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EXTENSION OF THE MODEL TO DURABLE COMMODITIES PRODUCTION, AND CORPORATIONS

In this chapter we extend the model of wealth allocation introduced in Chapter 1 to allow for commodity storage and production as vehicles, in addition to the purchase of securities, for the carrying of wealth from period to period. This extension is straightforward so long as the firms holding or producing the commodities are taken as owned and managed by a single individual. But serious problems in defining an appropriate criterion function arise as soon as we try to allow for multiowner corporations in which the decision-making power has largely been delegated to managers. Fortunately, however, there are at least some circumstances in which the existence of well-functioning markets for corporate securities permits a simple solution to these problems. And equally important from the present point of view, it turns out that much of the standard theory of valuation and corporate finance emerges naturally in the process of developing and interpreting the solution.

I. DURABLE COMMODITIES AND INVESTMENT

I.A. Representation of the Carry-over Opportunities Provided by Commodity Storage and Production

Even in simple two-period cases, the number and kind of opportunities for carrying over resources by means of commodities are quite large. The decision maker may choose, for example, to store commodities, such as wheat, in the first period and then either sell or consume them in the second. Or he may purchase a durable good, such as an automobile, and either rent out or consume its services and subsequently dispose of it by resale or scrapping. Or he may own some firm or other productive opportunity that he can exploit by transforming labor services and raw materials in period 1 into some finished product for sale or consumption in period 2.

Despite the differences in outward form, we can represent any such opportunities in general terms by the implicit function $T(K_1, K_2) = 0$, where K_1 represents the dollar value of the consumption possibilities provided by the opportunity in period 1 and K_2 the maximum value of these possibilities in period 2 consistent with having K_1 in period 1. The precise shape and position of T depend, of course, on the nature of the opportunity.

In the case of commodity storage, for example, the relationship between K_2 and K_1 summarized by the function T is a straight line if we assume that the commodity is traded in a perfect market and that the per unit cost of storage is independent of the quantity stored. Such a case is shown in Figure 2.1. The point $K_1^* = p_1 q_i$ is the current market value of a holding

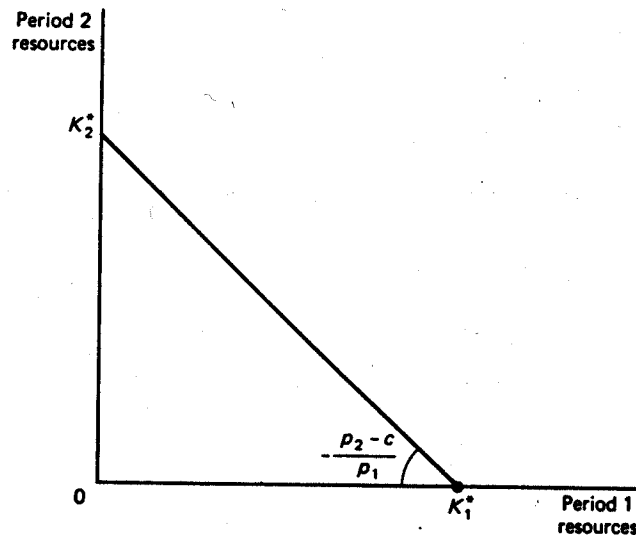


Figure 2.1 Opportunity Set for the Storage Case

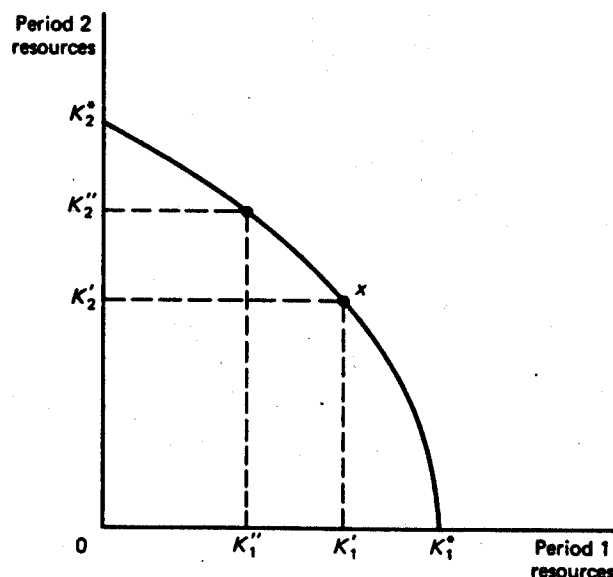


Figure 2.2 Opportunity Set for the Production Case

consisting of q_j units of some storable commodity j with current market price p_1 . If the entire stock were carried over to period 2, its value would be $K_2^* = (p_2 - c)q_j$, where c is the unit cost of storage. The straight line through these points shows all the possible intermediate strategies involving both some storage and some immediate liquidation of the stock; an explicit representation of the T function in this case would be

$$K_2 = K_2^* - \frac{p_2 - c}{p_1} K_1.$$

Figure 2.2 shows a T function that might be appropriate where the carry-over was by way of production or where the costs of storage increased with the quantity stored. The point K_1^* represents the current market value of initial resources. For concreteness, we may think of these resources as consisting of a given number of bushels of corn that may be either sold currently or planted to yield a further crop of corn next period. The point x represents a strategy involving the immediate sale of part of the stock yielding K_1' dollars and a planting of the remaining stock, that is, an investment of $K_1^* - K_1'$ dollars. The net value at period 2 of the crop obtained from this investment, after paying for any productive services used, is K_2' .

Note that, as drawn, the slope of the T function—the marginal rate of transformation of K_1 into K_2 —falls steadily in absolute value as we advance along the curve from K_1^* toward K_2^* . Or using more context-oriented terms, the marginal rate of yield or marginal rate of return on investment declines

steadily as the level of investment increases; that is, an investment of $K_1^* - K_1'$ yields a return of K_2' , but a further investment of exactly the same amount $K_1' - K_1'' = K_1^* - K_1'$ yields only $K_2'' - K_2' < K_2'$, and so on.

In drawing the curve with this shape and position, we are, of course, making a number of implicit assumptions about the underlying conditions of production and sale, as well as about the policies followed by the owner in his price and output decisions. Precisely what these assumptions are need not concern us at the moment; in Chapter 3 we take up the relations between the transformation function and the optimal price and output decisions as treated in the standard theory of the firm. For the present, we simply take all this substructure as somehow given and assume that the optimal solution to these decision problems yields a transformation function for the production carry-over case with the concavity property pictured. It is not important for our purposes at this point that we cannot state this function in explicit form.

1.B. The Opportunity Set When Both Commodity Carry-overs and Capital Markets Are Available

The construction of the opportunity set when both commodity carry-overs and capital markets are available is a simple enough matter in the linear storage case. Figure 2.3, for example, shows two such storage possibilities. In the first case, represented by the line $K_1^*K_2^*$, the returns from storage—price change minus storage costs—are less than the returns from

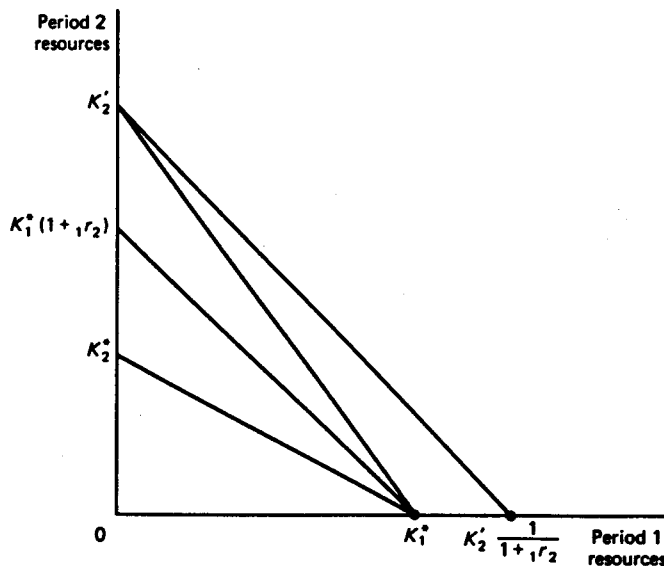


Figure 2.3 Storage and Capital Market Opportunities Combined

securities; that is, the slope of the line $K_1^*K_2^*$ is less in absolute value than that of the present value line $K_1^*K_1^*(1 + r_2)$. Because the present value line through K_1^* completely dominates the possibilities along the storage line,¹ the combined efficient set is attained by selling out the stock K_1^* immediately and relying on the capital market for any desired carry-over. In the second case, shown by the line $K_1^*K_2'$, the returns from storage are greater than from holding securities, and the reverse policy would be indicated; that is, the entire stock K_1^* should be stored to obtain K_2' at period 2, and any consumption needs during the current period met by borrowing against K_2' , that is, by borrowing down the present value line $K_2'K_2'[1/(1 + r_2)]$. Actually, of course, the latter case is considered for purposes of illustration only; its occurrence is ruled out when capital and commodity markets are perfect. For if it arose, a sure profit could always be made by the time arbitrage, so to speak, of buying the commodity currently—borrowing, if necessary—and reselling one period later. The assumption of perfect capital and perfect commodity markets, in other words, implies that the real returns from storage can never exceed the market rate of interest (but they may fall below the market rate, in which event the commodity is not stored at all).

I.C. The Case of Production and Investment

I.C.1. The double-tangency solution

For the case of productive opportunities, the steps in the construction of the combined opportunity set are shown in Figure 2.4. Consider first the point K_1^* , corresponding to an immediate and complete liquidation of the productive resources. The combinations of c_1 and c_2 that can then be obtained by lending part of K_1^* in the capital market are represented by the present value line $K_1^*K_1^*(1 + r_2)$, which is obtained by passing a line with slope $-(1 + r_2)$ through K_1^* . Consider next an investment policy leading to some higher point along the productive efficient set, such as the point x . The consumption combination that could be attained by using the capital market to reallocate the resource combination provided by x is represented by the present value line $K_1^*K_1^*(1 + r_2)$ through x , which clearly everywhere dominates that for the policy of complete liquidation, that is, the policy of paying out K_1^* in period 1. By repeating this process for all the possible investment strategies along the productive frontier, we can see that the investment policy that dominates all others is the point y , where the relevant present value line is exactly tangent to the productive efficient

¹ The notion of dominance used here is, of course, an implication of the nonsatiation axiom. For a given level of resources in period 1, the consumer always prefers more to less resources in period 2. And similarly, given the level of resources in period 2, more resources in period 1 are always preferred to less.

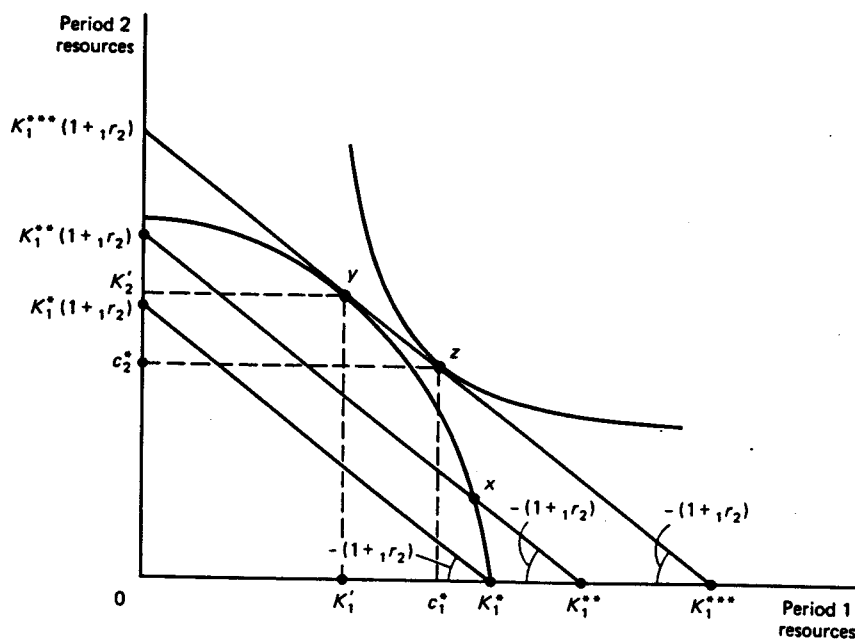


Figure 2.4 Joint Equilibrium of Production and Consumption Decisions

frontier. Or to use the rate terminology, the dominant investment policy is the one for which the marginal internal (one-period) rate of return is exactly equal to the (one-period) market rate of interest.

Given this fundamental result, the complete solution, showing the decision maker's simultaneous choices as to both investment and consumption patterns, can be indicated merely by adding in the consumption indifference curves. In the case shown in Figure 2.4, for example, the preferred position is the point z (for a decision maker assumed to have no other resources than the productive opportunity). This point represents the following combined set of choices:

1. The immediate liquidation and withdrawal of K_1' of the K_1^* dollars' worth of productive resources currently held in the firm ;
2. The investment of the remaining $K_1^* - K_1'$ dollars' worth of resources in the productive process to yield withdrawable proceeds of K_2' next period ;
3. The consumption of c_1^* this period and the borrowing of $c_1^* - K_1'$ dollars to finance the gap between consumption and the available proceeds from the immediate withdrawal ;
4. The consumption of c_2^* next period, repaying the loan plus interest $(c_1^* - K_1')(1 + r_2) = K_2' - c_2^*$ from the proceeds of the previously scheduled production in the firm ,

I.C.2. *Some remaining issues*

We have so far developed the model of allocation over time in terms of a single individual decision maker. To the extent that this individual had opportunities to carry over his resources by investing in productive assets, it was assumed that these opportunities arose in a business firm of which the individual was the sole owner and manager.

A model of a single owner-manager is certainly not devoid of interest in its own right, because the unincorporated business sector of the economy is by no means trivial. But the main concern here is with financial policy and investment decisions in the corporate sector. And before we can come to grips with these problems, we must show how the basic model can be extended or reinterpreted to accommodate two key facts of corporate life: (1) instead of a single owner whose preferences can be described by a single utility function, the typical corporation has many owners whose utility functions must be presumed to be different; and (2) the day-to-day business decisions, including most financial and investment decisions, are made, not directly by the stockholders, but by professional managers. It is true that for some large or important decisions, such as issues of new shares, a plebiscite of the shareholders, weighted by size of holding, may have to be taken; and the dividend decision is technically always made by the board of directors, who are in a legal sense the direct representatives of the shareholders. As a practical matter, however, and despite inevitable exceptions, effective control over decisions in large, widely held corporations is exercised by the management.

The next section in the present chapter focuses on these and related problems and, in particular, attempts to show how an operational criterion for management decisions can be developed within the framework of the model already presented. Before turning to this task, however, we first sketch out the mathematical solution to the two-constraint problem for the general n -period case.

*I.D. The Solution in the n -Period Case

As compared with the n -period choice problem considered in Chapter 1, the major change for the present problem is that we now have two sets of decision variables—the K_t as well as the c_t —and two opportunity sets, one representing borrowing and lending possibilities in the capital markets and the other investment and withdrawal opportunities from the production possibilities. The essence of the latter can be captured effectively by a simple implicit-form constraint written as

$$T(K_1, K_2, \dots, K_N) = 0. \quad (2.1)$$

To see how the former can be obtained, begin again with the last period, and note that the maximum amount of consumption possible in period

N is now

$$c_N = a_N + y_N + K_N.$$

Working backwards again, we obtain for a_N

$$a_N = (a_{N-1} + y_{N-1} + K_{N-1} - c_{N-1})(1 + {}_{N-1}r_N),$$

and similarly until after the successive substitutions and rearrangements we obtain the following as the general capital market constraint:

$$\begin{aligned} c_1 + \frac{c_2}{1 + {}_1r_2} + \frac{c_3}{(1 + {}_1r_2)(1 + {}_2r_3)} + \cdots + \frac{c_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})} \\ = a_1 + y_1 + \frac{y_2}{1 + {}_1r_2} + \frac{y_3}{(1 + {}_1r_2)(1 + {}_2r_3)} + \cdots + \frac{y_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})} \\ + K_1 + \frac{K_2}{1 + {}_1r_2} + \frac{K_3}{(1 + {}_1r_2)(1 + {}_2r_3)} + \cdots + \frac{K_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})}. \quad (2.2) \end{aligned}$$

In words, only those patterns of consumption are efficient for which the present value of the consumption stream exactly equals the sum of the present value of the future earned incomes plus the present value of the withdrawals from the productive firm or from other commodity carry-over opportunities.

The decision maker's preferred choice can then be represented as the solution to the problem

$$\max_{c_1, \dots, c_N; K_1, \dots, K_N} U(c_1, c_2, \dots, c_N) \quad (2.3)$$

subject to the constraints (2.1) and (2.2). To obtain a more explicit characterization, we form the lagrangian expression

$$\begin{aligned} U(c_1, c_2, \dots, c_N) - \lambda_1 T(K_1, K_2, \dots, K_N) \\ + \lambda_2 \left(K_1 + \frac{K_2}{1 + {}_1r_2} + \cdots + \frac{K_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})} \right. \\ \left. - c_1 - \frac{c_2}{1 + {}_1r_2} - \cdots - \frac{c_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})} \right. \\ \left. + a_1 + y_1 + \frac{y_2}{1 + {}_1r_2} + \cdots + \frac{y_N}{\prod_{i=1}^{N-1} (1 + {}_ir_{i+1})} \right), \end{aligned}$$

differentiate successively with respect to λ_1 , λ_2 , and each of the $2N$ decision variables, and obtain as the first-order conditions for a maximum (2.1), (2.2), and the relations

$$\begin{aligned}
 U'_1 - \lambda_2 &= 0 \\
 U'_2 - \lambda_2 \frac{1}{1 + r_2} &= 0 \\
 &\dots\dots\dots \\
 U'_i - \lambda_2 \frac{1}{\prod_{\tau=1}^{i-1} (1 + r_{\tau+1})} &= 0 \\
 &\dots\dots\dots \\
 U'_N - \lambda_2 \frac{1}{\prod_{\tau=1}^{N-1} (1 + r_{\tau+1})} &= 0
 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
 \lambda_2 - \lambda_1 T'_1 &= 0 \\
 \lambda_2 \frac{1}{1 + r_2} - \lambda_1 T'_2 &= 0 \\
 &\dots\dots\dots \\
 \lambda_2 \frac{1}{\prod_{\tau=1}^{i-1} (1 + r_{\tau+1})} - \lambda_1 T'_i &= 0 \\
 &\dots\dots\dots \\
 \lambda_2 \frac{1}{\prod_{\tau=1}^{N-1} (1 + r_{\tau+1})} - \lambda_1 T'_N &= 0.
 \end{aligned} \tag{2.5}$$

The set (2.4) is the same as that obtained for the allocation problem considered in Section II.B.6 in Chapter 1 and leads to the same conclusion; namely, between any two periods, adjacent or not, we have

$$-\frac{U'_i}{U'_{i+n}} = -\prod_{\tau=i}^{i+n-1} (1 + r_{\tau+1});$$

that is, the preferred allocation of consumption between the two periods is characterized by a point of tangency between an indifference curve and the capital market opportunity line given in this case by Equation (2.2). As

for the set (2.5), between the same two periods we have

$$-\frac{T'_t}{T'_{t+n}} = -\prod_{\tau=t}^{t+n-1} (1 + r_{\tau+1}),$$

implying that the preferred combination of investment and withdrawals from the productive opportunity between the two periods is characterized by a point of tangency between the productive opportunity set and the capital market opportunity line. As indicated by the analysis of Figure 2.4, however, only by coincidence do the two tangency points coincide.

II. EXTENSION TO THE CASE OF CORPORATIONS

II.A. Management Objectives and Stockholder Preferences

In Chapter 1 it was shown that the general process of choice or decision making could be represented as one of maximizing a given utility function subject to constraints. Managerial decision making is no exception; the problem is, however, that if the decision-making power is considered as having been delegated to management by the shareholders, it is not immediately clear what is the utility function that management should be seeking to maximize or how it could be constructed.

If the firm had only a single owner, it might seem plausible that the appropriate criterion function should be that of the owner. Although hardly to be considered a practical approach, it is at least possible to imagine that the owner might interrogate himself about his preferences and communicate the results, along with a list of all his other transformation opportunities, to his management so as to give an unambiguous decision criterion covering all eventualities. Indeed, on a small scale within organizations today operations researchers and other technicians very frequently assist those responsible for decisions in stating their preferences in the form of explicit utility functions and even sometimes take the initiative in suggesting possible alternative utility functions for the problem at hand subject to final selection by the officer ultimately responsible.²

Practicality aside, however, there are also some serious conceptual problems involved. In particular, it is by no means clear how such a direct approach is to be implemented when we allow, as we must, for the possibility of many owners with possibly very different preferences, and other opportunities. To see the nature of the difficulties involved in attempting to construct an aggregate preference function from the separate individual

² See, for example, C. J. Grayson, "The Use of Statistical Techniques in Capital Budgeting," in *Financial Research and Management Decisions*, A. Robichek, ed. New York: John Wiley & Sons, Inc., 1967.

functions, consider the following simple example. Suppose that we have a firm owned by three shareholders A, B, and C each with an equal one-third interest. Suppose further that the firm faces an investment or other decision that can be reduced to a choice of one among three mutually exclusive proposals 1, 2, and 3. The stockholders, after due reflection, inform management that the order of their preferences among these alternatives is as follows:

Stockholder	First Choice	Second Choice	Third Choice
A	1	2	3
B	2	3	1
C	3	1	2

And suppose finally that management attempted to combine the preferences and make the choice on the basis of "majority rule." If management first considered proposal 1 versus proposal 2, then proposal 1 would be the choice, because it is preferred to 2 by A and C. If the next confrontation is between the previous winner 1 and proposal 3, then the winner is 3, because it is preferred to 1 by C and B. If now one last check is made, comparing 3 with the previously rejected 2, then 2 emerges as the preferred alternative, because it is preferred to 3 by A and B. Thus the results of the majority-rule combination of the individual preference functions of the owners is either indecisive and circular if all two-way comparisons are made or completely capricious and arbitrary if an alternative, once rejected, is not given its second chance, because in this case the final outcome depends entirely on the order in which the comparisons happened to be made.

Nor is this result in any way freakish or merely an artifact of the particular combination rule used. It can be shown, more generally, that if we impose certain minimal standards of sensible behavior for any aggregation rule, there exists no sensible nondictatorial aggregation rule that can guarantee the avoidance of the kind of inconsistency or intransitivity encountered in the example. Or to put it in terms more closely related to the basic framework, there is, in general, no way of directly combining the preference or utility functions of the individual shareholders into a single global preference function which meets all the axioms of choice and hence which a management could use as an unambiguous criterion for making decisions "in the best interests of the owners."³

³ The proposition was first presented and proved in Kenneth J. Arrow, *Social Choice and Individual Values*. New York: John Wiley & Sons, Inc., 1951. Note the qualification that "in general" no unambiguous composite utility function exists. It is possible, however, to construct special cases in which all the properties of the individual utility functions do carry through to an aggregate representation of preference.

II.B. The Market Value Criterion

Fortunately, however, there is at least one important class of circumstances in which we can avoid these and related difficulties of constructing a decision criterion for management directly from stockholders' preferences. Where there exist organized capital markets in which shares can be freely bought and sold and where these markets are perfect in the sense defined in Chapter 1, it is possible to develop an objective, operational decision criterion for management that (1) does not involve stockholder utility functions directly but (2) leads to precisely the same investment and operating decisions that each stockholder would make if he were running the firm himself.

Important and fundamental as this result is for the theory of corporation finance, its proof requires little more than a reinterpretation of results that we have already obtained in Section I for the owner-managed firm. Recall that an overall optimum of consumption, saving, borrowing, and investment by a single owner-manager was characterized by the simultaneous satisfaction of two conditions: (1) tangency between the transformation curve of his firm and the capital market opportunity line and (2) tangency between the capital market opportunity line and an indifference curve. The points y and z in Figure 2.4 represent such an optimum, with $K_1^* - K_1'$ the amount invested in the firm, K_1' the amount withdrawn, $c_1^* - K_1'$ the amount borrowed or equivalently the amount of securities sold, and c_1^* and c_2^* the amounts allocated to consumption in periods 1 and 2, respectively.

Note that the achievement of the first of the two optimality conditions—that of tangency between the transformation function and the market opportunity line—does not require any of the “subjective” information summarized in the owner's indifference curves. It involves only the “objective” knowledge of the technology of the firm and of the ruling rate of interest. Hence finding a tangency point, such as y , is a subtask that can in principle be delegated to a manager.

Moreover, this delegation can readily be implemented by a decision rule that conforms directly to the standard model of the theory of choice presented in Chapter 1. Recall that market opportunity lines may also be interpreted as present value lines, or market value lines, because with perfect capital markets, current market value necessarily equals present value; that is, the intercept on the period 1 axis of any such line with slope $-(1 + r_2)$ shows the value in terms of current resources of any resource combination lying on the line, the value K_1^{**} in Figure 2.4, for example, thus being the present or market value of the point x . Hence the injunction to management to find the point of tangency y is equivalent to the injunction to maximize the current market value of the withdrawals to be provided by the firm to its current owners, subject to the technological constraints imposed by the transformation function. This criterion is henceforth often referred to as the “market value rule.”

Having exploited the opportunities of transformation inherent in the firm and thereby having increased the owner's wealth from its initial value of K_1^* to its maximum possible value of K_1^{***} , management can safely leave the rest of the full solution to the owner himself. Given the optimum investment and withdrawal pattern represented by the point y , the owner can then exploit the capital market on his own account, in this case by borrowing $c_1^* - K_1^*$ to reach his personal maximum of utility at z .

11.8.1. The case of many owners

Not only does the market value rule provide a utility-maximizing surrogate criterion for the managers of a single-owner firm, but it should be clear from the nature of the argument that the essential reasoning applies to the same extent and with equal force regardless of the number of individual shareholders or the differences in their preferences or other opportunities. To illustrate, consider a firm with two owners each owning a half interest and each with no further outside resources. For each owner, we could draw a figure similar to Figure 2.4, but because we assume their ownership to be equal and the firm to be their sole opportunity beyond those of the capital markets, we can combine them both into a single graph, as in Figure 2.5. As before, each owner, if entrusted with the complete power

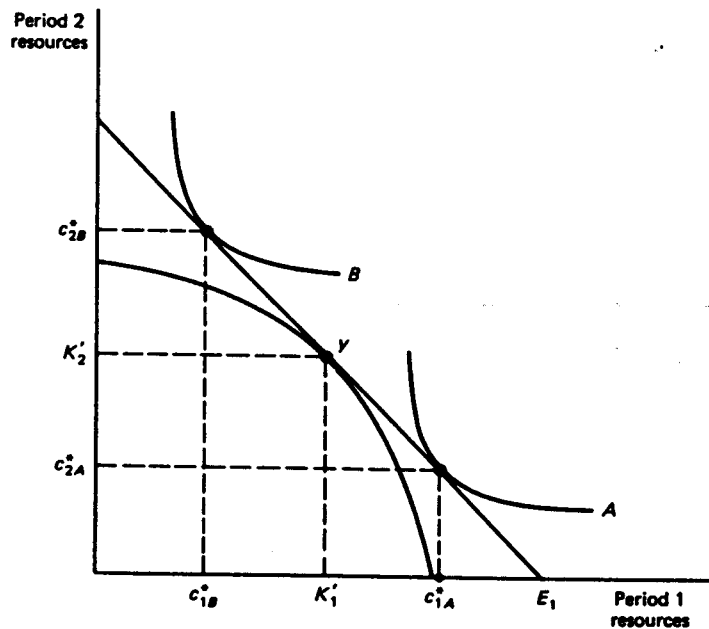


Figure 2.5 Equilibrium Production and Consumption Decisions for the Multiowner Firm

to manage, would choose the point y as the investment-withdrawal combination for the firm, the same point that would be chosen by a separate management following the market value rule. The differences in their tastes show up only in the second stage of the maximization by way of the capital market opportunities. Mr. A, with a strong preference for current consumption, supplements his "dividend" of K'_1 from the firm by borrowing the additional amount $c^*_{1A} - K'_1$; and Mr. B, who prefers to devote more of his wealth to future consumption, supplements the carry-over by way of the firm by investing $K'_1 - c^*_{1B}$ of his dividend in securities in the market. But these subsequent personal details are of no concern to the management. They can regard their obligations to the owners as having been satisfied by having achieved the point y and thereby having maximized the value of the resources available to the owners.

The relevance of the market value criterion for managerial decisions is in no way restricted to the case of two time periods or to the special assumptions that we have been making about the shape of the transformation function. Rather the decisive elements in the argument are the assumption of perfect capital markets and the nonsatiation axiom. It is the former that permits the results of all economically relevant management actions in any time period, current or future, to be expressed in terms of a common denominator, current wealth. More specifically, as we showed in Chapter 1, in a perfect capital market, and for given values of current and future one-period interest rates, a consumer's current wealth is sufficient information to describe all his consumption opportunities over time, because in such a market any sequence of dollar consumptions with a given market value can be exchanged for any other sequence with the same market value. Moreover, in a perfect capital market any changes in the configuration of consumption possibilities K , that an owner can derive from a firm can be converted uniquely and unambiguously to a present wealth equivalent by discounting at the rates of interest ruling in the capital markets. Thus if the change in the owner's current wealth implied by a given management decision is positive, he must be better off, because he would then be able to increase his consumption in some time periods without having to reduce his consumption in any other period.⁴

⁴ Although each individual owner always wants the firm to maximize the current market value of his holdings, it is possible to imagine (somewhat artificial) cases in which this does not imply an unambiguous decision rule for the firm. There are no problems as long as there is only one type of security outstanding, so that each unit of the security implies the same proportionate share of payoffs provided by the firm in all periods. Then increasing the current market value of the holdings of one owner always implies increasing the values of the holdings of the others.

But suppose, for example, that instead of a single type of security the firm has issued n different types each of which represents a share only in the payoffs for a given

*II.C. The Market Value Criterion in the General n -Period Case

A more formal proof that maximizing the utility of the owners implies maximizing the market value of their holdings in the firm is easily provided. We consider first a one-owner firm and then generalize the results to the multiowner case.

The choice problem presented by a one-owner firm has already been stated in mathematical form for the n -period case in Section I.D. To restate it for current purposes in more compact and transparent form, let $C = (c_1, c_2, \dots, c_n)$ represent the entire vector of consumption expenditures per period, K the vector of withdrawals from the firm, $V(C)$ the present value of lifetime consumption, $V(Y)$ the present value of lifetime earnings, plus any initial financial assets, and $V(K)$ the present value of future withdrawals from the firm, or for short, the present value of the firm. The system of Equations (2.1) to (2.3) then can be written

$$\max_{C, K} U(C) \quad (2.6)$$

subject to

$$V(C) = V(Y) + V(K), \quad (2.7)$$

$$T(K) = 0. \quad (2.8)$$

This constrained maximum problem can be written in entirely equivalent form as the unconstrained maximum problem

$$\max_{C, K} [U(C) - \lambda_1(V(C) - V(Y) - V(K)) - \lambda_2 T(K)], \quad (2.9)$$

where λ_1 and λ_2 are lagrangian multipliers.

Because the utility function $U(C)$ is assumed to be monotone-increasing, by virtue of the nonsatiation axiom, the multipliers λ_1 and λ_2 , which measure the effect on the utility index of loosening the constraints, must be strictly

period with no share in the payoffs of other periods. Then it is easy to imagine situations in which a production decision increases the total current market value of the firm, which in this case is best thought of as the sum of the market values of the payoffs in each period, but reduces the market values of the payoffs for some periods. Thus some of the firm's security holders are better off but others are worse off with the decision. As long as the total market value of the firm has increased, however, those who are better off receive more than enough to compensate in turn those who are worse off and so induce the latter to go along with the decision.

As indicated above, such problems are somewhat artificial in the context of a perfect certainty model. They will be less artificial, however, when we come to a world of uncertainty where multisecurity firms, for example, bonds and common stock, are common and the firm can undertake (unforeseen) actions that sacrifice the interests of one group to those of another.

positive. Hence we may regroup the C and K terms in Equation (2.9) and rewrite it

$$\max_C [U(C) - \lambda_1(V(C) - V(Y))] + \max_K [\lambda_1 V(K) - \lambda_2 T(K)]. \quad (2.10)$$

Let $\lambda_3 = \lambda_2/\lambda_1$. Then we can further reexpress Equation (2.10) as

$$\begin{aligned} \max_C [U(C) - \lambda_1(V(C) - V(Y))] + \lambda_1 \max_K (V(K) - \lambda_3 T(K)) \\ = \max_C \{U(C) - \lambda_1(V(C) - V(Y) - \max_K (V(K) - \lambda_3 T(K)))\}. \end{aligned} \quad (2.11)$$

Thus, because Equation (2.11) is merely a rearrangement of Equation (2.9), we have two completely equivalent ways of finding the (constrained) maximum of the owner's utility: either (1) a one-pass solution of the full system (2.6) to (2.8) by way of (2.9), as was done in Section I.D, or (2) a two-pass procedure in which we first solve the subsidiary maximum problem

$$\max_K [V(K) - \lambda_3 T(K)],$$

which is to say, maximize the value of the owner's interest in the firm, and then complete the solution by solving

$$\max_C \{U(C) - \lambda_1(V(C) - V(Y) - V^*(K))\},$$

where $V^*(K)$ is the maximum value of the owner's interest obtained in the first pass.

Generalizing to the case of a multiowner firm simply involves a re-interpretation of the preceding analysis. Now we let $V(K)$ be the share of an individual owner in the firm's total current market value, and likewise the constraint $T(K) = 0$ represents his implicit share of the firm's production possibilities. From the preceding arguments we then conclude immediately that for any one, and thus each, of the owner's, optimal production decisions by the firm involve maximizing the current market value of withdrawals.

II.D. The Market Value Criterion: Some Problems and Limitations

With the market value criterion in hand, we now have the essential starting point for a systematic attack on the problems of corporation, as opposed to individual, finance. Before moving in this direction, however,

we pause very briefly to take a closer look at some of the implications and limitations of the market value criterion.

II.D.1. The market value rule and profit maximization

The market value rule and the separation that it implies between management decisions and owners' tastes is by no means peculiar to the field of finance. On the contrary, it is merely the application to investment and financing decisions of the general principle of "decentralizing" decisions by exploiting markets and market prices—a principle running through all the standard economic theory of value. In the ordinary theory of production, for example, a firm considering whether to produce oranges or bananas is not presumed to base this decision on which of the two the owners happen to prefer for breakfast. The decision problem is resolved, rather, on the basis of the market prices for the alternative outputs, on the costs of the various factor inputs, and the technological possibilities of production. Once a production plan has been determined so as to maximize the net return to owners, they can spend their shares of the proceeds on whatever pattern of fruit or breakfasts appeals to them the most. The situation is exactly analogous in finance where the "commodities" being produced are generalized consumption possibilities at various points in time.

For some students of finance, coming fresh from an introductory course in price theory, this essential similarity is often missed at first, because of a difference in the terminology used in the two courses. The market value of the owner's equity in the firm, in particular, is rarely, if ever, mentioned in the ordinary theory of the firm, most of which is expounded in terms of flow concepts, such as profits and profit maximization.⁵ It is important to emphasize, therefore, that properly interpreted, there is no conflict between the stock and the flow form of the criterion. When we speak of maximizing profits in ordinary static price theory, we really mean profits in every unit of time; and clearly if we increase profits in every unit of time, we raise the value of the firm, which is essentially just a weighted sum of the returns per period. Lack of exact correspondence between the two criteria arises only when we deal with particular problems in which increases in returns for some periods involve reductions in other periods; and when this is so, the correct solution requires falling back on the more cumbersome, but universally applicable, market value criterion. Failure to realize that the simple flow criterion is not adequate in problems in which the timing of the returns is of the essence has been responsible for much unnecessary confusion, such as the frequently heard argument that the profit maximization criterion, and

⁵ Here and throughout we use the term "profits" in its loose conventional sense of net returns, and not in its Knightian sense of an unanticipated windfall.

hence most of economic theory, does not apply to the business world as we know it, because most large firms could certainly raise prices and profits in the short run.

II.D.2. The market value rule and management motivation

On the other hand, there are some genuine and more serious difficulties that have to be considered before attempting to apply the market value criterion. In particular, the reader may have sensed a subtle inconsistency between the criterion of individual choice as developed in Chapter 1 and that of managerial choice as presented here. Originally, we assumed that the individual decision maker was maximizing his own utility function. Because managers are also individuals, what right have we now to assume that they will put aside their own utility functions and maximize instead the market value of the owners' equity in the firm?

This is a good question and one for which economists have not yet been able to supply a completely satisfactory answer. Instead it is usually assumed that there are sufficient additional processes of "control" or motivation to remove any conflicts between the individual and the corporate criterion functions of the managers. Examples of such processes would be incentive schemes, such as bonuses and stock options, that make management's compensation a direct function of stockholders' compensation, or at the other extreme, sanctions, such as stockholder revolts or outside takeover bids, that could be invoked, or threatened, to remove a management failing to act in the best interests of the owners. Still other examples could be derived from elsewhere in the literature of accounting and management control, because the stockholder-management confrontation is merely one special case of the general problem of administration and delegation between levels in the hierarchy that exists throughout the whole firm.

Throughout the book we, too, shall assume in dealing with management decisions both that the stockholders have found a way to impose the market value rule on top management and that they in turn have been able to impose subcriteria to be applied farther down the line that are completely consistent with and equivalent to the master criterion. No alternative approach has yet achieved sufficient coherence or sufficient acceptance in the profession to merit extensive discussion in an introductory book of this kind. Nor despite many years of controversy, has it yet been demonstrated that the market value rule leads to predictions that are so widely at variance with observed management behavior as to rule it out, even as a first approximation, for the class of decision problems to which we shall here be applying it.⁶

⁶ A recent survey of the issues and the evidence is that of Robert J. Larner, *Management Control and the Large Corporation*. New York: Dunellen Publishing Co., 1970.

II.D.3. The criterion problem under imperfect capital markets

At this point, some readers may well be prepared to accept our working criterion of value maximization under conditions of perfect capital markets but are also impatiently waiting for guidance as to how to extend this criterion to the case of imperfect capital markets. If so, we are sorry to have to inform them that once the assumption of perfect capital markets is abandoned, so, in general, is the "separation" or "decentralization" principle that permits the present or market value of the firm, or, for that matter, any other "objective" magnitude, to serve as a proxy for the utility of the owners. We are back once again with the problems discussed in Section II.A.

Some further insight into the nature of these problems can be obtained from Figure 2.6, which has been designed to illustrate a case in which

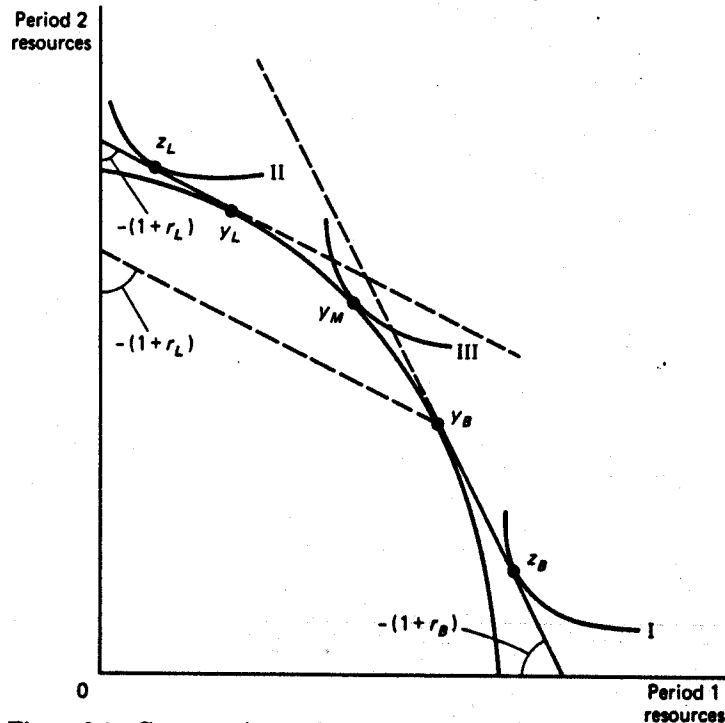


Figure 2.6 Consumption and Production Possibilities When Borrowing and Lending Rates Differ

transaction costs or some other imperfection in the capital markets has led to a situation in which the borrowing rate r_B is considerably higher than the lending rate r_L .⁷ For borrowers the optimal production or investment

⁷ For example, the stated market interest rate r_s might be the same for both borrowers and lenders, but because of brokerage fees the borrower does not realize the full amount of a loan and the lender must pay the broker a fee in addition to the face value of the loan. In this case the effective borrowing and lending rates are such that $r_L < r_s < r_B$.

decision in Figure 2.6 is the point y_B , where the slope of the transformation curve is $-(1 + r_B)$; for lenders the optimal production decision is the point y_L , where the slope of the transformation curve is $-(1 + r_L)$.

Note that because r_B is a borrowing rate, one can move only down along the line from y_B with slope $-(1 + r_B)$; the (dashed) extension of the line upward from y_B does not represent feasible points. If one wished to use the capital market to move up, the (dashed) line from y_B with slope $-(1 + r_L)$ is relevant; but points along this line are clearly inefficient. On the other hand, because r_L is a lending rate, one can move only up from y_L along the line with slope $-(1 + r_L)$; the dashed extension of this line downward from y_L does not represent feasible points.

Thus when borrowing and lending rates differ, there is no longer a unique production decision that would be made by any current stockholder regardless of his tastes. Stockholders with preferences such as those indicated by the indifference curve I would choose the point y_B and borrow down to z_B . Those with preferences II would choose y_L and lend additional amounts up to z_L . Nor is it simply a matter of replacing one point by two. Stockholders with tastes such as III would choose neither of these points but rather an intermediate point, such as y_M . Hence, without some further assumptions as to stockholders' tastes and how they are compromised, all that we can make is the relatively weak statement that the preferred position lies somewhere on the frontier between the points y_B and y_L .⁸

⁸ Because no market can ever be literally perfect, we must, of course, always imagine ourselves as having to face some indeterminacy of this form in discussing real-world markets. The perfect market assumption, then, should be interpreted as saying merely that this indeterminacy is small enough to be neglected for the problem under discussion. When imperfections are so large or pervasive as to rule out the perfect market model even as an approximation, one simply tries to approach the problem in some other way, usually very much tailored to the specific circumstances of the case. The hope is that the special formulation adopted, although it may not be strictly optimal, does at least capture the essence of the problem sufficiently well to yield insights that carry beyond the immediate and literal context. And in principle, this is all that we can really hope to do even in the perfect market case—a case that would surely not be worth studying if the “optimal” solutions so derived applied only to situations in which the markets were literally perfect.

But although there is thus certainly nothing impossible or illegitimate in analyzing problems of imperfect as opposed to perfect capital markets, there are important qualitative differences in the kind of theory that emerges in the two cases. Because a perfect market is a single and well-defined concept, we get a body of theory that can subsume a wide variety of seemingly very different problems in a single, unified framework. We merely change the labels on the axes, so to speak. Imperfect markets, by contrast, can be of many different kinds, and for each a very different kind of approach is likely to be appropriate. Hence we have not an organized or unified body of results that can be called a theory of decision under imperfect capital markets but rather a series of separate special cases with very little carry-over of concepts or methods among them.

III. MARKET VALUE, DIVIDENDS, AND STOCKHOLDER RETURNS

Up to this point we have been referring to the market or present value of the equity in a firm without any distinction between the value of the stream of earnings as generated within the firm and the value of the stream of payments actually flowing to the owners. But the multiowner corporation is in law and in fact an entity separate from the owners with entirely separate financial accounts. And more to the point, the owners have access only to that part of the firm's earnings which the directors of the firm choose to declare as dividends. How then does the dividend policy of the directors affect the value of the equity? What would constitute an optimal dividend policy from the standpoint of the current owners? What relations must hold between the stream of corporate earnings and the stream of stockholder returns? It is to these and related questions, which lie at the heart of the field of corporation finance, that we address ourselves in the remainder of the present chapter before going on, in Chapter 3, to consider some of the standard applications of the market value criterion to specific management decisions.⁹

III.A. The Effect of Dividend Policy

III.A.1. Assumptions, notation, and timing conventions

To isolate the dividend decision from all the other managerial decisions impinging on the value of the firm, we assume provisionally that the managers have somehow programmed all the future production, marketing, and investment decisions for the firm and have disclosed this information to the investing public. The question then is, given some such set of future production, marketing, and investment decisions, what effect do the firm's current and future dividend decisions have on the market value of current shareholder equity?

To help remind readers which magnitudes are on a per share basis and which are totals for the firm as a whole we use lowercase letters for the former and capital letters for the latter. As for timing, we continue the convention in Chapter 1 that all payments are made as of the beginning of the period and accrue to the owner of the asset as of the start of the previous period. In particular, $d_i(t)$ designates the dividend per share paid at the start of period t to holders of record as of the start of period $t - 1$. $R_i(t)$ designates the firm's receipts from operations at t , $W_i(t)$ are wages and similar outlays for the services of factors of production not owned by the firm, $I_i(t)$ are outlays on capital account, that is, gross investment, at t , and $R_i(t) - W_i(t) - I_i(t)$ is "net cash flow" at t .

⁹ We have, of course, already said a great deal about dividend policy both directly and by implication. Basically, we shall here be merely restating previous results in more general, algebraic form.

III.A.2. *The equal rate of return principle once again*

Turning first to the stream as it appears in the market, we showed in Chapter 1 that in equilibrium, under perfect capital markets, the one-period returns on all securities must be the same in any given period. Otherwise, investors would hold only the securities with the highest return, which is of course inconsistent with equilibrium in the sense of market clearing. Thus, although the stream of cash dividends on a share may stretch on indefinitely, we must have for any period t

$$r_{i,t+1} \equiv \frac{d_i(t+1) + v_i(t+1) - v_i(t)}{v_i(t)} = r_{t+1}, \quad \text{for all } i. \quad (2.12)$$

Or equivalently, in present value form,

$$v_i(t) = \frac{1}{1 + r_{t+1}} [d_i(t+1) + v_i(t+1)], \quad (2.13)$$

where $v_i(t)$ is the price, quoted without or "ex" any dividend at t , of a share in firm i at the start of period t . (Note that because it leads to no ambiguity, we hereafter drop the firm subscript i .)

The same principle extends, of course, to the value of all the shares, that is, to the current owners' total equity in the firm. Thus if there are $n(t)$ shares at the start of period t and if we define $D(t+1) = n(t)d(t+1)$ as total dividends paid by the firm at $t+1$ and $V(t) = n(t)v(t)$ as the total value of all shares outstanding at the start of t , we can reexpress Equation (2.13) in terms of these total values as

$$V(t) = \frac{1}{1 + r_{t+1}} [D(t+1) + n(t)v(t+1)]. \quad (2.14)$$

Note, however, that we cannot simply write $V(t+1)$ for $n(t)v(t+1)$ in the expression in brackets. The reason is, of course, that $V(t+1)$ is equal to the product of $v(t+1)$ and $n(t+1)$ and the latter does not equal $n(t)$ if new shares are issued at $t+1$. To allow for the possibility of such outside financing, let $m(t+1)$ be the number of new shares, if any, issued at the beginning of period $t+1$ at the then ruling price $v(t+1)$.¹⁰ Then

$$V(t+1) = n(t+1)v(t+1) = n(t)v(t+1) + m(t+1)v(t+1),$$

¹⁰ Here and throughout the rest of this chapter we assume that all outside financing takes the form of new issues of shares rather than of "borrowing." Nothing essential is involved in this assumption, because there is, effectively, only one class of securities possible under conditions of certainty and it matters little what name we give it. Only in Part II, after we have developed a framework for dealing with uncertainty, will we be able to consider problems encompassing both debt and equity securities. Note also that no restriction is placed on the sign of $m(t+1)$. A negative value would mean simply that the firm is buying back its own shares.

and on substituting for $n(t)v(t+1)$ in Equation (2.14), we obtain

$$V(t) = \frac{1}{1 + r_{t+1}} [D(t+1) + V(t+1) - m(t+1)v(t+1)]. \quad (2.15)$$

III. B. The Effects of Dividend Policy on the Market Value of the Shares

To complete the conversion to per firm totals, still one further simplifying substitution is possible. Because sources of funds must equal uses of funds, we know that

$$R(t+1) + m(t+1)v(t+1) = D(t+1) + W(t+1) + I(t+1). \quad (2.16)$$

If then we substitute $R(t+1) - D(t+1) - W(t+1) - I(t+1)$ for $-m(t+1)v(t+1)$ in Equation (2.15), the $D(t+1)$ cancels, and we are left with

$$V(t) = \frac{1}{1 + r_{t+1}} [R(t+1) - W(t+1) - I(t+1) + V(t+1)]. \quad (2.17)$$

In words, the total market value of the shares outstanding at the start of period t is completely independent of the dividend to be declared by the directors at period $t+1$. It depends only on (1) the rate of interest for the period, which is a given, market-determined parameter completely external to any single firm under the assumption of perfect capital markets, on (2) the firm's operating earnings and investment outlays, both assumed to have been determined before any dividend declaration for the period, and finally, on (3) the market value of the shares at the start of $t+1$, which, like any other market value, depends only on events subsequent to the date of the valuation.

Actually, we can make an even stronger statement about dividends and valuation. By repeating the reasoning above, we can show that $V(t+1)$, and hence $V(t)$, does not depend on $D(t+2)$, that $V(t+2)$, and hence $V(t)$ and $V(t+1)$, does not depend on $D(t+3)$, and so on, as far into the future as we care to look. Thus we may conclude that given a firm's operating policies, the particular dividend policy that the directors adopt has no effect whatever on the current market value of the shares or the current wealth of the shareholders.

The first time encountered, this proposition on the irrelevance of dividend policy often has an air of paradox about it. This feeling undoubtedly arises in many cases because the qualification "given a firm's operating policies" tends to be overlooked. An additional cash receipt, after all, would always seem to be welcome to the shareholder, or at least to his wife. But if the firm's production and investment plans for the period have really been

firmly and finally determined, the only way for the firm to finance this additional dividend distribution would be to obtain the funds from external sources by selling off rights to part of the future dividends to be paid by the firm. Hence although the shareholder would indeed have received a larger dividend check, he would also find that each of his shares was worth less, ex the dividend, than if the additional dividend had not been paid. Under perfect capital markets, the equal returns principle in Equation (2.12) requires these opposing effects to cancel, the higher dividend being exactly offset by the lower capital gain.

More formally, because the receipts of the firm $R(t+1)$ and the disbursements $W(t+1)$ and $I(t+1)$ are assumed to be given, we see from Equation (2.16) that any change in total dividends $D(t+1)$ implies an equal change in $m(t+1)v(t+1)$, the market value of new securities that must be issued. But the effects of these changes on Equation (2.15) are precisely offsetting: Every increase in $D(t+1)$ implies an equal decrease in $V(t+1) - m(t+1)v(t+1)$, which, it will be recalled, is $n(t)v(t+1)$, the part of the total value of the firm at $t+1$ that accrues, or belongs, to the shares outstanding from t . Or in more familiar terms, changes in the dividends to these shareholders at $t+1$ are matched by equal but offsetting changes in their capital gains at $t+1$, so that the total market value of their holdings, and of the firm, at t is completely unaffected by the dividend decision at $t+1$. The same analysis applies to the receipts, disbursements, and market values of each future period, so that in fact $V(t)$ is independent of the dividend decisions for all future periods.

This result is of critical importance. If current shareholder wealth is independent of dividend decisions for any given set of operating decisions, it follows that the firm's operating decisions can be made in accordance with the market value rule and without concern for its financing decisions. Thus, the assumption of perfect capital markets has led to another fundamental independence proposition or separation principle. Earlier we saw that as long as shareholder tastes conform to the nonsatiation axiom, maximizing shareholder wealth is equivalent to maximizing shareholder utility, so that operating decisions by the firm can be made independently of the details of shareholder tastes. Now we have seen that for a given set of production-investment decisions, the market value of the firm and thus of shareholder wealth is independent of the firm's dividend-financing decisions, so that operating and financing decisions can be made independently of each other and of shareholders' tastes.

III.C. A Graphical Illustration

A graphical illustration of the irrelevance of dividend policy is provided in Figure 2.7.¹¹ The curve X_1Z , as before, represents the two-period trans-

¹¹ An extensive numerical illustration is provided in the appendix to the present chapter.

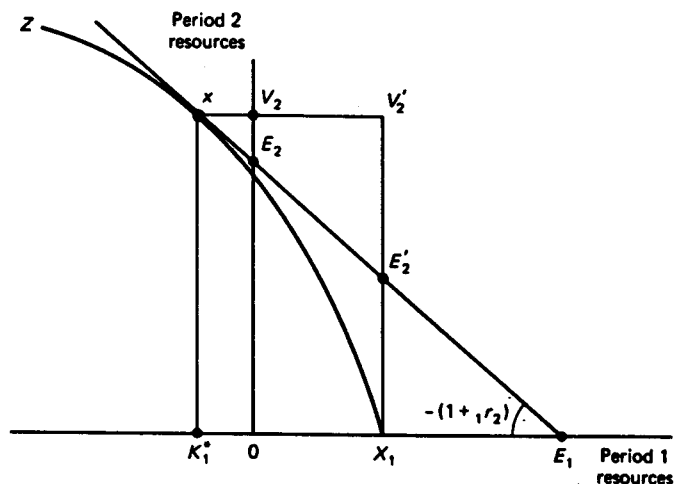


Figure 2.7 Effects of Dividend Policy on Value of Owners' Equity

formation possibilities, with the distance OX_1 representing the resources available in the firm in the first period.¹² Suppose that at the indicated market rate of interest r_2 , the point x represents the set of operating policies chosen by the management. As drawn, this happens to be a tangency point and also to involve a total investment at period 1 of $K_1^*X_1$ that is greater than the OX_1 of resources available from "internal" sources. Some outside financing is thus essential, but management still has considerable choice as to precisely how much.

At one extreme, for example, the management might decide to pay no dividends whatever from current resources and to plow the entire amount OX_1 back into the firm.¹³ This would leave the amount K_1^*O to be raised by the sale of shares to outsiders. To see the implications of this decision for shareholders, recall that the current market value of the equity in the firm is completely specified once we know the precise pattern of payments that

¹² Starting the curve at X_1 and allowing it to continue into the negative quadrant provides a "corporate" interpretation to the graph, but in fact the analysis is essentially the same as that in Fig. 2.5.

¹³ Note that in referring to this withdrawal as a dividend paid to the period 1 owners, we are using a different timing convention for dividends from that in the algebraic treatments in Sections III.A.2 and III.B. There, it will be recalled, we assumed that dividends would be paid to holders of record as of the previous period, which in this case would be period 0 and which hence would require adding another axis to the graph.

Although the use of different timing conventions at various points throughout the text may require some abrupt adjustments on the part of the reader from time to time, these shifts at least serve to emphasize that the various timing rules are indeed nothing but conventions and that none of the substantive propositions under discussion is in any way dependent on the particular one being used.

will accrue to the current owners in the two periods. Whatever this pattern may be, we can always convert it to a current period 1 value by passing a present value line through the point and recording the intercept on the period 1 axis.

In the present instance, the first coordinate of the payment pattern is clearly zero by virtue of the no-dividend decision. As for the second coordinate, note that the distance OV_2 represents the value of the entire firm as of period 2. The present owners, however, are not entitled to all this; they issued K_1^*O of additional shares to finance the firm's operations. Because the new shareholders must receive the market rate of interest on their investment, the value of their holdings at period 2 must be $E_2V_2 = K_1^*O(1 + r_2)$, leaving OE_2 for the present owners. (In essence, the firm, on behalf of its current shareholders, has borrowed down the present value line from point x to point E_2 .) The current market value of the payment pattern to the original owners provided by point E_2 is indicated by point E_1 —the same point, of course, as obtained by passing a present value line through the management decision point x .

Suppose now that instead of paying no dividends during period 1, the management had decided to go the other extreme and to pay out to the current shareholders the entire amount of currently available resources OX_1 . Then to achieve the required level of investment $K_1^*X_1$, it would be necessary to float exactly this much in a new issue of shares. But this would imply, in turn, that the claim of the new shareholders next period will be $K_1^*X_1(1 + r_2) = E_2'V_2'$. (In essence, the firm, on behalf of its current shareholders, has borrowed down the present value line from point x to point E_2' .) In this other extreme case as well, the payment pattern to the current shareholders lies on the present value line through the decision point x and hence represents exactly the same value OE_1 for the equity of the current shareholders. For this case, the market value OE_1 is made up of a dividend OX_1 plus the capital value X_1E_1 , whereas in the other case there was no current dividend, and OE_1 was simply the value of the shares.

Clearly the same reasoning would apply to any dividend policy between these two extremes; and clearly also the fact that all dividend policies lead to the same present value OE_1 for the equity of the original owners depends only on the assumption that the investment decision indicated by point x be taken as given.¹⁴

¹⁴ Note that the market value of the firm at period 1 implied by the investment decision x is $V_1 = V_2/(1 + r_2)$, which could be obtained geometrically by drawing a present value line from point V_2 in Fig. 2.7 to the horizontal axis. (Alternatively, V_1 is the value of the equity of the original owners OE_1 , plus the minimum quantity K_1^*O that must be raised to finance the decision implied by x .) The decision x maximizes the value of the equity of the original shareholders at period 1 but does not maximize the value of the firm at period 1. The latter maximum would be obtained by pushing investment

III.D. The Independence Proposition Further Considered

III.D.1. *The bird-in-the-hand argument*

If despite these explanations, a feeling of paradox still remains, it may be due to the seeming conflict between the independence proposition and the very sensible proverb that a bird in the hand is worth two in the bush. Any part of the current period earnings not paid out as dividends is presumably being invested by the management in assets that will yield, hopefully, a further stream of future earnings. Under real-world conditions, however, there is always considerable uncertainty attached to this future stream, and it thus seems natural to suppose that there may well be cases in which, given the investments that management has in mind, the stockholders would prefer the certainty of an immediate cash payment to the problematical future gains that might be obtained from entrusting these funds to the management.

Because all formal consideration of uncertainty has been assigned to Part II, we must, of course, defer until then the presentation of a constructive proof that the value of the shares remains independent of its dividend policy, given the firm's operating policies, even outside the framework of perfect certainty.¹⁵ Actually, however, the full apparatus should not be necessary for readers to see why the bird-in-the-hand analogy breaks down. A review of the assumptions and the key steps in the proof should make clear that the independence proposition does not require stockholders to be indifferent as between a present dividend and a future dividend or capital gain. It says, rather, that once management is committed to undertake and finance a given investment program, an increase in the dividends in any period will simply lead to a corresponding reduction in the ex-dividend value of the shares in the same period. It may be, for example, that even with the

to the point where the marginal return, that is, the slope of the transformation function, was zero. Thus the market value rule for current decisions within a firm applies to the market value of the equity of the current owners and not to the total market value of the firm.

¹⁵ To anticipate a bit, the proof depends critically on precisely the same two assumptions as in the certainty case. The first is nonsatiation of wants, that is, the assumption that more wealth is preferred to less, regardless of the form in which it accrues. (In some of the literature on valuation for which this book is intended to serve as an introduction, this assumption is referred to as "rational behavior," but we have here tried to avoid such emotion-laden terms.) The second is the assumption of perfect capital markets, in the broad sense of the term in which tax differentials on different sources of income are also ruled out. In practice, of course, many countries, including the United States, place substantially lower taxes on capital gains than on dividends. Where this is true, the value of the shares is not independent of the dividend policy, but it is still true at least that if the other assumptions continue to hold, paying or raising a dividend cannot increase the value of the shares, given the firm's investment policy.

old dividend, the stockholders were destined to suffer capital losses for the period. If so, raising the dividend would make these losses larger than they would otherwise have been by the exact amount of the dividend increase (or equivalently, by the value of the additional shares that would have to be sold to raise the funds to pay the additional dividends).¹⁶

III.D.2. Dividend policy and the internal rate of return

Still another reason for the seemingly paradoxical flavor of the independence proposition is its denial of the existence of a unique "optimal" dividend policy for the firm. This denial seems to fly in the face of the commonsensical notion that, although the directors might well be justified in withholding funds when the firm had higher-yielding investment alternatives than the shareholders, they could hardly be acting optimally in withholding funds when the owners actually had, or even only thought that they had, the better opportunities. The point having been made, it is often then illustrated by reference to the celebrated case of a well-known, autocratically managed mail-order house in the years immediately following the Second World War. This company is supposed to have plowed back its earnings into low-yield liquid assets in anticipation of an imminent major depression and to have persisted in this policy long after its competitors and the investing public generally had come to regard such an event as improbable in the extreme. Eventually, however, a palace revolution occurred in which the old management was toppled. The new management promised to end the hoarding and to follow a more generous dividend policy whereupon, at least according to legend, the price of the shares rose from its depressed and discouraged level to a glorious new high.

Quite apart from any questions of fact, and from the always treacherous business of *post hoc, ergo propter hoc* reasoning about stock prices, it should be amply clear by now that tales of this kind, even if true, would in no way conflict with the irrelevance proposition. There is only an appearance of conflict because of the failure once again to distinguish adequately between an optimal dividend policy and an optimal production and investment policy. For if the stock of the mail-order company was indeed depressed by the policies of the management, it was surely the policy of investing in unproductive liquid hoards that should bear the onus. The unwarrantedly

¹⁶ If the ex-dividend value of the shares did not drop off *pari passu* with the increase in the dividends, given the firm's investment policy, then Ponzi schemes could be successful. For the benefit of readers under fifty, the notorious Charles Ponzi was a "performance-minded" money manager of the 1920s, who promised and, for a time, apparently delivered very high cash returns to investors in his company. He did it, until the bubble burst, by using the proceeds of sales of new securities to pay the interest and dividends on the old.

low price, in other words, cannot be blamed on a nonoptimal, overconservative dividend policy unless, of course, one is also prepared to argue that the shares would have been less depressed if management had chosen to finance the same level of hoards from new stock issues rather than from retained earnings.

III.E. Some Equivalent Alternative Valuation Formulas

The analysis in the previous sections can also serve to make clear the essential emptiness of the long-standing controversy in the finance literature over whether the market "really" capitalizes the dividends received by the shareholder or the earnings generated by the firm. For the two approaches, provided only that they are consistently carried through, are easily seen to be entirely equivalent.

III.E.1. The stream of dividends approach

If, for example, we wish to focus on the stream of dividend payments, we can use Equation (2.13) as the basis of the valuation formula. Because this equation holds for all t , setting $t = 0$ permits us to express $v(0)$ in terms of $d(1)$ and $v(1)$, which in turn can be expressed in terms of $d(2)$ and $v(2)$, and so on, up to any arbitrary terminal period T .¹⁷ Carrying out these substitutions, and assuming, for simplicity only, that the one-period market rate of interest has the same value r in all periods, we obtain

$$v(0) = \sum_{t=1}^T \frac{d(t)}{(1+r)^t} + \frac{v(T)}{(1+r)^T} \quad (2.18)$$

as an expression for the present or market value of a share. Equilibrium of the capital markets requires, of course, that $v(0)$ converge to a finite limit as T approaches infinity. But this requires that the remainder term $v(T)/(1+r)^T$ converge to zero, which is to say nothing more than that beyond some point in time $v(T)$, if it continues to grow at all, it does so at a rate less than r per period.¹⁸ With these additional restrictions, the

¹⁷ Note that for the remainder of the present chapter we follow the practice of most of the literature on valuation and take the "present" as being period 0 rather than period 1 as in Chapter 1.

¹⁸ Suppose, on the contrary, that $v(T)$ grows at a rate $q > r$. Then

$$\frac{v(T)}{(1+r)^T} = \frac{v(0)(1+q)^T}{(1+r)^T} = v(0) \left(\frac{1+q}{1+r} \right)^T,$$

which approaches infinity as $T \rightarrow \infty$, and the price of a share is infinite. The economic implications of the statement that in equilibrium q must be less than r , that is, the price of a share must be finite, are considered in more detail later.

expression for the present value of a share simplifies to

$$v(0) = \sum_{t=1}^{\infty} \frac{d(t)}{(1+r)^t}. \quad (2.19a)$$

We earlier showed that the market value of the firm and that of a share at the beginning of any period are independent of the dividend decisions of all subsequent periods. In terms of Equation (2.13), a change in $d(t+1)$ is matched by an equal but offsetting change in $v(t+1)$. In terms of the dividend approach to valuation, this translates into the statement that a change in $d(t+1)$ is accompanied by an equal but offsetting change in the present value of dividends subsequent to $t+1$, because $v(t+1)$ is, after all, just the present value of these subsequent dividends. Or in other words, dividend policy affects the "time shape" of the dividend stream but not its present value.

Note also that the total value of the firm $V(0)$ is

$$V(0) = n(0)v(0) = \sum_{t=1}^{\infty} \frac{n(0)d(t)}{(1+r)^t}, \quad (2.19b)$$

the present value of the total dividends to be paid in future periods to the shares outstanding at period 0, and not to total dividends paid in all future periods.

III.E.2. Cash flow and earnings approaches

Alternatively, if we wish to focus attention on the stream of earnings generated in the firm, we can start from Equation (2.17). Making the corresponding successive substitutions for the $V(t)$ and adopting the same convention with respect to the vanishing of the remainder, we obtain

$$V(0) = \sum_{t=1}^{\infty} \frac{R(t) - W(t) - I(t)}{(1+r)^t} \quad (2.20)$$

as an expression for the present, or market, value of the shares currently outstanding in terms of the stream of "net cash flows" generated in the firm.¹⁹ For further compactness of notation, as well as to facilitate comparison with some of the analogous treatments in the standard literature,

¹⁹ The substitutions involved in obtaining Equation (2.20) from Equation (2.17) are precisely those presented verbally on pp. 80–81 following Equation (2.17), where they were used to derive the conclusion that, given the firm's production-investment decisions, its current market value is independent of all future dividend decisions. This conclusion is, perhaps, now easier to grasp directly from Equation (2.20), which includes only the variables $R(t)$, $W(t)$, and $I(t)$ generated by the firm's production-investment decisions, and the market interest rate r , which, in a perfect capital market, is unaffected by any actions of the firm.

we define $X(t) \equiv R(t) - W(t)$ as the "net operating cash flow" at period t . Substituting, we then have the familiar

$$V(0) = \sum_{t=1}^{\infty} \frac{X(t) - I(t)}{(1+r)^t} \quad (2.21)$$

as the basic formula for the current market value of a firm.

To summarize, the dividend valuation formula (2.19) is derived from the market equilibrium condition (2.13); the net cash flow valuation formula (2.21) is derived from Equation (2.17), which is in turn derived from Equation (2.13). Thus the dividend and net cash flow approaches must lead to equivalent expressions for the market value of the firm.

Note, however, that Equation (2.21) does not say that the value of the firm can be expressed simply as the sum of discounted "cash" earnings $X(t)$. The reason for the failure of the latter approach is not, as sometimes asserted, the fact that the corporation is an entity entirely separate from the owners and whose earnings cannot be withdrawn at will by the owners. Nor does the difficulty arise from the fact that the earnings in Equation (2.21) are the cash earnings $X(t) = R(t) - W(t)$, rather than the accounting earnings. Because the latter concept would differ from the former in the present context only by the amount of the depreciation and similar arbitrary accounting adjustments, we can easily convert Equation (2.21) to run explicitly in terms of accounting rather than cash earnings. In particular, if we let $Z(t)$ = the depreciation estimate of the firm's accountants at period t , $A(t) = R(t) - W(t) - Z(t)$ = accounting earnings, and $N(t) = I(t) - Z(t)$ = net investment in the sense of the net change in the accounting book value of assets, we can reexpress Equation (2.21) as

$$V(0) = \sum_{t=1}^{\infty} \frac{A(t) - N(t)}{(1+r)^t} \quad (2.22)$$

Rather, the difficulty with an earnings approach arises from the fact that in order to obtain the indicated future earnings stream, by either definition, additional resources must be committed over time to the production process. If these resources are obtained from the sale of shares to outsiders, the current owners obviously have to compensate the newcomers for their capital contributions by turning over to them a part of the future earnings stream. And if, on the other hand, the existing shareholders supply the funds themselves by way of reduced dividends or fully subscribed preemptive issues, they must offset against the future earnings the opportunity cost in the form of the interest income that might otherwise have been earned on the funds committed. Under perfect certainty and perfect capital markets the value to the owners of the sacrifice required to obtain the additional capital resources in any period t is precisely equal to the value of these resources $I(t)$, regardless of how they may happen to have

been obtained. Thus the relevant quantity to discount in obtaining the market value of the firm is the net cash flow $X(t) - I(t)$.²⁰

III.E.3. Investment opportunities, growth, and valuation

Some important additional insights into the valuation process and especially the relation between corporate earnings and investor returns can be obtained by considering certain simplified special cases of the valuation formula (2.21). In particular, suppose that the total investment $I(t)$ made by the management of a firm at the beginning of any period t generated a uniform stream of earnings in perpetuity thereafter at the rate of $100r^*(t)$ percent per period.²¹ In terms of previous notation and timing conventions this would mean that we could express the elements of the stream $X(t)$ successively as

$$X(2) = X(1) + r^*(1)I(1),$$

$$X(3) = X(2) + r^*(2)I(2) = X(1) + r^*(1)I(1) + r^*(2)I(2),$$

and so on, or more compactly in general form as

$$X(t) = X(1) + \sum_{\tau=1}^{t-1} r^*(\tau)I(\tau), \quad t = 2, 3, \dots, \infty. \quad (2.23)$$

Substituting this expression for $X(t)$ in Equation (2.21) and regrouping

²⁰ Once again, we leave the formal proof as an exercise for readers who feel that they could benefit from some additional practice in manipulating present value expressions. *Hint:* Consider first the case in which the currently existing owners plan to supply all the investment funds themselves. Then the opportunity cost to them of the $I(t)$ that they supply at period t is the loss of earnings of $rI(t)$ per period thereafter, starting, by our conventions, in period $t + 1$. But what will be the present value as of the start of t of the perpetual annuity $rI(t)$ at an interest rate of r per period?

²¹ We refer to $r^*(t)$ hereafter as the "rate of return on investment" without pausing at this point to explain its relation to the various other possible senses of this term. These subjects come up in due course in Chap. 3. It may be useful, however, at least to reassure readers that no necessary conflict exists between the market value rule for management decisions and our use of examples in which $r^*(t)$ is not equal to r . Our $r^*(t)$ should be thought of as the average rate of return on the total investment budget $I(t)$, which budget a management following the market value rule would push to the point where the rate of return on the last dollar just equaled the market rate of interest.

Note also that by virtue of the assumption that real assets generate uniform perpetual returns no "depreciation" adjustment is required. Hence, net cash flow from operations and accounting earnings are the same, and we use the terms interchangeably. This perpetuity assumption is, of course, much less restrictive in this context than it may seem. Under certainty and perfect capital markets, it is always possible to find an equivalent perpetuity, that is, one with the same present value, for the firm's earnings no matter what the actual pattern of the cash flow or the age composition of the assets.

the terms yield

$$V(0) = \sum_{t=1}^{\infty} \frac{X(1)}{(1+r)^t} + \sum_{t=1}^{\infty} I(t) \left(\sum_{r=t+1}^{\infty} \frac{r^*(t)}{(1+r)^r} - \frac{1}{(1+r)^t} \right). \quad (2.24)$$

The first term is nothing more than the present value of a uniform perpetuity of $X(1)$ and is immediately evaluated as $X(1)/r$. As for the second term, consider first the inner summation

$$\sum_{r=t+1}^{\infty} \frac{r^*(t)}{(1+r)^r}.$$

This, too, is a perpetual uniform annuity but one whose start is deferred to the beginning of period $t+1$, which is to say that we can reduce it to standard form as follows:

$$\sum_{r=t+1}^{\infty} \frac{r^*(t)}{(1+r)^r} = \frac{1}{(1+r)^t} \sum_{r=1}^{\infty} \frac{r^*(t)}{(1+r)^r} = \left(\frac{r^*(t)}{r} \right) \frac{1}{(1+r)^t}.$$

The second summation in Equation (2.24) thus becomes

$$\sum_{t=1}^{\infty} I(t) \left[\left(\frac{r^*(t)}{r} \right) \frac{1}{(1+r)^t} - \frac{1}{(1+r)^t} \right] = \sum_{t=1}^{\infty} I(t) \left[\frac{r^*(t) - r}{r} \right] \frac{1}{(1+r)^t},$$

and the complete present value expression can then be written as simply

$$V(0) = \frac{X(1)}{r} + \sum_{t=1}^{\infty} I(t) \left(\frac{r^*(t) - r}{r} \right) \frac{1}{(1+r)^t}. \quad (2.25)$$

In words, the market value of any firm, under our simplifying specifications, has been expressed as the sum of two components. The first is the capitalized value of the earnings stream produced by the assets that the firm currently holds. We now show that the second is the market value of any opportunities that the firm may have to make additional investments in real assets in the future at terms more favorable than those available to investors in the capital markets.²² There are many kinds of circumstances that might produce such opportunities to earn "economic rents."²³ Our

²² We always speak here of the internal opportunities as being as or more favorable than the external, simply because this is normally the more relevant circumstance from the economic point of view. Formally, however, an expression such as Equation (2.25) would remain valid even for the case of $r^* < r$, a case that would imply an inefficient, nonoptimal investment policy by the management, as shown in Chap. 3.

²³ Students are often bothered by the assumption of an r^* different from r on the grounds that this conflicts with the equal rate of return principle on which the analysis of valuation under certainty is based. Remember, however, that the assumption of perfect markets, from which the equal rate of return principle was derived, applies to the

concern here, however, is not with their origins but their implications for valuation.

By assumption, any such opportunities in period t generate earnings thereafter of $r^*(t)I(t)$ in perpetuity, a stream whose present value as of period t is $I(t)[r^*(t)]/r$. Subtracting out the cost of the resources necessary to produce this stream of earnings, we have as the net present value or "good will" of these opportunities as of the start of period t the quantity

$$I(t) \left(\frac{r^*(t)}{r} \right) - I(t) = I(t) \left(\frac{r^*(t) - r}{r} \right).$$

The value now (period 0) of these eventual opportunities in period t is thus

$$I(t) \left(\frac{r^*(t) - r}{r} \right) \frac{1}{(1+r)^t},$$

and the value of all such opportunities in all future periods is the expression given by the summation term in Equation (2.25).

Expression (2.25) might thus be viewed as representing the result of an "investment opportunities" approach to the valuation of the firm. But note that it was initially derived from Equation (2.21), which might be called the net cash flow approach, and Equation (2.21) in turn was shown to be equivalent to the discounted dividend formula (2.19). By now it should be clear that the equivalence of these various valuation formulas is due to the fact that all are ultimately direct implications of the equal rate of return principle expressed by Equation (2.12), which in turn is a direct implication of the assumption of perfect certainty and especially perfect capital markets.²⁴

capital market and requires only that returns on all securities for any period are equal. It does not necessarily imply the absence of imperfections in the markets for goods or for other resources of a kind that would permit rates of return on investment opportunities within the firm to be greater than the market interest rates on securities. In fact, a major purpose of the analysis here is to show that the market value of a firm's securities always fully reflects the market value of any extraordinary production-investment opportunities, with the result that returns on these securities are always in conformance with the equal rate of return principle.

²⁴ For those already somewhat familiar with the terminology of the capital budgeting literature, note that for the case $r_{t+1} =$ a constant r for all t , expression (2.25) also demonstrates why the "cost of capital," in the sense of the minimum rate of return for acceptable investment projects, is always equal to r . From (2.25) it is easy to see that projects with rates of return greater than r increase the current market value of the firm and that projects with rates of return less than r reduce the current value of the firm, because in this case the second term in (2.25) is negative. We emphasize, however, that the notion of a cost of capital as a cutoff rate for investment is only meaningful when one-period interest rates are constant over time. Otherwise there is no single rate with which the firm can compare the rates of return on its investment opportunities, so that in making its investment decisions, the firm must rely directly on the market value rule.

In thus relating future investment opportunities to current valuations, formula (2.25) should help, among other things, to clarify the essential meaning of those much abused terms in the valuation literature, "growth company" and "growth stocks." If a growth company is defined as one on which the market places a high value in relation to current earnings, that is, one for which $V(0)/X(1) > 1/r$, what puts it in this category cannot be simply the fact that its assets and earnings are expected to grow in the future. It is also necessary that the returns on the additional assets to be acquired by the firm be greater than those obtainable by purchasing outstanding shares in the market. For if the yield r^* on the additional investment is no greater than r , the second term in (2.25) is zero, and the firm's price/earnings ratio remains an unglamorous $1/r$ no matter how rapid the expansion in the size of the company.

Note also that in defining a growth company as one with substantial opportunities to earn above-normal returns, the returns being spoken of are those at the level of the firm, not of the investor. For no matter how profitable the future investment opportunities of the firm may be, the equal return principle in Equation (2.12) continues to hold in the capital market. The full value of the special opportunities is reflected in the current price of the shares, and investors buying into these growth companies earn no more, and no less, than if they had bought into less fortunately situated firms.²⁵

III.F. Growth Potential and Stockholder Returns

It should, we hope, be easy enough by now to accept the seemingly paradoxical conclusion that stockholder returns will be the same regardless of the firm's growth potential or actual earnings stream, provided only that all the relevant facts about the stream are fully known to the investing public. Nevertheless, it may be useful at this point to take a further and more detailed look at the connections between the streams of returns at the two levels. In doing so, it is convenient to make use of a simple and popular, but also very treacherous, specialization of Equation (2.21) that may be called the constant growth model.

III.F.1. The constant growth model

Specifically, we assume that in every period t the firm's investment $I(t)$ bears some given proportion k to its earnings $X(t)$ for the period and that

²⁵ On the other hand, somebody at some point in time must have received a windfall gain, presumably the promoters when the firm was organized or at least first went public. There is, however, no real inconsistency with the equal rate of return principle as long as we continue to treat such windfalls as unique events which have already occurred in the past and which will not occur again during the time span to be covered by our analysis.

the yield of every period's investment is a constant $100r^*$ percent per period in every period thereafter. These definitions imply in turn that we may express the earnings in any period t as

$$\begin{aligned} X(t) &= X(t-1) + r^*I(t-1) \\ &= X(t-1)(1 + kr^*) \\ &= X(1)(1 + kr^*)^{t-1}, \end{aligned} \quad (2.26)$$

where kr^* is the (constant) rate of growth of total firm earnings per period. Substituting $kX(t)$ for the $I(t)$ of the valuation equation (2.25), we have

$$\begin{aligned} V(0) &= \frac{X(1)}{r} + \sum_{t=1}^{\infty} kX(t) \left(\frac{r^* - r}{r} \right) \frac{1}{(1+r)^t} \\ &= \frac{X(1)}{r} + kX(1) \left(\frac{r^* - r}{r} \right) \sum_{t=1}^{\infty} \frac{(1 + kr^*)^{t-1}}{(1+r)^t} \\ &= \frac{X(1)}{r} \left[1 + \frac{k(r^* - r)}{1 + kr^*} \sum_{t=1}^{\infty} \left(\frac{1 + kr^*}{1+r} \right)^t \right]. \end{aligned} \quad (2.27)$$

As long as the growth rate kr^* is less than the rate of interest r —and recall in this connection the discussion in Section II.E.1 of the vanishing remainder term—the infinite summation in the brackets converges to a finite limit that can easily be shown to be $(1 + kr^*)/(r - kr^*)$.²⁶ Making this substitution and simplifying, we thus obtain

$$\begin{aligned} V(0) &= \frac{X(1)}{r} \left[1 + \frac{k(r^* - r)}{r - kr^*} \right] \\ &= \frac{X(1)(1 - k)}{r - kr^*} \end{aligned} \quad (2.28)$$

as an expression for the value of the currently outstanding shares in terms of the rate of growth of the firm's earnings.²⁷

²⁶ Readers who wish additional practice in working with present value and related formulas will again find it a useful exercise to derive this expression. By way of a hint note that if we define a new variable $\beta = (1 + kr^*)/(1 + r)$, it is easy to see that the summation in question is nothing more than the sum of a simple geometric progression. Those who have forgotten the formula should be able to reconstruct it after rereading Section II.B.2 in Chap. 1.

²⁷ Note that, because the value of the firm must be finite, in the constant growth model the condition $kr^* < r$ is a necessary condition of market equilibrium. It simply says that in equilibrium the market rate of interest must be such that no firm has opportunities into the indefinite future to invest the proportion k of each period's earnings at a rate r^* so that $kr^* > r$. Alternatively, because Equation (2.28) applies to the market value

Note that Equation (2.28), perhaps even more dramatically than its more general counterpart (2.25), makes clear the fundamental distinction between mere expansion ($k > 0$) and true "growth potential" ($r^* > r$). For when $r^* = r$, the denominator of Equation (2.28) is $r(1 - k)$; the $(1 - k)$ thus cancels in the numerator and denominator, leaving only $V(1) = X(1)/r$, although the rate of expansion kr of the firm's earnings, and of its total value, may be quite substantial.

III.F.2. The growth of total earnings and the growth of dividends and price per share

Up to this point, the valuation formulas involving growth have focused on events at the level of the firm. It is a relatively straightforward matter, however, to develop the corresponding formulas running in terms of the growth of dividends and the price of individual shares. In particular, consider a share of stock of the constant growth firm whose dividend payment at period 1 is $d(1)$ and whose dividend grows thereafter at a constant rate of g per period. Substituting $d(1)(1 + g)^{t-1}$ for each $d(t)$ in the stream of dividends valuation formula (2.19a), we obtain

$$\begin{aligned} v(0) &= \sum_{t=1}^{\infty} \frac{d(1)(1 + g)^{t-1}}{(1 + r)^t} = \frac{d(1)}{1 + g} \sum_{t=1}^{\infty} \left(\frac{1 + g}{1 + r} \right)^t \\ &= \frac{d(1)}{r - g} \end{aligned} \quad (2.29)$$

as an expression for the value of a share in terms of the current dividend and its rate of growth, assuming, of course $g < r$, so that the summation is finite.

Because an expression of the form (2.29) holds for the price of the share in every time period and because the dividend term in the numerator grows over time at the rate of g per period, it follows that the price per share also grows at the rate of g per period. To say that the dividend and the price per share grow at the same rate is not to suggest, of course, that dividends and capital gains contribute equally to investor returns. On the contrary, as can readily be seen by rewriting Equation (2.29) in the form

$$r = \frac{d(1)}{v(0)} + g, \quad (2.30)$$

of each period, in general

$$V(t) = \frac{X(t+1)(1 - k)}{r - kr^*} = \frac{X(1)(1 + kr^*)^t(1 - k)}{r - kr^*} = V(0)(1 + kr^*)^t,$$

so that the condition $kr^* < r$ says that in equilibrium the rate of interest must be such that the market value of any constant growth firm increases at a rate $kr^* < r$.

the relative contribution of dividends and capital gains to investor returns depends on the relation between g and r . When g is large relative to r , the current price is large relative to the current dividend, and the relative contribution of the immediate cash payment to the total yield is correspondingly small. As g gets smaller in relation to r , the cash component looms larger; and in the special case of $g = 0$, the yield per period would consist entirely of the cash return. But all this is, of course, just the equal rate of return principle again.

As for what determines the value of g in relation to r , we know in a general way from the previous discussion that the key factors are the growth potential of the firm and the financial, that is, dividend, policy that the firm chooses to follow. More precisely, let k_r = the proportion of total earnings retained by the firm in each period. Hence, for dividends we have $D(t) = X(t)(1 - k_r)$, where $(1 - k_r)$ is the so-called "dividend payout ratio." And let $k_e = k - k_r$ = the amount of external capital raised per period by the flotation of new shares, expressed as a proportion of total earnings. If we let $n(0)$, as before, represent the number of shares outstanding at the start of period 0, then by multiplying both sides of Equation (2.29) by $n(0)$, we obtain as one expression for the value of all the currently outstanding shares

$$V(0) = n(0)v(0) = \frac{n(0)d(1)}{r - g} = \frac{D(1)}{r - g} = \frac{X(1)(1 - k_r)}{r - g}.$$

But from Equation (2.28) we also have

$$V(0) = \frac{X(1)(1 - k)}{r - kr^*} = \frac{X(1)(1 - (k_e + k_r))}{r - kr^*}.$$

Equating the two expressions for $V(0)$ and solving for g yields

$$g = kr^* \frac{1 - k_r}{1 - k} - k_e r \frac{1}{1 - k} \quad (2.31)$$

as an explicit expression for the growth rate of dividends, and also the capital gain yield on the shares per period, in terms of the firm's growth rate and its dividend policy.

Note that in the extreme case in which all financing is internal ($k_e = 0$ and $k_r = k$), we have simply $g = kr^*$; that is, the growth rate of dividends per share is exactly the same as the growth rate of the firm itself. In all other cases, however, the growth rate of dividends is less than that of the firm. The reason, of course, is that if new shares are to be issued each period to finance the growth of the firm, the current stockholders must give up part of their claim to the future stream of earnings that the investment generates. They get a higher initial dividend than if the directors had elected to finance all the firm's growth from retained earnings, but their

dividend income grows more slowly as the future pie of total dividends gets shared among a larger and larger group of stockholders. The minimum value for g is reached at the other extreme, where all financing is external and where Equation (2.31) thus reduces to

$$g = \frac{k}{1-k} [r^* - r]. \quad (2.32)$$

Note that for a true growth company, that is, one for which $r^* > r$, g is positive, and the stream of dividends per share grows over time, although the firm is paying out all its earnings in dividends.

III.F.3. Corporate earnings and investor returns

A further question of some interest is that of the relation between the total earnings of the firm in any period and the total return, dividends plus capital gains, to the shareholders. If we let $G(t+1)$ be the total capital gains during period t to stockholders of record as of the start of t , we know that

$$p_{t+1}V(t) \equiv D(t+1) + G(t+1) = X(t+1)(1 - k_r) + gV(t). \quad (2.33)$$

Substituting the expression for g in Equation (2.31) and the expression for $V(t)$ implied by Equation (2.28) into the expression above and simplifying yields

$$D(t+1) + G(t+1) = X(t+1) \left[\frac{r(1-k)}{r - kr^*} \right]. \quad (2.34)$$

Stockholder returns in the market, in other words, are not in general the same as the returns $X(t+1)$ generated within the firm. Equality between $X(t+1)$ and $D(t+1) + G(t+1)$ occurs only in the case in which the firm has no special growth opportunities in our sense. Where such growth opportunities do exist, however, the expression in brackets on the right-hand side of Equation (2.34) is greater than unity, and total stockholder returns are greater than the earnings of the corporation.

Further insight into this seemingly paradoxical conclusion can be obtained by focusing on the relation between the earnings retained at the corporate level and the capital gains accruing at the shareholder level. Subtracting $D(t+1)$ from the left-hand side of Equation (2.34) and $(1 - k_r)X(t+1)$ from the right, we obtain

$$G(t+1) = k_r X(t+1) + kX(t+1) \left[\frac{r^* - r}{r - kr^*} \right]. \quad (2.35)$$

The first term $k_r X(t+1)$ is just retained earnings at $t+1$. But total

shareholder capital gains are equal to retained earnings only when the firm has no growth opportunities ($r^* = r$). Otherwise, when $r^* > r$, total shareholder capital gains are in excess of retained earnings by the quantity $kX(t+1)(r^* - r)/(r - kr^*)$, which, interestingly and perhaps not unexpectedly, is the interest on the total market value at t of the firm's future investment opportunities.²⁸ Finally, note that if there is growth, capital gains are still earned even in the event that all the firm's earnings are declared in dividends, that is, $k_r = 0$.

IV. SUMMARY

In this chapter, we have sought to extend the simple wealth allocation model in Chapter 1 to allow for the carry-over of resources by way of commodities as well as by securities, and, in particular, by the investment of resources in firms. The extension would have been relatively straightforward if all firms were simple, owner-managed enterprises. But difficulties arose as soon as we turned to questions of decision making in corporations in which the decision-making power had been delegated to managers and in which there were typically many different owners with very different tastes and resources.

We saw, however, that in at least one special set of circumstances it was possible to surmount these difficulties and develop an "objective" decision-making criterion for a management presumed to be acting on behalf of and in the best interests of the owners. In particular, we showed that under perfect capital markets, a policy of maximizing the current market value of the shares held by the present owners would lead to the same set of operating and investment decisions that each owner would have adopted if he had taken responsibility for the decisions himself.

After discussing briefly some of the implications and limitations of this solution to the criterion problem for management decisions, we turned our attention to the market value itself and the factors that determine it. Special emphasis was given to the role of dividend policy as the nexus between the firm and its owners, and we derived and discussed at some

²⁸ To see this result note that, from Equation (2.28), in the constant growth model the value of the firm at the beginning of t can be written

$$V(t) = \frac{X(t+1)}{r} + \frac{X(t+1)}{r} \left[\frac{k(r^* - r)}{r - kr^*} \right].$$

The first term is the market value at t of earnings produced by investments made at t and earlier periods; the second term is the market value of investments in later periods. Multiplying the second term by the rate of interest r gives $kX(t+1)(r^* - r)/(r - kr^*)$ as the interest on the market value of future investment opportunities.

length the fundamental, but somewhat paradoxical, proposition that dividend policy per se is irrelevant to the current market value of the shares.

We then went on to discuss what does count in valuation and to develop a variety of useful alternative approaches to valuation and valuation formulas. With these results on valuation in hand, on the theoretical level there remains only to illustrate how the market value criterion can be applied to specific problems of managerial decision making. It is to this task that we turn in Chapter 3 and with it conclude the discussion of certainty models and their applications in finance. First, however, we present in the appendix two numerical examples that should provide additional experience and insights into the preceding theoretical analyses.

REFERENCES

The fundamental double-tangency solution in Section I for an individual with both capital market and productive opportunities is due to

- ✓ 1. Fisher, Irving, *The Theory of Investment*. New York: Augustus M. Kelley, Publishers, 1965. Reprinted from the original 1930 edition.

Some of the problems discussed in Section II involving the extension of the basic model to corporations in which decision-making powers have been delegated to managers were first raised by

- ✓ 2. Berle, A. A., and Gardiner C. Means, *The Modern Corporation and Private Property*. New York: The Macmillan Company, 1932.

Among the leading recent critics of the market value rule as descriptive of actual management behavior have been

- ✓ 3. Baumol, William J., *Business Behavior, Value and Growth*. New York: The Macmillan Company, 1959.

- ✓ 4. Marris, Robin, *The Economic Theory of "Managerial" Capitalism*. London: Macmillan & Co., Ltd., 1964.

and

- ✓ 5. Williamson, Oliver E., "Managerial Discretion and Business Behavior," *American Economic Review*, vol. 53 (December 1963), pp. 1032-1057.

A recent survey of the issues and the evidence on the descriptive validity of the market value rule is that of

- ✓ 6. Lerner, Robert J., *Management Control and the Large Corporation*. New York: Dunellen Publishing Co., 1970.

The treatment of market valuation and dividend policy in Section III is based on that in

- ✓ 7. Miller, Merton H., and Franco Modigliani, "Dividend Policy, Growth and the Valuation of Shares," *Journal of Business*, vol. 34, no. 4 (October 1961), pp. 411-432.

The leading academic critique of the dividend independence proposition in Section III.A is that of

8. Gordon, Myron J., *The Investment, Financing and Valuation of the Corporation*. Homewood, Ill.: Richard D. Irwin, Inc., 1962. ✓

A briefer and more sharply focused presentation of his bird-in-the-hand argument is given in

9. Gordon, Myron J., "Optimal Investment and Financing Policy," *Journal of Finance*, vol. 18 (May 1963), pp. 264-272. ✓

There have been innumerable empirical studies of the effects of dividend policy on the value of shares. A survey stressing some of the pitfalls is that of

10. Friend, Irwin, and Marshall Puckett, "Dividends and Stock Prices," *American Economic Review*, vol. 44, no. 3 (September 1964), pp. 656-682. ✓

The literature on growth and valuation is also extensive. A typical practitioner-oriented treatment with an extensive further bibliography is that of

11. Wendt, Paul F., "Current Growth Stock Valuation Methods," *Financial Analysts Journal*, vol. 21, no. 2 (March-April 1965), pp. 91-103. ✓

APPENDIX

Valuation Formulas: Some Numerical Illustrations

I. FINANCIAL POLICY, RETURNS, AND VALUATION

Consider a firm with initial, and perpetual, earnings $X(1) = \$100$ per period and suppose, further, that its investment policy involves investing \$100 at the beginning of each of periods 1, 2, and 3. Each year's investment produces perpetual annual cash earnings of \$20 per period, with the earnings commencing at the beginning of the period following the investment. Thus the cash earnings of the firm will be \$120 at period 2, \$140 at period 3, and \$160 in all subsequent periods. The one-period market interest rates for periods 0 to 2 are, respectively, 1 percent, 5 percent, and 10 percent; for all subsequent periods the rate is 20 percent.

Because for $t > 3$, $r_{t+1} = 0.2$, and $X(t+1) - I(t+1) = X(t+1) = \160 , the market value of the firm at the beginning of period 3, and all subsequent periods, can easily be computed from Equation (2.21) as¹

$$V(3) = \sum_{\tau=1}^{\infty} \frac{[X(3+\tau) - I(3+\tau)]}{(1.2)^{\tau}} = \sum_{\tau=1}^{\infty} \frac{160}{(1.2)^{\tau}} = \frac{160}{0.2} = \$800.$$

¹ Recall that

$$\sum_{\tau=1}^{\infty} \frac{1}{(1+r)^{\tau}} = \frac{1}{r}.$$

From Equation (2.17) the value of the firm at the beginning of period 2 can then be computed as

$$\begin{aligned} V(2) &= \frac{1}{1 + r_3} [X(3) - I(3) + V(3)] \\ &= \frac{1}{1.1} [140 - 100 + 800] = \$763.63, \end{aligned}$$

and, of course, Equation (2.17) can also be used to compute $V(1)$ and $V(0)$.

We now examine the return each period to a share of stock in the firm under two extreme assumptions concerning financial or dividend policy. The relevant numbers are presented in Table 1. Table 1a shows the values each period of variables that are not affected by financial policy. At some point the reader should indeed convince himself that the values of $V(t)$, $t = 0, 1, 2, \dots$, are the same when computed by means of Equation (2.19) as when computed by Equation (2.21). Table 1b presents the values of the variables needed to compute the return each period on a share under the assumption that all investment is financed internally, that is, with retained earnings; Table 1c presents the values of the same variables under the assumption that all earnings are paid out as dividends, so that new investments are financed entirely externally, that is, by new shares.

When investments are financed entirely with retained earnings, the computation of the return on a share each period is simple. Total dividends $D(t)$ are just earnings minus investment, $X(t) - I(t)$. Because no new shares are issued, $m(t) = 0$ and $n(t) = 1000$ for all t . Thus for each period the price per share $v(t) = V(t)/1000$ and dividends per share $d(t) = D(t)/1000$.

The arithmetic becomes more involved, however, when investments are financed entirely with new shares. Total dividends $D(t)$ are then just total earnings $X(t)$. The price of a share at the beginning of any period can be obtained by noting that

$$v(0) = \frac{V(0)}{n(0)} = \frac{738.92}{1000} = \$0.7389,$$

$$m(t) \cdot v(t) = I(t),$$

and
$$v(t+1) = \frac{V(t+1) - m(t+1)v(t+1)}{n(t)}.$$

The number of new shares issued at the beginning of period $t+1$ can then be computed as

$$m(t+1) = \frac{m(t+1)v(t+1)}{v(t+1)} = \frac{I(t+1)}{v(t+1)}.$$

The first important point to note is that for both financial policies considered in Table 1 the price per share at the beginning of period 0, $v(0)$, is \$0.7389, although the time shape of the stream of dividends per share is much different for the two policies. Indeed when investments are financed with retained earnings, the undiscounted sum of dividends per share for a long period of time greatly exceeds the dividends paid on a share when investments are financed with new shares. With internal (retained earnings

TABLE 1a

The Initial Data

$t =$	0	1	2	3	≥ 4
$I(t)$		100	100	100	0
$X(t)$		100	120	140	160
t^*t+1	0.01	0.05	0.10	0.20	0.20
$V(t)$	738.92	746.31	763.63	800	800

TABLE 1b

Results When Investments Are Financed through Retained Earnings

$D(t)$		0	20	40	160
$v(t)$	0.7389	0.7463	0.7636	0.80	0.80
$m(t)v(t)$		0	0	0	0
$n(t)$	1000	1000	1000	1000	1000
$m(t)$		0	0	0	0
$d(t)$		0	0.0200	0.0400	0.1600
$v(t+1) - v(t)$	0.0074	0.0173	0.0364	0	0

TABLE 1c

Results When Investments Are Financed Entirely through New Shares

$D(t)$		100	120	140	160
$v(t)$	0.7389	0.6463	0.5747	0.5268	0.5268
$m(t)v(t)$		100	100	100	0
$n(t)$	1000	1154.7	1328.7	1518.5	1518.5
$m(t)$		154.7	174.0	189.8	0
$d(t)$		0.1000	0.1039	0.1054	0.1054
$v(t+1) - v(t)$	-0.0926	-0.0716	-0.0479	0	0

financing, however, dividends per share in the early periods are much lower than with external (new share) financing, so that at the beginning of time period 0 the present values, that is, market prices $v(0)$, of the dividend streams obtained under the two financing alternatives are equal.

In addition, it is important to note, and the reader should check, that under both financial policies the one-period return on a share during any given period is equal to the market rate of interest r_{t+1} for the period; that is,

$$\frac{d(t+1) + v(t+1) - v(t)}{v(t)} = r_{t+1}.$$

Thus the returns under both financial policies are 0.01 in period 0, 0.05 in period 1, 0.1 in period 2, and 0.2 thereafter. Moreover, the returns on the shares are equal to the market interest rates in spite of the fact that the firm has opportunities to invest, at least for three periods, at rates of return greater than market interest rates. With perfect capital markets, the market prices at all times take account of future investment opportunities. As a result, the returns on the shares are just equal to the market interest rates.

Although the rates of return period by period are the same with the two financial policies, in the early periods the distribution of returns between dividends and capital gains is different. In fact, for the particular example being considered, with external financing there are only capital losses and never gains. Because the initial price per share $v(0)$ is the same under both financial policies, this implies that, for $t > 0$, $v(t)$ is lower when investments are financed externally than when financed internally. This results, of course, from the fact that when new shares are issued, the part of the total value of the firm in future periods that accrues to the current shareholders is lower than when investments are financed with retained earnings.

The reader can convince himself, however, that by relending the difference between dividends per share in each of the first three periods under the two financial policies, the stream of dividends and capital gains obtained when investments are financed internally can be transformed into the stream obtained when investments are financed with new shares. Or, vice versa, by selling part of his holdings at the end of each of the first three periods, the stream of dividends and capital gains obtained when investments are financed internally can be transformed into the stream of dividends and capital losses obtained when investments are financed externally. But these opportunities are, of course, just implications of the fact that with perfect certainty and perfect capital markets, any stream of net cash flows with present value \$0.7389 at the beginning of period 0 can, by borrowing or lending in the market, be transformed into any other stream with present value \$0.7389.

II. AN EXAMPLE USING THE CONSTANT GROWTH MODEL

We now consider a numerical example based on the constant growth model. The example involves three different firms, all with the same level of current earnings $X(1) = 100$, the same proportion of income invested each period, $k = 0.4$, the same return on investment opportunities, $r^* = 0.2$, but different financial policies. The purpose of the example is to examine the effects of financial policy on dividends, stock prices, and returns to shareholders. The relevant information is summarized in Table 2.

Expression (2.28) can be used to compute the market value of each of the three firms at the beginning of any time period. Because (2.28) does not contain variables that depend on financial policy, the market values of the three firms are equal at any point in time. For example, at the beginning of time period $t = 0$,

$$V(0) = \frac{X(1)(1-k)}{r - kr^*} = \frac{100(0.6)}{0.10 - (0.4)(0.2)} = \frac{60}{0.02} = \$3000$$

for all three firms.

We now examine the returns per share provided by each firm. Firm I follows the policy of financing all investment with retained earnings; that is, $k_r = k = 0.4$ and $k_e = 0$. Investment at the beginning of period 1 is

$$I(1) = kX(1) = 0.40(100) = \$40.$$

Because firm I finances all its investment with retained earnings, total dividends and dividends per share are respectively

$$D(1) = X(1) - I(1) = 100 - 40 = \$60,$$

$$\text{and} \quad d(1) = \frac{D(1)}{n(0)} = \frac{60}{1000} = \$0.06.$$

TABLE 2

Firm	I	II	III
$X(1)$	100	100	100
r	0.10	0.10	0.10
r^*	0.20	0.20	0.20
k	0.40	0.40	0.40
k_r	0.40	0	0.20
k_e	0	0.40	0.20
$V(0)$	\$3000	\$3000	3000
$I(1)$	\$ 40	\$ 40	\$ 40
$n(0)$	1000	1000	1000
$v(0)$	\$ 3.00	\$ 3.00	\$ 3.00

For all three firms total earnings at the beginning of period 2 are

$$X(2) = X(1) + kr^*X(1) = 100 + 8 = \$108,$$

so that
$$V(1) = \frac{X(2)(1-k)}{r-kr^*} = \frac{108(0.6)}{0.10-0.4(0.2)} = \frac{64.8}{0.02} = \$3240.$$

Thus for firm I the price of a share of stock at the beginning of $t = 1$ is

$$v(1) = \frac{V(1)}{n(1)} = \frac{3240}{1000} = \$3.24,$$

and the one-period return from holding the stock during time period 0 is

$$\frac{d(1) + v(1) - v(0)}{v(0)} = \frac{0.06 + 3.24 - 3.00}{3.00} = 0.10 = r.$$

Thus when the firm finances its investment opportunities with retained earnings, the shareholder does indeed earn the market rate of interest on his investment. Moreover, the shareholder earns only r , although the return on the firm's investment is $r^* > r$. The reason, of course, is that with perfect certainty and perfect capital markets the price of a share at every point in time fully reflects the market value of current and future investment opportunities, so that from period to period the shareholder earns only his opportunity costs, the market rate of interest r .

The rate of growth of share prices and dividends is given by Equation (2.31). In this case, however, because $k_s = 0$, g is the same as the rate of growth of the firm, $kr^* = 0.4(0.2) = 0.08$.

Firm II is just the opposite of firm I. It pays out all its earnings as dividends and finances new investment entirely by issuing new shares; that is, $k_r = 0$, but $k = k_s = 0.40$. Thus dividends per share at the beginning of time $t = 1$ are $X(1)/n(0) = \$0.10$. The price of a share in firm II at the beginning of time period $t = 1$ can be computed as follows. The value of the firm at the beginning of $t = 1$ is, from Equation (2.25), $V(1) = \$3240$, the same as for firm I, because Equation (2.25) does not contain variables dependent on financial policy. However, $V(1)$ does not accrue in full to the shares outstanding at the beginning of time $t = 0$. At period 1 new shares were issued that had total value

$$m(1)v(1) = I(1) = \$40.$$

The price of a share of stock at the beginning of time period 1 is then

$$v(1) = \frac{V(1) - m(1)v(1)}{n(0)} = \frac{\$3240 - 40}{1000} = \$3.20,$$

and the return on a share held during $t = 0$ is

$$\frac{d(1) + v(1) - v(0)}{v(0)} = \frac{0.10 + 3.20 - 3.00}{3.00} = 0.10 = r,$$

which is, of course, the market rate of return. The larger dividends per share of firm II relative to firm I are exactly balanced by the smaller capital gains of firm II.

The value of $v(1)$ computed above can be obtained by a different approach. We saw earlier that in the constant growth model

$$v(t) = v(0)[1 + g]^t,$$

where g is the rate of growth of price per share. For firm II, according to Equation (2.31),

$$\begin{aligned} g &= 0.4(0.2) \frac{1}{0.6} - 0.4(0.1) \frac{1}{0.6} \\ &= \frac{0.04}{0.6} = 0.0667, \end{aligned}$$

$$\text{so that } v(1) = 3.00[1 + 0.0667] = \$3.00 + 0.20 = \$3.20,$$

which is exactly the price as previously computed. Because g is also the rate of growth of dividends per share, we see that, although the dividends per share of firm II are initially higher than those of firm I, their rate of growth is lower (0.0667 as compared with 0.08).

Unlike firms I and II, firm III uses both external and internal sources, in equal parts, to finance its investment projects; that is, $k = 0.40$, as before but $k_r = k_s = 0.20$. By reasoning similar to case II we find that

$$v(1) = \frac{V(1) - m(1)v(1)}{n(0)},$$

$$m(1)v(1) = I(1) - [X(1) - D(1)] = 40 - [100 - 80] = \$20,$$

$$v(1) = \frac{3240 - 20}{1000} = \$3.22,$$

$$\text{and } \frac{d(1) + v(1) - v(0)}{v(0)} = \frac{0.08 + 3.22 - 3.00}{3.00} = 0.10 = r.$$

Again the return is r , the market rate of interest.

For firm III the rate of growth of price and dividends per share is

$$\begin{aligned} g &= 0.08 \frac{0.80}{0.60} - 0.2(0.1) \frac{1}{0.60} \\ &= 0.10667 - 0.0333 = 0.07334. \end{aligned}$$

Thus $v(1) = v(0)[1 + g] = 3.00 + 3.00(0.7334) = \3.22 , just as we saw by another route above.

Thus for all three firms the rate of return from holding a share of stock during $t = 0$ is precisely equal to the market interest rate $r = 0.10$. The rates of return are the same for all firms in spite of the fact that widely different policies are followed in financing new investment and in spite of the fact that the new investment has an average rate of return $r^* > r$.

Finally the market values of the three firms at any point in time can be computed from Equation (2.29), the formula for the dividend approach to valuation when applied to the constant growth model. The results are the same as those obtained earlier in the present section with Equation (2.28). Thus for firm I with Equation (2.29) we get

$$V(0) = \frac{D(1)}{r - g}$$

$$V(0) = \frac{60}{0.10 - 0.08} = \frac{60}{0.02} = \$3000;$$

for firm II,

$$V(0) = \frac{100}{0.10 - 0.0667} = \frac{100}{0.0333} = \$3000;$$

and for firm III,

$$V(0) = \frac{80}{0.10 - 0.07334} = \frac{80}{0.02666} = \$3000.$$

The reader should now be in a position to carry on the numerical example on his own. In particular, he should be able to show that, although dividends per share grow at different rates for the three firms, total dividends paid per period grow at the same rate for all firms. To test his understanding, the reader will also find it a useful exercise to carry out the computations for an additional time period and to convince himself that by borrowing or lending in the market, the dividends and capital gains obtained from any one of the firms can be transformed into the dividends and capital gains obtained from either of the others.