alpha^1, \ldots, x_\alpha^m$, suppose that the coordinate functions are ordered in such a way that

$$\Omega|_U = f dx^1 \wedge \cdots \wedge dx^m.$$

Then for any two coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with coordinate functions $x_\alpha^1, \ldots, x_\alpha^m$ and $x_\beta^1, \ldots, x_\beta^m$, respectively, such that $U_\alpha \cap U_\beta \neq \emptyset$

$$\Omega|_{U_\alpha} = f_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m,$$

for some positive function $f_\alpha \in C^\infty(U_\alpha)$, and

$$\Omega|_{U_\beta} = f_\beta dx_\beta^1 \wedge \cdots \wedge dx_\beta^m,$$

for some positive function $f_\beta \in C^\infty(U_\beta)$. By coordinate transformation rule for differential forms

$$f_\beta = f_\alpha \sum_{\sigma \in S_m} sgn(\sigma) \frac{\partial x_\alpha^1}{\partial x_\beta^{\sigma(1)}} \cdots \frac{\partial x_\alpha^m}{\partial x_\beta^{\sigma(m)}} = f_\alpha det(\frac{\partial x_\alpha^i}{\partial x_\beta^j})$$

on $U_\alpha \cap U_\beta$. Since $f_\alpha > 0$ and $f_\beta > 0$, we have

$$det(\frac{\partial x_\alpha^i}{\partial x_\beta^j}) > 0,$$

and hence M is orientable. Conversely, if M is orientable with an oriented smooth maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$, choose a partition of unity $\{f_\lambda\}$ subordinate to the cover $\{U_\alpha\}$, and for each f_λ, U_λ such that $supp(f_\lambda) \subset U_\lambda$. In particular, $\{U_\lambda\}$ is an open cover of M. Define an m-form Ω on M by

$$\Omega = \sum_\lambda f_\lambda dx_\lambda^1 \wedge \cdots dx_\lambda^m.$$

This form is nowhere vanishing. At any point $p \in M$

$$\Omega_p = \sum_\lambda f_\lambda(p) dx_{\lambda p}^1 \wedge \cdots dx_{\lambda p}^m$$

where $f_\lambda(p) \geq 0$, and $f_\lambda(p) \neq 0$ for all but finitely many λ. Furthermore, $f_\lambda(p) \neq 0$ for at least one λ since $\sum_\lambda f_\lambda(p) = 1$. Meanwhile, for any two connected coordinate neighborhoods $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with coordinate functions $x_\alpha^1, \ldots, x_\alpha^m$ and $x_\beta^1, \ldots, x_\beta^m$, respectively, such that $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$dx_{\alpha p}^1 \wedge \cdots \wedge dx_{\alpha p}^m = det(\frac{\partial x_\alpha^i}{\partial x_\beta^j})(\varphi_\beta(p))\, dx_{\beta p}^1 \wedge \cdots \wedge dx_{\beta p}^m,$$

where $det(\frac{\partial x_\alpha^i}{\partial x_\beta^j})(\varphi_\beta(p)) > 0$. This completes the proof. $\square$

Next, we define the integral of an m-form on M. In this regard, let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented smooth maximal atlas on M. Then consider an m-form ω on M with compact support, namely, $supp(\omega) := \overline{\{p \in M \mid \omega_p \neq 0\}}$ is compact. Fix a partition of unity $\{f_\lambda\}$ subordinate to the cover $\{U_\alpha\}$ and for each f_λ, U_λ such that $supp(f_\lambda) \subset U_\lambda$. Since $supp(\omega)$ is compact and $\{supp(f_\lambda)\}$ is a locally finite cover of M, $supp(\omega) \cap supp(f_\lambda) = \emptyset$ for all but finitely many λ. Therefore, we can define

$$\int_M \omega := \sum_\lambda \int_{U_\lambda} f_\lambda \omega,$$

where $\int_{U_\lambda} f_\lambda \omega$ is the Riemann integral

$$\int_{\varphi_\lambda(U_\lambda)} (f_\lambda \circ \varphi_\lambda^{-1}) \cdot (h_\lambda \circ \varphi_\lambda^{-1}) dx_\lambda^1 \dots dx_\lambda^m,$$

and $\omega|_{U_\lambda} = h_\lambda \, dx_\lambda^1 \wedge \cdots \wedge dx_\lambda^m$. Next, we show that the integral $\int_M \omega$ is independent of the choice of the cover $\{U_\lambda\}$ and the choice of the partition of unity $\{f_\lambda\}$. Suppose $\{U'_\lambda\}$ is another cover of M such that $supp(f_\lambda) \subset U'_\lambda$. In particular, $supp(f_\lambda) \subset U_\lambda \cap U'_\lambda$. Then $\omega|_{U'_\lambda} = h'_\lambda \, dx'^1_\lambda \wedge \cdots \wedge dx'^m_\lambda$, and on $U_\lambda \cap U'_\lambda$

$$h'_\lambda = h_\lambda \cdot det(\frac{\partial x^i_\lambda}{\partial x'^j_\lambda}),$$

where $det(\frac{\partial x^i_\lambda}{\partial x'^j_\lambda}) > 0$ by virtue of the fact that the atlas $\{(U_\alpha, \varphi_\alpha)\}$ is coherently oriented. By change of variables formula for Riemann integrals

$$\begin{aligned}&\int_{\varphi_\lambda(U_\lambda \cap U'_\lambda)} (f_\lambda \circ \varphi_\lambda^{-1}) \cdot (h_\lambda \circ \varphi_\lambda^{-1}) dx_\lambda^1 \dots dx_\lambda^m \\ &= \int_{\varphi'_\lambda(U_\lambda \cap U'_\lambda)} (f_\lambda \circ \varphi'^{-1}_\lambda) \cdot (h_\lambda \circ \varphi'^{-1}_\lambda) \cdot |det(\frac{\partial x^i_\lambda}{\partial x'^j_\lambda})| dx'^1_\lambda \dots dx'^m_\lambda.\end{aligned}$$

Since $det(\frac{\partial x^i_\lambda}{\partial x'^j_\lambda}) > 0$, the right-hand side is equal to

$$\begin{aligned}&\int_{\varphi'_\lambda(U_\lambda \cap U'_\lambda)} (f_\lambda \circ \varphi'^{-1}_\lambda) \cdot (h_\lambda \circ \varphi'^{-1}_\lambda) \cdot det(\frac{\partial x^i_\lambda}{\partial x'^j_\lambda}) dx'^1_\lambda \dots dx'^m_\lambda \\ &= \int_{\varphi'_\lambda(U_\lambda \cap U'_\lambda)} (f_\lambda \circ \varphi'^{-1}_\lambda) \cdot (h'_\lambda \circ \varphi'^{-1}_\lambda) dx'^1_\kappa \dots dx'^m_\lambda,\end{aligned}$$

and hence

$$\sum_\lambda \int_{U_\lambda} f_\lambda \omega = \sum_\lambda \int_{U_\lambda \cap U'_\lambda} f_\lambda \omega = \sum_\lambda \int_{U'_\lambda} f_\lambda \omega.$$

On the other hand, if $\{g_\kappa\}$ is another partition of unity subordinate to $\{U_\alpha\}$, and $\{U'_\kappa\}$ is an open cover of M such that $supp(g_\kappa) \subset U'_\kappa$, then

$$\begin{aligned}\sum_\lambda \int_{U_\lambda} f_\lambda \omega &= \sum_\lambda \int_{U_\lambda} \sum_\kappa g_\kappa (f_\lambda \omega) = \sum_\lambda \sum_\kappa \int_{U'_\kappa \cap U_\lambda} g_\kappa f_\lambda \omega \\ &= \sum_\kappa \int_{U'_\kappa} \sum_\lambda g_\kappa f_\lambda \omega = \sum_\kappa \int_{U'_\kappa} \sum_\lambda f_\lambda (g_\kappa \omega) \\ &= \sum_\kappa \int_{U'_\kappa} g_\kappa \omega.\end{aligned}$$

Homework-2
Due 3/6/14

(1) Let M be a compact smooth m-dimensional manifold. Then every smooth vector field X on M generates a smooth action of $\mathbb{R}$ on M by diffeomorphisms, i.e. there exists $\rho : M \times \mathbb{R} \to M$ smooth in the sense of Theorem 3.1 (see [2, IV-Section 5]). Such vector fields are called *complete*, and the corresponding smooth map ρ is called the *flow* of X. Prove that for $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$,

$$\frac{d}{dt}\rho(\cdot,t)^*\omega = \rho(\cdot,t)^*\mathcal{L}_X\omega,$$

and that for a smooth 1-parameter family of k-forms $\{\omega_t\}_{t\in\mathbb{R}}$,

$$\frac{d}{dt}\rho(\cdot,t)^*\omega_t = \rho(\cdot,t)^*(\mathcal{L}_X\omega_t + \frac{d\omega_t}{dt}).$$

More generally, a smooth time-dependent vector field X^t on a compact smooth m-dimensional manifold M generates an isotopy of M, i.e. a smooth 1-parameter family of diffeomorphisms $\{\rho_t\}_{t\in\mathbb{R}}$ of M such that $\rho_0 = id_M$ and $\frac{d\rho_t(p)}{dt} = X^t_{\rho_t(p)}$. The second formula holds when $\rho(\cdot,t)$ is replaced by ρ_t and X is replaced by X^t.

(2) Let M be a smooth 3-dimensional manifold, and Δ be a smooth 2-dimensional distribution on M defined as the kernel of some 1-form λ. Prove that Δ is *completely non-integrable*, i.e. there is no point $p \in M$ where $[X,Y]_p \in \Delta_p$ for any smooth vector fields X, Y tangent to Δ in a neighborhood of p, if and only if $\lambda \wedge d\lambda$ is a volume form on M. (Hint: Use Corollary 4.2.)

A smooth distribution such as Δ is called a contact structure on M, and a 1-form such as λ is called a contact form.

(3) Given a contact form λ on a smooth 3-dimensional manifold M, show that there exists a unique smooth vector field R on M, called the *Reeb vector field*, such that $\iota_R\lambda = 1$ and $\iota_R d\lambda = 0$. Then use the time-dependent version of the second formula in Question (1) and the Cartan formula to show that if $\{\lambda_t\}_{t\in[0,1]}$ is a smooth path of contact forms on a closed smooth 3-dimensional manifold M, then there exists an isotopy $\{\rho_t\}_{t\in[0,1]}$ such that $\rho_t^*\lambda_t = f_t\,\lambda_0$ where $f_t \in C^\infty(M)$.

(4) Prove the following theorem:

Theorem 4.10. *Let M be a smooth oriented m-dimensional manifold.*

(a) *Let $-M$ denote the smooth manifold M with reverse orientation. Then $\int_{-M}\omega = -\int_M \omega$ for any $\omega \in \Omega^m(M)$ with compact support.*

(b) *Let $F : M_1 \to M_2$ be a diffeomorphism, and $\omega \in \Omega^m(M_2)$ be compactly supported. Then*

$$\int_{M_1} F^*\omega = \pm\int_{M_2}\omega$$

where the sign on the right-hand side depends on whether the Jacobian of F has positive or negative determinant.

5. February 24–February 28

Definition. An m-dimensional manifold with boundary is smooth if it admits a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$ where U_α are open subsets of M and φ_α are homeomorphisms of U_α onto open subsets of the upper half space H^m such that:

(1) $\{U_\alpha\}$ is an open cover of M,
(2) If $U_\alpha \cap U_\beta \neq \emptyset$, then the *transition map* $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism.

Note that if M is a smooth m-dimensional manifold with boundary, then ∂M is a smooth $(m-1)$-dimensional manifold without boundary. Given a maximal smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ as in the above definition, a smooth atlas on ∂M is defined by $\{(U_\alpha \cap \partial M, \varphi_\alpha|_{U_\alpha \cap \partial M})\}$.

If M is a smooth oriented m-dimensional manifold with boundary, then ∂M admits a natural orientation induced by the orientation on M, called the *boundary orientation*. Locally, in a coordinate neighborhood (U, φ) of a boundary point with coordinate functions $x^1, \ldots, x^m$ such that $U \cap \partial M$ is diffeomorphic to the locus $x^m = 0$, if the orientation of the manifold is determined by the form $dx^1 \wedge \cdots \wedge dx^m$, then the boundary orientation is determined by the form $(-1)^m dx^1 \wedge \cdots \wedge dx^{m-1}$.

Theorem 5.1 (Stokes's Theorem)**.** *Let M be a smooth oriented m-dimensional manifold with boundary, and $\omega \in \Omega^{m-1}(M)$ with compact support. Then*

$$\int_M d\omega = \int_{\partial M} i^*\omega,$$

where $i : \partial M \hookrightarrow M$ is the inclusion map, and it is understood that ∂M is endowed with the boundary orientation.

Proof. See [2, VI-Theorem 5.1] or [4, Theorem 4.2] for a proof. □

We can also define integrals of differential forms over images of smooth maps from simplices or polyhedra into the manifold. For example, given a k-form ω and a smooth k-simplex $\sigma : \Delta^k \to M$, we define

$$\int_\sigma \omega := \int_{\Delta^k} \sigma^*\omega.$$

Stokes's Theorem can be generalized to integration over smooth simplices. With the preceding understood, an interesting application of Stokes's Theorem is the following:

Theorem 5.2. *Let $\omega \in \Omega^1(M)$ such that $d\omega = 0$, $p, q \in M$ be two points, and γ_1, γ_2 be two homotopic piecewise smooth paths in M joining p to q. Then*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

Proof. If the homotopy between the two paths is smooth, then the theorem follows from an application of Theorem 5.1. If not, we break the homotopy into smooth pieces and apply Theorem 5.1 to each piece (see [2, VI-Theorem 6.6] for details). □

Corollary 5.3. *Let M be a smooth simply connected manifold, and $\omega \in \Omega^1(M)$ such that $d\omega = 0$. Then $\omega = df$ for some $f \in C^\infty(M)$.*

Proof. Fix a basepoint point $p \in M$ and define

$$f(q) := \int_\gamma \omega,$$

where γ is a piecewise smooth path from p to q. By Theorem 5.2, f is a well-defined smooth function. Moreover, changing the basepoint changes the function by an additive constant. Let (U, φ) be a coordinate neighborhood around p with coordinate functions $x^1, \ldots, x^m$ such that $\varphi(p) = (0, \ldots, 0)$, and write

$$\omega = \sum_{i=1}^m \omega_i dx^i,$$

where $\omega_i \in C^\infty(U)$. Then

$$\frac{\partial f}{\partial x^i}(\varphi(p)) = \frac{d}{dt}|_{t=0} \int_0^t \gamma_i^* \omega = (\gamma_i^* \omega)_0 = \omega_i(p),$$

where $\gamma_i : (-\epsilon, \epsilon) \to U$ such that $\varphi \circ \gamma_i(t) = (0, \ldots, 0, \underbrace{t}_{ith\ place}, 0, \ldots, 0)$. Hence $df_p = \omega_p$. Since changing the base only changes the function by an additive constant, $df_q = \omega_q$ at any $q \in M$. □

Property (3) in Theorem 3.9 indicates that we can define a chain complex with $\Omega(M)$ being the underlying vector space and the exterior derivative being the boundary operator. Since the exterior derivative increases the grading by 1, the result is a cohomology theory, called *de Rham cohomology*. The de Rham cohomology groups of M are defined as follows:

$$\begin{aligned} Z^k_{dR}(M) &:= \{\omega \in \Omega^k(M) \mid d\omega = 0\} \\ B^k_{dR}(M) &:= \{\omega \in \Omega^k(M) \mid \omega = d\eta \text{ for some } \eta \in \Omega^{k-1}(M)\}. \end{aligned}$$

Then

$$H^k_{dR}(M) := \frac{Z^k_{dR}(M)}{B^k_{dR}(M)}.$$

A differential form in $Z^k_{dR}(M)$ is called *closed*, while a differential form in $B^k_{dR}(M)$ is called *exact*. Note that, by Property (2) in Theorem 3.9,

$$H^*_{dR}(M) := \oplus_k H^k_{dR}(M),$$

is an $\mathbb{R}$-algebra, and by Theorem 4.3, a smooth map $F : M \to N$ induces an algebra homomorphism $F^* : H^*_{dR}(N) \to H^*_{dR}(M)$ that maps each $H^k_{dR}(N)$ linearly into $H^k_{dR}(M)$. Moreover, given two smooth maps $F : M \to N'$ and $G : N' \to N$, we have $(G \circ F)^* = F^* \circ G^*$ as a result of Theorem 1.1. In particular, the algebra homomorphism induced by the identity diffeomorhism of M is the identity map. Therefore, if M_1 and M_2 are diffeomorphic manifolds, then $H^*_{dR}(M_1)$ and $H^*_{dR}(M_2)$ are isomorphic.

To sum up, de Rham cohomology is a contravariant functor from the category *Diff* of smooth manifolds with morphisms as smooth maps between smooth manifolds to the category of $\mathbb{R}$-algebras.

Theorem 5.4 (de Rham Theorem)**.** *Let M be a smooth m-dimensional manifold. Then $H^k_{dR}(M) \cong H^k(M;\mathbb{R})$.*

Rough idea. Define a map from $\Omega(M)$ to the space of smooth singular cochains by sending $\omega \in \Omega^k(M)$ to the map that sends a smooth singular k-simplex σ to $\int_\sigma \omega$. This map induces an isomorphism between the respective homology groups. See [7] for details. $\square$

Remark. In fact, the isomorphism between de Rham and singular cohomology theories is *natural*, i.e. it respects homomorphisms induced by smooth maps.

One can still prove key results about de Rham cohomology without using Theorem 5.4:

Proposition 5.5. *Let M be a smooth m-dimensional manifold with b connected components. Then $H^0_{dR}(M) \cong \mathbb{R}^b$.*

Proof. A smooth function whose exterior derivative is zero has to be constant on each connected component of M. Therefore, $H^0_{dR}(M) = Z^k_{dR}(M) \cong \mathbb{R}^b$. $\square$

Corollary 5.6. *Let M be a smooth simply connected m-dimensional manifold. Then $H^0_{dR}(M) \cong \mathbb{R}$ and $H^1_{dR}(M) \cong \{0\}$.*

Proof. This follows from Proposition 5.5, and Corollary 5.3. $\square$

Now, let U be an open subset of $\mathbb{R}^m$ with coordinates $x^1, \ldots, x^m$. Then

Definition. The *homotopy operator* $H : \Omega^{k+1}(U \times \mathbb{R}) \to \Omega^k(U \times \mathbb{R})$ is a real linear map defined by

$$H\omega = \Big(\int_0^t f(x,s)ds\Big)dx^{i_1} \wedge \cdots dx^{i_k},$$

if $\omega = f(x,t)dt \wedge dx^{i_1} \wedge \cdots dx^{i_k}$, and $H\omega = 0$ if $\iota_{\frac{\partial}{\partial t}}\omega = 0$.

Proposition 5.7. *The homotopy operator induces a chain homotopy between the chain maps induced by the identity map and the map $i_0 \circ \pi$ on $U \times \mathbb{R}$ where π is the projection onto the first factor and $i_0 : U \to U \times \mathbb{R}$ is the inclusion sending x to $(x, 0)$.*

Proof. We need to show that

$$dH\omega + Hd\omega = \omega - (i_0 \circ \pi)^*\omega,$$

for any $\omega \in \Omega^{k+1}(U \times \mathbb{R})$. If $\omega = f(x,t)dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$, then

$$\begin{aligned}
dH\omega &= 0 \\
Hd\omega &= H[\sum_{i=1}^{m} \frac{\partial f(x,t)}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} + \frac{\partial f(x,t)}{\partial t} dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}] \\
&= \big(\int_0^t \frac{\partial f(x,s)}{\partial s} dt\big) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} \\
&= (f(x,t) - f(x,0)) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}
\end{aligned}$$

and hence

$$dH\omega + Hd\omega = \omega - (i_0 \circ \pi)^* \omega.$$

On the other hand, if $\omega = f(x,t)dt \wedge dx^{i_1} \wedge \cdots dx^{i_k}$, then

$$\begin{aligned}
dH\omega &= d\big(\int_0^t f(x,s)ds\big) dx^{i_1} \wedge \cdots dx^{i_k} = f(x,t)dt \wedge dx^{i_1} \wedge \cdots dx^{i_k} \\
&\qquad + \sum_{i=1}^{m} \big(\int_0^t \frac{\partial f(x,s)}{\partial x^i} ds\big) dx^i \wedge dx^{i_1} \wedge \cdots dx^{i_k} \\
Hd\omega &= H[-\sum_{i=1}^{m} \frac{\partial f(x,t)}{\partial x^i} dt \wedge dx^i \wedge dx^{i_1} \wedge \cdots dx^{i_k}] \\
&= -\sum_{i=1}^{m} \big(\int_0^t \frac{\partial f(x,s)}{\partial x^i} ds\big) dx^i \wedge dx^{i_1} \wedge \cdots dx^{i_k},
\end{aligned}$$

and

$$dH\omega + Hd\omega = \omega.$$

Meanwhile, $(i_0 \circ \pi)^* \omega = 0$ since $i_0^* \omega = 0$. This completes the proof. □

Corollary 5.8 (Poincaré Lemma)**.** *For any $m \geq 0$ we have*

$$H^k_{dR}(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It suffices to show that this holds for $m = 0$, the case of a point. The rest follows by induction on m using Proposition 5.7. In this regard, the only non-trivial differential forms on a point are the 0-forms, or smooth functions on a point. These are simply identified with constants in $\mathbb{R}$, which are clearly closed and non-exact. Hence

$$H^k_{dR}(pt) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

More generally:

Theorem 5.9. *Let $\{\rho_t\}_{t\in[0,1]}$ be a smooth 1-parameter family of diffeomorphisms of M. Then $\rho_t^* : H^*_{dR}(M) \to H^*_{dR}(M)$ is the same map for every $t \in [0,1]$.*

Proof. Let X_t be the time-dependent vector field corresponding to the isotopy $\{\rho_t \circ \rho_0^{-1}\}$, and $\omega \in \Omega^k(M)$. Then

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*(d\iota_{X_t}\omega + \iota_{X_t}d\omega).$$

If $d\omega = 0$, the above equation says that $\frac{d}{dt}\rho_t^*\omega = \rho_t^* d\iota_{X_t}\omega = d(\rho_t^*\iota_{X_t}\omega)$ is exact. Then

$$\rho_t^*\omega - \rho_0^*\omega = \int_0^t \frac{d}{ds}\rho_s^*\omega \, ds$$

is also exact. This completes the proof. □

Theorem 5.10. *Let $\{\rho_t : M \to N\}_{t\in[0,1]}$ be a smooth 1-parameter family of smooth maps. Then $\rho_t^* : H^*_{dR}(N) \to H^*_{dR}(M)$ is independent of $t \in [0,1]$. In other words, if $F, G : M \to N$ are smooth maps that are smoothly homotopic, then $F^* = G^*$.*

Proof. Extend $\{\rho_t\}_{t\in[0,1]}$ to $\{\rho_t\}_{t\in\mathbb{R}}$. Regard $\{\rho_t\}_{t\in\mathbb{R}}$ as a smooth map $\psi : M \times \mathbb{R} \to N$, and denote by i_t the inclusion of M into $M \times \mathbb{R}$ as the t-slice. Then $\rho_t = \psi \circ i_t$. Meanwhile, consider the diffeomorphism $\Psi_t : M \times \mathbb{R} \to M \times \mathbb{R}$ defined by sending (p, s) to $(p, s+t)$. Then $i_t = \Psi_t \circ i_0$, and hence $\rho_t = \psi \circ \Psi_t \circ i_0$. Since, by Theorem 5.9, Ψ_t^* is independent of t, so is $\rho_t^* = i_0^* \circ \Psi_t^* \circ \psi^*$. □

Corollary 5.11. *If M is a smooth m-dimensional manifold that is smoothly contractible, then*

$$H^k_{dR}(M) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}.$$

Proof. This follows from Theorem 5.10, and that the de Rham cohomology of a point is as in Corollary 5.8. □

Smooth Vector Bundles

We start with some definitions.

Definition. Let M be a smooth m-dimensional manifold. A smooth manifold E together with a smooth map $\pi : E \to M$ is called a smooth real *vector bundle of rank n* if

1. The map π is onto, and $\pi^{-1}(p)$ is isomorphic to $\mathbb{R}^n$ for each $p \in M$,
2. Every point $p \in M$ has an open neighborhood U such that there exists a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$, called a *local trivialization of E over U*, that fits into the following commutative diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[d, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "\pi_1"] \\ U \arrow[r, "id_U"] & U \end{tikzcd}$$

where π_1 is the projection onto the first factor, and $\Phi_U|_{\pi^{-1}(p)}$ is an isomorphism of real vector spaces for each $p \in U$.

The manifold E is called the *total space*, M is called the *base space*, and $E_p = \pi^{-1}(p)$ for each $p \in M$ is called a *fiber* of the vector bundle. The map π is called the *bundle projection.*

Note that if $(U_\alpha, \Phi_{U_\alpha})$ are local trivializations of E such that $\{U_\alpha\}$ covers M, then whenever $U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\Phi_{U_\alpha} \circ \Phi_{U_\beta}^{-1}(p, v_p) = (p, g_{\alpha\beta}(p)(v_p)),$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$, called a *transition function*, is smooth such that

- $g_{\alpha\alpha}(p) = id$ for any $p \in U_\alpha$,
- $g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) = id$ for any $p \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

The above two conditions are called the *cocycle conditions*, and the group $GL(n, \mathbb{R})$ is called the *structure group* of the vector bundle. Conversely, if $\{U_\alpha\}$ is an open cover of M and $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ is a collection of smooth maps that satisfy the cocycle conditions, then there exists a smooth real vector bundle of rank n with $\{g_{\alpha\beta}\}$ as its transition functions. To construct such a vector bundle, patch $\{U_\alpha \times \mathbb{R}^n\}$ together using the maps $\{g_{\alpha\beta}\}$. To be more explicit, define

$$E := \bigsqcup_\alpha U_\alpha \times \mathbb{R}^n \Big/ (p, v) \sim (p, g_{\alpha\beta}(p)(v))$$

where $p \in U_\alpha \cap U_\beta$, $(p, v) \in U_\beta \times \mathbb{R}^n$, and $(p, g_{\alpha\beta}(p)(v))) \in U_\alpha \times \mathbb{R}^n$. The bundle projection is defined in the obvious way.

A smooth complex vector bundle of rank n is defined in exactly the same way as above but replacing $\mathbb{R}$ by $\mathbb{C}$.

Example 5. The tangent and cotangent bundles of M are both smooth real vector bundles of rank m with transition functions as the Jacobians of the coordinate transition maps.

Definition. A smooth *section* of a smooth real (complex) vector bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$. A rank k *subbundle* $\pi' : E' \to M$ of $\pi : E \to M$ is a smooth real (complex) vector bundle of rank k such that E' is a submanifold of E and $\pi' = \pi|_{E'}$.

Let $\pi : E \to M$ and $\pi' : E' \to N$ be two smooth real (complex) vector bundles. Then a *bundle homomorphism* $\tilde{F} : E \to E'$ is a smooth map that descends to a smooth map $F : M \to N$ which fit in the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\tilde{F}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

such that $\tilde{F}|_{E_p} : E_p \to E'_{F(p)}$ is a homomorphism of real (complex) vector spaces for each $p \in M$. In particular, a *bundle isomorphism* between two smooth real (complex) vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ is a bundle homomorphism $F : E \to E'$ that is a diffeomorphism which fits in the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\tilde{F}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

A smooth real (complex) vector bundle $\pi : E \to M$ of rank n is *trivial* if it is isomorphic to the trivial vector bundle $\pi_M : M \times \mathbb{R}^n \to M$ ($\pi_M : M \times \mathbb{C}^n \to M$). Note that a smooth real (complex) vector bundle $\pi : E \to M$ of rank n is trivial if and only if it admits n linearly independent nowhere vanishing sections. To be more explicit, if $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$ ($\mathbb{C}^n$), then $s_i : M \to M \times \mathbb{R}^n$ ($s_i : M \to M \times \mathbb{C}^n$) defined by sending p to (p, e_i) yield n linearly independent nowhere vanishing sections of the trivial vector bundle, and we can carry these sections onto any other smooth vector bundle $\pi : E \to M$ isomorphic to the trivial vector bundle via a bundle isomorphism $\tilde{F} : M \times \mathbb{R}^n \to E$ ($\tilde{F} : M \times \mathbb{C}^n \to E$), namely, take $\tilde{F} \circ s_i$.

Example 6. A smooth vector field on M is a smooth section of the tangent bundle of M. More generally, a smooth tensor field on M is a smooth section of the tensor bundle on M. The tangent bundle of the spheres S^1, S^3, and S^7 are trivial.

Definition. A smooth real vector bundle $\pi : E \to M$ of rank n is orientable if there exist transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ where $\{U_\alpha\}$ is an an open cover of M such that $g_{\alpha\beta}(p)$ has positive determinant for each $p \in U_\alpha \cap U_\beta$. When such a collection of transition functions exist, we say that the structure group of the bundle can be *reduced* to $GL^+(n, \mathbb{R})$. A choice of transition functions $\{g_{\alpha\beta}\}$ with $g_{\alpha\beta} \in GL^+(n, \mathbb{R})$ is called an *orientation* of the vector bundle. In general, if there exist a collection of transition functions into a subgroup H of $GL(n, \mathbb{R})$, then we say that the structure group of the bundle can be reduced to H.

Example 7. The tangent bundle of an orientable manifold M is orientable.

Operations on Vector Bundles

Let M be a smooth m-dimensional manifold and $\pi : E \to M$ be a smooth real (complex) vector bundle of rank n. Given a submanifold N of M, we can *restrict* the vector bundle $\pi : E \to M$ to a vector bundle $\pi : E|_N \to N$ where $E|_N := \pi^{-1}(N)$ is a submanifold of E and π is the original bundle projection. To be more explicit, if $\{g_{\alpha\beta}\}$ are transition functions for the vector bundle $\pi : E \to M$, then $\{g_{\alpha\beta}|_{U_\alpha \cap U_\beta \cap N}\}$ are transition functions for the vector bundle $\pi : E|_N \to N$.

More generally, if $F : N \to M$ is a smooth map, then the *pull-back* vector bundle $\pi^F : F^*E \to N$ is defined so as to result in the following commutative diagram:

$$\begin{tikzcd} F^*E \arrow[r, "\pi_2"] \arrow[d, "\pi^F"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "F"] & M \end{tikzcd}$$

where $F^*E := N \times_M E = \{(p, e) \mid F(p) = \pi(e)\} \subset N \times E$ and π_2 is the projection onto the second factor. To be more explicit, if $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ ($\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})\}$) are transition functions for the vector bundle $\pi : E \to M$ where $\{U_\alpha\}$ is an an open cover of M, then $\{g_{\alpha\beta} : F^{-1}(U_\alpha \cap U_\beta) \to GL(n, \mathbb{R})\}$ ($\{g_{\alpha\beta} : F^{-1}(U_\alpha \cap U_\beta) \to GL(n, \mathbb{C})\}$) are transition functions for the vector bundle $\pi^* : F^*E \to N$ where $\{F^{-1}(U_\alpha)\}$ is an an open cover of N.

The dual of a smooth real (complex) vector bundle $\pi : E \to M$ of rank n is a smooth real (complex) vector bundle $\pi^* : E^* \to M$ of rank n obtained by replacing the fibers of the former bundle by their real (complex) duals. Given a collection of transition functions $\{g_{\alpha\beta}\}$ for the vector bundle $\pi : E \to M$, a collection of transition functions for the dual vector bundle is $\{{g_{\alpha\beta}^T}^{-1}\}$.

Let $\pi : E \to M$ and $\pi' : E' \to M$ be two smooth real (complex) vector bundles of respective ranks n and k. Then their *direct sum* $\pi \oplus \pi' : E \oplus E' \to M$ and *tensor product* $\pi \otimes \pi' : E \otimes E' \to M$ are smooth real (complex) vector bundle of rank $n + k$ and $n \cdot k$, respectively, defined by taking fiberwise direct sum and tensor product. More precisely, if $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition functions for the vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ respectively, then $\{g_{\alpha\beta} \oplus g'_{\alpha\beta}\}$ and $\{g_{\alpha\beta} \otimes g'_{\alpha\beta}\}$ are transition functions for the direct sum bundle $\pi \oplus \pi' : E \oplus E' \to M$ and the tensor product bundle $\pi \otimes \pi' : E \otimes E' \to M$, respectively. Here,

$$g_{\alpha\beta} \oplus g'_{\alpha\beta} = \left[\begin{array}{cc} g_{\alpha\beta} & 0 \\ 0 & g'_{\alpha\beta} \end{array} \right],$$

and $g_{\alpha\beta} \otimes g'_{\alpha\beta}$ is the matrix with $n \times n$ blocks of $k \times k$ matrices with the (u, v)-entry of the (i, j)-block being $(g_{\alpha\beta})^i_j \cdot (g'_{\alpha\beta})^u_v$. The smooth vector bundle $\pi^* \otimes \pi : E^* \otimes E \to M$ is the *endomorphism bundle* of $\pi : E \to M$, whose fibers are the vector spaces of real (complex) endomorphisms of the fibers of $\pi : E \to M$.

The kth exterior power $\Lambda^k\pi : \Lambda^k E \to M$ is a smooth real (complex) vector bundle of rank $\frac{n!}{k! \cdot (n-k)!}$ obtained by taking fiberwise kth exterior power. Given a collection of transition functions $\{g_{\alpha\beta}\}$ for the vector bundle $\pi : E \to M$, a collection of transition functions for the vector bundle $\Lambda^k\pi : \Lambda^k E \to M$ is defined by $k \times k$ minors of $g_{\alpha\beta}$. More precisely, let $S = \{I = (i_1, \ldots, i_k) \mid 1 \le i_1 < \cdots < i_k \le n\}$ and put the lexicographic order on it. Then the (I, J) entry of the transition maps is the $k \times k$ minor of $g_{\alpha\beta}$ obtained by taking the elements at the intersections of the $i_1, \ldots, i_k$ rows and the $j_1, \ldots, j_k$ columns.

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Given two smooth real vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ of rank n, a bundle isomorphism between them is a smooth section of the homomorphism bundle $Hom(E, E') \cong E^* \otimes E'$. A bundle isomorphism is a section of the homomorphism bundle consisting of isomorphisms of fibers.

Theorem 6.1. *Let $F_0, F_1 : M \to N$ be two smoothly homotopic maps and $\pi : E \to N$ be a smooth vector bundle. Then $F_0^*\pi : F_0^*E \to M$ and $F_1^*\pi : F_1^*E \to M$ are isomorphic bundles.*

Proof. We will sketch the argument in the case M is compact, and the general case follows from using the fact that a topological manifold is paracompact. (See [3, Theorem 6.8] for more details.) Let $F : M \times [0,1] \to N$ be a smooth homotopy between F_0 and F_1. Consider the pull-back bundles F^*E and $\pi_M^*(F_0^*E)$ where $\pi_M : M \times [0,1] \to M$ is the projection map. Then $Hom(F^*E, \pi_M^*(F_0^*E))$ has a smooth section over $M \times \{0\}$ restricting to the identity isomorphism of $F_0^*E_p$ for each $p \in M$. Since any linear map near an isomorphism is also an isomorphism, and M is compact, we can extend this section to over $M \times [0, \epsilon)$. Finally, by compactness of the interval $[0,1]$, we can extend the section to the whole of $M \times [0,1]$. But then, the restriction of this section to $M \times \{1\}$ is a bundle isomorphism between F_0^*E and F_1^*E. $\square$

An immediate corollary of the above theorem is the following:

Corollary 6.2. *Any smooth vector bundle over a (smoothly) contractible manifold is trivial.*

Metrics on Vector Bundles

A smooth metric on a smooth real vector bundle $\pi : E \to M$ of rank n is a smooth section $(\cdot,\cdot)$ of $(E \otimes E)^*$ that defines a positive-definite inner product on every fiber. A smooth Hermitian metric on a smooth complex vector bundle $\pi : E \to M$ of rank n is a bundle homomorphism $\langle\cdot,\cdot\rangle$ of $(E \otimes_{\mathbb{R}} E)^*$ that defines a non-degenerate Hermitian inner product on every fiber. Every smooth real vector bundle admits a smooth metric, and every smooth complex vector bundle admits a smooth Hermitian metric. In order to see this, take a collection $\{(U_\alpha, \Phi_\alpha)\}$ of local trivializations of the vector bundle $\pi : E \to M$, and use a partition of unity subordinate to the open cover $\{U_\alpha\}$ to construct a metric or a Hermitian metric on the vector bundle using the standard inner product on $\mathbb{R}^n$ or the standard Hermitian inner product on $\mathbb{C}^n$.

Example 8. A Riemannian metric on M is a smooth metric on the tangent bundle of M.

Proposition 6.3. *The structure group of a smooth real vector bundle $\pi : E \to M$ of rank n can be reduced to $O(n)$, and it can be further reduced to $SO(n)$ if the vector bundle is orientable.*

Proof. Having fixed a smooth metric on the vector bundle, we can find transition functions into the group $O(n)$ by the Gram-Schmidt process. $\square$

Proposition 6.4. *A smooth real vector bundle $\pi : E \to M$ of rank n is orientable if and only if $\Lambda^n E \cong \underline{\mathbb{R}}$.*

Proof. For the only if direction, it follows from Proposition 6.3 that we can find a smooth metric on and a collection of transition functions $\{g_{\alpha\beta}\}$ for the vector bundle $\pi : E \to M$ with $g_{\alpha\beta} \in SO(n)$. Then the transition functions for the vector bundle $\Lambda^n\pi : \Lambda^n E \to M$ become $\{det(g_{\alpha\beta}) = 1\}$. As a result, we can define a unit length section of the bundle $\Lambda^n\pi : \Lambda^n E \to M$ which yields a trivialization of that bundle. As for the if direction, start with a collection of local trivializations $\{(U_\alpha, \Phi_\alpha)\}$ for the vector bundle $\pi : E \to M$ such that the transition functions $g_{\alpha\beta} \in O(n)$. Since $\Lambda^n E \cong \underline{\mathbb{R}}$, it is orientable. Therefore, we can modify each local trivialization by an element of $O(n)$ so as to make every transition function $g_{\alpha\beta} \in SO(n)$. This completes the proof. $\square$

Remark. If $\pi : E \to M$ is a smooth complex vector bundle of rank n, then it is orientable when regarded as a smooth real vector bundle of rank $2n$. This is because the group $GL(n, \mathbb{C})$ is a subgroup of $GL^+(2n, \mathbb{R})$. To see this, identify $X + iY \in GL(n, \mathbb{C})$ with the real matrix

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix},$$

and note that the determinant of the latter is $\det(X + iY) \cdot \overline{\det(X + iY)}$.

Example 9. Let $(x : y : z)$ denote the homogeneous coordinates on $\mathbb{R}\mathrm{P}^2$, and $E = \mathbb{R}\mathrm{P}^2 \setminus \{(0 : 0 : 1)\}$. The coordinate neighborhoods (U_x, φ_x) and (U_y, φ_y) defined by

$$\begin{aligned} U_x &:= \{(x : y : z) \mid x \neq 0\} \\ U_y &:= \{(x : y : z) \mid y \neq 0\} \\ \varphi_x(x : y : z) &:= (\frac{y}{x}, \frac{z}{x}) \\ \varphi_y(x : y : z) &:= (\frac{x}{y}, \frac{z}{y}) \end{aligned}$$

cover E, and are also local trivializations of E over $\mathbb{R}\mathrm{P}^1 := \{(x : y : z) \mid z = 0\} \subset \mathbb{R}\mathrm{P}^2$ as a smooth real vector bundle of rank 1, called the *Möbius bundle*. Here the bundle projection $\pi : E \to \mathbb{R}\mathrm{P}^1$ is defined by $\pi(x : y : z) := (x : y)$, and the transition function on $\pi(U_x \cap U_y)$ is defined by $g(x : y)(v) := \frac{y}{x}v$, which is orientation preserving if $\frac{y}{x} > 0$, and is orientation reversing otherwise. This indicates that the Möbius bundle is not orientable. We can prove this by showing that the Möbius bundle is not trivial. This is clear since a bundle isomorphism would map $E \setminus \mathbb{R}\mathrm{P}^1$ to $(\mathbb{R}\mathrm{P}^1 \times \mathbb{R}) \setminus \mathbb{R}\mathrm{P}^1 \times \{0\}$, which is a contradiction since the former is connected, while the latter is not.

Similarly, $\mathbb{C}\mathrm{P}^2 \setminus \{(0 : 0 : 1)\}$ admits the structure of a smooth complex vector bundle of rank 1 over $\mathbb{C}\mathrm{P}^1$. The latter is orientable but not trivial.

The Thom Isomorphism

Definition. Let $\pi : E \to M$ be a smooth real oriented vector bundle of rank n, and $\Omega_{cv}(E)$ denote the space of differential forms on E with compact support in the vertical direction. Since $\Omega_{cv}(E)$ is closed under exterior differentiation, it is a subcomplex of $\Omega(E)$. The cohomology groups arising from this complex are denoted by $H^*_{cv}(E)$, called the *compact vertical cohomology.*

Let U be an open subset of $\mathbb{R}^m$. We define a real linear map $\pi_* : \Omega^k_{cv}(U \times \mathbb{R}^n) \to \Omega^{k-n}(U)$, called the *push-forward map.* Denote by $x^1, \ldots, x^m$ the coordinates on U and by $u^1, \ldots, u^n$ the coordinates on $\mathbb{R}^n$. Then a differential form $\omega \in \Omega_{cv}(U \times \mathbb{R}^n)$ is a real linear combination of two types of forms:

(1) $(\pi^*\eta) \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell}$ where $1 \le i_1 < \cdots < i_l \le n$ and $\ell < n$,
(2) $(\pi^*\eta) \wedge f(x,u) du^1 \wedge \cdots \wedge du^n$,

where η is a differential form on U and $f(x,\cdot)$ is a smooth compactly supported function on $\mathbb{R}^n$ at each $x \in U$. Define $\pi_*\omega$ to be zero if ω is of type (1), or else

$$\pi_*\omega = \Big(\int_{\mathbb{R}^n} f(x,u) du^1 \cdots du^n \Big) \eta.$$

If $\pi : E \to M$ is a smooth real oriented vector bundle of rank n, then define the push-forward map $\pi_* : \Omega^k_{cv}(E) \to \Omega^{k-n}(M)$ so as to agree with the above definition given on local trivializations.

Exercise. Check that the map π_* is well-defined globally.

Proposition 6.5. *The map π_* commutes with the exterior derivative.*

Proof. It suffices to prove the proposition in a local trivialization. Let $\omega \in \Omega_{cv}(U \times \mathbb{R}^n)$ be of type (1). Then $\omega = (\pi^*\eta) \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell}$, and

$$\begin{aligned}
d\pi_*\omega &= 0 \\
\pi_* d\omega &= \pi_*\Big(d\pi^*\eta \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell} \\
&\quad + (-1)^{deg(\eta)} \sum_{i=1}^{m} \pi^*\eta \wedge dx^i \wedge \frac{\partial f(x,u)}{\partial x^i} du^{i_1} \wedge \cdots \wedge du^{i_\ell} \\
&\quad + (-1)^{deg(\eta)} \sum_{i=1}^{n} \pi^*\eta \wedge \frac{\partial f(x,u)}{\partial u^i} du^i \wedge du^{i_1} \wedge \cdots \wedge du^{i_\ell} \Big) \\
&= (-1)^{deg(\eta)} \sum_{i=1}^{n} \pi_*\big(\frac{\partial f(x,u)}{\partial u^i} du^i \wedge du^{i_1} \wedge \cdots \wedge du^{i_\ell} \big) \eta \\
&= 0.
\end{aligned}$$

The last equality follows immediately if $du^i \wedge du^{i_1} \wedge \cdots \wedge du^{i_\ell} \neq \pm du^1 \wedge \cdots \wedge du^n$, and it follows from

$$\pi_*\big(\frac{\partial f(x,u)}{\partial u^i} du^i \wedge du^{i_1} \wedge \cdots \wedge du^{i_\ell}\big) = \pm \int_{\mathbb{R}^n} \frac{\partial f(x,u)}{\partial u^i} du^1 \cdots du^n = 0,$$

if $du^i \wedge du^{i_1} \wedge \cdots \wedge du^{i_\ell} = \pm du^1 \wedge \cdots \wedge du^n$, since $f(x,\cdot)$ is compactly supported.

On the other hand, if $\omega \in \Omega_{cv}(U \times \mathbb{R}^n)$ is of type (2), then

$$\omega = (\pi^*\eta) \wedge f(x,u) du^1 \wedge \cdots \wedge du^n,$$

and

$$\begin{aligned}
d\pi_*\omega &= d\Big(\int_{\mathbb{R}^n} f(x,u) du^1 \cdots du^n\Big)\eta \\
&= \sum_{i=1}^m \Big(\int_{\mathbb{R}^n} \frac{\partial f(x,u)}{\partial x^i} du^1 \cdots du^n\Big) dx^i \wedge \eta + \Big(\int_{\mathbb{R}^n} f(x,u) du^1 \cdots du^n\Big) d\eta \\
\pi_* d\omega &= \pi_*\Big((\pi^* d\eta) \wedge f(x,u) du^1 \wedge \cdots \wedge du^n \\
&\quad + (-1)^{deg(\eta)} \sum_{i=1}^m \pi^*\eta \wedge dx^i \wedge \frac{\partial f(x,u)}{\partial x^i} du^{i_1} \wedge \cdots \wedge du^{i_\ell}\Big) \\
&= \Big(\int_{\mathbb{R}^n} f(x,u) du^1 \cdots du^n\Big) d\eta + (-1)^{deg(\eta)} \sum_{i=1}^m \Big(\int_{\mathbb{R}^n} \frac{\partial f(x,u)}{\partial x^i} du^1 \cdots du^n\Big) \eta \wedge dx^i \\
&= \Big(\int_{\mathbb{R}^n} f(x,u) du^1 \cdots du^n\Big) d\eta + \sum_{i=1}^m \Big(\int_{\mathbb{R}^n} \frac{\partial f(x,u)}{\partial x^i} du^1 \cdots du^n\Big) dx^i \wedge \eta.
\end{aligned}$$

This completes the proof. $\square$

Proposition 6.6. *Let $\pi : E \to M$ be a smooth real oriented vector bundle of rank n, $\mu \in \Omega(M)$, and $\omega \in \Omega_{cv}(E)$. Then*

(a) $\pi_*((\pi^*\mu) \wedge \omega) = \mu \wedge \pi_*\omega$,
(b) *If M is oriented, μ is compactly supported, and $\pi^*\mu \wedge \omega \in \Omega^{m+n}(E)$, then*

$$\int_E \pi^*\mu \wedge \omega = \int_M \mu \wedge \pi_*\omega.$$

Proof. For suffices to prove claim (a) in a local trivialization. Let $\omega \in \Omega_{cv}(U \times \mathbb{R}^n)$ be of type (1). Then $\omega = (\pi^*\eta) \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell}$, and

$$\begin{aligned}
\pi_*((\pi^*\mu) \wedge \omega) &= \pi_*\big((\pi^*\mu) \wedge (\pi^*\eta) \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell}\big) \\
&= \pi_*\big(\pi^*(\mu \wedge \eta) \wedge f(x,u) du^{i_1} \wedge \cdots \wedge du^{i_\ell}\big) \\
&= 0.
\end{aligned}$$

Meanwhile, $\mu \wedge \pi_*\omega = 0$ since $\pi_*\omega = 0$. Next, suppose that ω is of type (2). Then $\omega = (\pi^*\eta) \wedge f(x,u) du^1 \wedge \cdots \wedge du^n$, and

$$\begin{aligned}
\pi_*((\pi^*\mu) \wedge \omega) &= \pi_*\big((\pi^*\mu) \wedge (\pi^*\eta) \wedge f(x,u)du^1 \wedge \cdots \wedge du^n\big) \\
&= \pi_*\big(\pi^*(\mu \wedge \eta) \wedge f(x,u)du^1 \wedge \cdots \wedge du^n\big) \\
&= \Big(\int_{\mathbb{R}^n} f(x,u)du^1 \cdots du^n\Big)\mu \wedge \eta \\
&= \mu \wedge \big(\Big(\int_{\mathbb{R}^n} f(x,u)du^1 \cdots du^n\Big)\eta\big) \\
&= \mu \wedge \pi_*\omega.
\end{aligned}$$

In order to prove claim (b), choose an oriented collection of local trivializations $\{(U_\alpha, \Phi_\alpha)\}$ of E, and choose a partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Then

$$\begin{aligned}
\int_E \pi^*\mu \wedge \omega &= \sum_\alpha \int_{E|_{U_\alpha}} (\rho_\alpha \circ \pi)\pi^*\mu \wedge \omega \\
&= \sum_\alpha \int_{U_\alpha \times \mathbb{R}^n} \pi^*\mu \wedge (\rho_\alpha \circ \pi)\omega
\end{aligned}$$

Direct computation proves that

$$\int_{U_\alpha \times \mathbb{R}^n} \pi^*\mu \wedge (\rho_\alpha \circ \pi)\omega = \int_{U_\alpha} \mu \wedge \rho_\alpha \pi_*\omega = \int_{U_\alpha} \rho_\alpha\, \mu \wedge \pi_*\omega,$$

and hence

$$\sum_\alpha \int_{U_\alpha \times \mathbb{R}^n} \pi^*\mu \wedge (\rho_\alpha \circ \pi)\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha\, \mu \wedge \pi_*\omega = \int_M \mu \wedge \pi_*\omega.$$

□

Theorem 6.7 (Thom Isomorphism)**.** *Let $\pi : E \to M$ be a smooth real oriented vector bundle of rank n. Then*

$$H^k_{cv}(E) \cong H^{k-n}_{dR}(M),$$

for each $k \geq 0$.

Proof. We start by proving the isomorphism in the case of the trivial vector bundle. In this regard, define a linear map

$$p_* : \Omega^k_{cv}(M \times \mathbb{R}^n) \to \Omega^{k-1}_{cv}(M \times \mathbb{R}^{n-1})$$

as follows:

$$p_*(\pi^*\eta \wedge f(x,u)du^{i_1} \wedge \cdots \wedge du^{i_\ell}) = \pi^*\eta \wedge \Big(\int_{\mathbb{R}} f(x,u)du^n\Big)du^{i_1} \wedge \cdots \wedge du^{i_{\ell-1}}$$

if $i_l = n$, and $p_*(\omega) = 0$, otherwise.

It is easy to check that the map p_* commutes with the exterior derivative, and hence it is a chain map. Now let $e := e(u^n)du^n$ be a compactly supported 1-form on $\mathbb{R}$ with $\int_{\mathbb{R}} e = 1$.

Then define $e_* : \Omega^{k-1}_{cv}(M \times \mathbb{R}^{n-1}) \to \Omega^k_{cv}(M \times \mathbb{R}^n)$ by

$$e_*(\eta) = \eta \wedge e.$$

Note that $p_* \circ e_* = id$. We claim that $e_* \circ p_*$ is chain homotopic to id. Define a homotopy operator $H : \Omega^{k+1}_{cv}(M \times \mathbb{R}^n) \to \Omega^k_{cv}(M \times \mathbb{R}^n)$ by

$$\begin{aligned} H(\pi^*\eta \wedge f(x,u)du^{i_1} \wedge \cdots \wedge du^{i_\ell}) &= (-1)^k \Big[\pi^*\eta \wedge \big(\int_{-\infty}^{u^n} f(x,u^1,\dots,u^{n-1},t)dt\big)du^{i_1} \wedge \cdots \wedge du^{i_{\ell-1}} \\ &\quad -\pi^*\eta \wedge \big(\int_{-\infty}^{u^n} e(t)dt\big)\big(\int_{\mathbb{R}} f(x,u)du^n\big)du^{i_1} \wedge \cdots \wedge du^{i_{\ell-1}}\Big], \end{aligned}$$

if $i_l = n$, and $H_*(\omega) = 0$, otherwise. Direct computation verifies that $dH\omega + Hd\omega = \omega - (e_* \circ p_*)\omega$ for any $\omega \in \Omega^{k+1}_{cv}(M \times \mathbb{R}^n)$. Hence

$$p_* : H^k_{cv}(M \times \mathbb{R}^n) \cong H^{k-1}_{cv}(M \times \mathbb{R}^{n-1}),$$

and by induction on n, we get

$$\pi_* : H^k_{cv}(M \times \mathbb{R}^n) \cong H^{k-n}(M),$$

For the general case, let $\{U_\alpha\}$ be an open cover of M by local trivializations. Fix U_α and U_β from this open cover, and consider the Mayer–Vietoris exact sequence

$$0 \to \Omega^*_{cv}(E|_{U_\alpha \cup U_\beta}) \xrightarrow{r} \Omega^*_{cv}(E|_{U_\alpha}) \oplus \Omega^*_{cv}(E|_{U_\beta}) \xrightarrow{s} \Omega^*_{cv}(E|_{U_\alpha \cap U_\beta}) \to 0,$$

where r has coordinates the restrictions of a differential form to $E|_{U_\alpha}$ and $E|_{U_\beta}$, respectively, while s is the difference of the restrictions of differential forms in $\Omega^*_{cv}(E|_{U_\alpha})$ and $\Omega^*_{cv}(E|_{U_\beta})$ to $E|_{U_\alpha \cap U_\beta}$. This induces a long-exact sequence

$$\cdots H^k_{cv}(E|_{U_\alpha \cup U_\beta}) \xrightarrow{r} H^k_{cv}(E|_{U_\alpha}) \oplus H^k_{cv}(E|_{U_\beta}) \xrightarrow{s} H^k_{cv}(E|_{U_\alpha \cap U_\beta}) \xrightarrow{d^*} H^{k+1}_{cv}(E|_{U_\alpha \cup U_\beta}) \cdots$$

where the map d^* can be described via the help of a partition of unity $\{f_\alpha, f_\beta\}$ subordinate to $\{U_\alpha, U_\beta\}$. Given $\omega \in \Omega_{cv}(E|_{U_\alpha \cap U_\beta})$, note that $s(\pi^* f_\alpha \omega, -\pi^* f_\beta \omega) = \omega$. If $d\omega = 0$, then $d^*[\omega]$ is represented by $d(\pi^* f_\alpha \omega)$ on $E|_{U_\alpha}$, and by $-d(\pi^* f_\beta \omega)$ on $E|_{U_\beta}$. Now consider the following diagram:

$$\begin{tikzcd} \cdots H^k_{cv}(E|_{U_\alpha \cup U_\beta}) \arrow[r, "r"] \arrow[d, "\pi_*"'] & H^k_{cv}(E|_{U_\alpha}) \oplus H^k_{cv}(E|_{U_\beta}) \arrow[r, "s"] \arrow[d, "\pi_*"] & H^k_{cv}(E|_{U_\alpha \cap U_\beta}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{k+1}_{cv}(E|_{U_\alpha \cup U_\beta}) \cdots \arrow[d, "\pi_*"] \\ \cdots H^{k-n}_{dR}(U_\alpha \cup U_\beta) \arrow[r, "r"] & H^{k-n}_{dR}(U_\alpha) \oplus H^{k-n}_{dR}(U_\beta) \arrow[r, "s"] & H^{k-n}_{dR}(U_\alpha \cap U_\beta) \arrow[r, "d^*"] & H^{k+1-n}_{dR}(U_\alpha \cup U_\beta) \cdots \end{tikzcd}$$

The commutativity of the two left-most rectangles is easy to check. As for the right-most rectangle, commutativity follows from Proposition 6.6:

$$\pi_* d(\pi^* f_\alpha \omega) = \pi_*(d(\pi^* f_\alpha) \wedge \omega) = \pi_*(\pi^* df_\alpha \wedge \omega) = df_\alpha \wedge \pi_* \omega = d(f_\alpha \pi_* \omega).$$

Since π_* has been proved to be an isomorphism for trivial bundles over U_α, U_β, and $U_\alpha \cap U_\beta$, it follows from the five-lemma that

$$\pi_* : H^k_{cv}(E|_{U_\alpha \cup U_\beta}) \to H^{k-n}_{dR}(U_\alpha \cup U_\beta)$$

is also an isomorphism. Using the partial ordering by inclusions of open subsets of M and applying Zorn's Lemma, the statement of the theorem then follows from induction on the cardinality of open covers by local trivializations. □

Let $\Omega_c(M)$ denote the subalgebra of compactly supported differential forms on M. This is a subcomplex of the de Rham complex. The resulting cohomology groups are denoted by $H^*_c(M)$. Theorem 6.7 implies the compactly supported version of the Poincaré Lemma:

Corollary 6.8. *For any $m \geq 0$ we have*

$$H^k_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from applying Theorem 6.7 to a vector bundle over a point. □

Definition. Denote the inverse of the isomorphisms π_* by $\mathfrak{T}$, called the *Thom isomorphism.* Then the image of $1 \in H^0_{dR}(M)$ under the isomorphism $\mathfrak{T}$ is called the *Thom class.*

By Proposition 6.6, for any $\mu \in H^k_{dR}(M)$ we have

$$\pi_*(\pi^*\mu \smile \tau) = \mu \smile \pi_*\tau = \mu.$$

Hence, $\mathfrak{T}(\mu) = \pi^*\mu \smile \tau$.

Proposition 6.9. *The Thom class of a smooth real oriented vector bundle $\pi : E \to M$ of rank n is characterized as the unique cohomology class in $H^n_{cv}(E)$ that restricts to the oriented generator of $H^n_c(E_p)$ on each fiber E_p of $\pi : E \to M$.*

Proof. Since $\pi_*\tau = 1$, the pull-back of τ to any fiber of the vector bundle is represented by a compactly supported n-form whose integral over the fiber is 1. Conversely, if $\tau' \in H^n_{cv}(E)$ is such that it restricts to the oriented generator of $H^n_c(E_p)$ on each fiber E_p of $\pi : E \to M$, then for any $\mu \in H^k_{dR}(M)$ we have $\pi_*(\pi^*\mu \smile \tau') = \mu \smile \pi_*\tau' = \mu$. Therefore, $\mathfrak{T}(\mu) = \pi^*\mu \smile \tau'$ and $\mathfrak{T}(1) = \tau'$. □

We denote the Thom class of the vector bundle $\pi : E \to M$ by $\tau(E)$. We define the *Euler class* of a smooth real oriented vector bundle $\pi : E \to M$ of rank n as follows:

Definition. Let $s_0 : M \to E$ be the zero section. Then, there exists a homomorphism

$$s_0^* : H^n_{cv}(E) \to H^n_{dR}(M).$$

The Euler class $e(E)$ is the cohomology class $s_0^*\tau(E) \in H^n(M; \mathbb{Z})$.

Homework-3
Due 3/25/14

(1) Let M be a smooth closed oriented m-dimensional manifold. Prove that $H^m_{dR}(M)$ is non-trivial. (Hint: Construct a homomorphism from $\Omega^m(M)$ to $\mathbb{R}$ that induces a surjective homomorphism from $H^m_{dR}(M)$ onto $\mathbb{R}$.)

(2) Prove the following:
   (a) Let $F : N \to M$ be a smooth covering map, and Ω be a volume form on M. Then $F^*\Omega$ is a volume form on N. In particular, N is also orientable.
   (b) $\mathbb{R}\mathrm{P}^n$ admits a volume form if and only if n is odd. (Hint: Observe the effect of the antipodal map on the volume form
$$\Omega = \sum_{i=1}^{n+1}(-1)^{i-1}x_i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$
   on S^n.)

(3) Let M be a smooth m-dimensional manifold. Prove that if M is connected and non-compact, then $H^0_c(M) = \{0\}$.

(4) Use the notation of Example 9, but for $\mathbb{C}\mathrm{P}^2$. Let (r_x, θ_x) and (r_y, θ_y) denote the polar coordinates on fibers of E over $\pi(U_x)$ and $\pi(U_y)$, respectively. Fix a partition of unity $\{f_x, f_y\}$ subordinate to the cover $\{\pi(U_x), \pi(U_y)\}$ of $\mathbb{C}\mathrm{P}^1$. With the preceding understood, note that we have a transition function $g_{xy} : \pi(U_x \cap U_y) \to U(1)$ defined by $g_{xy}(x : y) = \frac{x|y|}{y|x|} = e^{i\phi_{xy}}$ where $0 \le \phi_{xy} < 2\pi$ and $\theta_y = \theta_x + \phi_{xy}$. Define $\eta_x = -\frac{f_x}{2\pi} d\phi_{xy}$ and $\eta_y = \frac{f_y}{2\pi} d\phi_{xy}$ on E_0, the complement of the zero section of E. **First, show that there exists a 1-form Ω on E_0 which restricts to $\frac{1}{2\pi} d\theta_x - \eta_x$ on $U_x \cap E_0$ and to $\frac{1}{2\pi} d\theta_y - \eta_y$ on $U_y \cap E_0$.** Next, consider a smooth non-decreasing function $\rho : [0, \infty) \to \mathbb{R}$ such that $\rho(r) = -1$ near $r = 0$, $\rho(r) = 0$ for $r > 2$, hence $\int_{[0,\infty)} d\rho = 1$. **Now show that $d(\rho \cdot \Omega)$ represents the Thom class of the Hopf bundle. Finally, verify that the Euler class of the Hopf bundle is represented on $\pi(U_x)$ by the differential form**
$$\frac{i}{2\pi} d\big(f_x \cdot d \ln g_{xy}\big).$$

(5) Prove the following properties of the Euler class:
- If $\pi : E \to M$ and $\pi' : E \to M$ are two smooth real oriented vector bundles, then $e(E \oplus E') = e(E) \smile e(E')$.
- If the orientation of the vector bundle $\pi : E \to M$ is reversed, then $e(E)$ changes sign.

7. March 10–March 14

Connections on Vector Bundles

Definition. Let M be a smooth m-dimensional manifold, and $\pi : E \to M$ be a smooth real vector bundle of rank n. Denote by V the kernel of $d\pi : TE \to TM$, which is a smooth vector bundle isomorphic to $\pi^* E$. A *linear Ehresmann connection* on E is a subbundle H of TE such that $TE = V \oplus H$ and H varies linearly in the fiber direction. With the direct sum decomposition $TE = V \oplus H$ in mind, there exists a bundle homomorphism $\pi^v : TE \to V$ obtained by projecting onto the kernel of $d\pi$.

To be more precise, let (U, Φ) be a local trivialization of E. Denote by $x^1, \cdots, x^m$ the coordinates on U and by $u^1, \ldots, u^n$ the coordinates on fibers of E over U. Then, $H|_{\pi^{-1}(U)}$ can be described as the kernel of linearly independent 1-forms $\theta^1, \ldots, \theta^n$ on $\pi^{-1}(U)$ that have the form

$$\theta^i = du^i + \sum_{k=1}^{m} \theta^i_k(x, u) dx^k$$

for some smooth functions θ^i_j on $\pi^{-1}(U)$ which are linear in the fiber coordinates. Therefore,

$$\theta^i_k(x, u) = \sum_{j=1}^{n} \Gamma^i_{jk}(x) u^j,$$

where Γ^i_{jk} are smooth functions on U, called the *connection coefficients*. The matrix of 1-forms A on U with entries $A^i_j = \sum_{k=1}^m \Gamma^i_{jk} dx^k$ is called the *connection matrix*. Consequently,

$$\theta^i = du^i + \sum_{j=1}^{n} A^i_j u^j.$$

In what follows, we use the *Einstein summation convention*, where super and sub indices with the same label indicate summing over the corresponding range of positive integers. Having said that, take two local trivializations (U_α, Φ_α) and (U_β, Φ_β) of E where U_α and U_β are two coordinate neighborhoods with $U_\alpha \cap U_\beta \neq \emptyset$. Denote by $x^1_\alpha, \ldots, x^m_\alpha, u^1_\alpha, \ldots, u^n_\alpha$ and $x^1_\beta, \ldots, x^m_\beta, u^1_\beta, \ldots, u^n_\beta$ the coordinates on $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$, respectively. Suppose $H|_{\pi^{-1}(U_\alpha)}$ can be described as the kernel of 1-forms $\theta^1_\alpha, \ldots, \theta^n_\alpha$ where

$$\theta^i_\alpha = du^i_\alpha + \Gamma^i_{jk} u^j_\alpha dx^k_\alpha.$$

Then

$$\begin{aligned}
\theta^i_\alpha &= d((g_{\alpha\beta})^i_r u^r_\beta) + \Gamma^i_{jk} (g_{\alpha\beta})^j_r u^r_\beta \frac{\partial x^k_\alpha}{\partial x^s_\beta} dx^s_\beta \\
&= (g_{\alpha\beta})^i_r du^r_\beta + \Big(\frac{\partial (g_{\alpha\beta})^i_r}{\partial x^s_\beta} + \Gamma^i_{jk} (g_{\alpha\beta})^j_r \frac{\partial x^k_\alpha}{\partial x^s_\beta} \Big) u^r_\beta dx^s_\beta
\end{aligned}$$

and

$$\begin{aligned}(g_{\beta\alpha})^t_i\theta^i_\alpha &= (g_{\beta\alpha})^t_i(g_{\alpha\beta})^i_r du^r_\beta + \Big((g_{\beta\alpha})^t_i\frac{\partial(g_{\alpha\beta})^i_r}{\partial x^s_\beta} + (g_{\beta\alpha})^t_i\Gamma^i_{jk}\frac{\partial x^k_\alpha}{\partial x^s_\beta}(g_{\alpha\beta})^j_r\Big)u^r_\beta dx^s_\beta \\ \theta^t_\beta &= du^t_\beta + \underbrace{\Big((g_{\beta\alpha})^t_i\frac{\partial(g_{\alpha\beta})^i_r}{\partial x^s_\beta} + (g_{\beta\alpha})^t_i\Gamma^i_{jk}\frac{\partial x^k_\alpha}{\partial x^s_\beta}(g_{\alpha\beta})^j_r\Big)}_{\Gamma^t_{rs}}u^r_\beta dx^s_\beta.\end{aligned}$$

Since $g_{\alpha\beta} = {g_{\beta\alpha}}^{-1}$, we can write

$$\Gamma^t_{rs} = (g_{\beta\alpha})^t_i\frac{\partial({g_{\beta\alpha}}^{-1})^i_r}{\partial x^s_\beta} + (g_{\beta\alpha})^t_i\Gamma^i_{jk}\frac{\partial x^k_\alpha}{\partial x^s_\beta}({g_{\beta\alpha}}^{-1})^j_r,$$

and

$$A^t_r = (g_{\beta\alpha})^t_i\frac{\partial({g_{\beta\alpha}}^{-1})^i_r}{\partial x^s_\beta}dx^s_\beta + (g_{\beta\alpha})^t_iA^i_j({g_{\beta\alpha}}^{-1})^j_r,$$

or shortly, $A_\beta = g_{\beta\alpha}A_\alpha {g_{\beta\alpha}}^{-1} + g_{\beta\alpha}d{g_{\beta\alpha}}^{-1} = g_{\beta\alpha}A_\alpha {g_{\beta\alpha}}^{-1} - (dg_{\beta\alpha}){g_{\beta\alpha}}^{-1}$. Hence, a linear Ehresmann connection on E is uniquely described by a collection of $n\times n$-matrix of 1-forms on its local trivializations which transform according to the former formula.

Next, let $s : M \to E$ be a smooth section. Then $\pi^v \circ ds : TM \to V|_{s(M)} \cong s^*V \cong E$ is a bundle homomorphism, called the *covariant derivative* associated to the linear Ehresmann connection on E. Note that $\pi^v\circ ds$ is a section of the homomorphism bundle $Hom(TM, E) \cong T^*M \otimes E$. Let (U, Φ) be a local trivialization of E as before. Then $s \in C^\infty(M; E)$ can be locally written as $s(x) = (s^1(x), \ldots, s^n(x))$ where $s^i(x) \in C^\infty(U)$, and $\pi^v \circ ds$ over U can be written as

$$\Big(\frac{\partial s^i}{\partial x^k} + s^j\Gamma^i_{jk}\Big)dx^k \otimes \frac{\partial}{\partial u^i}.$$

More generally:

Definition. A *covariant derivative operator* on E is an $\mathbb{R}$-linear map

$$\nabla : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$$

such that

(1) For any $s_1, s_2 \in C^\infty(M; E)$,

$$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2.$$

(2) (Leibniz rule) For any $f \in C^\infty(M)$ and $s \in C^\infty(M; E)$,

$$\nabla(fs) = df \otimes s + f\nabla s.$$

If $X \in \mathfrak{X}(M)$ and $s \in C^\infty(M; E)$, then the covariant derivative of s along X is given by

$$\nabla_X s := \langle \nabla s, X\rangle.$$

Note that we also have

- For any $X, Y \in \mathfrak{X}(M)$ and $s \in C^\infty(M; E)$,
$$\nabla_{X+Y} s = \nabla_X s + \nabla_Y s.$$
- For any $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $s \in C^\infty(M; E)$,
$$\nabla_{fX} s = f \nabla_X s.$$

Using the trivialization (U, Φ), we can write

$$\begin{aligned} \nabla s|_U &= \nabla(s^i \frac{\partial}{\partial u^i}) \\ &= ds^i \otimes \frac{\partial}{\partial u^i} + s^j \nabla \frac{\partial}{\partial u^j}. \end{aligned}$$

Hence, writing

$$\nabla \frac{\partial}{\partial u^j} = \Gamma^i_{jk} dx^k \otimes \frac{\partial}{\partial u^i}$$

where $\Gamma^i_{jk} \in C^\infty(U)$, we get

$$\begin{aligned} \nabla s|_U &= ds^i \otimes \frac{\partial}{\partial u^i} + s^j \Gamma^i_{jk} dx^k \otimes \frac{\partial}{\partial u^i} \\ &= \frac{\partial s^i}{\partial x^k} dx^k \otimes \frac{\partial}{\partial u^i} + s^j \Gamma^i_{jk} dx^k \otimes \frac{\partial}{\partial u^i} \\ &= \Big(\frac{\partial s^i}{\partial x^k} + \Gamma^i_{jk} s^j \Big) dx^k \otimes \frac{\partial}{\partial u^i}. \end{aligned}$$

Moreover, the functions Γ^i_{jk} transform in exactly the same way as the connection coefficients.

Exercise. Prove the last claim.

This proves that every covariant derivative on a smooth real vector bundle $\pi : E \to M$ is induced by a linear connection on E. Therefore, the notions of linear connection and covariant derivative are equivalent. It remains to show that covariant derivatives always exist:

Theorem 7.1. *Every smooth real vector bundle admits a covariant derivative.*

Proof. Let (U_α, Φ_α) be an open cover of M by local trivializations of E. Fix a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. On each $E|_{U_\alpha}$, define a covariant derivative ∇^α by

$$\nabla^\alpha s := ds^i \otimes \frac{\partial}{\partial u^i}.$$

Then,

$$\nabla s := \sum_\alpha \nabla^\alpha (f_\alpha s) = \sum_\alpha f_\alpha \nabla^\alpha (s|_{U_\alpha}),$$

where the last equality follows from the Leibniz rule and the fact that $\sum_\alpha f_\alpha = 1$. It is easy to check that ∇ is a covariant derivative on E. $\square$

If ∇ and ∇' are two covariant derivatives on E described by connection 1-forms A and A' over local trivializations, then

$$A_\beta - A'_\beta = g_{\beta\alpha}(A_\alpha - A'_\alpha)g_{\beta\alpha}^{-1}.$$

Therefore, $\nabla - \nabla' \in C^\infty(M; T^*M \otimes End(E))$, and the space of covariant derivatives on E is an affine space over the vector space $C^\infty(M; T^*M \otimes End(E))$.

Proposition 7.2. *Let ∇ be a covariant derivative on E, and $p \in M$. Then there exists a coordinate neighborhood (U, φ) of p and a local trivialization of E over U such that the corresponding connection matrix is trivial at p.*

Proof. Fix a coordinate neighborhood (U, φ) around p with coordinate functions $x^1, \ldots, x^m$ such that $x^i(p) = 0$ for all $i = 1, \ldots, m$, and a local trivialization of E over U with coordinates $u^1, \ldots, u^n$. Then write the connection matrix in this local trivialization as

$$A^i_j = \Gamma^i_{jk} dx^k,$$

and consider the linear transformation $g \in GL(n, \mathbb{R})$ defined by

$$g^i_j = \delta^i_j + \Gamma^i_{jk}(p) x^k,$$

in a possibly smaller neighborhood of p. Note that $g^i_j(p) = \delta^i_j$. Furthermore, $A_p = dg_p$. Therefore,

$$A' = gAg^{-1} - (dg)g^{-1}$$

satisfies $A'_p = 0$. □

Given a linear connection with associated covariant derivative ∇ on a smooth real vector bundle $\pi : E \to M$, we can uniquely extend the covariant derivative operator to an $\mathbb{R}$-linear operator

$$d_\nabla : C^\infty(M; \Lambda^k(M) \otimes E) \to C^\infty(\Lambda^{k+1}(M) \otimes E)$$

such that

$$d_\nabla(\mu \wedge s) := d\mu \wedge s + (-1)^{k_1} \mu \wedge d_A s,$$

for any $\mu \in \Omega^{k_1}(M)$ and $s \in C^\infty(M; \Lambda^{k_2}(M) \otimes E)$. Locally, we can write

$$d_\nabla s = ds + A \wedge s,$$

for any $s \in C^\infty(M; \Lambda^k(M) \otimes E)$. Then, over a local trivialization (U, Φ)

$$\begin{aligned} d_\nabla \nabla s &= d_\nabla(ds + As) \\ &= (dds + A \wedge ds) + ((dA)s - A \wedge \nabla s) \\ &= A \wedge ds + (dA)s - A \wedge ds - A^i_j \wedge s^j A^k_i \frac{\partial}{\partial u^k}) \\ &= (dA)s + A^k_i \wedge A^i_j s^j \frac{\partial}{\partial u^k} = (dA + A \wedge A)s \end{aligned}$$

where $\Omega = dA + A \wedge A$ is called the *curvature matrix* of the connection. Note that the transformation rule for the connection matrix yields

$$\Omega_\beta = g_{\beta\alpha}\Omega_\alpha g_{\beta\alpha}^{-1}.$$

Therefore, Ω is a globally defined 2-form with values in $C^\infty(M; End(E))$. More precisely, if $X, Y \in \mathfrak{X}(M)$, the curvature matrix defines a linear map $R(X,Y)$ from $C^\infty(M;E)$ to itself by

$$R(X,Y)(s)|_U = \Omega^i_j(X,Y)|_U s^j$$

where $s = (s^1, \ldots, s^n)$ over a local trivialization (U, Φ). Note that

- $R(X,Y) = -R(X,Y)$,
- $R(fX,Y) = fR(X,Y)$, and $R(X,Y)(fs) = fR(X,Y)(s)$.

Furthermore,

Theorem 7.3. *Let $X, Y \in \mathfrak{X}(M)$. Then*

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Proof. Exercise. □

Theorem 7.4 (Bianchi identity)**.** *The curvature matrix Ω satisfies*

$$d\Omega = \Omega \wedge A - A \wedge \Omega,$$

over any local trivialization.

Proof. Over a local trivialization:

$$\begin{aligned} d\Omega &= d(dA + A \wedge A) \\ &= dA \wedge A - A \wedge dA \\ &= (\Omega - A \wedge A) \wedge A - A \wedge (\Omega - A \wedge A) \\ &= \Omega \wedge A - A \wedge \Omega. \end{aligned}$$

□

Definition. A connection whose curvature matrix vanishes is called *flat.*

A smooth section s of the vector bundle $\pi : E \to M$ is called *parallel* if

$$\nabla s = 0.$$

More specifically,

Definition. Let γ be a smooth parametrized curve in M, and X denote the tangent vector field to γ. Then a smooth section s of the vector bundle $\pi : E \to M$ is called *parallel along* γ if

$$\nabla_X s = 0.$$

Note that the above condition amounts to solving a system of ordinary differential equations on the pull-back vector bundle γ^*E. Since the latter is the trivial vector bundle, these equations read:

$$\nabla_X s = \langle \nabla s, X\rangle = \Big(\frac{ds^i}{dt} + \Gamma^i_{jk} s^j X^k(t)\Big)\frac{\partial}{\partial u^i} = 0,$$

where $s(t) = (s^1(t), \ldots, s^n(t))$ and $u^1, \ldots, u^n$ are the fiber coordinates. Given a smooth parametrized curve $\gamma : (\epsilon, \epsilon) \to M$ through a point $p \in M$, a point $e \in E$ such that $\pi(e) = p$, and a vector $v \in E_p$, we can parallel transport v along γ by solving the initial value problem for the above system of equations. Having fixed a point $p \in M$ and a loop γ starting and ending at p, we can parallel transport all vectors in the fiber E_p to obtain a linear map from E_p to itself. Moreover, this linear map is invertible. Identifying E_p with $\mathbb{R}^n$, we can regard this automorphism of E_p as an element of $GL(n, \mathbb{R})$, called the *holonomy of the connection around* γ. Holonomy around the composition of two loops is the composition of the automorphisms around the individual loops. Therefore, holonomies around loops based at p form a subgroup $Hol_p(A)$ of $GL(n, \mathbb{R})$, called the *the holonomy group*, and we have an epimorphism

$$\pi_1(M, p) \to Hol_p(A)/Hol^\circ_p(A),$$

where $Hol^\circ_p(A)$ is the subgroup of $Hol_p(A)$ consisting of holonomies around contractible loops. A linear connection is flat if and only if $Hol^\circ_p(A)$ is trivial.

Given a linear connection with covariant derivative ∇ on a smooth real vector bundle $\pi : E \to M$, we can construct a linear connection on E^* with covariant derivative ∇^* satisfying

$$d\langle s^*, s\rangle = \langle \nabla^* s^*, s\rangle + \langle s^*, \nabla s\rangle.$$

If $F : N \to M$ is a smooth map, the covariant derivative associated to the induced connection on the pullback bundle F^*E satisfies

$$(F^*\nabla)_X F^* s = \nabla_{dF(X)} s.$$

If $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are two smooth real vector bundles with linear connections with covariant derivatives ∇^1 and ∇^2, respectively, the vector bundles $E_1 \oplus E_2$ and $E_1 \otimes E_2$ have induced covariant derivatives $\nabla^\oplus$ and $\nabla^\otimes$ defined by

$$\begin{aligned} \nabla^\oplus(s_1 \oplus s_2) &:= \nabla^1 s_1 \oplus \nabla^2 s_2 \\ \nabla^\otimes(s_1 \otimes s_2) &:= \nabla^1 s_1 \otimes s_2 + s_1 \otimes \nabla^2 s_2. \end{aligned}$$

Recall that given a (Hermitian) metric on the smooth real (complex) vector bundle $\pi : E \to M$, we can find an open cover of M by orthogonal (unitary) trivializations. With this understood, an orthogonal (unitary/Hermitian) connection on E is one whose associated covariant derivative ∇ satisfies

$$\nabla(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2).$$

$(\nabla\langle s_1, s_2\rangle = \langle\nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$.) The connection and curvature matrices of an orthogonal (unitary) connection are skew-symmetric (skew-Hermitian).

Exercise. Prove the last claim.

8. March 24–March 28

Lie Groups and Lie Algebras

Definition. A *Lie group* G is a smooth manifold without boundary that has the structure of a group such that the group operation and inversion are both smooth maps.

Note that the maps

- $L_g : G \to G$,
- $R_g : G \to G$,
- $\psi^g : G \to G$,

defined by $L_g(h) = gh$, $R_g(h) = hg$, and $\psi^g(h) = ghg^{-1}$, respectively, are diffeomorphisms.

Example 10. Examples of multiplication Lie groups include matrix groups such as:

- $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) \neq 0\}$,
- $GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$,
- $O(n) = \{A \in GL(n, \mathbb{R}) \mid AA^T = I = A^T A\}$,
- $SO(n) = \{A \in O(n) \mid det(A) = 1\}$,
- $U(n) = \{A \in GL(n, \mathbb{C}) \mid AA^* = I = A^* A\}$,
- $SU(n) = \{A \in U(n) \mid det(A) = 1\}$.

In particular, $SU(2)$ is the Lie group diffeomorphic to S^3. In order to see this, note that an element of $SU(2)$ is written as
$$\begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix},$$
where $z, w \in \mathbb{C}$ such that $|z|^2 + |w|^2 = 1$. Meanwhile, $\mathbb{C}^2$ has a multiplicative group structure $\mathbb{H}$ defined by quaternion multiplication: $(z = z_1 + iz_2, w = w_1 + iw_2)$ is identified with the quaternion $z_1 + iz_2 + jw_1 - kw_2$. Inside $\mathbb{H}$, the 3-sphere sits as the subgroup of unit quaternions. With respect to these identifications the matrix multiplication in $SU(2)$ and quaternion multiplication in S^3 coincide.

Definition. A Lie group homomorphism $F : H \to G$ between two Lie groups is a smooth map that is also a group homomorphism. A Lie group isomorphism is a Lie group homomorphism that is a diffeomorphism. A Lie subgroup H of a Lie group G is a subgroup that is also a submanifold.

Lemma 8.1. *Let G be a Lie group, and $H \subset G$ be an open subgroup. Then H is a Lie subgroup consisting of connected components of G.*

Proof. The assertion that H is a Lie subgroup is immediate from the definition. As for the assertion that H is a union of connected components of G, we show that H is a closed subset of G. To see this, note that $G \setminus H$ is a union of cosets of H not equal to itself. Since H is an open subset of G, each of these cosets are also open. Hence $G \setminus H$ is open, and H is closed. This completes the proof. □

Proposition 8.2. *Let G be a Lie group and U be an open neighborhood of the identity in G. Then U generates an open Lie subgroup of G. Furthermore, if U is connected, it generates an open connected Lie subgroup of G.*

Proof. Denote by U_k the collection of elements of G that can be written as a product of k elements from U or U^{-1}. Then $U_1 = U \cup U^{-1}$ is open and each $U_k = U_1 U_{k-1}$ can be written as a union of diffeomorphic copies of U_{k-1}. Therefore, each U_k is open by induction, and their union $H := \bigcup_k U_k$ is also open. It follows from Lemma 8.1 that H is an open subgroup of G that is a union of connected components of G. If U is connected, then so are U^{-1} and $U \cap U^{-1}$, since $U \cap U^{-1} \neq \emptyset$. Meanwhile, the map $\cdot : U_1 \times U_{k-1} \to G$ is smooth with image U_k. Hence by induction each U_k is connected, and their union is also connected since the identity element of G is contained in each U_k. □

Definition. The connected component of a Lie group G containing the identity is called the *identity component* of G. We denote the identity component of G by $G_\circ$.

Proposition 8.3. *Let G be a Lie group. Then $G_\circ$ is a normal Lie subgroup of $G_\circ$, and it is the only open connected Lie subgroup of G. Any connected component of G is diffeomorphic to $G_\circ$.*

Proof. Let $h \in G$ and consider the subgroup $hG_\circ h^{-1}$, which is a connected Lie subgroup of G isomorphic to $G_\circ$. Then $hG_\circ h^{-1} \subset hG_\circ h^{-1} \cup G_\circ = G_\circ$. Since the latter holds for any $h \in G$, $G_\circ$ is a normal subgroup of G.

Next, let $H \subset G$ be an open connected Lie subgroup. Then by Lemma 8.1, H is a connected component of G. Since $H \cap G_\circ \neq \emptyset$, we conclude $H = G_\circ$. Finally, cosets of $G_\circ$ constitute all connected components of G since they are all connected and disjoint. □

Definition. A smooth vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if $L_{g_*} X = X$ for any $g \in G$. The space of all left-invariant vector fields on G is called the *Lie algebra* of G and is denoted by $\mathfrak{g}$. The Lie algebra of G is equipped with the usual Lie bracket of smooth vector fields on G.

Theorem 8.4. *Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. Then $\mathfrak{g}$ is isomorphic to $T_e G$ where e denotes the identity element of G.*

Proof. We will show that the map $\Phi : \mathfrak{g} \to T_e G$ defined by $\Phi(X) := X_e$ is an isomorphism. To see this, first check that this is a linear map. Then note that a left-invariant vector field X on G is uniquely characterized by its value at the identity element since $X_g = L_{g_*} X_e$. This completes the proof. □

Corollary 8.5. *Any Lie group is parallelizable.*

Example 11. Consider the Lie group $GL(n,\mathbb{R})$. The Lie algebra of $GL(n,\mathbb{R})$ can be determined as follows: note that $GL(n,\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$. Then $T_I GL(n,\mathbb{R}) \cong M_n(\mathbb{R})$. Given $A \in M_n(\mathbb{R})$, a smooth parametrized curve through the identity matrix I where its tangent is the matrix A is defined by the matrix exponential

$$e^{tA} := \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k,$$

and the corresponding left-invariant vector field $X_A \in \mathfrak{gl}(n,\mathbb{R})$ is defined by $X_A(g) = L_{g_*}A = gA$ for any $g \in GL(n,\mathbb{R})$. Meanwhile, the 1-parameter group of diffeomorphisms $\{\rho^{X_A}(\cdot,t)\}$ generated by X_A is defined by $\rho^{X_A}(g,t) = ge^{tA}$. Therefore, for any $A, B \in M_n(\mathbb{R})$ we have

$$[X_A, X_B](I) = \frac{d}{dt}|_{t=0} e^{tA} B e^{-tA} = AB - BA = [A,B].$$

Analogous statements hold for the Lie group $GL(n,\mathbb{C})$ as well. Next we describe the Lie algebra of $SU(2)$. By definition, $U(2)$ is the pre-image of the identity under the map from $GL(n,\mathbb{C})$ to $M_n(\mathbb{C})$ sending A to AA^*. This map is smooth and its differential at the identity is

$$\lim_{t\to 0} \frac{e^{tA}e^{tA^*} - I}{t} = A + A^*.$$

Setting the differential equal to zero gives $T_I U(2)$ which is the space of skew-Hermitian matrices. Meanwhile, the identity $det(e^{tA}) = e^{tr(tA)}$ implies that $T_I SU(2)$ is the space of traceless skew-Hermitian matrices. Then the Lie algebra $\mathfrak{su}(2)$ is a real vector space of dimension 3 generated by the Pauli spin matrices:

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

equipped with the commutator bracket.

Theorem 8.6. *A Lie group homomorphism $F : H \to G$ induces a Lie algebra homomorphism $F_* : \mathfrak{h} \to \mathfrak{g}$.*

Proof. Let Y be left-invariant vector field on H. Then denote by $X_{e_G} \in T_{e_G}G$ the tangent vector $dF_{e_H}(Y_{e_H})$. This tangent vector corresponds to a left-invariant vector field X on G by $X_g = L_{g_*}X_{e_G}$. First, we claim that for any $h \in H$

$$dF_h Y_h = X_{F(h)}.$$

To see this, note that $F \circ L_h = L_{F(h)} \circ F$ and hence $F_* \circ L_{h*} = L_{F(h)_*} \circ F_*$. As a result,

$$dF_h Y_h = F_*(L_{h*}Y_{e_H}) = L_{F(h)_*}(F_* Y_{e_H}) = L_{F(h)_*}X_{e_G} = X_{F(h)}.$$

To finish the proof, we need to show that the map F_* respects the Lie brackets. Let Y_1 and Y_2 be left-invariant vector fields on H, then

$$F_*[Y_1, Y_2] = [F_*Y_1, F_*Y_2].$$

This is because for any $f \in C^\infty(G)$, we have

$$(F_*[Y_1, Y_2])f = df(F_*[Y_1, Y_2]) = (F^*df)([Y_1, Y_2]) = d(f \circ F)([Y_1, Y_2]) = [Y_1, Y_2](f \circ F).$$

and

$$\begin{aligned} [Y_1, Y_2](f \circ F) &= Y_1Y_2(f \circ F) - Y_2Y_1(f \circ F) \\ &= Y_1((F_*Y_2 f) \circ F) - Y_2((F_*Y_1 f) \circ F) \\ &= ((F_*Y_1F_*Y_2)f) \circ F - ((F_*Y_2F_*Y_1)f) \circ F \\ &= ([F_*Y_1, F_*Y_2]f) \circ F \end{aligned}$$

as is easily verified by a computation using local coordinates. □

Next, we generalize the above example to more general Lie groups and their Lie algebras. In this regard, remember that a smooth vector field is complete if the maximal integral curve through any point is defined for all time.

Lemma 8.7. *Every left-invariant vector field on a Lie group is complete.*

Proof. Let X be a left-invariant vector field on G, and ρ denote the local 1-parameter family of diffeomorphisms as provided by Theorem 3.1. Then

$$L_g \circ \rho(\cdot, t) = \rho(\cdot, t) \circ L_g,$$

for any $g \in G$. To see this, let $h \in G$ and $\gamma = \rho(h, \cdot)$ be the maximal integral curve of X through h. Then $\hat{\gamma} = L_g \circ \gamma$ is an integral curve of X through gh since

$$\begin{aligned} \frac{d}{dt}|_{t=s}\hat{\gamma}(t) &= \frac{d}{dt}|_{t=s}L_g \circ \rho(h, t) \\ &= L_{g*}(\frac{d}{dt}|_{t=s}\rho(h, t)) \\ &= L_{g*}(X_{\rho(h,s)}) = X_{g\cdot\rho(h,s)} \end{aligned}$$

and by uniqueness of ρ the desired equality follows. Now suppose $\gamma(t)$ is defined for $t \in (a, b)$ where $b < \infty$, and $\rho(e, t)$ is defined for $t \in (-\epsilon, \epsilon)$. Then choose $t_\circ \in (b - \epsilon, b)$ and define

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in (a, b), \\ L_{\rho(h,t_\circ)}(\rho(e, t - t_\circ)) & \text{if } t \in (t_\circ - \epsilon, t_\circ + \epsilon). \end{cases}$$

It is easy to check using $L_g \circ \rho(\cdot, t) = \rho(\cdot, t) \circ L_g$ that the two lines of the definition of the curve $\tilde{\gamma}$ agree on $(t_\circ - \epsilon, b)$. Furthermore, by virtue of the fact that X is a left-invariant vector field, we have

$$\frac{d}{dt}|_{t=s}\tilde{\gamma}(t) = X_{\tilde{\gamma}(s)},$$

for $s \in (t_\circ - \epsilon, t_\circ + \epsilon)$, hence the $\tilde{\gamma}$ is an integral curve of X through h. But this contradicts the maximality of γ since $t_\circ + \epsilon > b$. Similar arguments can be used to derive a contradiction assuming $a > -\infty$. □

Definition. Let G be a Lie group. A 1-parameter subgroup of G is a Lie group homomorphism from $\mathbb{R}$ into G.

Lemma 8.8. *Let G be a Lie group and X be a left-invariant vector field on G. Then the integral curve of X through the identity element is a 1-parameter subgroup of G.*

Proof. Let ρ denote the 1-parameter group of diffeomorphisms (or flow) of X, which is complete by Lemma 8.7. Then $\rho(e, \cdot) : \mathbb{R} \to G$ is the integral curve of X through the identity element. It follows from $L_g(\rho(e,s)) = \rho(g,s)$ and $\rho(\rho(e,t),s) = \rho(e,t+s)$ that

$$\rho(e,t) \cdot \rho(e,s) = \rho(e,t+s).$$

This proves the claim. □

Theorem 8.9. *Given a 1-parameter subgroup of a Lie group G, there exists a unique left-invariant vector field on G generating it. Consequently, there is a one-to-one correspondence between 1-parameter subgroups of G and left-invariant vector fields on G.*

Proof. Let $\gamma : \mathbb{R} \to G$ be a 1-parameter subgroup of G. The vector field $\frac{d}{dt}$ is the left-invariant vector field on $\mathbb{R}$. Meanwhile, $d\gamma_0(\frac{d}{dt}|_{t=0}) = X_e \in T_eG$ yields a left invariant vector field X on G by Theorem 8.4. It suffices to prove that γ is an integral curve of X through e. This follows from

$$\gamma'(s) = d\gamma_s(\frac{d}{dt}|_{t=s}) = d\gamma_s(L_{s*}\frac{d}{dt}|_{t=0}) = L_{\gamma(s)_*}(d\gamma_0(\frac{d}{dt}|_{t=0})) = L_{\gamma(s)_*}X_e = X_{\gamma(s)}.$$

□

Definition. Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by sending a left-invariant vector field X to $\gamma(1)$ where γ is the 1-parameter subgroup of G generated by X.

Theorem 8.10. *Let G be Lie group and $\mathfrak{g}$ denote its Lie algebra. Then*

(1) *The exponential map is smooth.*
(2) *Given a left-invariant vector field X on G, $\gamma(t) = \exp(tX)$ is the 1-parameter subgroup generated by X. Consequently, $\exp((t+s)X) = \exp(tX) \cdot \exp(sX)$.*
(3) *The differential at the identity $d\exp_0 : T_0\mathfrak{g} \to T_eG$ is the identity map under the canonical identifications of $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$. Moreover, the restriction of the exponential map to some open neighborhood of 0 in $\mathfrak{g}$ is a diffeomorphism onto its image.*

(4) *Given a Lie group homomorphism $F : H \to G$, we have a commutative diagram*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "F_*"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ H \arrow[r, "F"] & G \end{tikzcd}$$

(5) *The flow ρ generated by a left-invariant vector field X is given via right multiplication by* $\exp(tX)$.

Proof. We prove the first three items and leave the last two as homework. First of all, (1) follows from an application of Theorem 3.1 to the vector field X defined on $G \times \mathfrak{g}$ by $\mathrm{X}_{(g,X)} = (X_g, 0)$, since Note that $\rho^{\mathrm{X}}((g,X),\cdot) = (\rho^X(g,\cdot), X)$. To prove (2), let ρ^t denote the flow generated by tX. To prove the first part of the claim, we need to show that $\rho^t(e,1) = \rho^1(e,t)$. In this regard, fix $t \in \mathbb{R}$ and let $\gamma : \mathbb{R} \to G$ be defined by $\gamma(s) = \rho^1(e, st)$. Then

$$\gamma'(s) = tX_{\gamma(s)},$$

by the chain rule, which shows that $\gamma(s) = \rho^t(e,s)$. The second part of the claim follows immediately from the first part.

To prove (3), let $X \in T_0\mathfrak{g} \cong \mathfrak{g}$. Consider the line $\ell : \mathbb{R} \to \mathfrak{g}$ defined by $\ell(t) = tX$. Then by (2), we have

$$d\exp_0 X = \frac{d}{dt}|_{t=0}\exp \circ \ell = \frac{d}{dt}|_{t=0}\exp(tX) = X.$$

This proves the first part of the claim. As for the second part, use the Inverse Function Theorem to find an open neighborhood of $0 \in \mathfrak{g}$ with the desired property. □

The exponential map is used to prove the following theorem due to Cartan:

Theorem 8.11 (Cartan's Theorem)**.** *A closed subgroup of a Lie group is a Lie subgroup.*

Cartan's Theorem then implies:

Proposition 8.12. *An embedded subgroup of a Lie group is closed if and only if it is a Lie subgroup.*

Homework-4
Due 4/10/14

(1) Prove Theorem 7.3.
(2) Prove that the the connection and curvature matrices of an orthogonal and unitary connection are skew-symmetric and skew-Hermitian, respectively.
(3) Find the Lie algebra of $SO(n)$.
(4) Prove (4) and (5) in Theorem 8.10. (Hint: Use (2) in Theorem 8.10)

9. March 31–April 4

A good reference for the following discussion is the book [6] by John M. Lee.

Proposition 9.1. *Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. Then there exists a one-to-one correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

Proof. Connected Lie subgroups of G define Lie subalgebras of $\mathfrak{g}$ by Theorem 8.6. To prove the converse, let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, and define a distribution Δ on G by $\Delta_g = L_{g*}\mathfrak{h}$. Then Δ is closed under Lie bracket since $[L_{g*}X, L_{g*}Y] = L_{g*}[X, Y]$. Therefore, it follows from the Frobenius Theorem (Theorem 4.6) that Δ is completely integrable, and there exists a maximal connected integral submanifold H of Δ through the identity element e in G. For any $h \in H$, $h^{-1}H$ is also an integral submanifold of Δ through e. By the maximality of H, we have $h^{-1}H \subset H$, and hence $h^{-1} \in H$. Now the same argument also proves that H is closed under the group operation. Therefore, H is a subgroup of G, and as is easily verified, the group operation and inversion on H are smooth maps. Finally, the claim that H is the unique connected Lie subgroup with Lie algebra $\mathfrak{h}$ follows from Lemma 8.1 and Proposition 8.2 since any two connected Lie subgroups H and H' of G with the same Lie algebra would agree on an open neighborhood of the identity in H by item (3) in Theorem 8.10. □

Definition. A *Lie group covering* is a Lie group homomorphism $F : H \to G$ that is also a covering map. Note that F is surjective with discrete kernel. In fact, the converse is also true (See [6, Proposition 9.30]).

Exercise. The universal cover of a connected Lie group is also a Lie group. To show this, lift the smooth structure and the group structure by the covering map having fixed an element in the pre-image of the identity element.

Theorem 9.2 (Corollary 20.16 in [6])**.** *Let H and G be simply connected Lie groups with isomorphic Lie algebras. Then H and G are isomorphic as Lie groups.*

Theorem 9.3 (Ado's Theorem)**.** *Every finite dimensional Lie algebra admits a faithful finite dimensional representation.*

Theorem 9.4. *There exists a one-to-one correspondence between isomorphism classes of finite dimensional Lie algebras and isomorphism classes of simply connected Lie groups.*

Proof. Given a finite dimensional Lie algebra $\mathfrak{g}$, Ado's Theorem (Theorem 9.3) implies that $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ with the commutator bracket. Then it follows from Proposition 9.1 that there exists a unique connected Lie subgroup G of $GL(n, \mathbb{R})$ with Lie algebra isomorphic to $\mathfrak{g}$. The universal cover $\widetilde{G}$ of G is a simply connected Lie group with Lie algebra isomorphic to $\mathfrak{g}$. Hence, by Theorem 9.2, $\widetilde{G}$ is the unique, up to isomorphism, Lie group with Lie algebra isomorphic to $\mathfrak{g}$. □

Theorem 9.5 (Theorem 20.21 in [6]). *Given a finite dimensional Lie algebra $\mathfrak{g}$, isomorphism classes of Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}$ are in one-to-one correspondence with G/Γ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, and Γ is a discrete central subgroup of G.*

Maurer–Cartan Form

Definition. Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. Then the *adjoint representation* of G by $T_eG \cong \mathfrak{g}$ is

$$Ad : G \to GL(\mathfrak{g})$$

defined by $Ad(g) := d\psi^g{}_e : T_eG \to T_eG$. Note that since $\psi^g = L_g \circ R_{g^{-1}}$, $d\psi^g{}_e = L_{g*} \circ R_{g^{-1}}{}_*$. The *adjoint representation* of $\mathfrak{g}$ is

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by $ad(X_e) := d(Ad)_e(X_e)$.

Example 12. If G is a Lie subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$, then

$$Ad(g)(B) = \frac{d}{dt}|_{t=0} g e^{tB} g^{-1} = gBg^{-1},$$

for any $g \in G$ and $B \in \mathfrak{g}$, and

$$ad(A)(B) \frac{d}{dt}|_{t=0} Ad_{e^{tA}} B = [A, B],$$

for any $A, B \in \mathfrak{g}$.

Definition. Let G be a Lie group and $\mathfrak{g}$ denote its Lie algebra. A 1-form ω on G is called *left-invariant* if for any $g \in G$, $L_g{}^*\omega = \omega$. The *Maurer–Cartan* form on G is the canonical left invariant 1-form with values in $T_eG \cong \mathfrak{g}$ defined by

$$\omega_g(v) := L_{g^{-1}}{}_* v,$$

for any $v \in T_gG$. It follows from the definition that if X is a left invariant vector field on G, i.e. $X_g = L_{g*}X_e$, then

$$\omega_g(X_g) = L_{g^{-1}}{}_* X_g = X_e,$$

for all $g \in G$. Moreover,

(1) $w_e : T_eG \to T_eG$ is the identity map,
(2) $R_g{}^*\omega_g = Ad(g^{-1})\omega_e$ for any $g \in G$.

The above properties uniquely characterize the Maurer–Cartan form.

Now, let X, Y be left-invariant vector fields on G, then by Corollary 4.2 we have

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]).$$

Meanwhile, $\omega([X,Y]) = [\omega(X), \omega(Y)]$ since

$$\begin{aligned}\omega_g([X,Y](g)) &= \omega_g(L_{g*}[X,Y](e)) \\ &= (L_g{}^*\omega_g)([X,Y](e)) \\ &= \omega_e([X,Y](e)) \\ &= [X_e, Y_e] \\ &= [\omega_g(X_g), \omega_g(Y_g)].\end{aligned}$$

Hence

$$(d\omega + \frac{1}{2}[\omega,\omega])(X,Y) = 0,$$

where $[\omega,\omega](X,Y) := [\omega(X),\omega(Y)] - [\omega(Y),\omega(X)]$. In fact, the above equation holds for any pair of smooth vector fields $X, Y \in \mathfrak{X}(G)$. Hence, we obtain the *Maurer–Cartan equation*:

$$d\omega + \frac{1}{2}[\omega,\omega] = 0.$$

Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g} \cong T_eG$, and $\theta^1, \ldots, \theta^n$ be the dual basis of left-invariant 1-forms. Then

$$[X_j, X_k] = \sum_{i=1}^{n} c^i_{jk} X_i,$$

where c^i_{jk} are called the *structure constants*, and with

$$d\theta^i(X_j, X_k) = -\theta^i(X_j, X_k) = -c^i_{jk}$$

the Maurer–Cartan equation reads:

$$\begin{aligned} d\theta^i + \frac{1}{2}\sum_{j,k} c^i_{j,k}\theta^j \wedge \theta^k &= 0 \\ c^i_{jk} + c^i_{kj} &= 0. \end{aligned}$$

The latter are called the *structure equations* for the Lie algebra $\mathfrak{g}$. By exterior differentiating the top equation, we find that the structure constants satisfy the Jacobi identity:

$$\sum_{j=1}^{n} (c^i_{sj}c^s_{k\ell} + c^i_{sk}c^s_{\ell j} + c^i_{s\ell}c^s_{jk}) = 0.$$

As we will see soon, the Maurer–Cartan form is the connection 1-form for the canonical principal connection on G regarded as the trivial principal G-bundle over a point.

Principal Bundles

A good reference for this subject is [5] by Dale Husemoller.

Definition. Let G be a Lie group and M be a smooth manifold. A *principal G-bundle* over M is a smooth manifold P with a smooth and free right G-action such that P/G is diffeomorphic to M, and a projection map $\pi : P \to M$ such that every point $p \in M$

admits an open neighborhood U and a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times G$, called a *local trivialization*, with the property that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi_U"] \arrow[d, "\pi"'] & U \times G \arrow[d, "\pi_1"] \\ U \arrow[r, "id_U"] & U \end{tikzcd}$$

where π_1 is the projection onto the first factor.

Example 13. The product principal G-bundle is

$$\pi_1 : M \times G \to M$$

where π_1 is the projection onto the first factor. Here the right G action on $M \times G$ is defined in the obvious way: $(x, g) \cdot h := (x, gh)$ for any $x \in M$ and $g, h \in G$.

Definition. A principal G-bundle $\pi : P \to M$ is said to be *trivial* if there exists a G-equivariant diffeomorphism $\phi : P \to M \times G$, called a *trivialization*, such that the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\pi"'] & M \times G \arrow[d, "\pi_1"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

A *homomorphism of principal bundles* from a principal G-bundle $\pi' : P' \to N$ to a principal G-bundle $\pi : P \to M$ is a smooth G-equivariant map $\widetilde{F} : P' \to P$ lifting a smooth map $F : N \to M$ in the sense that the following diagram commutes:

$$\begin{tikzcd} P' \arrow[r, "\widetilde{F}"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ N \arrow[r, "F"] & M \end{tikzcd}$$

An *isomorphism of principal bundles* between principal G-bundles $\pi' : P' \to M$ and $\pi : P \to M$ is a G-equivariant diffeomorphism $\phi : P' \to P$ such that the following diagram commutes:

$$\begin{tikzcd} P' \arrow[r, "\phi"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

Remark. Any homomorphism of principal G-bundles over the same base manifold lifting the identity map on the base is an isomorphism of principal G-bundles.

A smooth *section* of a principal G-bundle $\pi : P \to M$ is a smooth map $s : M \to P$ such that $\pi \circ s = id_M$. Unlike the case of vector bundles, existence of a smooth section is sufficient to say that a principal G-bundle is trivial.

Proposition 9.6. *A principal G-bundle $\pi : P \to M$ is trivial if and only if it admits a smooth section.*

Proof. If $\pi : P \to M$ is a trivial principal G-bundle, then there exists a G-equivariant diffeomorphism $\phi : P \to M \times G$ as in the above definition. Therefore, it suffices to find a section s of $\pi_1 : M \times G \to M$. A section can be defined by $s(x) := (x, e)$. Composing the latter with ϕ^{-1} gives a smooth section of $\pi : P \to M$. Conversely, if there exists a smooth section s of $\pi : P \to M$, we can define a smooth map $\phi : P \to M \times G$ by $\phi(s(x) \cdot g) = (x, g)$. It is easy to check that this is a principal bundle isomorphism between $\pi : P \to M$ and the product principal G-bundle. □

Given a principal G-bundle $\pi : P \to M$, one can find a cover of M by G-equivariant local trivializations. To see this, start by taking a cover of M by local trivializations and fix a smooth section of each local trivialization. Then define a cover of M by G-equivariant local trivializations defined by these sections as in the proof of Proposition 9.6.

Note that a G-equivariant local trivialization (U, Φ_U) has the form $\Phi_U(p) = (\pi(p), g_U(p))$ where $g_U : \pi^{-1}(U) \to G$ is a smooth G-equivariant map. If (U_α, Φ_α) and (U_β, Φ_β) are two G-equivariant local trivializations with $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G$$

has the form $\Phi_\alpha \circ \Phi_\beta^{-1}(x, g) = (x, g_{\alpha\beta}(p) \cdot g)$ where $p = \Phi_\beta^{-1}(x, g)$ and $g_{\alpha\beta}(p) = g_\alpha(p) \cdot g_\beta^{-1}(p)$. The G-equivariant condition proves

$$\begin{aligned} g_{\alpha\beta}(p \cdot g) &= g_\alpha(p \cdot g) \cdot g_\beta^{-1}(p \cdot g) \\ &= g_\alpha(p) \cdot g \cdot g^{-1} \cdot g_\beta^{-1}(p) \\ &= g_{\alpha\beta}(p), \end{aligned}$$

and hence $g_{\alpha\beta}$ reduces to a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. Now, if we choose a cover $\{(U_\alpha, \Phi_\alpha)\}$ of M by G-equivariant local trivializations, then the functions $\{g_{\alpha\beta}\}$ satisfy the cocycle conditions:

- $g_{\alpha\alpha}(x) = e$ for any $x \in U_\alpha$,
- $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = e$ for any $x \in U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

The functions $\{g_{\alpha\beta}\}$ are called the *transition functions*, and the Lie group becomes the *structure group* of the principal bundle. As in the case of vector bundles, a principal G-bundle can be recovered from the data of its transition functions. In this regard, two sets of transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define isomorphic principal G-bundles if and only if there exist smooth maps $\{h_\alpha : U_\alpha \to G\}$ such that

$$g'_{\alpha\beta} = h_\alpha^{-1} \cdot g_{\alpha\beta} \cdot h_\beta.$$

Exercise. Prove the last claim. To do this, note that an isomorphism of the local trivialization (U_α, Φ_α) is uniquely determined by a smooth map $h_\alpha : U_\alpha \to G$.

Similar to vector bundles, we can pull-back principal G-bundles via smooth maps. Let $\pi : P \to M$ be a principal G-bundle and $F : N \to M$ be a smooth map. Then

$$F^*P := N \times_M P = \{(x, p) \mid F(x) = \pi(p)\}$$

admits a smooth right G-action defined by $(x, p) \cdot g := (x, p \cdot g)$. This action is free since the G-action on P is free. Meanwhile, the map $F^*\pi : F^*P \to N$ defined by $F^*\pi(x, p) = x$ fits into the following commutative diagram:

$$\begin{tikzcd} F^*P \arrow[r, "\pi_2"] \arrow[d, "F^*\pi"'] & P \arrow[d, "\pi"] \\ N \arrow[r, "F"] & M \end{tikzcd}$$

where π_2 is the projection onto the second factor. With the preceding understood, it is easily verified that $F^*\pi : F^*P \to N$ is a principal G-bundle.

Definition. Let G be a Lie group and $\rho : G \to GL(n, \mathbb{R})$ be a real finite dimensional representation. Then given a principal G-bundle $\pi : P \to M$, there exists an associated smooth real vector bundle of rank n with total space

$$P \times_\rho \mathbb{R}^n := P \times \mathbb{R}^n/(p, v) \simeq (p \cdot g, \rho(g^{-1})(v)),$$

base M, and bundle projection $\pi^\rho : P \times_\rho \mathbb{R}^n \to M$ defined by $\pi^\rho[p, v] := \pi(p)$. Moreover, the transition functions of this vector bundle are the images under ρ of the transition functions for $\pi : P \to M$. Similarly, a complex finite dimensional representation leads to a smooth complex vector bundle.

Suppose H is another Lie group and $\rho : G \to Aut(H)$ is a group homomorphism, then the above recipe produces a principal H-bundle.

Example 14. Let G be a Lie group, $\mathfrak{g}$ denote its Lie algebra, and $\pi : P \to M$ be a principal G-bundle. Recall that the adjoint representation $Ad : G \to GL(\mathfrak{g})$ is defined by $Ad(g) = d\psi^g{}_e$. We denote the associated smooth vector bundle $P \times_{Ad} \mathfrak{g}$ by $Ad(P)$, called the *adjoint bundle*. Now consider the homomorphism $\Psi : G \to Aut(G)$ defined by $\Psi(g) = \psi^g$. Without ambiguity, we also call the associated principal G-bundle $P \times_\Psi G$ the *adjoint bundle*.

Definition. Let G be a Lie group and M be a smooth manifold. A *gauge transformation* of a principal G-bundle $\pi : P \to M$ is an automorphism of it. In other words, a gauge transformation is a G-equivariant diffeomorphism $u : P \to P$ which fits into the following commutative diagram:

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

Clearly, gauge transformations form a group $\mathcal{G}$, called the *gauge group.* Given an element $u \in \mathcal{G}$, there exists a smooth map $g_u : P \to G$ defined by $u(p) = p \cdot g_u(p)$. The fact that u is G-equivariant implies that

$$p \cdot g_u(p) \cdot g = u(p) \cdot g = u(p \cdot g) = p \cdot g \cdot g_u(p \cdot g),$$

and hence, $g_u(p \cdot g) = g^{-1} \cdot g_u(p) \cdot g$. Therefore, we can regard u as a smooth section of the adjoint bundle $P \times_\Psi G$.

Theorem 9.7. *Let G be a Lie group and P be a principal G-bundle over a smooth manifold M. Suppose that either P is trivial or that G is Abelian. Then the gauge group is isomorphic to the group $C^\infty(M, G)$.*

Proof. The case when P is the trivial principal G-bundle, i.e. $P \cong M \times G$, follows from the observation that any smooth G-equivariant map from P to itself is determined by its restriction to $M \times \{e\}$, and hence is induced by a smooth map from M to G. To be more explicit, a gauge transformation u of $M \times G$ is determined by a smooth map $\phi_u : M \to G$ such that $g_u : M \times G \to G$ is given by $g_u(x, g) = g^{-1} \cdot \phi_u(x) \cdot g$, and $u(x, g) = (x, g \cdot g_u(x, g)) = (x, \phi_u(x) \cdot g)$. As for the case when G is Abelian, the result follows from the observation that any smooth map $g_u : P \to G$ satisfying $g_u(p \cdot g) = g^{-1} \cdot g_u(p) \cdot g$ is invariant under the right G-action on P. $\square$

10. April 7–April 11

Connections on Principal Bundles

Definition. Let G be a Lie group of dimension n, M be a smooth manifold of dimension m, and $\pi : P \to M$ be a principal G-bundle. Denote by V the real vector bundle of rank n that is the kernel of $d\pi : TP \to TM$. The fibers of V are tangent spaces to the fibers of P. A *principal Ehresmann connection* on P is a subbundle H of TP such that $TP = H \oplus V$ and $R_{g*} H_p = H_{p \cdot g}$ for $g \in G$ and $p \in P$, i.e. H is G-invariant.

Denote by $\mathfrak{g}$ the Lie algebra of G, i.e. $T_e G$. For any $p \in P$ consider the smooth G-equivariant map $i_p : G \to P$ defined by $i_p(g) = p \cdot g$. Then $i_{p*} : \mathfrak{g} \to V_p \subset T_p P$ is an isomorphism since G acts freely on P. We can describe H_p as the kernel of a linear surjective map $\omega_p : TP \to \mathfrak{g}$ that is the composition of the projection map from $T_p P = H_p \oplus V_p$ to V_p and $i_{p*}^{-1} : V_p \to \mathfrak{g}$. Because H is a smooth vector bundle over P, it is then described by a 1-form ω on P with values in $\mathfrak{g}$. This 1-form satisfies

(1) $\omega_p(i_{p*} X_e) = X_e$ for any $p \in P$ and $X_e \in \mathfrak{g}$,
(2) $R_g{}^* \omega_{p \cdot g} = Ad(g^{-1}) \omega_p$ for any $p \in P$ and $g \in G$.

Having written any $v \in T_pP$ uniquely as $v = v^h + i_{p_*}X_e$ where $v^h \in H_p$ and $X_e \in \mathfrak{g}$, the first property follows immediately from the definition. As for the second property:

$$\begin{aligned} R_g{}^*\omega_{p\cdot g}(v) &= \omega_{p\cdot g}(R_{g_*}(v^h + i_{p_*}X_e)) = \omega_{p\cdot g}(R_{g_*}i_{p_*}X_e) = \omega_{p\cdot g}(i_{p\cdot g_*}(Ad(g^{-1})X_e) \\ &= Ad(g^{-1})X_e = Ad(g^{-1})\omega_p(i_{p_*}X_e) = Ad(g^{-1})\omega_p(v), \end{aligned}$$

since $R_g \circ i_p(h) = i_{p\cdot g}(g^{-1}hg)$. These properties imply that $i_p{}^*\omega$ is the Maurer–Cartan form on G for any $p \in P$.

Notation. Given a Lie group G with Lie algebra $\mathfrak{g}$, and a smooth manifold M, we shall denote the space of k-forms on a principal G-bundle $\pi : P \to M$ by $\Omega^k(P;\mathfrak{g})$. To be more specific, these are smooth sections of the vector bundle $\Lambda^k(P) \otimes \mathfrak{g}$.

Conversely, a Lie algebra valued 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ satisfying the above properties defines a principal Ehresmann connection on P by its kernel. To see this, note that the kernel of A is a subbundle $H \subset TP$ of rank m, and for any $v^h \in H_p$, $\omega_{p\cdot g}(R_{g_*}v^h) = Ad(g^{-1})\omega_p(v^h) = 0$ by the second property above. Hence, H is G-invariant. A 1-form $\omega \in \Omega^1(P;\mathfrak{g})$ with the above properties is called a *connection form* on P.

Let (U_α, Φ_α) be a G-equivariant local trivialization of P such that $\Phi_\alpha(p) = (\pi(p), g_\alpha(p))$ where $g_\alpha : \pi^{-1}(U_\alpha) \to G$ is G-equivariant, and $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ be the local section canonically associated to this trivialization, i.e. $\Phi_\alpha(s_\alpha(x)) = (x, e)$. Then $A_\alpha := s_\alpha{}^*\omega \in \Omega^1(U_\alpha;\mathfrak{g})$ satisfies

$$\omega|_{\pi^{-1}(U_\alpha)} = Ad(g_\alpha^{-1})\pi^*A_\alpha + g_\alpha{}^*\omega^{MC}.$$

To see this, first note that since for any $p \in \pi^{-1}U_\alpha$ we have $g_\alpha \circ i_p = L_{g_\alpha(p)}$, the first property above is satisfied by the right-hand side of the equation:

$$\begin{aligned} Ad(g_\alpha(p)^{-1})(\pi^*A_\alpha)_p(i_{p_*}X_e) + (g_\alpha{}^*\omega^{MC})_p(i_{p_*}X_e) &= Ad(g_\alpha(p)^{-1})(A_\alpha)_{\pi(p)}(\pi_*(i_{p_*}X_e)) \\ &\quad + \omega^{MC}_{g_\alpha(p)}(g_{\alpha *}(i_{p_*}X_e)) \\ &= \omega^{MC}_{g_\alpha(p)}(X_{g_\alpha(p)}) \\ &= X_e. \end{aligned}$$

As for the second property above, note that $s_\alpha \circ \pi \circ R_g = s_\alpha \circ \pi$, and since g_α is G-equivariant we have $Ad((g_\alpha \circ R_g)^{-1}) = Ad(g^{-1}) \circ Ad(g_\alpha^{-1})$ and $R_g{}^*g_\alpha{}^*\omega^{MC} = g_\alpha{}^*R_g{}^*\omega^{MC} = Ad(g^{-1})g_\alpha{}^*\omega^{MC}$. Therefore, it suffices to prove the above equality at $p \in \pi^{-1}(U_\alpha)$ where $p = s_\alpha(x)$ for some $x \in U_\alpha$. In this regard, any $v \in T_pP$ is uniquely written as $v = (s_\alpha \circ \pi)_*(v) + i_{p_*}X_e$. Having said that, since $g_\alpha \circ s_\alpha = e$

$$\begin{aligned} Ad(g_\alpha(p)^{-1})(\pi^*A_\alpha)_p(v) + (g_\alpha{}^*\omega^{MC})_p(v) &= (\pi^*s_\alpha^*\omega)_p(v) + \omega_e^{MC}(g_{\alpha *}(v)) \\ &= \omega_p((s_\alpha \circ \pi)_*(v)) + \omega_e^{MC}(g_{\alpha *}i_{p_*}X_e) \\ &= \omega_p((s_\alpha \circ \pi)_*(v)) + \omega_p(i_{p_*}(g_{\alpha *}i_{p_*}X_e) \\ &= \omega_p((s_\alpha \circ \pi)_*(v)) + \omega_p(i_{p_*}X_e) = \omega_p(v) \end{aligned}$$

If (U_β, Φ_β) is another G-equivariant local trivialization of P such that $U_\alpha \cap U_\beta \neq \emptyset$, $\Phi_\beta(p) = (\pi(p), g_\beta(p))$ where $g_\beta : \pi^{-1}(U_\beta) \to G$ is G-equivariant, and $s_\beta : U_\alpha \to \pi^{-1}(U_\beta)$ is the local section canonically associated to this trivialization, then $g_\alpha \circ s_\beta = g_{\alpha\beta}$ and

$$\begin{aligned} A_\beta = {s_\beta}^*\omega &= {s_\beta}^*\omega|_{\pi^{-1}(U_\alpha)} \\ &= {s_\beta}^*\big(Ad(g_\alpha^{-1})\pi^* A_\alpha + {g_\alpha}^*\omega^{MC}\big) \\ &= Ad(g_{\alpha\beta}^{-1})A_\alpha + {g_{\alpha\beta}}^*\omega^{MC}. \end{aligned}$$

Conversely, given a cover $\{U_\alpha\}$ of M by G-equivariant local trivializations $\{(U_\alpha, \Phi_\alpha)\}$ of P, a collection $\{A_\alpha \in \Omega^1(U\alpha; \mathfrak{g})\}$ obeying the above transformation rule defines a connection 1-form $\omega \in \Omega^1(P; \mathfrak{g})$.

Exercise. Let P be a smooth manifold, G be a Lie group, and $g : P \to G$ be a smooth map. Note that ${g^*\omega^{MC}}_p = {L_{g(p)}}_* \circ dg_p$. Next, let G be the Lie group $GL(n, \mathbb{R})$. Show that $g^*\omega^{MC} = g^{-1}dg$, and the connection form of a principal connection on a principal G-bundle transform according to the rule:

$$A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} + g_{\beta\alpha} d g_{\beta\alpha}^{-1}.$$

Existence of principal connections on principal G-bundles is proved in a similar fashion to the existence of connections on a vector bundle using a partition of unity and patching pull-backs of Maurer–Cartan forms over G-equivariant local trivializations. Given a pair of connection forms ω_0 and ω_1 on a principal G-bundle $\pi : P \to M$ with G-equivariant local trivializations $\{(U_\alpha, \Phi_\alpha)\}$, if (U_α, Φ_α) has canonically associated section $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$, then

$$(\omega_1 - \omega_0)|_{\pi^{-1}(U_\alpha)} = Ad(g_\alpha^{-1})\pi^*(A_{1\alpha} - A_{0\alpha}),$$

where $A_{0\alpha} = {s_\alpha}^*\omega_0$ and $A_{1\alpha} = {s_\alpha}^*\omega_1$. Moreover

$$A_{1\beta} - A_{0\beta} = Ad(g_{\alpha\beta}^{-1})(A_{1\alpha} - A_{0\alpha}).$$

Hence, the collection $\{A_{1\alpha} - A_{0\alpha}\}$ defines a 1-form on M with values in $Ad(P)$. Conversely, having fixed a principal connection on P, any 1-form on M with values in $Ad(P)$ can be used to define a new principal connection. Therefore, the space $\mathcal{A}(P)$ of principal connections on P is an affine space modeled on $\Omega^1(M; Ad(P))$.

The gauge group acts on the space of principal connections as follows: let ω be a connection form on P and $u \in \mathcal{G}$ be a gauge transformation such that $u(p) = p \cdot g_u(p)$ for some smooth map $g_u : P \to G$. The action of u on ω is defined as follows:

$$u \cdot \omega = (u^{-1})^*\omega.$$

Given a G-equivariant local trivialization $(U_\alpha, \Phi_\alpha = (\pi, g_\alpha))$ of P with canonically associated section $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$, we have

$$\omega|_{\pi^{-1}(U_\alpha)} = Ad(g_\alpha^{-1})\pi^* A_\alpha + {g_\alpha}^*\omega^{MC},$$

where $A_\alpha = {s_\alpha}^*\omega$ as before. Then

$$\begin{aligned}
(u^{-1})^*\omega|_{\pi^{-1}(U_\alpha)} &= Ad((g_\alpha \circ u^{-1})^{-1})(u^{-1})^*\pi^* A_\alpha + (u^{-1})^* {g_\alpha}^*\omega^{MC} \\
&= Ad((g_\alpha \circ u^{-1})^{-1})(\pi \circ u^{-1})^* A_\alpha + (g_\alpha \circ u^{-1})^*\omega^{MC} \\
&= Ad((g_\alpha \circ u^{-1})^{-1})\pi^* A_\alpha + ((\phi_\alpha^{-1} \circ \pi) \cdot g_\alpha)^*\omega^{MC} \\
&= Ad(((\phi_\alpha^{-1} \circ \pi) \cdot g_\alpha)^{-1})\pi^* A_\alpha + {g_\alpha}^*\omega^{MC} - Ad(g_\alpha^{-1} \cdot (\phi_\alpha^{-1} \circ \pi)^{-1})(\phi_\alpha \circ \pi)^*\omega^{MC} \\
&= Ad(g_\alpha^{-1} \cdot (\phi_\alpha \circ \pi))(\pi^* A_\alpha - (\phi_\alpha \circ \pi)^*\omega^{MC}) + {g_\alpha}^*\omega^{MC},
\end{aligned}$$

where $\Phi_\alpha \circ u \circ \Phi_\alpha^{-1}(x, g) = (x, \phi_\alpha(x) \cdot g)$. Consequently, on any local trivialization (U_α, Φ_α) where ω is represented by $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$,

$$u \cdot A_\alpha = Ad(\phi_\alpha)(A_\alpha - \phi_\alpha^*\omega^{MC}).$$

Example 15. The Lie algebra of $U(1)$ is identified with $i\mathbb{R}$. The adjoint representation of $U(1)$

$$Ad : U(1) \to GL(i\mathbb{R})$$

is given by $Ad(e^{i\theta}) = 1$. Therefore, if $\pi : P \to M$ is a principal $U(1)$-bundle, then $Ad(P) = M \times i\mathbb{R}$, and principal connections on P are in one-to-one correspondence with $\Omega^1(M; i\mathbb{R})$. Meanwhile, since $U(1)$ is Abelian, the gauge group of P is $C^\infty(M, U(1))$, and the action of $C^\infty(M, U(1))$ on $\mathcal{A}(P)$ is given by

$$u \cdot A = A - u^{-1}du.$$

Definition. Let $\pi : P \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$, and $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection form. Then the *curvature form* of the connection defined by the kernel of ω is given by

$$F_\omega := d\omega + \frac{1}{2}[\omega, \omega].$$

and $F_\omega \in \Omega^2(P, \mathfrak{g})$.

Before we proceed, note that any $Z \in T_pP$ can be written as $Z = Z^h + Z^v$ where $Z^h \in H_p = ker(\omega_p)$ and $Z^v \in V_p$. Then F_ω satisfies:

(1) ${R_g}^* F_\omega = Ad(g^{-1})F_\omega$ for any $g \in G$,
(2) $F_\omega(Z_1, Z_2) = -\omega([Z_1^h, Z_2^h])$ for any $Z_1, Z_2 \in \mathcal{P}$.

We see the first property as follows:

$$\begin{aligned}
{R_g}^* F_\omega &= {R_g}^*(d\omega + \frac{1}{2}[\omega, \omega]) \\
&= d{R_g}^*\omega + \frac{1}{2}[{R_g}^*\omega, {R_g}^*\omega] \\
&= dAd(g^{-1})\omega + \frac{1}{2}[Ad(g^{-1})\omega, Ad(g^{-1})\omega] \\
&= Ad(g^{-1})(d\omega + \frac{1}{2}[\omega, \omega]) \\
&= Ad(g^{-1})F_\omega.
\end{aligned}$$

To see the second property, note that $F_\omega(Z_1^v, Z_2^v) = 0$ since the restriction of ω to any fiber is the Maurer–Cartan form ω^{MC}, and $d\omega^{MC} + \frac{1}{2}[\omega^{MC}, \omega^{MC}] = 0$. Therefore,

$$\begin{aligned} F_\omega(Z_1, Z_2) &= F_\omega(Z_1^h, Z_2^h) + F_\omega(Z_1^h, Z_2^v) + F_\omega(Z_1^v, Z_2^h) \\ &= d\omega(Z_1^h, Z_2^h) + d\omega(Z_1^h, Z_2^v) + d\omega(Z_1^v, Z_2^h). \end{aligned}$$

Now by Corollary 4.2,

$$\begin{aligned} d\omega(Z_1^h, Z_2^h) &= -\omega([Z_1^h, Z_2^h]) \\ d\omega(Z_1^h, Z_2^v) &= -\omega([Z_1^h, Z_2^v]) = 0 \\ d\omega(Z_1^v, Z_2^h) &= -\omega([Z_1^h, Z_2^v]) = 0, \end{aligned}$$

where the last two are due to the fact that the Lie bracket of a horizontal vector field and a vertical vector field is zero (Exercise). As a result, the curvature form detects whether the horizontal distribution defining the principal connection is integrable.

Theorem 10.1 (Bianchi identity)**.** *The curvature form F_ω satisfies*

$$dF_\omega = [F_\omega, \omega].$$

Proof. Simply take the exterior derivative of the curvature form:

$$\begin{aligned} dF_\omega &= d(d\omega + \frac{1}{2}[\omega, \omega]) \\ &= \frac{1}{2}([d\omega, \omega] - [\omega, d\omega]) \\ &= [d\omega, \omega] \\ &= [F_\omega - \frac{1}{2}[\omega, \omega], \omega] \\ &= [F_\omega, \omega], \end{aligned}$$

since $\frac{1}{2}[[\omega, \omega], \omega] = 0$ by the Jacobi identity via $\frac{1}{2}[[\omega, \omega], \omega](Z_1, Z_2, Z_3) = [[\omega(Z_1), \omega(Z_2)], \omega(Z_3)] + [[\omega(Z_2), \omega(Z_3)], \omega(Z_1)] + [[\omega(Z_3), \omega(Z_1)], \omega(Z_2)]$. □

Next, let $\{(U_\alpha, \Phi_\alpha)\}$ be a cover of M by G-equivariant local trivializations of P with canonical local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$. Define, as before, $A_\alpha = {s_\alpha}^*\omega$, and $F_{A_\alpha} := {s_\alpha}^* F_\omega$. Then,

$$F_{A_\alpha} = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha],$$

and it follows from the transformation rule for A_α, together with the Maurer–Cartan equation, that

$$F_{A_\beta} = Ad(g_{\alpha\beta}^{-1}) F_{A_\alpha}.$$

Hence, $\{F_{A_\alpha}\}$ defines a 1-form on M with values in $Ad(P)$, i.e. $F_A \in \Omega^1(M; Ad(P))$.

Definition. Let $\pi : P \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$, and $\omega \in \Omega^1(P; \mathfrak{g})$ be a connection form. The connection defined by the kernel of ω is called *flat* if $F_\omega = 0$.

Definition. Let $\pi : P \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$, $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection form, and $H := ker(\omega)$. Fix $x, x_0 \in M$. Suppose $\gamma : [0,1] \to M$ is a smooth parametrized curve from x_0 to x in M. Denote by Z_γ the tangent vector field to this curve. We can lift Z_γ to a unique horizontal vector field Z^h on $\gamma^* P$, i.e. $Z^h_p \in H_p$ for any $p \in \gamma^* P$. Having fixed a point $p \in \gamma^* P$, we can *parallel transport* along γ by integrating the vector field Z^h to get a smooth horizontal parametrized curve $\tilde{\gamma}$ in P starting from the point p. Since H is G-invariant, this results in a G-equivariant diffeomorphism

$$\mathfrak{h}_\gamma : \pi^{-1}(x_0) \to \pi^{-1}(x).$$

defined by $\mathfrak{h}_\gamma(p \cdot g) = \tilde{\gamma}(1) \cdot g$. In particular, if $x = x_0$, then parallel transport yields a G-equivariant diffeomorphism

$$\mathfrak{h}_\gamma : \pi^{-1}(x_0) \to \pi^{-1}(x_0),$$

called the *holonomy of ω around γ*. The *holonomy group* of ω at p is defined as

$$Hol_p(\omega) := \{h \in G \mid \mathfrak{h}_\gamma(p) = p \cdot h \text{ for some } \gamma\}.$$

Note that the latter is a subgroup of G and $Hol_{p \cdot g}(\omega) = g^{-1} Hol_p(\omega) g$. Denote by $Hol^\circ_p(\omega)$ the subgroup of $Hol_p(\omega)$ consisting of holonomies around contractible loops based at x_0. It is easy to see that the latter is a normal subgroup of $Hol_p(\omega)$. Therefore, we have a group homomorphism:

$$\mathfrak{h} : \pi_1(M, x_0) \to Hol_p(\omega)/Hol^\circ_p(\omega).$$

Moreover, $Hol^\circ_p(\omega)$ is path-connected as a subspace of G. Note that $\mathfrak{hol} := \{X_e \mid X_e = \frac{d}{dt}|_{t=0} x(t) \text{ where } x : [0,1] \to Hol^\circ_p(\omega) \text{ with } x(0) = e\}$ is a vector subspace. Moreover, $\mathfrak{hol}$ is a subalgebra of $\mathfrak{g}$ since for any $X_e, Y_e \in \mathfrak{hol}$ with corresponding paths $x(t), y(t)$ in $Hol^\circ_p(\omega)$ we have $[X_e, Y_e] = \frac{d}{dt}|_{t=0} x(\sqrt{t}) y(\sqrt{t}) x(\sqrt{t})^{-1} y(\sqrt{t})^{-1}$. Now, by Theorem 9.1, the Lie subalgebra $\mathfrak{hol}$ corresponds to a connected Lie subgroup H of G. Meanwhile, it follows from Theorem 8.10 that H and $Hol^\circ_p(\omega)$ agree on an open neighborhood of the identity, and hence, $H \subset Hol^\circ_p(\omega)$. On the other hand, $Hol^\circ_p(\omega) \subset H$ since for any path $x(t) \subset Hol^\circ_p(\omega)$ and $t_0 \in [0,1]$, $\frac{d}{dt}|_{t=t_0} x(t) \in L_{x(t_0)_*} \mathfrak{hol}$. Therefore, $Hol^\circ_p(\omega)$ is a Lie subgroup of G. The proof that $Hol_p(\omega)$ is a Lie subgroup of G then follows from the fact that $Hol^\circ_p(\omega)$ is a Lie subgroup of G and $Hol_p(\omega)$ is a union of disjoint cosets of $Hol^\circ_p(\omega)$. In fact, $Hol^\circ_p(\omega)$ is the identity component of $Hol_p(\omega)$.

Theorem 10.2 (Ambrose–Singer [1])**.** *Let $\pi : P \to M$ be a principal G-bundle with Lie algebra $\mathfrak{g}$, and $\omega \in \Omega^1(P; \mathfrak{g})$ be a connection form. Then the Lie algebra of $Hol^\circ_p(\omega)$ is generated by $F_{\omega p}(Z_1, Z_2)$ for any $Z_1, Z_2 \in T_p P$.*

Theorem 10.2 implies that if ω is a connection form with $F_\omega = 0$, then we have a group homomorphism, called the *holonomy representation*:

$$\mathfrak{h} : \pi_1(M, x_0) \to Hol_p(\omega) \subset G.$$

Homework-5
Due 4/28/14

(1) Recall the complex vector bundle from Example 9: $\pi : \mathbb{C}\mathrm{P}^2 \smallsetminus \{(0:0:1)\} \to \mathbb{C}\mathrm{P}^1$ defined by $\pi(x:y:z) = (x:y)$. Another way to describe the dual/conjugate of this vector bundle geometrically is as follows. Consider all complex lines through the origin in $\mathbb{C}^2$. These are parametrized by points in $\mathbb{C}\mathrm{P}^1$. Then consider the complex line bundle with total space the set

$$\mathcal{O}(-1) := \{((x:y),(\lambda x, \lambda y)) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}^2 \mid \lambda \in \mathbb{C}\}.$$

The projection map $\pi : \mathcal{O}(-1) \to \mathbb{C}\mathrm{P}^1$ is defined by $\pi((x:y),(\lambda x, \lambda y)) = (x:y)$. The latter is also called the *tautological* line bundle over the complex projective line. There is a natural complex vector bundle isomorphism between the two descriptions:$F : \mathcal{O}(-1) \to \overline{\mathbb{C}\mathrm{P}}^2 \smallsetminus \{(0:0:1)\}$ defined by $F((x:y),(a,b)) := (x:y:\bar{a}x+\bar{b}y)$ lifts the identity map on $\mathbb{C}\mathrm{P}^1$. The standard Hermitian inner product on $\mathbb{C}^2$ is used to define a Hermitian metric on $\mathcal{O}(-1)$. With this metric, consider in each fiber the subset of unit length vectors. **Show that the resulting space is diffeomorphic to S^3, and the restriction of the bundle projection to this subset is a principal $U(1)$-bundle.** The latter is nothing but the *Hopf fibration* from S^3 to S^2. *When blow-up a point in $\mathbb{C}^2$, one replaces the point with the total space of the tautological bundle. With the above understood, one can think of this operation in the case of smooth oriented 4-manifolds as taking connected sum with the oriented 4-manifold $\overline{\mathbb{C}\mathrm{P}}^2$.*

(2) Let $\pi : P \to M$ be a principal G-bundle, and $\rho : G \to GL(V)$ be a smooth finite dimensional representation where V is either a real or complex vector space of dimension n. Denote by E the associated vector bundle $P \times_\rho V$. Then a principal connection H on P defines a linear connection H^ρ on E as follows: given $v \in V$ let $i_v : P \to E$ be defined by $i_v(p) := [p,v]$. Then $H^\rho_{[p,v]} = i_{v*}H_p$. **First show that the principal connection is well-defined, i.e. it does not depend on a representative of the equivalence class $[p,v]$.**

Next, let $x \in M$ and $(U_\alpha, \Phi_\alpha = (\pi, g_\alpha))$ be a G-equivariant local trivialization of P with canonical section s_α. Let $H = ker(\omega)$, and $s_\alpha{}^*\omega = A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$. **Show that $H_{s_\alpha(x)} = \{(\Phi_\alpha)_*^{-1}(w, -A_{\alpha\,x}(w)) \mid w \in T_xM\}$, and that $H^\rho_{[p,v]} = \{(\Phi^\rho_\alpha)_*^{-1}(w, -\rho_*(A_{\alpha\,x}(w))v) \mid w \in T_xM\}$ where $\Phi^\rho_\alpha[p,v] = (\pi(p), \rho(g_\alpha(p))v)$. Finally, verify that the latter implies**

$$(\nabla^\rho_Z s)(x) = ds_x(Z_x) + \rho_*(A_{\alpha\,x}(Z_x))s(x),$$

where $Z \in \mathfrak{X}(U_\alpha)$ and $s \in C^\infty(U_\alpha; V)$, defines the covariant derivative corresponding to the linear connection H^ρ.

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