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OUTPUT DECISION UNDER DEMAND UNCERTAINTY WITH STOCHASTIC PRODUCTION FUNCTION: A CONTINGENT CLAIMS APPROACH*

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This paper presents a contingent claims analysis of output decisions for the firm facing technological and demand uncertainty. The paper reveals that: (i) the optimal output level increases with the higher interest rate when the firm is subject to demand (and technological) uncertainty; (ii) the effect of demand volatility and production lead time on the optimal output level could be either positive or negative; (iii) the optimal project value decreases with the higher demand volatility; (iv) the optimal project value decreases with the longer production lead time when the firm is subject to demand uncertainty; and (v) the optimal project value decreases with the higher interest rate when the firm is subject to demand uncertainty; it increases with the higher interest rate, however, when the firm is subject to both demand and technological uncertainty. Some important managerial implications are discussed.
(OUTPUT DECISION; CONTINGENT CLAIMS ANALYSIS; UNCERTAINTY)

1. Introduction

The analysis of the firm's behavior under uncertainty and the valuation of risky assets are basic concerns of contemporary economics and financial theorists. This paper studies a classic problem faced by firms of determining the level of output before demand is known, which has been known as the "Mills' firm" in the economics literature (see, e.g., Baron 1971, Leland 1972, and Mills 1962) and as the "newsboy problem" in the operations research literature (see, e.g., Atkinson 1979 and Lau 1980). If actual demand is less than the level of output, any items left over may be sold at a loss, and if demand exceeds the quantity produced, the firm may turn customers away. Firms in these studies are typically viewed as maximizing the von Neumann-Morgenstern expected utility of profit where the utility function involved represents either the collective will of shareholders or management's perception of it.

Recognizing that these studies ignore the role of financial markets, and individuals are not characterized as holding multiple asset portfolios, Magee (1975) uses the Sharpe-Lintner-Mossin (1964, 1965, 1966) market equilibrium model (CAPM) for the evaluation of the firm's output decision in the context of cost-volume-profit analysis.[1] More recently, Anvari (1987) and Kim and Chung (1989) also employ the CAPM framework for the analysis of the firm's output decision under demand uncertainty.

However, the application of the CAPM valuation to the analysis of optimal output decision suffers from some potential problems since the assumption of normality should allow the possibility of negative demand and the alternative assumption of quadratic

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[1] This study, however, does not properly address the stochastic demand problem because of a rather strong and hard-to-defend assumption on the relationship between the demand for the firm's output and the market return. Specifically, Magee assumes that $E(r_m/X) = a + bX$, where r_m is the one-period random market return, X is the firm's random demand, E is the expected value operator, and a and b are constants (see p. 262). Generally, there are three possible sources of uncertainty in the theory of the firm: future demand for the firm's output, future selling price of the firm's output, and production costs. For the traditional theory of the competitive firm facing price uncertainty, see Sandmo (1971), and for production costs uncertainty, see Adar et al. (1977), respectively.

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utility function implies negative marginal utility, which was shown to lead to an unacceptable result (see Gonzales, Litzenberger and Rolfo 1977). Consequently, results derived using the CAPM valuation have some limitation.

In order to avoid these pitfalls, this paper presents a contingent claims analysis of output decisions for the firm facing uncertain demand and uncertain production technology (i.e., stochastic production function) using the option pricing models of Black and Scholes (1973), Smith (1976), Fischer (1978), and Margrabe (1978).[2] This study should be viewed in the spirit of recent strands of research which employ the contingent claims analysis for the evaluation of various real asset investment decisions. Baldwin, Mason and Ruback (1983), Brennan and Schwartz (1983), and Kester (1984) provide the methods of explicitly incorporating the value of operating flexibility into the capital budgeting process. Myers and Majd (1983) provide a contingent claims approach to the valuation of the option to abandon a project. McDonald and Siegel (1985) and Brennan and Schwartz (1985) develop a methodology for valuing risky investment projects, where there is an option to temporarily and costlessly shut down production. McDonald and Siegel (1986) and Majd and Pindyck (1987) evaluate the option value of being able to delay irreversible investment expenditures. Most recently, Pindyck (1988) examines the implications of the irreversibility of investment for the firm's capacity choice and expansion decisions.[3]

The contingent claims approach is well suited for the analysis of the firm's output decisions under uncertainty because the payoff from the firm's output decision is a contingent claim with its value dependent upon uncertain future demand and/or uncertain output volume. In essence, this paper demonstrates that: (i) the optimal output level increases with the higher interest rate when the firm is subject to demand (and technological) uncertainty; (ii) the effect of demand volatility and production lead time on the optimal output level could be either positive or negative; (iii) the optimal project value decreases with the higher demand variability; (iv) the optimal project value decreases with the longer production lead time when the firm is subject to only demand uncertainty; and (v) the optimal project value decreases (increases) with the higher interest rate when the firm is subject to demand (demand and technological) uncertainty.

The paper is organized as follows. §2 analyzes the firm's output decisions under demand uncertainty with nonstochastic production function. §3 examines the effect of technological uncertainty on the firm's output decisions under the assumption of nonstochastic

[2] The pitfall of using the CAPM to price a truncated cash flow can also be resolved by employing the intertemporal CAPM of Merton (1973) (see, e.g., Constantinides 1978). But this approach has the drawback that we need to assume the validity of Merton model. By employing the Black and Scholes (1973) model we implicitly assume that the futures price or the spot commodity is a traded asset. It is important to note, however, that Rubinstein (1976) shows that Black and Scholes result can also be obtained from a discrete time framework. As Brennan (1979) lucidly summarizes, a major advantage of the discrete time option pricing model is the absence of the assumption of the ability to form a riskless portfolio with both the contingent claim and the underlying asset. Such a portfolio can be formed only under strong assumptions that the underlying asset and the contingent claim be not only traded assets but may be purchased and sold continuously in any proportion. The discrete time model, on the other hand, imposes no such requirements and thus extends the applicability of option pricing model to the valuation of a broad class of assets for which one or other of these assumptions is not met. Rubinstein's approach, however, is not without its drawbacks; it requires strong assumptions on the investor preference (i.e., the constant proportional risk aversion) and on the return generating process (i.e., the return on the underlying asset and the return on aggregate wealth follow an arbitrary bivariate lognormal distribution).

[3] Other authors (see e.g., Brock, Rothschild and Stiglitz 1982, Paddock, Siegel and Smith 1982, Pindyck 1980, and Tourinho 1979) have used the contingent claims framework for the analysis of a natural resource exploitation. On the other hand, Brennan and Schwartz (1982a, 1982b) used stochastic optimal control theory for the analysis of investment decisions in the regulated public utility firm. See Brennan (1979) and Mason and Merton (1985) for the general discussion of potential applicability of the option pricing model to the broad class of real assets investment decisions.

demand. §4 deals with the most general case where the firm is subject to both technological and demand uncertainty. Finally, §5 presents the summary and concluding remarks.

2. Output Decision under Demand Uncertainty with Nonstochastic Production Function

2.1. *Assumptions and Valuation*

Suppose that the firm continuously receives orders from its customers for the delivery at time T. It is assumed that stochastic changes in demand (i.e., the cumulative order) are spanned by existing assets, that is, there is an asset or dynamic portfolio of assets whose price is perfectly correlated with the demand. This assumption will hold for most commodities which are traded on both spot and futures markets. With the spanning assumption, we can determine the output decision that maximizes the project value using the contingent claim valuation which avoids assumptions regarding risk preferences or discount rates. We assume that the cumulative order level at time t, $x(t)$, changes in the time interval $(t, t + dt)$ by

$$dx(t) = x(t)[(\mu + \delta)dt + \sigma dw]; \tag{1}$$

where μ is the instantaneous equilibrium rate of return on a security or dynamic portfolio of assets whose price is perfectly correlated with $x(t)$, $\mu + \delta$ is the instantaneous expected demand growth rate, σ is the instantaneous standard deviation of the demand growth rate, and dw is a Wiener process. It is assumed that δ could be either positive or negative. A positive δ implies that the growth rate of demand is greater than the equilibrium rate of return on a security or dynamic portfolio of assets that has the same risk as $x(t)$. On the other hand, a negative δ implies that the growth rate of demand is less than the equilibrium rate of return.[4] We will denote $x = x(0)$ and $X = x(T)$. Hence x denotes the size of outstanding orders at time zero which we define as the last day to notify the firm's production manager of the output quantity for the shipment at time T (i.e., the firm's production lead time is assumed to be T). Likewise X denotes the cumulative orders (i.e., demand) at time T.

We assume that the firm's production process is described by the following form:

$$Q = q^{\alpha}, \tag{2}$$

where q is the input, Q is the output. This provides a convenient characterization of technology in terms of returns to scale as represented by α. It is well known that under certainty decreasing returns to scale (i.e., $0 < \alpha < 1$) is necessary for the existence of a competitive optimum for the firm. Although, as will be shown later, decreasing returns to scale is not necessary for the existence of an optimal output level under uncertainty, we will assume that the production function exhibits nonincreasing returns to scale (i.e., $0 < \alpha \leq 1$) for the simplicity of exposition in the remainder of the paper.

If the demand, X, is greater than the output produced, Q, the only implicit cost is assumed to be the profit lost on unsatisfied demand. If the demand is less than the output produced, unsold units are salvaged at a lower price than their original selling price.[5]

[4] Since some customers may withdraw their previously made orders, the cumulative order level is not monotonic in time. Uncertainty in demand can alternatively be modelled with stochastic output price (see Brennan and Schwartz 1985 and McDonald and Siegel 1985). For the discussion on the determinant of δ, see McDonald and Siegel (1985, pp. 337–338) and Majd and Pindyck (1987, p. 13). In general, δ could be either positive or negative, depending on the situation.

[5] Examples of such goods are many. Certain types of goods are subject to obsolescence, whether it be in technology or in consumer tastes. A change in technology may make an electronic component worthless. A change in style may cause a retailer to sell fashion goods at substantially reduced prices. Other goods, such as agricultural products, are subject to physical deterioration. With deterioration, the goods will have to be sold at lower prices.

The firm's objective is to determine the output level (Q) such that the present value of cash flows is maximized. We assume that the firm has no initial stock of inventory when it makes the production decision at time zero. We also assume that there is no fixed cost of production. These last two assumptions are relaxed in Appendix A.

According to the usual convention that cash flows are realized at the end of period (i.e., at time T), cash flows at time T will be $PQ - Cq$ if the demand is greater than the quantity produced, or $PX + p(Q - X) - Cq$ if the demand is less than the quantity produced; where P is the selling price per unit, C is the marginal factor cost ($C < P$), and p is the salvage value per unit ($p < C$). (See also Figure 1 for the graphical description of the end of period cash flows.) Then we can express the time T cash flows as

$$\text{Min}\ [PQ - Cq,\ PX + p(Q - X) - Cq]. \tag{3}$$

If we factor out $PQ - Cq$ from the bracket, (3) can be rewritten as

$$PQ - Cq + (P - p)\ \text{Min}\ [0, X - Q]. \tag{4}$$

However, note that

$$\text{Min}\ [0, X - Q] + \text{Max}\ (0, X - Q] = X - Q; \qquad \text{or equivalently,} \tag{5}$$

$$\text{Min}\ [0, X - Q] = (X - Q) - \text{Max}\ [0, X - Q]. \tag{6}$$

Substituting (6) into (4) and after rearrangement, the cash flow at time T can be expressed as

$$(pQ - Cq) + (P - p)X - (P - p)\ \text{Max}\ [0, X - Q]. \tag{7}$$

Then using the theorem in Smith (1976, p. 16), it can be shown that the present value of the last term in (7) is

$$(P - p)\{xe^{\delta T}N(d_1) - Qe^{-rT}N(d_2)\}; \tag{8}$$

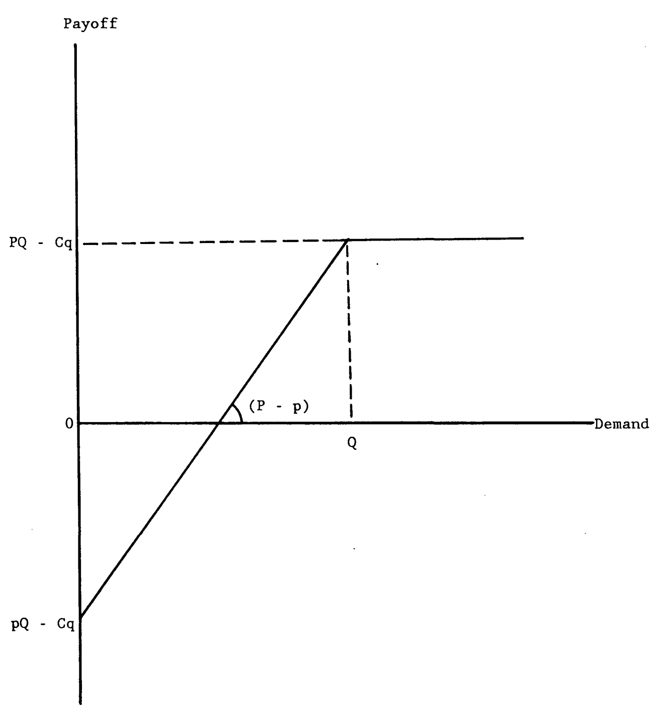

FIGURE 1. Payoff Pattern as a Function of the Firm's Output Decision.

where

$$d_1 = [\{\ln(x/Q) + (r+\delta)T\}/\sigma\sqrt{T}] + (1/2)\sigma\sqrt{T},$$
$$d_2 = [\{\ln(x/Q) + (r+\delta)T\}/\sigma\sqrt{T}] - (1/2)\sigma\sqrt{T},$$

$N(\ )$ = the cumulative distribution function of the standard normal distribution,
σ = the instantaneous standard deviation of dx/x, and
r = the instantaneous risk-free rate.

The present value of $(P-p)X$ is given by $E_0[(P-p)X]e^{-\mu T}$, where E_0 is the conditional expectation given the information available at time zero. Since

$$E_0[(P-p)X] = (P-p)xe^{(\mu+\delta)T},$$

the present value becomes

$$(P-p)xe^{(\mu+\delta)T}e^{-\mu T} = (P-p)xe^{\delta T}.^{6}$$

Finally, noting that the present value of $pQ - Cq$ is $(pQ - Cq)e^{-rT}$, the net present value of cash flows (NPV hereafter) is defined as

$$\text{NPV} = (pQ - Cq)e^{-rT} + (P-p)xe^{\delta T} - (P-p)[xe^{\delta T}N(d_1) - Qe^{-rT}N(d_2)]. \quad (9)$$

2.2. *Optimality Conditions and Comparative Statics Analysis*

This section examines the optimality condition and comparative statics properties of the model. Letting $dNPV/dQ = 0$, and after simplification, the optimality condition can be expressed as[7]

$$PN(d_2) + p\{1 - N(d_2)\} = \{(1/\alpha)Q^{(1/\alpha)-1}\}C. \quad (10)$$

The economic intuition underlying equation (10) is obvious. Recall that P and p, respectively, are the selling price per unit and the salvage value per unit. Also notice that $N(d_2)$ represents the probability of under-production and $\{1 - N(d_2)\}$ represents the probability of over-production.[8] Since the firm will incur sales revenues of P with the probability of $N(d_2)$ and p with the probability of $\{1 - N(d_2)\}$, the left-hand side of equation (10) measures the *expected* marginal revenue. Since the right-hand side of (10) is the marginal factor cost, the optimal output decision occurs when the firm produces up to the point at which the expected marginal revenue equals marginal cost, a standard result in microeconomics.[9]

[6] The author would like to thank Mark Rubinstein (the referee) for helping him clarify this point. See Majd and Pindyck (1987, p. 12) for a somewhat different application of the same method. Majd and Pindyck find the present value of continuous future sales when the price of the output follows the diffusion process (i.e., equation (1)).

[7] Notice that $N(d_1)$ and $N(d_2)$ are also a function of Q, which makes the differentiation complicated. This and some following results in the paper involve lengthy and tedious algebraic operations. We present only the final expressions for brevity. The details of derivations are available from the author upon request.

[8] See Copeland and Weston (1988, p. 276) for the discussion of intuitive interpretation of $N(d_2)$. It should be noted that, in general, $N(d_2)$ represents the probability of under-production only if investors are risk-neutral.

[9] An alternative intuition underlying the optimality condition can be obtained if equation (10) is rewritten as

$$[P - \{(1/\alpha)Q^{(1/\alpha)-1}\}C]N(d_2) = [\{(1/\alpha)Q^{(1/\alpha)-1}\}C - p]N(-d_2). \quad (10a)$$

Note first that $[P - \{(1/\alpha)Q^{(1/\alpha)-1}\}C]$ represents the marginal cost of under-production (i.e., lost profit per unit) and $N(d_2)$ represents the probability of under-production. Thus the left-hand side of equation (10a) measures the expected marginal cost of shortage. Similarly, note that $[\{(1/\alpha)Q^{(1/\alpha)-1}\}C - p]$ represents the implicit marginal cost of each unsold unit (i.e., the difference between the marginal cost and salvage value), and $N(-d_2)$ is the probability of over-production. Thus the right-hand side of equation (10a) measures the expected marginal cost of over-production. The optimal output decision occurs when the firm produces up to the point at which the expected marginal cost of shortage equals the expected marginal cost of excess production.

If the production function exhibits constant returns to scale (i.e., $\alpha = 1$), the optimality condition simplifies to

$$PN(d_2) + p\{(1 - N(d_2))\} = C. \tag{11}$$

It is interesting to observe that the firm with the production function exhibiting decreasing returns to scale will produce less than the firm with constant returns to scale since the marginal cost with decreasing returns to scale is greater than that with constant returns to scale (i.e., $\{(1/\alpha)Q^{(1/\alpha)-1}\}C = \{q \ln(q)/Q \ln(Q)\}C > C$, given $0 < \alpha < 1$). This should be obvious since with $P > p$, the larger marginal cost results in a larger *optimal* $N(d_2)$ which, in turn, implies a smaller optimal Q since

$$d_2 = [\{\ln(x/Q) + (r + \delta)T\}/\sigma\sqrt{T}] - (1/2)\sigma\sqrt{T}.$$

In order to obtain a closed-form solution for the optimal output level, Q^*, first note from (11) that

$$d_2 = N^{-1}[(C - p)/(P - p)],$$

where $N^{-1}[\ \ ]$ denotes the inverse cumulative distribution function of the standard normal distribution. Then, since

$$d_2 = [\{\ln(x/Q) + (r + \delta)T\}/\sigma\sqrt{T}] - (1/2)\sigma\sqrt{T},$$

we obtain

$$N^{-1}[(C - p)/(P - p)] = [\{\ln(x/Q) + (r + \delta)T\}/\sigma\sqrt{T}] - (1/2)\sigma\sqrt{T}. \tag{12}$$

Rearrangement of (12) yields the following closed-form expression for the optimal output level:

$$Q^* = \exp[\ln(x) - \{N^{-1}(\Phi) + .5\sigma\sqrt{T}\}\sigma\sqrt{T} + (r + \delta)T], \tag{13}$$

where $\Phi = (C - p)/(P - p)$.

Finally, the following second derivative insures the sufficiency condition given nonincreasing returns to scale (i.e., $0 < \alpha \leq 1$):

$$d^2\mathrm{NPV}/dQ^2 = -e^{-rT}[(P - p)f(d_2)(1/\sigma\sqrt{T}Q) + \beta C] < 0, \tag{14}$$

where $\beta = [(1 - \alpha)/\alpha^2]Q^{(1/\alpha)-2}$. Notice that nonincreasing returns to scale is sufficient but not necessary for the concavity of the net present value function. This should be obvious since the net present value function can still be concave everywhere with $\alpha > 1$ if β is sufficiently small in absolute magnitude. Thus increasing returns to scale is not a sufficient condition for the nonexistence of an optimal output level under uncertainty. The firm may have a competitive optimum even with increasing returns to scale. Although this is an interesting result which has also been observed by other authors in the somewhat different context (see, e.g., Leland 1970 and Sandmo 1971), we assume nonincreasing returns to scale in this paper for the succinctness of presentation.

We now examine the effect of the interest rate, demand variability, and production lead time on the optimal output level.[10]

THEOREM 1. (i) *The optimal output level increases with higher interest rate.*

(ii) *The effects of demand volatility and production lead time on the optimal output level could be either positive or negative.*

PROOF. (i) Partially differentiating Q^* with respect to r, we have

$$Q_r^* = \frac{(1/\sigma)\sqrt{T}f(d_2)}{\beta\{C/(P - p)\} + (1/Q)f(d_2)(1/\sigma\sqrt{T})}, \tag{15}$$

[10] Somewhat less interesting but intuitively clear partials are: $Q_P^* > 0$, $Q_C^* < 0$, $Q_p^* > 0$, and $Q_x^* > 0$.

where the subscript (r) indicates the partial derivative. If the production function exhibits nonincreasing returns to scale (i.e., $0 < \alpha \leq 1$), then $\beta \geq 0$, and thus $Q_r^* > 0$.

(ii) Partially differentiating Q^* with respect to σ, we obtain

$$Q_\sigma^* = -\frac{[\{\ln (x/Q) + (r+\delta)T\}/(\sqrt{T}\sigma^2) + (1/2)\sqrt{T}]f(d_2)}{\beta\{C/(P-p)\} + (1/Q)f(d_2)(1/\sigma\sqrt{T})}. \tag{16}$$

If $x > Q$, ln (x/Q) is positive, and thus the partial effect is always negative, i.e., the riskier demand leads to the lower optimal output, given $0 < \alpha \leq 1$. However, if $x < Q$, then ln (x/Q) is negative, and the partial effect could be either positive or negative, depending on the relative magnitudes of T, x, Q, r, α, δ, and σ.

Partially differentiating Q^* with respect to T, we obtain

$$Q_T^* = \frac{[(r+\delta)/\sqrt{T}\sigma - \{\ln (x/Q) + (r+\delta)T\}/(2T\sqrt{T}\sigma) - (\sigma/4)\sqrt{T}]f(d_2)}{\beta\{C/(P-p)\} + (1/Q)f(d_2)(1/\sigma\sqrt{T})}. \tag{17}$$

The sign of this partial will be determined by relative magnitudes of T, x, Q, r, α, δ, and σ. *Q.E.D.*

The positive relationship between the interest rate and optimal output level is somewhat surprising since it seemingly runs counter to a basic tenet of classical economic thought that firms invest up to the point where the marginal efficiency of (i.e., the marginal rate of return on) capital is equal to the interest rate, which implies a *lower* investment with a higher interest rate. It should be remembered however that the situation we have modelled in this paper is vastly different from that of the classical model of firms' investment decision. That is, we seek the optimal amount of output given the probability distribution of the output demand, whereas the classical analysis seeks the optimal amount of investment given the investment opportunity set. Hence implications of the models could be different.

The economic intuition underlying the positive relationship between the interest rate and optimal output level goes as follows. Note that d_2 can be rewritten as

$$d_2 = [\{\ln (xe^{\delta T}) - \ln (Qe^{-rT})\}/\sigma\sqrt{T}] - (1/2)\sigma\sqrt{T}.$$

Since a higher interest rate implies a larger d_2, an increase in the interest rate results in the higher probability of under-production. The intuition of this observation is simple. The higher interest rate, in essence, implies a smaller time-zero-equivalent magnitude of the output level (i.e., Qe^{-rT}). Thus, *ceteris paribus,* the *probability* that the magnitude of cumulative order turns out to be greater than the output produced will be greater with the higher interest rate. As a result, the higher interest rate implies the higher *expected* marginal cost of under-production. Facing with the higher expected marginal cost of under-production, the firm will increase its output to move to its new optimality point, *ceteris paribus.*[11] This result has an important managerial implication. If the interest rate is high, management should be more aggressive in determining the volume of operation. If, on the other hand, the interest rate is low, management must be conservative in setting output level.

[11] This explanation, of course, requires the risk-neutrality assumption. An alternative economic intuition can be obtained if we rewrite

$$d_2 = [\ln \{xe^{(r+\delta)T}\} - \ln (Q)]/\sigma\sqrt{T} - (1/2)\sigma\sqrt{T}.$$

Since the growth rate of demand will, in equilibrium, approximate the growth rate of economy as a whole (i.e., the real interest rate), a higher interest rate implies a higher demand growth rate. Thus, given a fixed output level, the higher interest rate, in essence, implies a larger cumulative order level at time T (i.e., $xe^{(r+\delta)T}$), which in turn implies a higher *probability* of under-production. Since the *expected* marginal cost of under-production will be greater with the higher probability of under-production, the firm will increase its output to move to its new optimality point, *ceteris paribus*.

The effect of demand variability on the optimal output suggests that the higher demand volatility does not necessarily lead to the smaller output. Thus a result of previous studies (see, e.g., Atkinson 1979, Baron 1971, Constantinides et al. 1981, Hawawini 1978, Leland 1972, and Sandmo 1971) that firms will reduce output in the presence of uncertainty does not apply to the "Mills' firm" problem analyzed in this paper.

2.3. *Sensitivity Analysis*

This section examines the effect of demand variability, interest rate, and production lead time on the project value given the assumption that the firm follows its optimal output decisions:[12]

THEOREM 2. (i) *The project value decreases with higher demand variability and higher interest rate.*

(ii) *The project value decreases with longer production lead time when the demand growth rate is less than or equal to the rate of return on the asset whose price is perfectly correlated with the demand.*

PROOF. (i) $\mathrm{NPV}_\sigma = -(P-p)Qe^{-rT}f(d_2)\sqrt{T} < 0.$

$$\mathrm{NPV}_r = TCQ^{1/\alpha}e^{-rT}\{1-(1/\alpha)\}.$$

Thus $\mathrm{NPV}_r \le 0$ given $0 < \alpha \le 1$.

(ii)

$$\mathrm{NPV}_T = -e^{-rT}Q(P-p)(\sigma/2\sqrt{T})f(d_2) + rCQ^{1/\alpha}e^{-rT}\{(1-(1/\alpha)\} + \delta(P-p)xe^{\delta T}.$$

Notice that

$$\mathrm{NPV}_T = (\sigma/2T)\mathrm{NPV}_\sigma + (r/T)\mathrm{NPV}_r + (\delta/T)\mathrm{NPV}_\delta.$$

Thus $\mathrm{NPV}_T < 0$ if $\delta \le 0$ since $\mathrm{NPV}_r \le 0$, $\mathrm{NPV}_\sigma < 0$, and $\mathrm{NPV}_\delta > 0$. *Q.E.D.*

An increase in the dispersion of possible outcome increases the probability that the demand will be less than the quantity produced, and thus lowers expected cash flow. The increased variance also increases the probability that the demand will be greater than the quantity ordered. However, since the firm has the maximum cash flow it can receive, $PQ - Cq$, the benefit of the increased variance will be less than the cost of it. Hence the net effect of the increase in variance on the firm value will be negative.[13] Managerial implication of this result is clear and perhaps in line with management's intuition. If the firm has several alternative product lines to choose from that have similar cost and revenue structures but with different demand volatility, the firm should choose the one with the least demand volatility, *ceteris paribus.*

The intuition underlying the negative effect of the interest rate on the project value is clear from (7). Note that the third term can be interpreted as that the firm in essence is

[12] Effects of other variables on the NPV are: $\mathrm{NPV}_P > 0$, $\mathrm{NPV}_C < 0$, $\mathrm{NPV}_p > 0$, and $\mathrm{NPV}_x > 0$.

[13] McDonald and Siegel (1985) find the similar result when the firm has an option to temporarily and costlessly shut down production whenever variable costs exceed operating revenues. Assuming that the firm owned by risk averse investors faces price (rather than quantity) uncertainty, they find that an increase in the variability of output price will lower the project value when the output price is highly correlated with the market return. It should be noted, however, that the similarity is due to a different reason since their result critically depends on investors' risk aversion, whereas the result of this paper is independent of investors' risk preference. Under the assumptions that investment is irreversible and future demand or cost conditions are uncertain, Pindyck (1988) finds that the firm is worth more the more volatile is demand since a larger variance implies a larger value for the firm's growth opportunities. Under somewhat different context, McDonald and Siegel (1986) find that, in general, an increase in demand volatility could either increase or decrease the value of the growth option. Their results differ from those of the present study since the situations they modelled are different. In particular, it should be remembered that the main feature of the present study is that the firm could under- or over-produce, whereas the firm in their studies could adjust its output level immediately to demand.

writing a call option against the future value of $(P - p)X$ with an exercise price of $(P - p)Q$. Since the call option value is higher with the higher interest rate, the higher interest rate results in the smaller third term. On the other hand, the higher interest rate results in the larger present value of the first term since $pQ - Cq$ is always negative. For the firm with decreasing returns to scale, the third term dominates the first, thus an increased output resulted from the higher interest rate will lead to a smaller project value than the project value with the previously lower interest rate.[14] The firm with constant returns to scale will also increase its output with the higher interest rate. Its project value, however, will be invariant to the change in the interest rate since the effect of the first term will exactly offset that of the third term.

Finally, the effect of the production lead time on the project NPV is a complex function of several variables. Notice first that there are three effects on the NPV of an increase in the production lead time: (1) it increases the variability of demand; (2) it has similar effect as an increase in the interest rate; and (3) it reduces the effective (i.e., time-risk adjusted) demand. Since all three effects have a negative impact on the NPV, the longer production lead time will lead to the lower project value.[15] Managerial implication is clear; if the firm has several mutually exclusive product lines with similar cost and revenue structures but with different production lead times and when the demand growth rate is less than or equal to the rate of return on the asset whose price is perfectly correlated with the demand, the firm should choose the one with the least production lead time.

3. Output Decision with Stochastic Production Function under Nonstochastic Demand

3.1. *Assumptions and Valuation*

In this section we analyze the situation where the firm manufactures a certain product in order to meet the predetermined customer orders. For instance, it is not unusual in the aircraft manufacturing industry that manufacturers initiate the production of a new line of aircraft after they receive customer orders. Specifically, suppose that the firm has a predetermined *certain* customer order of X units for the delivery at time T. We assume that the firm's production process is described by the following stochastic production function:

$$Y = q^{\alpha} z(T), \tag{18}$$

where Y is the random output at time T, q is the input, $0 < \alpha \leq 1$, and $z(T)$ is the term capturing technological uncertainty of the firm's production process. We assume that $z(t)$ for $0 \leq t \leq T$ follows the stochastic process

$$dz(t) = z(t)[\Omega + \tau)dt + \phi dv], \tag{19}$$

where Ω is the instantaneous equilibrium rate of return on a security or dynamic portfolio of assets whose price is perfectly correlated with $z(t)$, $\Omega + \tau$ is the instantaneous expected output growth rate, ϕ is the instantaneous standard deviation of the output growth rate, and dv is the Wiener process. It is assumed that $z(t)$ has an initial value $z(0) = 1$. Hence q^{α} is the quantity the firm can produce with the technology prevailing at time zero. As time elapses (i.e., $t > 0$), however, the firm's output can only be described by the stochastic variable $q^{\alpha} z(t)$ due to technological uncertainty.[16]

Then cash flows at time T will be $PY - Cq$ if the realized output level is less than the customer orders, or $PX + p(Y - X) - Cq$ if the realized output level is greater than the

[14] McDonald and Siegel (1985, p. 344) make a similar observation in a somewhat different context.

[15] McDonald and Siegel (1985, p. 341) find that the time to production has an ambiguous effect on the value of the firm when the equilibrium return on a financial asset which has the same covariance with the market as the commodity price is greater than the expected growth rate of the output price.

[16] Uncertainty in the production process can alternatively be modelled with the stochastic production costs (see McDonald and Siegel 1985).

customer orders (see also Figure 2 for the graphical description of the end of period cash flows).[17] The firm's objective is to determine a "target" output level (Q) such that the present value of cash flows is maximized. Note that the cash flows at time T can be expressed as

$$\text{Min}\,[PY - Cq,\ PX + p(Y - X) - Cq]. \tag{20}$$

If we factor out $PY - Cq$ from the bracket, (20) can be rewritten as

$$PY - Cq + (P - p)\,\text{Min}\,[0,\ X - Y]. \tag{21}$$

Using (6) and after rearrangement, the cash flow at time T can be expressed as

$$(pY - Cq) + (P - p)X - (P - p)\,\text{Max}\,(0,\ X - Y]. \tag{22}$$

Then, using the theorem in Smith (1976, p. 16), the present value of the last term in (22) can be evaluated as

$$(P - p)\{Xe^{-rT}N(d_1) - Qe^{\tau T}N(d_2)\}; \tag{23}$$

where

$$d_1 = [\{\ln(X/Q) - (r + \tau)T\}/\phi\sqrt{T}] + (1/2)\phi\sqrt{T},$$

$$d_2 = [\{\ln(X/Q) - (r + \tau)T\}/\phi\sqrt{T}] - (1/2)\phi\sqrt{T}, \qquad \text{and}$$

$$Q = q^{\alpha}.$$

Also note that the present values of pY, Cq, and $(P - p)X$ are $pQe^{\tau T}$, Cqe^{-rT}, and $(P - p)Xe^{-rT}$, respectively. Hence the net present value of cash flows is defined as

$$\text{NPV} = pQe^{\tau T} - Cqe^{-rT} + (P - p)Xe^{-rT} - (P - p)[Xe^{-rT}N(d_1) - Qe^{\tau T}N(d_2)]. \tag{24}$$

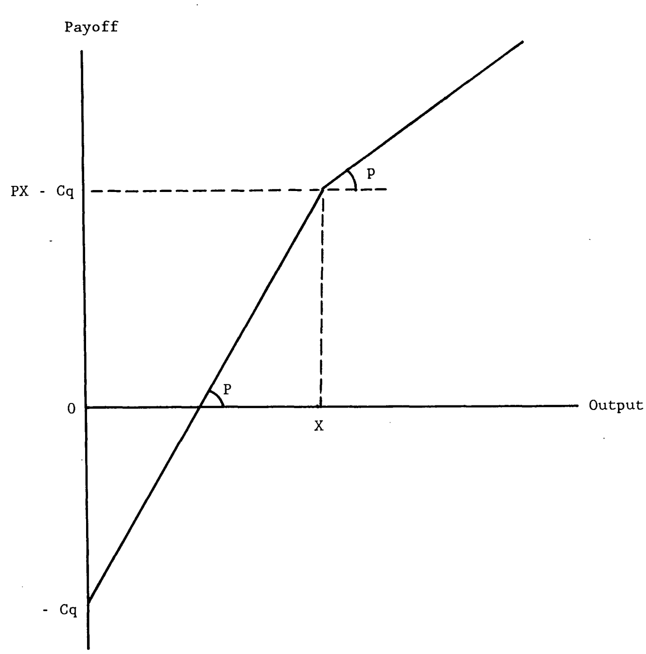

FIGURE 2. Payoff Pattern as a Function of the Firm's Output.

[17] In this and the following sections, we assume that there is no fixed cost of production. The results with the fixed cost, however, can be easily obtained by following the analytical framework presented in Appendix A.

3.2. *Optimality Conditions and Comparative Statics Analysis*

Letting $d\text{NPV}/dQ = 0$, and after simplification, the optimality condition can be expressed as

$$PN(d_2) + p\{1 - N(d_2)\} = \{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T}. \tag{25}$$

If the production function exhibits constant returns to scale, the optimality condition simplifies to

$$PN(d_2) + p\{1 - N(d_2)\} = Ce^{-(r+\tau)T}. \tag{26}$$

Rearrangement of (26) yields the following closed-form solution for the optimal target output level:

$$Q^* = \exp[\ln(X) - \{N^{-1}(\Phi_1) + .5\phi\sqrt{T}\}\phi\sqrt{T} - (r + \tau)T], \tag{27}$$

where $\Phi_1 = \{Ce^{-(r+\tau)T} - p\}/(P - p)$.

Finally, the following second derivative insures the sufficiency condition for $0 < \alpha \leq 1$:

$$d^2\text{NPV}/dQ^2 = -[(P - p)e^{\tau T}f(d_2)(1/\phi\sqrt{T}Q) + \beta e^{-rT}C] < 0, \tag{28}$$

where $\beta = [(1 - \alpha)/\alpha^2]Q^{(1/\alpha)-2}$.

We now examine the effect of the interest rate, demand variability, and production lead time on the optimal output level.

THEOREM 3. *The effects of the interest rate, demand volatility, and production lead time on the optimal output level could be either positive or negative.*

PROOF. Partially differentiating Q^* with respect to r, we obtain

$$Q_r^* = \frac{\{(1/\alpha)Q^{(1/\alpha)-1}\}e^{-rT}C - (1/\phi)\sqrt{T}f(d_2)}{\beta\{C/(P - p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\phi\sqrt{T})} \gtreqless 0. \tag{29}$$

Partially differentiating Q^* with respect to ϕ, we obtain

$$Q_\phi^* = -\frac{[\{\ln(X/Q) - (r + \tau)T\}/(\sqrt{T}\phi^2) + (1/2)\sqrt{T}]f(d_2)}{\beta\{C/(P - p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\phi\sqrt{T})}. \tag{30}$$

If $X > Q$, $\ln(X/Q)$ is positive, and thus the partial effect is always negative, given $0 < \alpha \leq 1$. However if $X < Q$, then $\ln(X/Q)$ is negative, and the partial effect could be either positive or negative, depending on the relative magnitudes of T, X, Q, r, α, τ, and ϕ.

Partially differentiating Q^* with respect to T, we obtain

$$Q_T^* = \frac{G - [(r + \tau)/(2\sqrt{T}\phi) + \{\ln(X/Q)/(2T\sqrt{T}\phi)\} + (\phi/4)\sqrt{T}]f(d_2)}{\beta\{C/P - p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\phi\sqrt{T})}, \tag{31}$$

where

$$G = [(r + \tau)\{(C/\alpha)Q^{(1/\alpha)-1}\}e^{-(r+\tau)T}]/(P - p).$$

The sign of this partial will also be determined by relative values of T, x, Q, r, α, τ, and ϕ. Q.E.D.

Differing from the case when demand is uncertain, the relationship between the interest rate and optimal output level is ambiguous when the firm's production technology is uncertain. In order to see the economic rationale for this result, first note that

$$d_2 = [\{\ln(Xe^{-rT}) - \ln(Qe^{\tau T})\}/\phi\sqrt{T}] - (1/2)\phi\sqrt{T}.$$

Thus a higher interest rate, in essence, implies a smaller time-zero-equivalent magnitude of the order size (i.e., Xe^{-rT}). Thus the probability that the magnitude of the output produced turns out to be smaller (greater) than the predetermined customer order will be lower (higher) with the higher interest rate. It is important to note however that the marginal cost of under-production (i.e., $P - \{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T}$) will be greater with the higher interest rate. Similarly, notice that the marginal cost of over-production (i.e., $\{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T} - p$) will be smaller with the higher interest rate. Thus the higher interest rate could imply either higher or lower *expected* marginal cost of under-production (over-production). Therefore the net effect of the interest rate change on the firm's optimal target output level is ambiguous.

3.3. *Sensitivity Analysis*

This section examines the effect of demand variability, interest rate, and production lead time on the project value given the assumption that the firm follows its optimal output decisions:

THEOREM 4. (i) *The project value decreases with higher production uncertainty.*
(ii) *The project value could either increase or decrease with higher interest rate and longer production time.*

PROOF. (i) $\mathrm{NPV}_\phi = -(P - p)e^{\tau T}Qf(d_2)\sqrt{T} < 0$.
(ii) $\mathrm{NPV}_r = \{CQ^{1/\alpha} - N(-d_1)(P - p)X\}Te^{-rT} \lesseqgtr 0$.

$$\mathrm{NPV}_T = -(P - p)e^{\tau T}Qf(d_2)\sqrt{T}(\phi/2\sqrt{T}) + r\{CQ^{1/\alpha} - N(-d_1)(P - p)X\}e^{-rT} + \tau pQe^{\tau T}.$$

Notice that NPV_T can be rewritten as

$$\mathrm{NPV}_T = (\phi/2T)\mathrm{NPV}_\phi + (r/T)\mathrm{NPV}_r + (\tau/T)\mathrm{NPV}_\tau.$$

Now note $\mathrm{NPV}_T \lesseqgtr 0$ since $\mathrm{NPV}_r \lesseqgtr 0$. Q.E.D.

The negative partial effect of technological uncertainty on the project value can be explained as follows. An increase in the dispersion of possible output level increases the probability that the quantity produced is less than the quantity ordered, and thus lowers the value of the expected cash flow. Of course, the increased variance also increases the probability that the quantity produced is greater than the quantity ordered. It is important to observe, however, that the marginal revenue (p) for the range of output $Y > X$ is smaller than the marginal revenue (P) for the range of output $Y < X$ (see Figure 2). Since the factor cost is constant (i.e., Cq) regardless of the output range, the net effect of the increase in variance on the project value will be negative. Managerial implication is clear; if the firm has several alternative production technologies to choose from, the firm should choose the one with the least production uncertainty, *ceteris paribus*.

Differing from the case with demand uncertainty, the effect of the interest rate on the project value is ambiguous. The reasons for this ambiguity is clear from (22). The third term of the equation can be interpreted as that the firm in essence is writing a put option against the future value of $(P - p)Y$ with an exercise price of $(P - p)X$. Since the put option value is smaller with the higher interest rate, the higher interest rate results in the larger third term. On the other hand, the higher interest rate results in the smaller present value of the second term. Therefore the net effect of the interest rate change on the project value could be either positive or negative.

Finally, there are three effects on the NPV of an increase in the production lead time: (1) it increases the technological uncertainty; (2) it has the similar effect as an increase in the interest rate; and (3) it either reduces or increases the effective (i.e., time-risk adjusted) output level, depending on the sign of τ. Since the second and third effects could be negative or positive. The net effect of the production lead time on the NPV could be either positive or negative.

4. Output Decision under Demand Uncertainty with Stochastic Production Function

4.1. *Assumptions and Valuation*

In this section, both demand and production processes are assumed to be stochastic. As is in §§2 and 3, the change in the cumulative order level and the firm's production process are described by equations (1) and (18), respectively. Then cash flows at time T will be $PY - Cq$ if the realized output level is less than the customer order, or $PX + p(Y - X) - Cq$ if the realized output level is greater than the customer orders.[18] The firm's objective is to determine the target output level, Q, such that the present value of cash flows is maximized. Note that the time T cash flows can be expressed as

$$\text{Min}\,[PY - Cq, PX + p(Y - X) - Cq]. \tag{32}$$

Then using (6) and after rearrangement, the cash flow at time T can be expressed as

$$(pY - Cq) + (P - p)X - (P - p)\,\text{Max}\,[0, X - Y]. \tag{33}$$

Notice that both X and Y are stochastic variables. Then using the techniques in Fischer (1978) and Margrabe (1978), the present value of the last term in (33) can be evaluated as

$$(P - p)\{xe^{\delta T}N(d_1) - Qe^{\tau T}N(d_2)\}; \tag{34}$$

where

$$d_1 = [\{\ln(x/Q) + (\delta - \tau)T\}/\theta\sqrt{T}] + (1/2)\theta\sqrt{T},$$

$$d_2 = [\{\ln(x/Q) + (\delta - \tau)T\}/\theta\sqrt{T}] - (1/2)\theta\sqrt{T},$$

$$Q = q^{\alpha},$$

$$\theta^2 = \sigma^2 - 2\sigma\phi r_{12} + \phi^2, \qquad \text{and}$$

r_{12} = the correlation between the Wiener processes dw and dv.

Also note that the present values of pY, Cq, and $(P - p)X$ are $pQe^{\tau T}$, Cqe^{-rT}, and $(P - p)xe^{\delta T}$, respectively. Hence the net present value of cash flows is defined as

$$\text{NPV} = pQe^{\tau T} - Cqe^{-rT} + (P - p)xe^{\delta T} - (P - p)[xe^{\delta T}N(d_1) - Qe^{\tau T}N(d_2)]. \tag{35}$$

4.2. *Optimality Conditions and Comparative Statics Analysis*

Letting $d\text{NPV}/dQ = 0$, and after simplification, the optimality condition can be expresses as

$$PN(d_2) + p\{1 - N(d_2)\} = \{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T}. \tag{36}$$

If the production function exhibits constant returns to scale, the optimality condition simplifies to

$$PN(d_2) + p\{1 - N(d_2)\} = Ce^{-(r+\tau)T}. \tag{37}$$

Rearrangement of (37) yields the following closed-form solution for the optimal output level:

$$Q^* = \exp[\ln(x) + (\delta - \tau)T - \{N^{-1}(\Phi_1) + .5\theta\sqrt{T}\}\theta\sqrt{T}], \tag{38}$$

where $\Phi_1 = \{Ce^{-(r+\tau)T} - p\}/(P - p)$.

Finally, the following second derivative insures the sufficiency condition for $0 < \alpha \leq 1$:

[18] The graphical description of the end of period cash flows of this case can be visualized by assuming different values of Q in Figure 1 or X in Figure 2.

$$d^2\text{NPV}/dQ^2 = -[(P-p)e^{\tau T}f(d_2)(1/\theta\sqrt{T}Q) + \beta e^{-rT}C] < 0, \tag{39}$$

where $\beta = [(1-\alpha)/\alpha^2]Q^{(1/\alpha)-2}$.

We now examine the effect of the interest rate, uncertainty, and production lead time on the optimal output level.

THEOREM 5. (i) *The effect of the interest rate on the optimal output level is positive.*
(ii) *The effects of uncertainty and the production lead time on the optimal output level could be either positive or negative.*

PROOF. (i) Partially differentiating Q^* with respect to r, we have

$$Q_r^* = \frac{\{(1/\alpha)Q^{(1/\alpha)-1}\}e^{-rT}C}{\beta\{C/(P-p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\theta\sqrt{T})} > 0. \tag{40}$$

(ii) Partially differentiating Q^* with respect to θ, we obtain

$$Q_\theta^* = -\frac{[(\ln(x/Q) + (\delta-\tau)T\}/(\sqrt{T}\theta^2) + (1/2)\sqrt{T}]f(d_2)}{\beta\{C/(P-p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\theta\sqrt{T})} \lesseqgtr 0. \tag{41}$$

Partially differentiating Q^* with respect to T, we obtain

$$Q_T^* = \frac{G - [\{\ln(x/Q) + (\delta-\tau)T\}/(2T\sqrt{T}\theta) + (\theta/4)\sqrt{T}]f(d_2)}{\beta\{C/(P-p)\}e^{-(r+\tau)T} + (1/Q)f(d_2)(1/\theta\sqrt{T})} \lesseqgtr 0, \tag{42}$$

where

$$G = [(r+\tau)\{(C/\alpha)Q^{(1/\alpha)-1}\}e^{-(r+\tau)T}]/(P-p). \quad \text{Q.E.D.}$$

Although we find the positive relationship between the interest rate and optimal output level as we did when there exists *only* demand uncertainty, the underlying economic force is somewhat different. Note first that, since d_2 is not a function of the interest rate, the probability of under-production is independent of the interest rate. Notice, however, that the marginal cost of under-production (i.e., $P - \{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T}$) will be greater with the higher interest rate. Similarly, notice that the marginal cost of over-production (i.e., $\{(1/\alpha)Q^{(1/\alpha)-1}\}Ce^{-(r+\tau)T} - p$) will be smaller with the higher interest rate. Thus the higher interest rate implies the higher (lower) *expected* marginal cost of under-production (over-production). Facing with the higher (lower) expected marginal cost of under-production (over-production), the firm will increase its output to move to its new optimality point, *ceteris paribus*.

4.3. *Sensitivity Analysis*

This section examines the effect of uncertainty, interest rate, and production lead time on the project value.

THEOREM 6. (i) *The project value decreases with higher uncertainty.*
(ii) *The project value increases with higher interest rate.*
(iii) *The project value can increase or decrease with longer production lead time.*

PROOF. (i) $\text{NPV}_\theta = -(P-p)e^{\tau T}Qf(d_2)\sqrt{T} < 0$.
(ii) $\text{NPV}_r = CQ^{1/\alpha}Te^{-rT} > 0$.
(iii)

$$\text{NPV}_T = -(P-p)e^{\tau T}Qf(d_2)\sqrt{T}(\theta/2\sqrt{T}) + rCQ^{1/\alpha}e^{-rT} + \delta(P-p)xe^{\delta T} + \tau pQe^{\tau T}.$$

Notice that NPV_T can be rewritten as

$$\text{NPV}_T = (\theta/2T)\text{NPV}_\theta + (r/T)\text{NPV}_r + (\delta/T)\text{NPV}_\delta + (\tau/T)\text{NPV}_\tau.$$

Since $\text{NPV}_\theta < 0$, $\text{NPV}_r > 0$, $\text{NPV}_\delta > 0$, and $\text{NPV}_\tau > 0$, $\text{NPV}_T \lesseqgtr 0$. Q.E.D.

Since $\theta^2 = \sigma^2 - 2\sigma\phi r_{12} + \phi^2$, where r_{12} is the correlation between the demand (dw) and production (dv) uncertainty, result (i) above implies that the project value will be large when random components of demand and output levels are highly positively correlated. Conversely, if the random components of the demand and production levels move in opposite directions, the project value will be small, *ceteris paribus*. Managerial implication of this resuls is clear. When the firm has choices among different factors of production, it pays to select and combine input factors in such a way that the uncertain portion of the firm's output is closely tied to the market demand of the firm's output.

It is interesting to note that the higher interest rate results in the greater project value. The underlying intuition of this result is clear upon inspection of equation (35). With the higher interest rate, the firm essentially reduces the present value of the production cost, and thus has the greater project value. In sum, therefore, the firm facing both demand and production uncertainty can increase its value by producing more when the interest rate rises.

Finally, the effect of the production lead time on the project NPV can be either positive or negative since the longer production lead time implies the higher uncertainty, the larger interest rate, the smaller effective demand. Since these factors have opposite effects on the NPV, the net effect of the production lead time on the NPV can be either positive or negative.

5. Summary and Concluding Remarks

Differing from the previous studies which typically viewed firms as maximizing the von Neumann-Morgenstern expected utility of profit, this study views firms as maximizing the present value of future cash flows. Specifically, this study uses the option pricing model for the evaluation of the firm's output decision under uncertainty.

The findings of this study are summarized in Table 1. A somewhat surprising but nonetheless interesting finding of the study is that the effects of demand and technological uncertainty on the optimal output level are generally ambiguous. It is also worth noting that the longer production lead time does not necessarily lead to the larger output volume. Some important managerial implications which follow from the results of the study are: (i) the management should be more aggressive in determining output volume when the interest rate is higher if the demand for the firm's output is uncertain; (ii) the firm should choose the product line with the least demand volatility; and (iii) the firm should choose

TABLE 1

Effect of Interest Rate, Uncertainty, and Production Lead Time on Optimal Output Level and Project Value

		Q^*			NPV		
		r	$\sigma^{\#}$	T	r	σ	T
Demand Uncertainty	$\alpha = 1$	+	I	I	0	−	−
	$\alpha < 1$	+	I	I	−	−	−
Production Uncertainty	$\alpha = 1$	I	I	I	I	−	I
	$\alpha < 1$	I	I	I	I	−	I
Demand and Production Uncertainty	$\alpha = 1$	+	I	I	+	−	I
	$\alpha < 1$	+	I	I	+	−	I

For production uncertainty, this should be replaced by ϕ and for both production and technological uncertainty by θ.

+ denotes the positive effect.

− denotes the negative effect.

0 denotes no effect.

I denotes indeterminate, i.e., the direction of effect is determined by relative magnitudes of other variables.

the product line with the least production lead time when the technological uncertainty is absent.[19]

[19] The author is grateful to Mark Rubinstein (the referee), an anonymous referee, Walter Kemmsies, Jinho Jeong, and Charlie Charoenwong for their insightful comments on earlier versions of this paper. The author is solely responsible for any remaining errors.

Appendix A

Optimal Output Decisions with Initial Stock and Fixed Cost

This appendix extends the basic model by relaxing the assumptions of zero initial stock of inventory and zero fixed cost. First let us assume that at time zero the firm has I units of beginning inventory. If we redefine Q as the number of units to be available at time T, the net cash flow at time T can be defined as follows:

$$\text{Min } [PQ - Cq, PX + p(Q - X) - Cq] + CI. \tag{A1}$$

Since CI is a constant, equation (10) yields the optimal solution to this problem provided that $Q^* \geq I$. If $Q^* < I$, the optimal decision is not to produce. Thus the optimal decision process can be summarized as follows:

(a) Find Q^* such that

$$[P - \{(1/\alpha)Q^{(1/\alpha)-1}\}C]N(d_2) = [\{(1/\alpha)Q^{(1/\alpha)-1}\}C - p]N(-d_2),$$

(b) If $Q^* \leq I$, do not produce, and

(c) If $Q^* > I$, produce $(Q^* - I)$ units.

If we further assume that there is a fixed cost of production, F, the net cash flow at time T will be

$$\text{Min } [PQ - Cq, PX + p(Q - X) - Cq] + CI - F. \tag{A2}$$

As is in equation (A1), the optimal output decision can be described by equation (10) because F is also a constant, provided that incurring of F is economically justifiable. If it is not, the optimal policy is not to produce any additional units. In that case, the sales revenue, $S(I)$, is defined as

$$S(I) = \text{Min } [PI, PX + p(I - X)]. \tag{A3}$$

If we denote $G(Q) = \text{Min } [PQ, PX + p(Q - X)]$, the relationship between Q and $G(Q) - Cq$ can be described as in Figure 3, where H is the value of Q that maximizes $G(Q) - Cq$ and L is the smallest value of Q for which $G(L) - CL^{1/\alpha} = G(H) - CH^{1/\alpha} - F$. In the case of $I > H$, it is evident that

$$G(Q) - Cq - F < S(I) - CI \qquad \text{for all } Q > I. \tag{A4}$$

Note that (A4) can be rewritten as

$$G(Q) - C(q - I) - F < S(I) \qquad \text{for all } Q > I, \tag{A5}$$

where the left-hand side of (A5) represents the net cash flow if the firm produces up to Q, and the right-hand side represents the net cash flow if the firm does not produce. Hence if $I > H$, the optimal policy is not to produce. If $L \leq I \leq H$, it is evident again from Figure 3 that equation (A5) holds. Thus the optimal decision is not to produce.

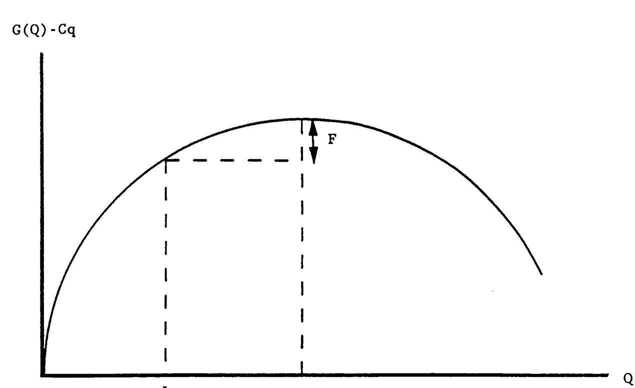

FIGURE 3. The Relationship Between Q and $G(Q) - C_q$.

Finally, if $I < L$, it follows from Figure 3 that

$$\operatorname*{Max}_{Q \geq I} [G(Q) - Cq - F] = G(H) - CH^{1/\alpha} - F > S(I) - CI, \tag{A6}$$

or alternatively,

$$\operatorname*{Max}_{Q \geq I} [G(Q) - Cq + CI - F] = G(H) - CH^{1/\alpha} + CI - F > S(I), \tag{A7}$$

so that it pays to produce additional units. The maximum value is obtained if the firm produces up to H. Thus the optimal output decision can be summarized as follows:

(a) Find H^* such that

$$[P - \{(1/\alpha)H^{(1/\alpha)-1}\}C]N(d_2) = [\{(1/\alpha)H^{(1/\alpha)-1}\}C - p]N(-d_2),$$

(b) Find the smallest value of L such that

$$G(L) - CL^{1/\alpha} = G(H) - CH^{1/\alpha} - F,$$

(c) If $I < L$, produce $H - I$ units, and
(d) If $I \geq L$, do not produce.

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