

1

A MODEL OF THE ACCUMULATION AND ALLOCATION OF WEALTH BY INDIVIDUALS

In this introductory chapter to Part I, we present one of the fundamental building blocks of the theory of finance, namely, the model of the accumulation and allocation of wealth over time by individuals under conditions of certainty and perfect capital markets. Once the model itself has been set forth in fairly general and abstract terms, we go on in subsequent chapters to consider its extensions to a variety of problems in finance. The extension of the model to the case of uncertainty is given in Part II.

Because the wealth allocation model itself is merely a special case of the more general economic theory of choice under certainty, we begin by reviewing briefly some of the main concepts and features of this theory.

I. THE ECONOMIC THEORY OF CHOICE

I.A. Opportunities and Preferences

The economic theory of choice, like any other body of theory, aims at establishing empirical generalizations

4 Certainty Models

about the class of phenomena under study. For the theory of choice, this means generalizations about the way in which choices change in response to changes in the circumstances surrounding the choice.

The first step in constructing the theory is the simple but important one of classification. We divide the many separate elements bearing on any choice into two classes. One class is called the "opportunity set" or "constraint set." As the name implies, it is the collection of possible choices available to the decision maker. Its content, depending on the context, may be determined by technological limitations, such as the impossibility of constructing a perpetual-motion machine, or by legal restrictions, such as those prohibiting selling oneself into slavery, or by market restrictions, such as the inability to buy a new automobile for \$5 because of the absence of sellers at this price.

The other main elements in the decision problem are the decision maker's "tastes" or preferences. For an individual choosing on his own behalf, these tastes depend on personality, upbringing, education, and so forth. The theory does not propose to say precisely how these factors enter separately into the choice but takes the individual and his preferences as given from outside the problem and proceeds from there. For individuals acting as managers, and hence presumably on behalf of others, the question of precisely whose preferences we are talking about, that is, the agent's or the principal's or some combination, must also be faced. We eventually try to do so, but to keep the presentation uncluttered, we defer this issue to Chapter 2. For the remainder of the present chapter, we are concerned only with a single individual acting entirely on his own behalf.

I.B. Representation of Preferences: The Utility Function and Indifference Curves

To develop interesting generalizations about choice behavior, we need a convenient way of representing tastes and opportunities. One possibility is to tabulate the decision maker's choices under laboratory conditions. We could present the subject with a series of bundles or boxes each containing some of the relevant objects of choice, carefully record the contents of each box, and then note which box he actually chose. By suitably varying the contents of the boxes, we could obtain an extensive picture of his likes and dislikes.

Such a table would certainly contain a great deal of information about the subject's tastes, but it would be difficult to work with. Any patterns in it would be too hard to see. The question arises, therefore, whether some simpler and more compact way exists for representing or at least approximating the data in the table. Such a representation is indeed possible, provided that the subject's preferences satisfy certain conditions.

I.B.1. The axioms of choice and the principle of maximum utility

These conditions collectively constitute the axioms of the economic theory of choice. The word *axiom* is not to be taken here in its old-fashioned sense of a self-evident truth. The axioms are to be regarded rather as provisional assumptions, plausible enough, perhaps, as approximations, but whose ultimate justification comes, not from their own truth or plausibility, but from the predictive and descriptive power of the conclusions to which they lead.¹

In discussing the axioms, we let the letters x , y , and z represent boxes of objects presented to the subject for choice. These objects, whatever their outward form, are referred to as "commodities" to which, instead of names, we assign numbers $1, 2, \dots, n$. Each box is completely specified by indicating the number of units q of each commodity that it contains; that is, the box x is represented by an n -tuple of the form $(q_1^{(x)}, q_2^{(x)}, \dots, q_n^{(x)})$.

Axiom 1 (Comparability). For every pair of boxes x and y the decision maker can tell us either (1) that he prefers x to y , or (2) that he prefers y to x , or (3) that he is indifferent to having x or y .

The function of the axiom of comparability is to rule out cases in which the decision maker refuses or is unable to make a choice, because, for example, he may regard the objects of choice as essentially different and hence incomparable.

Axiom 2 (Transitivity). Whenever the decision maker prefers x to y and y to z , he also prefers x to z . Likewise, if he is indifferent to having x or y and to having y or z , he is also indifferent to having x or z . Basically, the subject behaves consistently in making his choices.

If the decision maker's tastes conform to these axioms, the following important proposition holds:

The subject's choice behavior may be characterized by saying that he behaves as if he were maximizing the value of a "utility function." This function assigns a numerical value or "utility index" to each box x, y, \dots ,

¹ The methodological principle that theories are to be judged by the empirical validity of their consequences rather than by that of their assumptions considered separately has come to be called "positivism." The classical statement of the positivist position in economics is that of Milton Friedman, "The Methodology of Positive Economics," in *Essays in Positive Economics*. Chicago: University of Chicago Press, 1956.

with the quantities of each commodity in the box as the arguments of the function.

To illustrate, suppose that we somehow knew that a utility function for a particular subject took the form

$$U = F(q_1, q_2) = q_1^2 + q_2^2,$$

where the letter U is the utility index and q_1 and q_2 are the quantities respectively of commodities 1 and 2 in any box. Suppose that we also knew that box x held 6 units of commodity 1 and 4 units of 2 and that box y held 8 units of commodity 1 and 2 units of 2. Which one does he prefer? Substituting the contents of box x into the utility function, we obtain an index of $6^2 + 4^2 = 52$; for the second box we have $8^2 + 2^2 = 68$. The assignment of a higher utility index to the second box is equivalent to saying that the subject prefers box y to box x .

Note that in the statement of the proposition we say that the subject behaves as if he were maximizing the value of a utility function. The economic theory of choice does not assert that the subject performs these calculations on his utility function in making his choices or even that the subject knows that he has a utility function. It says merely that if his preferences are complete and consistent, the utility function provides us, as outside observers, with a way of representing his choices. In what follows we sometimes gloss over this distinction and speak, say, of a decision maker's increasing or decreasing his utility by some action. But this should always be considered a stylistic device and must not be taken literally.

The reasonableness of the proposition in relation to the axioms is easily seen. The axioms of comparability and transitivity permit us to rank all possible boxes in increasing order of preferability. This rank ordering, in turn, can always be expressed by some device that assigns numbers to each box so that higher numbers go with higher ranks. The utility function is just such a device.²

It should also be clear that any specific representation of a subject's utility function, such as in this illustration, is not unique. If one such function exists, so do many others, because what matters is whether the number assigned to a box is higher or lower than that of another box. How much higher or lower is of no consequence so long as the ranking is preserved. Thus multiplying by a positive constant or performing any other monotone-increasing transformation of a utility function also yields an

² The more advanced mathematical treatments in the economics literature are also concerned with the continuity of the utility function and the additional requirements necessary to assure it. Readers interested in a complete and rigorous discussion of these and related issues can find it in the classic treatment of the subject by Gerard Debreu, *The Theory of Value*. New York: Wiley, 1959.

equivalent and perfectly consistent representation of the subject's preferences.³

Given the utility function to represent the general structure of preferences, the next step in the search for interesting, and testable, generalizations about choice behavior is to invoke certain additional assumptions with respect to the form and properties of this function. Some of these assumptions are mainly for mathematical convenience and are noted here, in footnotes, only for the sake of completeness.⁴ Others, however, are more substantive and merit further discussion. In introducing and explaining them, it is helpful first to show how choices and utility functions can be represented in geometrical or graphical form.

1.B.2. Indifference curves and the geometrical representation of preferences

In representing the utility function $U = U(q_1, q_2, \dots, q_n)$ graphically, we are, of course, limited to at most three dimensions. Because we must reserve one dimension for U , the value of the utility index, we are thus restricted to only two commodities in making up the boxes between which the subject is to choose. Although this collapsing down from n distinct commodities to only two may seem a very drastic simplification, the loss of generality is actually quite small. For establishing the kinds of propositions that are our main concern, the two-commodity case is almost always adequate.

One way of representing a utility function in three dimensions on a two-dimensional page is with figures of the kind used in solid geometry. Because it is difficult, and expensive, to draw the figures in correct perspective, however, we use a projective method that is considerably simpler but requires, at first, a somewhat greater effort in interpretation on the part of the reader. To see how it works, imagine a three-dimensional solid representation of the utility function $U = U(q_1, q_2)$, with q_1 and q_2 on two axes in a horizontal plane and U measured on a third axis vertical, and perpendicular, to the other two. Imagine now that we pick some particular value

³ Utility functions with this property are said to be "ordinal" functions, in contrast to "cardinal" utility functions, which are unique up to a "linear" transformation and which hence can convey some indication of how much higher one bundle is on the subject's scale of preferences than another. We have no need for the stronger, cardinal functions until we reach Part II and introduce uncertainty.

⁴ In particular, we assume that the arguments of the utility function, q_i , can be represented as continuous variables, that is, that all commodities are infinitely divisible, that the utility function itself is a continuous function of its arguments, and that it has continuous derivatives of whatever order we happen to need.

Another assumption that we have already made implicitly by our distinction between tastes and opportunities is that the two can in fact be separated in the sense that each can be defined independently of the other.

of U , say, $U = 7$, and pass a knife horizontally through the surface at this level. The cut being horizontal, the outline is necessarily always a two-dimensional figure. Now project, that is, drop, this horizontal outline down onto the horizontal q_1q_2 plane. If we repeated this process for many different values of U , we should obtain a whole set of such two-dimensional figures; and by labeling each such figure or contour with its value of U , we could mentally reconstruct the entire three-dimensional figure.

An example of a utility function represented by such contours is shown in Figure 1.1a. From the numbering of the contours it can be seen that the utility surface has the form of an irregular hill rising to a summit at the point labeled $U = 40$. The indentations in the contours on the southwest slope imply a trough or draw at the lower levels running in the northeast-southwest direction. The circular area labeled $U = 20$ lying between the contours $U = 10$ and $U = 15$ implies a small isolated knoll on the northeast face. And so on for the various other curls and wrinkles that have been drawn into the map.

In addition to the utility surface itself, the figure also shows the representation of some boxes offered to the subject for choice and the pattern of his preferences among them. These boxes are indicated by the points labeled w, x, y , and z —the box w containing $q_1^{(w)}$ units of commodity 1 and $q_2^{(w)}$ of commodity 2, and similarly for the others. As drawn, the points w, x , and y lie on the same contour, $U = 10$. That these boxes all have the same utility index is the same as saying that the subject was completely

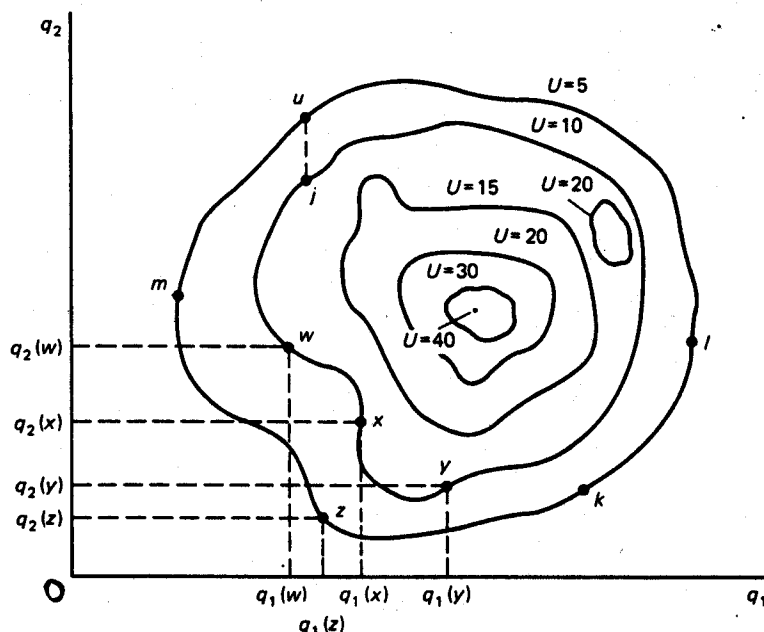


Figure 1.1a Iso-utility Contours

indifferent when presented with a choice between them—hence the term “indifference curves,” which is customarily used in the theory of choice to refer to an iso-utility contour line. Box z , however, lies on an indifference curve with a lower utility index, which is to say that it would be rejected by the subject in any choice involving w , x , or y .

1.B.3. The axioms of nonsatiation and convexity

We turn now to the final two axioms or assumptions about choice behavior.

Axiom 3 (Nonsatiation of Wants). The subject would always prefer, or at worst would be indifferent, to have more of any commodity if at the same time he did not have to take less of any other commodity.

This assumption serves to rule out any positively sloped segments of indifference curves, such as the segments mu or kl on curve $U = 5$ in Figure 1.1a. For where such segments exist, there are points, such as j , having the same amount of q_1 as box u but less q_2 and yet preferred to u , as shown by the higher utility index of its indifference curve. A map of a utility surface that does conform to the axiom—a hill with no summit or northeast face—is shown in Figure 1.1b.

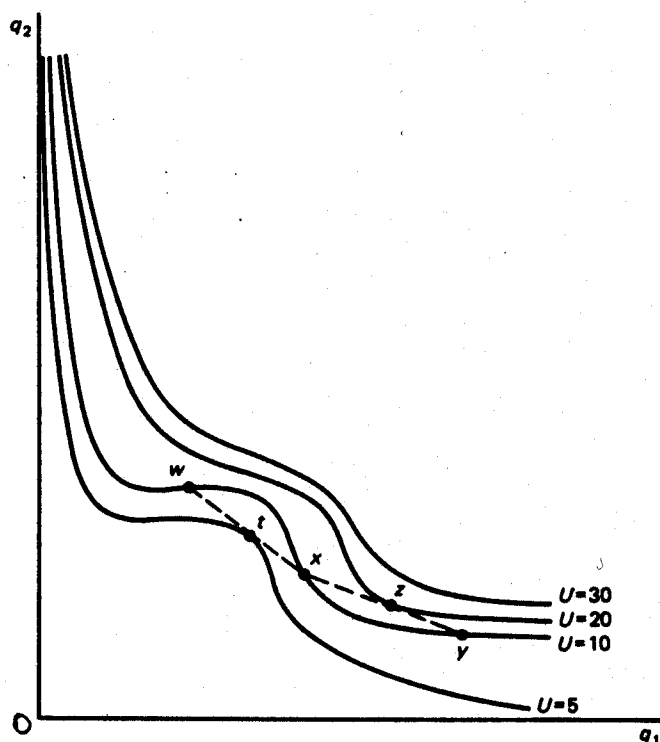


Figure 1.1b Convexity and Concavity of Indifference Curves

Axiom 4 (Convexity). If x and y are two boxes such that $U(x) = U(y)$ and if z is a combination of boxes x and y of the form $z = \alpha x + (1 - \alpha)y$, $0 \leq \alpha \leq 1$, then $U(z) \geq U(x) = U(y)$. In words, if we have two boxes between which the subject is indifferent, and we construct a new box whose contents, commodity by commodity, are a weighted average of those of x and y , where the weights are positive and sum to unity, the subject never chooses either x or y in preference to the combination box z .⁵

Examples of (local) regions of the utility surface that do and do not meet this convexity assumption are shown in Figure 1.1b. If we draw a straight line connecting the points w and x lying on the same indifference curve $U = 10$, this straight line representing all the combination boxes along the line $\alpha x + (1 - \alpha)w$ for $0 \leq \alpha \leq 1$, we see that a combination box, such as t , would not be preferable to x or w . It lies on a lower indifference curve with index, say, $U = 5$, as we have happened to draw it. In this region, then, the indifference curve does not meet the convexity assumption. By contrast, the intermediate points on a straight line between x and y all lie on higher indifference curves, and in this region the indifference curve for $U = 10$ is convex.

To see in more concrete terms what this assumption of convexity implies about choice behavior, imagine that the boxes contain two commodities, such as books and theater tickets. Suppose that we start with a box containing 20 books and 3 tickets and then withdraw 1 ticket. Books and tickets both being desirable commodities for the subject, we should have to add, say, 2 additional books to the box to keep him on the same indifference curve—the additional books, in effect, compensating for the ticket withdrawn. If now we subtract another ticket from the 2 remaining, the convexity assumption implies that we should have to add at least another 2 books and possibly more, to keep the utility index of the box unchanged. When the removal of the second item requires more compensation than that of the first, the indifference curve is said to be “strictly convex.” When the compensation required is the same on successive removals, the curve is linear, which is the limiting extreme case of convexity.⁶

In most subsequent applications of the theory to problems of finance we work with indifference maps of the kind shown in Figure 1.1c. Such maps have a number of important features and properties that will be invoked

⁵ As an illustration of the construction of z from x and y , suppose that x has 8 units of commodity 1 and 6 units of commodity 2 and y has 6 units of 1 and 10 units of 2; then by taking $\alpha = \frac{1}{2}$, z would be a box with $\frac{1}{2}(8) + \frac{1}{2}(6) = 6\frac{1}{2}$ units of commodity 1 and $\frac{1}{2}(6) + \frac{1}{2}(10) = 9$ units of commodity 2.

⁶ For those readers who find it difficult to accept the convexity axiom, even provisionally, it may be well to point out that we use it here mainly to simplify the graphical presentations and to avoid the complication of multiple or “corner” solutions. All the really essential conclusions can be derived from the first three axioms only.

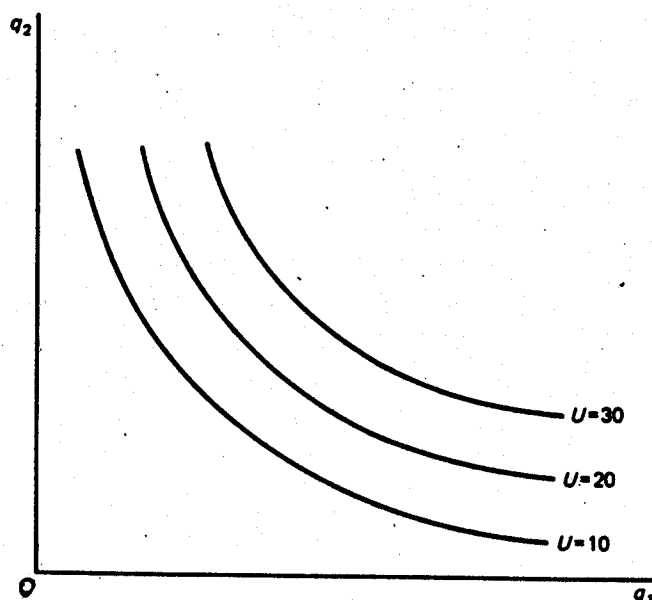


Figure 1.1c A Standard Indifference Map

repeatedly in deriving propositions about choice behavior. To summarize, the slope of an indifference curve, often called the marginal rate of substitution of q_2 for q_1 , is always negative. The marginal rate of substitution rises in algebraic or falls in absolute value as we move from left to right. Curves take on higher utility indexes as we move upward on the map. And there are never any crossings of indifference curves. To test their understanding, readers for whom this apparatus is unfamiliar may find it useful to imagine the contrary of each of these properties and then show that such cases would be inconsistent with one or more of the basic axioms of comparability, transitivity, nonsatiation, or convexity.⁷

⁷ Because these concepts are used repeatedly throughout the book, it is well to note more formally here the distinction between convex "sets" and convex and concave "functions" and "curves." A set of points is said to be convex if for any two points x and y in the set, the point $z = \alpha x + (1 - \alpha)y$, $0 \leq \alpha \leq 1$, is also in the set. Geometrically, a set is convex if all points on a straight line between any points in the set are also in the set.

On the other hand, a function f is convex if for any two points x and y in the domain of f

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \quad 0 \leq \alpha \leq 1,$$

and the function is concave if

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y), \quad 0 \leq \alpha \leq 1.$$

Geometrically, a function or curve is convex if a line between any two points on the

1.C. The Opportunity Set

As noted earlier, the class of choice problems of most concern in economics as opposed to, say, psychology or aesthetics is that in which the decision maker's choices are limited by external restrictions. A graphical representation of such restrictions for a two-commodity case is shown in Figure 1.2. As indicated by the shading, the feasible boxes or combinations of commodities 1 and 2 are restricted to those lying within, or on the boundaries of, the irregular figure outlined by OAB . Any box not lying within this opportunity set is excluded from consideration, no matter how desirable it might be for the subject to have it.

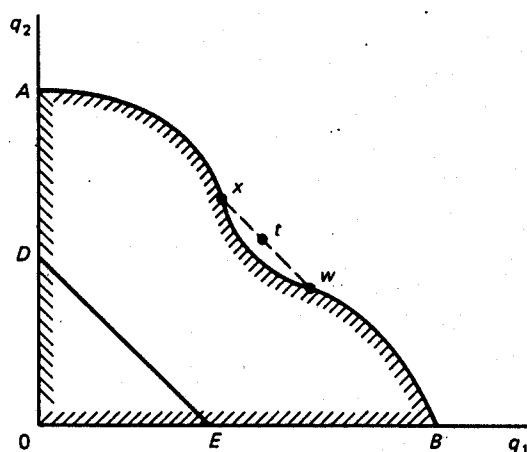


Figure 1.2 Opportunity Sets

The precise content or shape of the opportunity set depends, of course, on the problem at hand. There is, however, one general restriction that is a feature of virtually all those to be considered here, namely, the assumption that the set is convex; that is, if x and y are any two boxes in the opportunity set, all boxes z of the form $z = \alpha x + (1 - \alpha)y$, $0 < \alpha < 1$, are also in the set. Thus, in terms of Figure 1.2, the property of convexity of the set

function lies everywhere on or above the function; concavity implies that a line between any two points is everywhere on or below the function.

Hence, when added to the nonsatiation axiom, the convexity axiom presented in the text is an assumption that commodity indifference curves are convex. But note carefully that it is not an assumption that the utility function U is convex. This function is still only ordinal; that is, the only restriction on the numbers assigned to successive indifference curves are that they are monotone-increasing, so that the curvature of the function across indifference curves is arbitrary.

rules out segments like that between w and x , where there are boxes like t lying on a line joining x and w , but not part of the allowable choices.⁸

Note that the definition of convexity includes sets with straight-line outer boundaries, such as the set bounded by ODE in Figure 1.2, as the limiting special case. In general, convex opportunity sets with curved boundaries usually arise when the constraints are imposed by "technology" and the curvature represents "diminishing returns" in the physical possibilities of transforming commodity 1 into commodity 2. The straight-line cases usually represent market exchange opportunities, the constant slope implying that the commodities can be exchanged for each other at given, fixed prices. In some cases, we consider problems with both types of constraints operating simultaneously.

I.D. Choice Subject to Constraints

To obtain a representation of a choice in the presence of constraints, there remains now only to bring together the two pieces of the problem, tastes and opportunities, that we have so far considered separately. A graphical illustration for a two-commodity case is shown in Figure 1.3. The

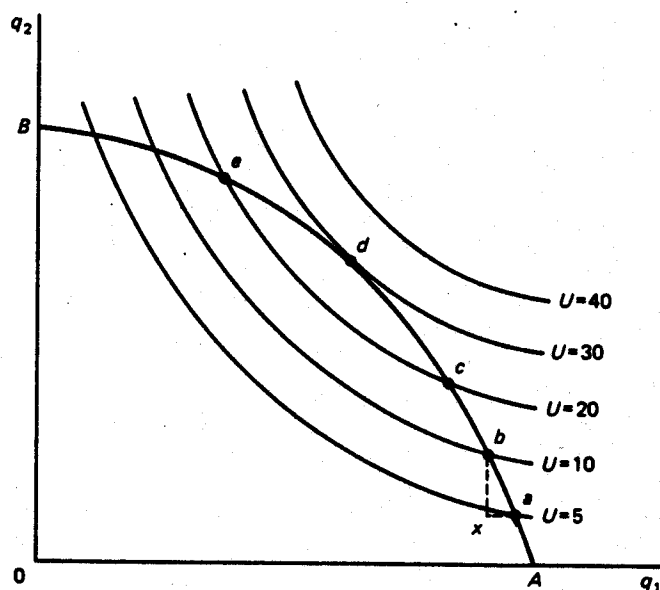


Figure 1.3 Representation of a Constrained Choice

figure shows a subject whose preferences meet the four axioms of choice and can therefore be represented by a utility map with convex indifference

⁸ As with the assumption of convexity of indifference curves (see footnote 6 above), the assumption of convex opportunity sets is much stronger than necessary for most of the important conclusions and is used throughout mainly to simplify the presentation.

curves increasing in utility index as one moves up and to the right. The possible boxes available to him lie within the figure OBA . Which one of the immense number of possibilities represents the one that he would actually choose?

We can, first of all, eliminate immediately a large number of boxes as possibilities. In particular, no box such as x in the interior of the opportunity set could possibly be his eventual choice, because the boxes lying between a and b on the boundary all have at least as much of one of the two commodities as x and more of the other. By virtue of the nonsatiation axiom, such dominated interior boxes are ruled out, and only the boxes along the right boundary need be considered. It is for this reason that the right boundary of the opportunity set is often referred to as the "efficiency frontier" or "efficient set".⁹

Having narrowed the range of possibilities to the efficient set, we need only sweep along it in some systematic manner. If, for example, we work up from the lower right-hand corner, steadily decreasing the amount of q_1 in the box and increasing the amount of q_2 , we produce a series of choices of the form a versus b , then b versus c , and so on. As drawn, the subject prefers b to a , indicated by the fact that b lies on a higher indifference curve, and prefers c to b . Eventually, by repeating this process, we find a point that is preferred to its neighbors on either side. In the figure as drawn, this is the point d , at which an indifference curve is tangent to the efficiency frontier. All other boxes, whether on the efficiency frontier or within it, lie on lower indifference curves.

This representation of a constrained choice completes our review of the fundamentals of the theory of choice. The importance of this theory for our purposes lies in its simplicity and in the fact that it is both context- and subject-free. No matter what kinds of commodities that we put in the boxes and no matter who the subject or what the details of his personal preferences, his choice can always be represented by a point on the efficiency frontier, provided only that his tastes meet Axioms 1 to 3. If, in addition, his tastes meet Axiom 4 and if the opportunity set is convex, we can narrow the possible choices along the frontier down to the single one where an indifference curve is tangent to the opportunity set. The four axioms plus the convexity of the opportunity set guarantee both that there must be such a point and that there is no more than one.¹⁰

⁹ Because the right boundary of the opportunity set is assumed to be negatively sloping, the assumption that the set is convex implies that this boundary must be a concave curve.

¹⁰ In what follows, we frequently refer to these tangency points as "optimal solutions" or "equilibrium points." As in the case of references to utility, however, such terms are normally to be taken in an as if sense and not as implying that the subject is consciously optimizing or equilibrating.

The task of much of the rest of the book is essentially a filling in of these "empty boxes" and a showing of how the very general and abstract theory of choice can be specialized to obtain meaningful generalizations about financial choices. But before turning to this task, we present for the record a very brief account of how a choice subject to constraints can be represented in mathematical form.

*I.E. The Solution in Mathematical Form¹¹

In particular, for an n -commodity case, the subject's choice can be expressed as the solution to the problem¹²

$$\max_{q_1, q_2, \dots, q_n} U(q_1, q_2, \dots, q_n)$$

subject to a constraint or opportunity set that may be written in general, implicit form as

$$T(q_1, q_2, \dots, q_n) = 0,$$

and which represents the points (q_1, q_2, \dots, q_n) that lie along the efficiency frontier.

With appropriate assumptions about the continuity of the derivatives of the U and T functions, we can use the methods of the differential calculus to study properties of the solution to this problem. Specifically, form the lagrangian function

$$L = U(q_1, q_2, \dots, q_n) - \lambda T(q_1, q_2, \dots, q_n), \quad (1.1)$$

and differentiate partially with respect to λ and each of the q_i . Setting these derivatives equal to zero yields the following $n + 1$ equations as first-order or necessary conditions for a maximum:¹³

$$\begin{aligned} U'_1 - \lambda T'_1 &= 0 \\ U'_2 - \lambda T'_2 &= 0 \\ &\vdots \\ U'_n - \lambda T'_n &= 0 \\ T(q_1, q_2, \dots, q_n) &= 0, \end{aligned} \quad (1.2)$$

¹¹ Starred sections here and throughout mark the places in the exposition where some knowledge of the calculus is required. Such sections may safely be skipped by readers lacking the necessary mathematical background, without fear of losing the main thread of the argument. Such readers may nevertheless find it helpful to skim through the sections, because the discussion surrounding the mathematical results may provide additional insights.

¹² The expression

$$\max_{q_1, q_2, \dots, q_n} U(q_1, q_2, \dots, q_n)$$

is read "choose values of q_1 to q_n that maximize the utility index."

¹³ In general, in the mathematical treatments of various maximization and minimiza-

where U'_i and T'_i are the partial derivatives of U and T with respect to q_i .

To see the relation between this representation and the graphical representation of the two-commodity case, observe that between any pair of commodities we can eliminate λ from the relevant equations of (1.2) to obtain, for example,

$$\frac{U'_1}{T'_1} = \frac{U'_2}{T'_2} \quad \text{or} \quad \frac{U'_1}{U'_2} = \frac{T'_1}{T'_2}. \quad (1.3)$$

The definition of an indifference curve is equivalent to the solution set satisfying the condition on the differential

$$dU = U'_1 dq_1 + U'_2 dq_2 + \cdots + U'_n dq_n = 0$$

for some specified level of U . Holding q_3 to q_n constant, that is, setting dq_3 to $dq_n = 0$, this condition implies that

$$\frac{dq_2}{dq_1} = - \frac{U'_1}{U'_2}.$$

Or, in words, the slope of an indifference curve at any point is equal to (minus) the ratio of the first partial derivatives evaluated at this point. Similarly, for the opportunity set we have

$$\frac{dq_2}{dq_1} = - \frac{T'_1}{T'_2}.$$

Thus the condition (1.3) is equivalent to the statement that at a maximum of U , the slope of an indifference curve is the same as that of the opportunity set, which is, of course, the familiar tangency condition in the geometric analysis.

II. THE APPLICATION OF THE THEORY OF CHOICE TO THE ALLOCATION OF FINANCIAL RESOURCES OVER TIME

II.A. The Two-Period Case

As noted earlier, the basic application of the theory of choice under certainty to the field of finance is the problem of the allocation of financial resources by individuals over time. The rest of the present chapter is

tion problems in this book, we do not present sufficient or second-order conditions, because they usually contribute little in the way of economic insight, which, after all, is the primary concern. Moreover, such "mathematical license" is often characteristic of our mathematical treatments in other ways as well. For example, we ignore, for the most part, the additional complications introduced by nonnegativity conditions on some or all of the variables.

devoted to the development of this application in its most general and abstract form. A number of extensions are then taken up in subsequent chapters.

II.A.1. The objects of choice: standards of living at different points in time

In studying the allocation of financial resources over time, the objects of choice can be defined in several ways, depending on how much detail is to be shown and precisely how the passage of time is to be represented. Taking the question of time first, we assume throughout that the passage of time is not continuous but occurs in discrete jumps one "unit of time" apart. Decisions are assumed to be taken and payments to be made only at the start of these discrete time periods. For generating the kind of qualitative generalizations about behavior that are our concern, no precise statement need be made about the length of a unit time period in terms of calendar time.¹⁴

As for the objects of choice, we could continue to work in the standard economic framework of individual commodities, giving each commodity a time index to indicate the period in which it was being presented to the subject. The utility function would then be of the form $V(q_{11}, q_{21}, \dots, q_{n1}, q_{12}, q_{22}, \dots, q_{n2}, \dots, q_{1t}, q_{2t}, \dots, q_{nt}, \dots)$. But although formally unobjectionable, and really no great complication mathematically, such an approach makes it difficult to focus sharply on the effects of time per se and on the purely financial aspects of the allocation decision. To highlight these aspects of the problem, therefore, we make the following simplifications in the utility function.

First, because the function in its most general form includes in principle commodities that represent income-earning activities, for example, hours of labor, we assume that these can be separated out and that a utility function can be defined over the set of consumer goods; that is, we assume that preferences between boxes of consumer goods depend only on the contents of the boxes and not on the amount or nature of the income-earning activities that might accompany them. This creates some problems with respect to what might be called "leisure goods," but they are minor for our purposes and are neglected. We also neglect the details of the decision with respect to occupational choice and hours of work. The subject's occupation and earnings are taken as given, determined somehow outside the model.

¹⁴ Despite assertions to the contrary occasionally encountered in the literature, there is little basis other than taste or convenience in choosing between a discrete-time formulation of the kind to be followed here and a continuous-time formulation of the kind often found in economists' expositions of capital theory. Substantive results that can be developed under the one convention can always be translated into the other under conditions of certainty (see, in this connection, footnote 27, below.)

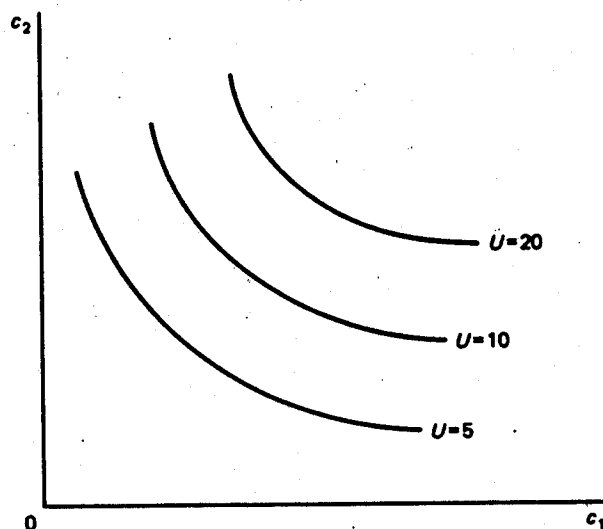


Figure 1.4 Indifference Curves for Patterns of Total Consumption

Second, we replace the individual consumer goods in any period t by a single composite commodity c_t , called the subject's "total consumption" or "standard of living" in period t . This composite commodity is defined as

$$c_t = p_{1t}q_{1t}^* + p_{2t}q_{2t}^* + \cdots + p_{nt}q_{nt}^*,$$

where the p_{it} are the prices of the commodities¹⁵ in period t , assumed to be known and fixed in our perfect certainty framework, and the q_{it}^* are the amounts of the commodities that would be purchased by the consumer, given the prices and a pattern of total resources over all periods that would permit him to spend c_t in period t . Operationally, in terms of the earlier imaginary choice experiments, we now picture ourselves confronting the subject with different patterns of standards of living, such as, say, \$10,000 for period 1 and \$8000 thereafter versus \$6500 for period 1, \$6000 for period 2, and \$9000 thereafter. The subject decides how he would allocate these amounts among commodities in each period and then announces his

¹⁵ Although we speak of prices and commodities, it should be remembered that the q_i represent not stocks of commodities but services rendered by the stocks. In the case of durable goods, for example, the p_{it} are to be interpreted not as the purchase prices but as the one-period rentals for the equipment.

Note also that the p_{it} are to be regarded as being measured in terms of some standard commodity or numéraire whose price per unit is arbitrarily set equal to 1 in period 1 and every period thereafter. We shall often refer to this standard commodity as "money" and hence speak of dollars of income or consumption. It is important to keep in mind, however, that this is again only a stylistic device, and the concept of money in our sense should not be equated with money in its more familiar sense, in which it serves as a medium of exchange and a store of value as well as the unit of account.

preference ordering for time sequences of standards of living. This preference ordering over time sequences of standards of living is summarized in a utility function as $U(c_1, c_2, \dots, c_t, \dots)$.

It is shown in the next (starred) section that this function, stated in terms of total consumptions optimally allocated, satisfies the axioms assumed for the underlying utility function defined in terms of the commodities q_{it} . This implies that the indifference map for such a utility function for a two-period case must have the general properties shown in Figure 1.4. In particular, the nonsatiation axiom implies that any given indifference curve for total consumption must be negatively sloped and that utility must increase as we move upward and to the right onto higher indifference curves. The axiom of transitivity implies that indifference curves cannot cross; the axioms of nonsatiation and convexity together imply convex indifference curves.

***II.A.2. The properties of $U(c_1, c_2, \dots, c_t, \dots)$**

For simplicity, the analysis is carried out for a two-period, two-commodity case. But the method is perfectly general and can readily be extended to the multiperiod, multicommodity case.

If U is a utility function for dollars of consumption and V a utility function for consumption commodities, the relationship between the two functions can be expressed as

$$U(c_1, c_2) = \max_{q_{11}, q_{21}, q_{12}, q_{22}} V(q_{11}, q_{21}, q_{12}, q_{22}) \quad (1.4a)$$

subject to

$$c_1 = p_{11}q_{11} + p_{21}q_{21} \quad \text{and} \quad c_2 = p_{12}q_{12} + p_{22}q_{22}. \quad (1.4b)$$

We first show that if the convexity axiom applies to V , it also applies to U .

Let $(q_{11}^*, q_{21}^*, q_{12}^*, q_{22}^*)$ be the optimal quantities of commodities consumed when the dollar levels of consumption are (c_1, c_2) , and let $(\hat{q}_{11}^*, \hat{q}_{21}^*, \hat{q}_{12}^*, \hat{q}_{22}^*)$ be optimal for (\hat{c}_1, \hat{c}_2) . Assume in addition that

$$V(q_{11}^*, q_{21}^*, q_{12}^*, q_{22}^*) = V(\hat{q}_{11}^*, \hat{q}_{21}^*, \hat{q}_{12}^*, \hat{q}_{22}^*)$$

so that

$$U(c_1, c_2) = U(\hat{c}_1, \hat{c}_2).$$

Thus (c_1, c_2) and (\hat{c}_1, \hat{c}_2) are on the same indifference curve.

For $0 \leq \alpha \leq 1$, let

$$(\hat{c}_1, \hat{c}_2) = (\alpha c_1 + (1 - \alpha)\hat{c}_1, \alpha c_2 + (1 - \alpha)\hat{c}_2)$$

$$(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22}) = (\alpha q_{11}^* + (1 - \alpha)\hat{q}_{11}^*, \dots, \alpha q_{22}^* + (1 - \alpha)\hat{q}_{22}^*),$$

so that

$$\hat{c}_1 = p_{11}\hat{q}_{11} + p_{21}\hat{q}_{21}, \quad \hat{c}_2 = p_{12}\hat{q}_{12} + p_{22}\hat{q}_{22},$$

and $(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22})$ is a feasible consumption pattern for (\hat{c}_1, \hat{c}_2) .

From the axiom of convexity, the utility function for consumption commodities satisfies

$$V(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22}) \geq \alpha V(q_{11}^*, q_{21}^*, q_{12}^*, q_{22}^*) + (1 - \alpha) V(\hat{q}_{11}^*, \hat{q}_{21}^*, \hat{q}_{12}^*, \hat{q}_{22}^*).$$

Or, equivalently,

$$V(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22}) \geq \alpha U(c_1, c_2) + (1 - \alpha) U(\hat{c}_1, \hat{c}_2).$$

Although feasible, the allocation of (\hat{c}_1, \hat{c}_2) implied by $(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22})$ is not necessarily optimal. The decision maker, if given \hat{c}_1 and \hat{c}_2 directly, might choose some other allocation among commodities in preference to that of

$$(\alpha q_{11}^* + (1 - \alpha) \hat{q}_{11}^*, \dots, \dots, \alpha q_{22}^* + (1 - \alpha) \hat{q}_{22}^*).$$

Because the more constrained choice can never be preferable to the less constrained one, we thus must have

$$U(\hat{c}_1, \hat{c}_2) \geq V(\hat{q}_{11}, \hat{q}_{21}, \hat{q}_{12}, \hat{q}_{22}).$$

$$\text{Hence } U(\hat{c}_1, \hat{c}_2) \geq \alpha U(c_1, c_2) + (1 - \alpha) U(\hat{c}_1, \hat{c}_2), \quad 0 \leq \alpha \leq 1,$$

which implies that the convexity axiom also applies to $U(c_1, c_2)$.

To show that the nonsatiation axiom also applies to $U(c_1, c_2)$, note that if the dollars available for consumption in either period are increased, consumption of at least one commodity can be increased without reducing consumption of any other commodity in either period. Thus, the nonsatiation of wants assumed in deriving the commodity utility function V must carry over directly to the utility function U . As in the case of commodity indifference curves, the nonsatiation and convexity axioms together imply convexity of the indifference curves.

It is also clear that the axiom of comparability carries over directly from V to U . Establishing that the preference ordering implied by $U(c_1, c_2)$ satisfies the transitivity axiom is left as an exercise for the reader.

Finally, it is well to note that the function $U(c_1, c_2)$, like the function $V(q_{11}, q_{21}, q_{12}, q_{22})$, provides only a rank ordering of consumption boxes; that is, it tells us only whether one box is preferred to another, so that U and V are ordinal utility functions, as distinct from cardinal functions that would also tell us unambiguously by how much one box is preferred to another.

II.A.3. Opportunities: resources and capital markets

The resources that an individual can draw on for his consumption in any period are of several kinds. Most households, for example, carry over stocks of durable consumer goods from previous periods, so that they may either consume the services of these stocks directly or may rent or sell the goods and consume the proceeds. For simplicity, however, we defer all

consideration of durable goods until Chapter 2, after we have first sketched the main features of the wealth allocation model. For concreteness, at this stage, the reader may perhaps find it helpful to think of the household as renting its housing, automobile, television set, and any other durables from specialized rental firms.

Also generally available to individuals to support their current consumption are, of course, any wages, salaries, or other similar payments that they receive as compensation for current labor services provided. We denote such payments during any period t as y_t , and refer to them, somewhat loosely, as the individual's "income." In addition to his current income, the individual typically can look forward to further income in future years. Future earnings obviously cannot be directly consumed today, but they may still be able to support current consumption, provided that the individual can arrange to transfer them to someone else in exchange for resources to be made available to him immediately.

In what follows, we assume that such exchanges can in fact be made and that they take place in a "capital market." The term *market*, of course, is to be taken, not in the narrow sense of a physical place where buyers and sellers gather, although some real-world capital markets have this property as well, but rather in its broader economic sense of the whole collection of legal, moral, and physical arrangements that make it possible to effect exchanges of current and future incomes.

The precise form that the household's opportunity set takes in the presence of a capital market depends on the additional specifying characteristics that we choose to attribute to the market. An extreme, but particularly fruitful, special set of attributes are those which constitute a "perfect capital market." In such a market we assume the following:

1. All traders have equal and costless access to information about the ruling prices and all other relevant properties of the securities traded.
2. Buyers and sellers, or issuers, of securities take the prices of securities as given; that is, they do, and can justifiably, act as if their activities in the market had no detectable effect on the ruling prices.
3. There are no brokerage fees, transfer taxes, or other transaction costs incurred when securities are bought, sold, or issued.¹⁶

Needless to say, no such market exists in the real world, nor could it. Rather, what we have here is an idealization of the same kind and function as that of a perfect gas or a perfect vacuum in the physical sciences. Such

¹⁶ Although perhaps not strictly an attribute of the market proper, we also assume that there are no income taxes on the earnings from securities; or, if there are, that there are at least no income tax differentials between income in the form of capital gains and dividends or interest.

idealizations permit us to focus more sharply on a limited number of aspects of the problem and usually greatly facilitate both the derivation and statement of the sought-for empirical generalizations. In the nature of the case, however, the generalizations so obtained can never be anything more than approximations to the real phenomena that they are supposed to represent. The question is whether, considered as approximations, they are close enough; and this, of course, is a question that can only be answered empirically and in the light of the specific uses to which the approximations are put.¹⁷

II.A.4. The opportunity set under perfect capital markets

An immediate implication of a perfect market, and one of the main reasons for using this concept, is that at any one time only one price may rule in the market. For if two different prices ruled simultaneously, no one whose preferences obey the nonsatiation axiom would be willing to sell at the lower of the two prices or buy at the higher. Only when a common, single price had been restored could transactions take place.¹⁸

In the capital markets the commodities currently being bought and sold are sums of money to be delivered at various future points in time. We regard the delivery contracts for each such future point as a separate (perfect) market and represent the single, current or spot price at the beginning of the τ th period for delivery of \$1 at the beginning of the t th period as $p_{\tau t}$; that is, $p_{\tau t}$ is the amount of money that must be paid at period τ for \$1 to be obtained, for certain, at t . The advantages of the double subscript notation become clear in later extensions of the analysis.¹⁹

Given these market prices or rates of exchange between future and current resources, what can be said about the form of the opportunity set confronting the decision maker? Let us consider first a two-period case, and ask how much the decision maker can consume in the second of the

¹⁷ To make use again of the analogy from physics, laws of motion derived under a perfect-vacuum assumption may be close enough approximations for many engineering purposes when dealing with heavy objects, but not for some light objects when the neglect of air resistance could lead to a breakdown of the mechanism.

¹⁸ An equivalent, alternative proof often useful in showing the implications of the perfect capital market assumption in more complicated cases involves the notion of "arbitrage" in the sense of a sure profit at no risk or expense; that is, if for ignorance or some other reason there did exist some willing sellers at the lower price and some willing buyers at the higher, knowledgeable arbitrageurs would buy and resell until one or the other group had been driven from the market.

¹⁹ The prices $p_{\tau t}$ are obviously somehow related to interest rates. But to stress the similarity between capital markets and other kinds of markets, it is convenient to present the analysis initially in terms of the $p_{\tau t}$. The formal relationships between these prices and interest rates are shown later.

two periods. Because the second period is the "last" period, no resources can be obtained in period 2 by drawing on future periods. The decision maker's resources would be limited to his income for period 2, y_2 , plus any financial assets carried over from period 1, a_2 , or minus any net liabilities incurred in period 1 that must now be repaid, in which case a_2 would be a negative number; that is, the period 2 consumption would be

$$c_2 = y_2 + a_2. \quad (1.5)$$

Of the two components y_2 is taken for our purposes as a fixed and unalterable amount determined somehow from outside the context of the problem, but a_2 , within limits at least, is under the decision maker's control. The less he consumes in period 1, the more funds he can bring to the capital market to purchase funds deliverable at the start of period 2. In particular, his net worth at period 2 will be

$$a_2 = [(y_1 + a_1) - c_1] \cdot \frac{1}{{}_1p_2}. \quad (1.6)$$

The term in brackets represents the difference between his resources in period 1 and his consumption in this period and is the amount used to purchase resources to be delivered at period 2. The total number of dollars to be received at period 2 is then just the amount invested divided by the price ${}_1p_2$.²⁰ Substituting Equation (1.6) for a_2 in the period 2 constraint (1.5), we obtain

$$c_2 = y_2 + (y_1 + a_1) \frac{1}{{}_1p_2} - c_1 \cdot \frac{1}{{}_1p_2} \quad (1.7)$$

as the relation describing the allowable, efficient combinations of c_1 and c_2 that can be obtained by an individual whose income plus initial wealth consists of $y_1 + a_1$ in the first period and whose income is y_2 in the second.

In graphical terms (see Figure 1.5) Equation (1.7) is a straight line in the c_1c_2 plane with a slope of $-1/{}_1p_2$ and running through the "endowment point" x whose coordinates are $((y_1 + a_1), y_2)$. A movement along this efficiency frontier away from this point to point v would represent a purchase of funds for future delivery, that is, lending, and from x to z a sale of future funds, that is, borrowing. The maximum attainable value of c_2 , that is,

²⁰ Alternatively, if the amount invested at period 1 is $[(y_1 + a_1) - c_1]$ and the price of a dollar to be delivered at period 2 is ${}_1p_2$, the number of period 2 dollars purchased at period 1 must be the value of a_2 such that

$$[(y_1 + a_1) - c_1] = {}_1p_2 a_2,$$

from which we easily obtain Equation (1.6). Thus $1/{}_1p_2$ is the number of dollars at period 2 that can be obtained by investing a dollar at period 1.

24 Certainty Models

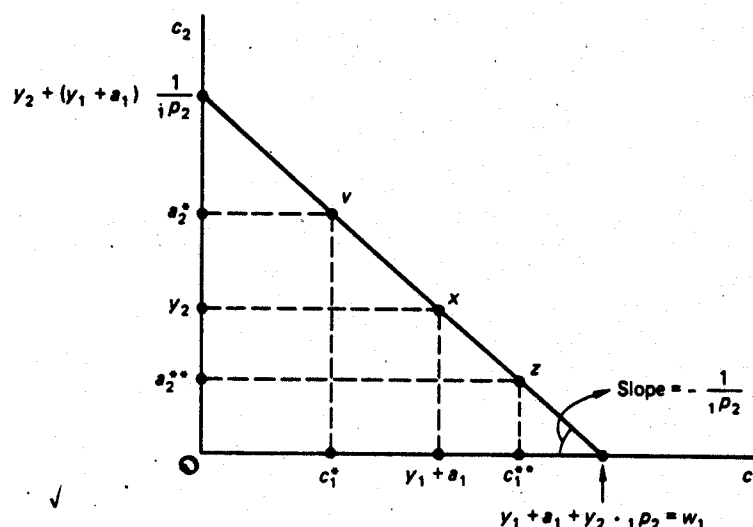


Figure 1.5 Opportunity Set in a Perfect Capital Market

the intercept on the c_2 axis, is

$$y_2 + (y_1 + a_1) \cdot \frac{1}{1p_2},$$

which occurs when all the resources available at period 1 are used to purchase dollars for delivery at period 2. The maximum attainable value of c_1 , that is, the intercept on the c_1 axis,

$$y_1 + a_1 + y_2 \cdot 1p_2,$$

occurs when all income to be received at period 2 is sold at the beginning of period 1. This maximum attainable c_1 can also be interpreted as the consumer's wealth w_1 at period 1; it is the market value of all his current and future resources.

II.A.5. Interest rates and present values

We have chosen to express the opportunities for carrying over resources in terms of current market prices for dollars to be delivered in the future partly to stress the fundamental similarity between capital markets and any of the other markets considered in economics and partly to lay the groundwork for future extensions in which this approach is really the only feasible one. For the present class of problems, however, there is another way of expressing the rate of exchange between current and future sums.

Suppose, for example, that we have the current sum of P dollars and ask what will be the number of dollars A that we shall have at the beginning of the next period if we purchase contracts for the future delivery of dollars

at the current market price of ${}_1p_2$. The answer is

$$A = P \frac{1}{{}_1p_2}. \quad (1.8)$$

But we can always express the terminal amount A as a sum of the initial value P , plus, or minus, the difference between A and P , say, ΔP . Hence we can rewrite Equation (1.8) as

$$\frac{1}{{}_1p_2} = \frac{A}{P} = \frac{P + \Delta P}{P} = 1 + \frac{\Delta P}{P}. \quad (1.9)$$

The term $\Delta P/P$ is the rate of growth of the capital sum invested during the period, to be denoted by ${}_1r_2$ and referred to as the one-period, spot rate of interest or, for short, just "rate of interest." The term $1 + \Delta P/P \equiv 1 + {}_1r_2$ is often referred to in the economics literature as the "force of interest" and in the actuarial literature as the one-period "accumulation factor" at the rate ${}_1r_2$.²¹

Note that in an equation such as (1.8) we can also perform the inverse

²¹ Here and throughout, unless otherwise noted, interest rates, such as ${}_1r_2$, are assumed to be those on securities denominated in the numéraire commodity, which we here call money. Nothing in the analysis, however, rules out the possibility that there might also exist securities denominated in terms of other commodities (or other currencies). If the prices, in terms of numéraire, of such commodities differ in different time periods, the rates of interest on such commodity-denominated securities are not the same as those on numéraire-denominated securities. Nevertheless, we need not take such securities explicitly into account in constructing the opportunity set. As long as the markets for both types of securities are perfect, arbitrage ensures that the effective yield in terms of numéraire is the same on both numéraire- and commodity-denominated contracts; that is, we must have, for any commodity i ,

$$(1 + {}_1r_2^i) \left[1 + \left(\frac{p_{i2} - p_{i1}}{p_{i1}} \right) \right] = (1 + {}_1r_2)$$

or, equivalently,

$${}_1r_2 = {}_1r_2^i + \left(\frac{p_{i2} - p_{i1}}{p_{i1}} \right) + ({}_1r_2^i) \left(\frac{p_{i2} - p_{i1}}{p_{i1}} \right),$$

where p_i is the unit price, in terms of numéraire, of commodity i and ${}_1r_2^i$ is the rate of interest on securities denominated in commodity i . The same expression rearranged as

$${}_1r_2^i = {}_1r_2 - \frac{p_{i2} - p_{i1}}{p_{i1}}$$

is, of course, familiar to economists as the principle that under certainty, the "real" rate of interest is equal to the money rate minus the rate of change of prices. The cross-product term $({}_1r_2^i) [(p_{i2} - p_{i1})/p_{i1}]$ is usually omitted, because in the economics literature time is typically treated in continuous rather than discrete terms.

operation and obtain the relation

$$P = A \cdot {}_1p_2 = A \frac{1}{1 + {}_1r_2}. \quad (1.10)$$

Here we are answering the question, What is the market value today of A dollars delivered next period? The sum P is called the "present value" of the deferred payment A , and the term

$$\frac{1}{1 + {}_1r_2} = {}_1p_2,$$

known as the one-period "present value factor" at the rate of interest ${}_1r_2$, provides an alternative interpretation of the price ${}_1p_2$.

These relations between present and future sums are shown graphically in Figure 1.6. The future sum A to which the present P accumulates at the rate ${}_1r_2$ is indicated by the intercept on the period 2 axis of a line drawn through the point P with slope equal to (minus) the accumulation factor. And, in the other direction, passing the line through a point such as A gives us the present value of the point as the point P , the intercept on the period 1 axis. The same procedure can, of course, also be used to find the present value, or, equivalently, current market value, of any combination of period 1 and 2 sums, such as the point x , whose present value is P' .

We can now easily restate the efficient opportunity set in terms of interest rates merely by substituting $(1 + {}_1r_2)$ for $1/{}_1p_2$ in Equation (1.7). We thus obtain

$$c_2 = y_2 + (y_1 + a_1)(1 + {}_1r_2) - c_1(1 + {}_1r_2) \quad (1.11)$$

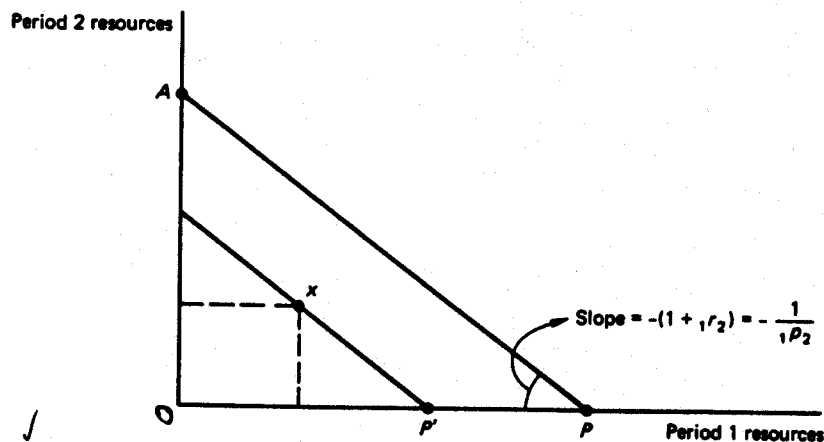


Figure 1.6 Present Value Lines

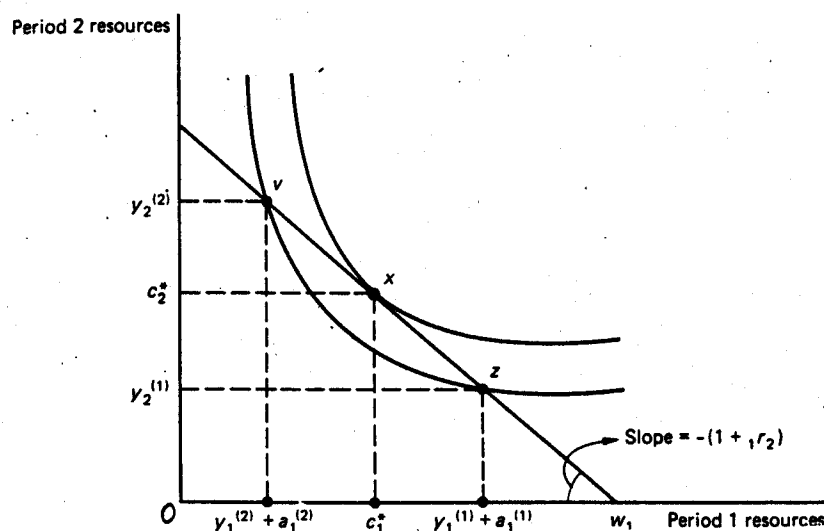


Figure 1.7 Preferred Allocation

or, rearranging,

$$c_1 + \frac{c_2}{(1 + {}_1r_2)} = y_1 + a_1 + \frac{y_2}{(1 + {}_1r_2)}. \quad (1.12)$$

In words, the efficient combinations of standards of living in the two periods, from among which our subject can and will choose, are those for which the present value of total consumption is equal to the present (\equiv market) value of total resources. His actual choice among these possibilities depends on his tastes, as summarized in his utility function and his indifference curves. A geometric illustration of an optimal allocation for a particular set of indifference curves is shown in Figure 1.7.

II.A.6. The preferred allocation

The pattern of standards of living chosen by the decision maker is indicated by the tangency point x , the total consumptions being c_1^* in the first period and c_2^* in the second. If his initial endowment were the point z , with first-period resources of $y_1^{(1)} + a_1^{(1)}$ and second-period income of $y_2^{(1)}$, he would have moved from z to the preferred point x by sacrificing some potential consumption in period 1, lending the proceeds $y_1^{(1)} + a_1^{(1)} - c_1^*$ on the capital markets at the rate ${}_1r_2$, and then adding the matured proceeds plus the interest on them to his second-period income of $y_2^{(1)}$ to reach c_2^* . If he had started from v , where his second-period resources were so much greater than those of the first, he would have moved to the preferred point by borrowing against these future resources to the extent of $c_1^* - (y_1^{(2)} + a_1^{(2)})$. At the start of the next period he would repay his loan

plus interest from his income $y_2^{(2)}$ and consume the balance of $c_2^* = y_2^{(2)} - (c_1^* - (y_1^{(2)} + a_1^{(2)}))(1 + r_2)$.

It is important to note that the decision maker is really indifferent as to whether his initial endowment is the point v or the point z in Figure 1.7. From either starting position, his ultimate consumption combination for the two periods is c_1^* and c_2^* . The initial endowment affects merely the market exchanges that he undertakes to obtain this consumption combination. Indeed, with a perfect capital market, c_1^* and c_2^* are the optimal combination of consumptions for any initial endowment, such as v or x or z , with market value w_1 at period 1. This follows from the fact that in a perfect capital market any endowment with a given market value can be exchanged for any consumption combination with the same market value.

Thus despite, or perhaps because of, the immense simplifications that have gone into its construction, the solution shown in Figure 1.7 does serve to point up one of the crucial functions that the capital markets perform in economic life. In the absence of such markets, the patterns of consumption that individuals could obtain would be rigidly tied to the patterns in which they earned their incomes. When these patterns are very irregular, individuals might find it preferable to shift out of such activities into those offering smoother patterns even at some loss in total earnings for them, and some consequent reduction in the total income potential for society as a whole. Capital markets, however, serve to reduce wastes and inefficiencies of this kind by making it possible, to a considerable extent, to separate the income-earning from the consumption-pattern decisions. Individuals are free to make their earning and occupation decisions largely on other grounds, relying on the capital markets to effect any desired degree of "smoothing" of the standard of living over time.²²

II.B. Extension to Three or More Time Periods

Although the simple two-period model is adequate for conveying the essence of the allocation problem, there are a number of applications for which it is essential to be able to work with more than two periods. An extension of this kind poses no great difficulty insofar as the utility function is concerned. As long as the subject has a finite planning horizon, which may be, and, in principle normally is, as long as his lifetime, we can simply incorporate into the utility function as many additional c_t as required to

²² This stress on consumption smoothing should also help make clear how transactions in capital assets can arise even without "differences of opinion." Although all households are assumed to have identical and entirely correct expectations as to market returns and might equally well have been assumed to have identical preferences, exchanges would still take place as long as the pattern of endowments is different. Further discussion and illustration of consumption smoothing and its economic significance is given in the appendix to the present chapter.

bring us to the horizon period N . The desire to leave an estate, although the transfer might occur after the individual's lifetime, also poses no difficulty and can be treated simply as an additional consumption term,

c_{N+1} .²³

The extension of the opportunity set to more than two periods, however, does pose, if not difficulties, at least some complications. For when we consider many periods, we also open up for the individual many different strategies for transferring funds between periods. In a three-period case, for example, an individual who wants to carry over funds for two periods might buy a two-period claim; or he might buy a one-period claim this period and then reinvest the proceeds next period in another one-period claim. We also have to admit the existence of "compound claims," such as coupon securities paying specified amounts in each of n time periods plus a further lump-sum amount in the n th period. The opportunity set must somehow allow for these and every other transfer possibility.

II.B.1. The structure of prices for claims

To see how this problem can be solved as well as to gain some additional insight into the structure of prices for claims under perfect capital markets, let us first consider a simple case involving only one- and two-period claims. In particular suppose that the current price of \$1 delivered at the start of period 2 is ${}_1p_2$ and that the price at the start of period 2 for \$1 to be delivered at period 3 is known to be ${}_2p_3$; that is, ${}_2p_3$ is next period's price for a one-period claim, a number to be taken as known now under perfect certainty. What can we say about the price now for a claim for \$1 to be delivered at the beginning of period 3, that is, ${}_1p_3$?

The answer is that under perfect capital markets we must have

$${}_1p_3 = {}_1p_2 \cdot {}_2p_3; \quad (1.13)$$

that is, the price of a two-period claim for \$1 must be equal to the product of the prices for one-period claims in the two periods. For suppose that such were not the case and that the price of a two-period claim were greater than this, say,

$${}_1p_3 = {}_1p_2 \cdot {}_2p_3 + \epsilon, \quad \epsilon > 0. \quad (1.14)$$

Then an individual, no matter what his resources, could always issue a two-period claim for ${}_1p_3$, that is, promise to repay \$1 at the end of two periods, and immediately invest the proceeds in a one-period claim. The

²³ In principle we could also handle cases in which the planning horizon is infinitely long. Where such an approach is taken, however—and it is quite commonly found in economic theory, particularly in connection with macro growth models—it is necessary to impose certain additional restrictions about the individual's "time preference" so as to guarantee a finite solution to the maximization of utility. The concept of time preference is discussed and illustrated in the appendix to the present chapter.

value of the one-period claim at the beginning of period 2 would be²⁴

$$\frac{{}_1p_3}{{}_1p_2} = \frac{{}_1p_2 \cdot {}_2p_3 + \epsilon}{{}_1p_2}.$$

When these claims came due at the beginning of period 2, he could again reinvest the proceeds in one-period claims, so that at the beginning of period 3 he would receive

$$\left(\frac{{}_1p_2 \cdot {}_2p_3 + \epsilon}{{}_1p_2} \right) \frac{1}{{}_2p_3} = \left(1 + \frac{\epsilon}{{}_1p_2 \cdot {}_2p_3} \right)$$

dollars. This would be sufficient for him to discharge his debt for \$1 on the two-period claim that he issued and still leave a net gain of $\epsilon/({}_1p_2 \cdot {}_2p_3)$ dollars as a pure arbitrage profit.

Similarly in the other direction, if ${}_1p_3 = {}_1p_2 \cdot {}_2p_3 - \epsilon$, an individual could buy such a two-period claim, which will yield him \$1 at the end of two periods, and finance the purchase by issuing a one-period claim, that is, by promising to pay

$$\frac{{}_1p_3}{{}_1p_2} = \frac{{}_1p_2 \cdot {}_2p_3 - \epsilon}{{}_1p_2}$$

at the beginning of period 2. When this claim came due, he could issue a second claim, that is, promise to repay

$$\left(\frac{{}_1p_2 \cdot {}_2p_3 - \epsilon}{{}_1p_2} \right) \frac{1}{{}_2p_3}$$

at the beginning of period 3. His total liability at period 3 would thus be

$$\left(1 - \frac{\epsilon}{{}_1p_2 \cdot {}_2p_3} \right),$$

which he could repay from the proceeds of \$1 coming due on his two-period claim and still have $\epsilon/({}_1p_2 \cdot {}_2p_3)$ as a pure arbitrage profit.²⁵

²⁴ Recall that when the price is ${}_1p_3$, the market value at period 2 of \$1 invested at period 1 is $1/{}_1p_3$. Similarly, when the price is ${}_2p_3$, the value at period 3 of \$1 invested at period 2 is $1/{}_2p_3$.

²⁵ The ratio of the two "spot prices," ${}_1p_3/{}_1p_2$, is often called the "implicit forward price" for a one-period claim, one period from now. The name comes from the fact that, even without making a direct or literal forward contract, an investor with, say, A_2 of funds available at the start of period 2 can always act now to guarantee himself the opportunity to purchase one-period claims with these funds next period at a price ${}_2p_3^F = {}_1p_3/{}_1p_2$ by borrowing "short" an amount equal to $(A_2 \cdot {}_1p_2)$ and lending it "long," thus yielding $(A_2 \cdot {}_1p_3)/{}_1p_2$. In terms of the implicit forward prices, an equivalent way of stating the equilibrium condition in Equation (1.13) would be ${}_2p_3^F = {}_2p_3$; that is, as the proposition that in perfect capital markets and perfect certainty, the implicit forward rate for any period must equal the spot rate expected to rule for the period.

The same reasoning can be applied to cases of three or four or indeed any number of periods, and starting from any period, so that we can write as the general rule for the price in period τ for an n -period claim for \$1, that is, a claim to be delivered at the beginning of period $\tau + n$,

$${}_{\tau}p_{\tau+n} = {}_{\tau}p_{\tau+1} \cdot {}_{\tau+1}p_{\tau+2} \cdots {}_{\tau+n-1}p_{\tau+n}, \quad (1.15)$$

or, more compactly,

$${}_{\tau}p_{\tau+n} = \prod_{t=\tau}^{\tau+n-1} {}_t p_{t+1}. \quad (1.16)$$

Note also that with Equation (1.15) or (1.16) we can immediately obtain the price of any compound claims, because with no transaction costs, any claim to pay or deliver sums in more than one period can always be duplicated by a set of separate one-period contracts. In particular, the market value $V(\tau)$ at the beginning of period τ of any claim, positive or negative, to $X(\tau + 1)$ dollars at the end of one period, $X(\tau + 2)$ at the end of two periods, and so on, down to $X(\tau + n)$ n periods later, can always be expressed under perfect capital markets as the simple sum of the market values of the component claims:

$$\begin{aligned} V(\tau) &= X(\tau + 1) {}_{\tau}p_{\tau+1} + X(\tau + 2) {}_{\tau}p_{\tau+2} + \cdots + X(\tau + n) {}_{\tau}p_{\tau+n} \\ &= X(\tau + 1) {}_{\tau}p_{\tau+1} + X(\tau + 2) {}_{\tau}p_{\tau+1} \cdot {}_{\tau+1}p_{\tau+2} \\ &\quad + \cdots + X(\tau + n) \prod_{t=\tau}^{\tau+n-1} {}_t p_{t+1}. \end{aligned} \quad (1.17)$$

Or, to put it somewhat more dramatically, in a perfect capital market the market value of any set of claims depends only on the amounts of the claims and the prices of one-period unit claims and not at all on how the claims happen to be "packaged."

II.B.2. An equivalent representation in terms of interest rates

Before applying these results to the problem of constructing the opportunity set, it is helpful once again to restate the essentials in terms of the more familiar interest rate and yield formulations in which discussions in finance are typically conducted. In particular, recall that we have previously defined the one-period rate of interest in terms of the price of one-period claims by the relations

$$1 + {}_{\tau}r_{\tau+1} = \frac{1}{{}_{\tau}p_{\tau+1}} \quad (1.18)$$

$$\text{or} \quad {}_{\tau}p_{\tau+1} = \frac{1}{1 + {}_{\tau}r_{\tau+1}}. \quad (1.19)$$

Substituting from Equation (1.19) into (1.17), we obtain for $V(\tau)$

$$V(\tau) = \frac{X(\tau+1)}{1+r_{\tau+1}} + \frac{X(\tau+2)}{(1+r_{\tau+1})(1+r_{\tau+2})} + \cdots + \frac{X(\tau+n)}{\prod_{i=\tau}^{\tau+n-1} (1+r_{i+1})}, \quad (1.20)$$

which is, of course, the familiar expression for the present value of an arbitrary stream of payments $X(\tau+1)$, $X(\tau+2)$, ..., $X(\tau+n)$.

Certain special cases of Equation (1.20) are frequently encountered in the literature, mainly because they permit simpler and more easily manipulated valuation formulas. If, for example, we assume that the one-period interest rates are constant over time at some given value r , products of accumulation factors over n periods of the form

$$\prod_{i=1}^n (1+r_{i+1})$$

can be expressed as powers of the force of interest of the form $(1+r)^n$ and similarly with the present value factors $1/(1+r)^n$. If in addition to assuming a constant one-period interest rate, we assume that the payments too have some constant value X over time, we obtain the familiar "annuity factors," and their inverses, the "capital recovery factors." In particular, Equation (1.20) reduces to

$$V = \frac{X}{1+r} + \frac{X}{(1+r)^2} + \cdots + \frac{X}{(1+r)^n}. \quad (1.21)$$

Multiplying both sides by $1/(1+r)$ and subtracting the resulting expression from Equation (1.21) yields

$$\begin{aligned} V - \frac{1}{1+r} V &= \frac{X}{1+r} + \frac{X}{(1+r)^2} + \cdots + \frac{X}{(1+r)^n} - \frac{X}{(1+r)^2} \\ &\quad - \frac{X}{(1+r)^3} - \cdots - \frac{X}{(1+r)^{n+1}} = \frac{X}{1+r} - \frac{X}{(1+r)^{n+1}}, \end{aligned} \quad (1.22)$$

from which it follows that

$$V = X \left[\frac{(1+r)^n - 1}{r(1+r)^n} \right]. \quad (1.23)$$

The expression in brackets in Equation (1.23) is the uniform annuity present value factor for n periods at a rate of $100r$ percent per period. It is sometimes also called the n -period "capitalization factor," because it tells

us by how much we have to multiply a given flow of payments to convert them to a stock of capital value, that is, to convert the income statement item to a balance sheet item. The inverse operation

$$X = V \left[\frac{r(1+r)^n}{(1+r)^n - 1} \right] \quad (1.24)$$

converts a capital sum V to an equivalently valued n -period flow of payments X at the rate r , a process familiar to any homeowner with a conventional, equal-payment mortgage. Still another interesting variant on the annuity formula is the case of a perpetual annuity or "perpetuity." In Equation (1.23), if $r > 0$, the term $-1/r(1+r)^n$ approaches zero as n approaches infinity. Hence the capitalization factor in brackets collapses in this case to the very simple and convenient expression²⁶

$$V = \frac{X}{r}. \quad (1.25)$$

Many other interesting and useful variations on these formulas can be developed, and some of these are in fact introduced at appropriate places in subsequent chapters. For present purposes, the important feature of the analysis is not the formulas themselves. Our concern has rather been to show where the formulas come from, and to call attention to the critical role of the concept of a perfect capital market²⁷ in their derivation.

²⁶ For many beginners there is something unnatural about a stream of payments supposed to continue forever. It should be emphasized, therefore, particularly because we make heavy use of such perpetuities in later analyses, that there is nothing particularly strange about such securities and that they do exist in considerable numbers and varieties in real-world capital markets. Land, common stocks, and most issues of preferred stocks are the most obvious examples of securities without any specific maturity date, but there are also some bonds of a similar kind, notably the celebrated "consols" issued by the British government in the nineteenth century, and so called because they were a "consolidation" of a series of previously issued debts. Another homely example would be the "perpetual maintenance" agreements assumed by many sellers of cemetery plots.

²⁷ Some of the reading for which this book is intended to serve as an introduction will be found to run in terms of continuous compounding rather than the discrete compounding that we have chosen to use throughout. Results and formulas developed under one convention for the case of constant interest rates may be easily translated into the other merely by interchanging $(1+r)^t$ with e^{rt} and interchanging integrals with the corresponding summations. Thus, for example, the annuity formula (1.23) can be obtained in a continuous formulation as

$$V = X \int_0^n e^{-rt} dt = \frac{X(e^{rn} - 1)}{re^{rn}}$$

and similarly for the rest of the family of simple formulas. The perpetuity capitalization factor $1/r$ remains the same, of course, for both cases.

11.B.3. Multiperiod rates of interest and the concept of the term structure

We have shown and made use of the fact that a one-to-one correspondence exists between the one-period rate of interest and the price of one-period claims. We can readily push this correspondence further and obtain, say, a two-period rate of interest that would be the similar equivalent to the price of a two-period claim.

Consider first the case of simple claims. In particular, suppose that we have now in period 1 a claim for \$1 payable at the beginning of period 3, whose current price is ${}_1p_3$, and suppose further that the period 1 and period 2 one-period interest rates are ${}_1r_2$ and ${}_2r_3$. Then from the product rule we have

$${}_1p_3 = {}_1p_2 \cdot {}_2p_3 = \frac{1}{(1 + {}_1r_2)} \cdot \frac{1}{(1 + {}_2r_3)} = \frac{1}{1 + {}_1\hat{r}_3},$$

where ${}_1\hat{r}_3$ is the rate over two periods that corresponds to the price ${}_1p_3$. We use the new symbol ${}_1\hat{r}_3$ to emphasize that ${}_1\hat{r}_3$ cannot be directly compared with one-period rates, such as ${}_1r_2$ or ${}_2r_3$, because the period of accumulation for ${}_1\hat{r}_3$ is twice as long. To obtain this comparability, however, we need merely ask what one-period rate, if earned in two successive periods, would leave the investor indifferent to the actual sequence of ${}_1r_2$ followed by ${}_2r_3$. By indifferent we mean, of course, that a present amount P would accumulate to the same amount A under either strategy, or that a future sum A would have the same present value in the two cases. In symbols, if we let ${}_1r_3$ be the uniform rate, we ask what value of ${}_1r_3$ satisfies

$$P(1 + {}_1r_3)(1 + {}_1r_3) = P(1 + {}_1r_3)^2 = P(1 + {}_1r_2)(1 + {}_2r_3) = A,$$

or, equivalently,

$$\frac{A}{(1 + {}_1r_3)(1 + {}_1r_3)} = \frac{A}{(1 + {}_1r_3)^2} = \frac{A}{(1 + {}_1r_2)(1 + {}_2r_3)} = P.$$

Solving either expression and keeping only the positive root, we obtain

$$(1 + {}_1r_3) = \sqrt[2]{(1 + {}_1r_2)(1 + {}_2r_3)}$$

or

$${}_1r_3 = \sqrt[2]{(1 + {}_1r_2)(1 + {}_2r_3)} - 1.$$

More generally, for an n -period simple claim

$${}_1r_{n+1} = \sqrt[n]{(1 + {}_1r_2)(1 + {}_2r_3) \cdots (1 + {}_nr_{n+1})} - 1.$$

In words, the n -period rate of interest that is equivalent to the sequence of n one-period rates is the geometric average of (unity plus) the future one-period rates (minus unity).

The entire set of n -period rates for all values of n in sequence, that is, ${}_1r_2, {}_1r_3, {}_1r_4, \dots, {}_1r_{n+1}$, constitutes what is often called the "term structure of interest rates." Some representations of various possible term structures are shown in Figure 1.8. In structure A, for example, the rates rise through-

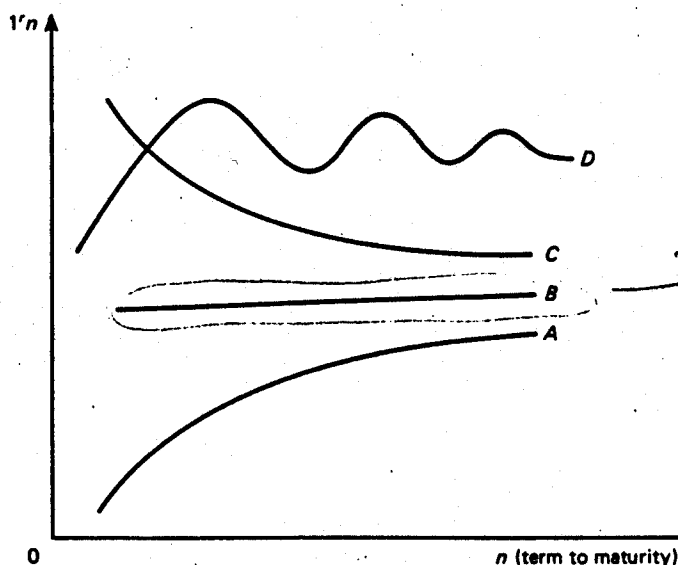


Figure 1.8 Term Structures

out; B is a flat term structure; C falls throughout. Structure D is of an irregularity far more severe than any term structures ever observed, but still entirely conceivable.

In relating the structure of these "long" rates to the one-period "short" rates of which they are averages, a useful concept is that of the "marginal rate," or, as it is sometime called, the implicit forward rate.²⁸ In particular, the marginal or forward rate for period t , r_{t+1}^m , may be defined as the rate at which the capital sum invested in a simple claim grows when invested for one more period, that is, when held through the period t . If we let $V(t+1)$ stand for the value at the start of period $t+1$ of \$1 invested at the start of period 1, and $V(t)$ for the value at the start of t , this marginal rate is clearly $V(t+1)/V(t) - 1 = r_{t+1}^m$. Under perfect capital markets we can replace the V 's by their corresponding accumulation factors and obtain

$$1 + r_{t+1}^m = \frac{V(t+1)}{V(t)} = \frac{(1 + r_{t+1})^t}{(1 + r_1)^{t-1}} = \frac{(1 + r_2)(1 + r_3) \cdots (1 + r_{t+1})}{(1 + r_2)(1 + r_3) \cdots (1 + r_t)} = 1 + r_{t+1};$$

that is, not surprisingly, under perfect certainty and perfect capital markets, the marginal or forward rate for any future period is equal to the (known) future spot rate for the period.

²⁸ Compare footnote 25 above. In much of the literature on the term structure of interest rates, the term *implicit* is usually omitted, for simplicity, in referring to forward rates. We stress it here merely to remind students of finance that true explicit forward contracts also exist and in fact are quite common, especially in connection with the financing of residential construction.

The relation between these marginal rates and the (geometric) average n -period, or long, rates of the term structure is the same as that between any other average and the series marginal to it. In particular when the marginal rate exceeds the average rate, the average rises; and when the marginal is below the average, the average falls. In Figure 1.8, for example, the rising yield curve A implies that future one-period spot rates will be higher than both the current one-period spot rate and current long rates, although not necessarily in any uniform or regular way. For curve C , the opposite is the case, and for the flat curve B , all future spot rates are constant and equal to the current rate. For curve D , the pattern is one of substantial fluctuation in the future short rates sometimes rising above and sometimes falling below the current rates.²⁹

11.3.4. The equal rate of return principle

Up to this point, we have been discussing yields and interest rates mainly in the context of simple claims, that is, claims in which there is only a single cash payment that occurs at the maturity date. There remains only to show the implications of the results obtained for the case of compound claims. In particular, what can we say about the yields on such claims in relation to those of simple claims?

Consider, for example, an n -period compound security that pays $X(t+1)$ in cash at the beginning of period $t+1$, and suppose that the one-period rate of interest for t is r_{t+1} . As before, we can define the one-period yield on the compound security during period t as the rate of growth of wealth initially invested in such a security. If we denote this initial investment by $V(t)$, this rate of growth, r_{t+1} , is

$$r_{t+1} = \frac{X(t+1) + V(t+1) - V(t)}{V(t)}, \quad (1.26)$$

where $V(t+1)$ is the market value of the claim as of the start of period $t+1$. Under the perfect market assumption we know that this one-period yield or rate of return on any compound security during any period must be exactly the same as what we have been calling the one-period rate of interest on a simple claim for the period. In such a market the investor is free to

²⁹ As drawn, curve D is meant to imply that some marginal rates, and hence future spot rates, will actually be negative. If we interpreted the rates as "real" rates in the sense of footnote 21, there would, of course, be nothing whatever anomalous about such negative rates. Their occurrence would simply mean that the rate of price increase of the commodity exceeded the nominal or money rate of interest. Even for contracts denominated in numéraire, however, there would be nothing in the present framework to prevent negative rates if we expressly ruled out the carry-over of any physical commodity, including, presumably the numéraire commodity itself. Once the carry-over of numéraire is allowed, of course, we do have a floor under money rates of interest, but this floor may still be below zero to the extent that there are storage or other costs of carry-over.

hold any of the securities available during a given time period, and to shift costlessly among securities from one period to the next. Thus if all securities are to be held, all must yield the same rate of return during any period. Equivalently, the existence of differential yields would imply the existence of costless arbitrage, which is, of course, inconsistent with a perfect market.

Some further insight into the meaning of this result can be obtained by separating the definitional expression (1.26) into two components as

$${}_t\rho_{t+1} = {}_tr_{t+1} = \frac{X(t+1)}{V(t)} + \frac{V(t+1) - V(t)}{V(t)}. \quad (1.27)$$

The first term $X(t+1)/V(t)$ is often called the "cash yield" and the second the "capital gain yield." Our analysis implies, in effect, that the total yield is independent of the packaging. A security with a high cash yield will have a correspondingly low capital gain yield, or even a capital loss; but whatever the package, the total is the same and is always equal to the market rate of interest for the period.

Because ${}_t\rho_{t+1} = {}_tr_{t+1}$ for every period t , it also follows that we can define an n -period holding yield

$${}_t\rho_{t+n} = \sqrt[n]{(1 + {}_t\rho_{t+1})(1 + {}_{t+1}\rho_{t+2}) \cdots (1 + {}_{t+n-1}\rho_{t+n})} - 1$$

exactly analogous to, and in terms of its value, exactly equal to, the n -period rate of interest ${}_tr_{t+n}$. As it turns out, there is relatively little need for this more general form of the holding period yield in the class of decision problems that we examine in this book. Its main function in the finance literature has been to serve as one basis for the *ex post* evaluation of investments in particular securities—a use in which it is often referred to as the yield "with reinvestment."³⁰

³⁰ To see the origin and justification of this characterization of the n -period yield, suppose that an investor held n_t units of a given compound security at the start of period t . If he reinvests its cash payment of $X(t+1)$ per unit at the then ruling market price of $V(t+1)$ per unit, he will have $n_t(1 + X(t+1)/V(t+1)) = n_{t+1}$ units at the start of period $t+1$. Hence the total value of his holdings of the security at that time will be

$$\begin{aligned} n_{t+1}V(t+1) &= n_t \left(1 + \frac{X(t+1)}{V(t+1)} \right) \cdot V(t+1) \\ &= n_t \left(\frac{V(t+1) + X(t+1)}{V(t+1)} \right) \cdot V(t+1) \\ &= n_t \left(\frac{V(t+1) + X(t+1)}{V(t)} \right) \cdot V(t) = n_t V(t) [1 + {}_t\rho_{t+1}]. \end{aligned}$$

At the start of $t+2$, his wealth will be

$$\begin{aligned} n_{t+2}V(t+2) &= n_{t+1}V(t+1)[1 + {}_{t+1}\rho_{t+2}] \\ &= n_t V(t) [1 + {}_t\rho_{t+1}][1 + {}_{t+1}\rho_{t+2}] = n_t V(t) [1 + {}_t\rho_{t+2}]^2 \end{aligned}$$

and so on.

11.B.5. The n -period opportunity set

With these results about the structure of prices and yields in perfect capital markets in hand, we may turn back now to the task of constructing the opportunity set for a decision maker in the multiperiod case. As before, consider first the total consumption possible for the subject during the last period N . Because we permit no carryover of resources to or from later periods, this maximum must be given by

$$c_N = y_N + a_N \geq 0,$$

that is, by the sum of the income for the period plus the proceeds of any securities accumulated from previous periods. The term y_N is taken as somehow given from outside the problem, but a_N can be further decomposed into

$$a_N = [y_{N-1} + a_{N-1} - c_{N-1}](1 + {}_{N-1}r_N).$$

Note that with our perfect market assumption, this expression for a_N applies regardless of the particular financial strategy that he may happen to adopt; that is, whatever may be the particular securities composing a_{N-1} and a_N , we know that the one-period yield on any resources invested during period $N - 1$ will be precisely ${}_{N-1}r_N$. Furthermore this will be the appropriate rate to apply regardless of whether he is a net borrower for the period—the case of $y_{N-1} + a_{N-1} < c_{N-1}$ and hence $a_N < 0$ ³¹—or a net lender, because with perfect markets the borrowing and lending rates are the same.

By repeated application of the same reasoning we can eliminate all the intermediate values of a_t , obtaining ultimately as the expression for the opportunity set

$$\begin{aligned} c_1 + \frac{c_2}{1 + {}_1r_2} + \frac{c_3}{(1 + {}_1r_2)(1 + {}_2r_3)} + \cdots + \frac{c_N}{\prod_{t=1}^{N-1} (1 + {}_tr_{t+1})} \\ = a_1 + y_1 + \frac{y_2}{(1 + {}_1r_2)} + \frac{y_3}{(1 + {}_1r_2)(1 + {}_2r_3)} + \cdots + \frac{y_N}{\prod_{t=1}^{N-1} (1 + {}_tr_{t+1})}; \end{aligned} \quad (1.28)$$

In addition to the yield with reinvestment, references are sometimes found to a "yield without reinvestment." This yield is the same as that of the so-called "discounted cash flow, internal rate of return," that we consider in Chapter 3. It is also the way in which the so-called "yield to maturity" on a coupon bond is customarily computed and quoted.

³¹ Note incidentally that the requirement that c_N be nonnegative implicitly puts a maximum on the amount that can be borrowed during $N - 1$ and, by extension, in all previous periods. Or, to put it another way, no one is allowed to die in debt.

that is, the efficient combinations of standards of living available to the decision maker under perfect capital markets are only those for which the present (=market) value of the sequence of consumptions is equal to the present value of his income plus any initial endowment a_1 . This set encompasses every financial strategy, security package, or portfolio composition available to the decision maker over the N -period horizon.

***II.B.6. The optimal allocation in the multiperiod case, mathematical treatment**

Putting together the expressions for the utility function and the opportunity set, we may now represent the optimal allocation of financial resources over time as the solution to the problem

$$\max_{c_1, c_2, \dots, c_N} U(c_1, c_2, \dots, c_N)$$

subject to the constraint expressed by Equation (1.28). To obtain the necessary conditions for a maximum, form the lagrangian function

$$L = U(c_1, c_2, \dots, c_N) - \lambda \left(c_1 + \frac{c_2}{1 + r_2} + \dots + \frac{c_N}{\prod_{t=1}^{N-1} (1 + r_{t+1})} - a_1 - y_1 - \frac{y_2}{1 + r_2} - \dots - \frac{y_N}{\prod_{t=1}^{N-1} (1 + r_{t+1})} \right),$$

and differentiate partially with respect to λ and to each of the c_i . Setting these derivatives equal to zero, the first-order conditions for a maximum are then the following N equations, plus Equation (1.28):

$$U'_1 - \lambda = 0$$

$$U'_2 - \lambda \frac{1}{1 + r_2} = 0$$

$$\vdots$$

$$U'_N - \lambda \frac{1}{\prod_{t=1}^{N-1} (1 + r_{t+1})} = 0,$$

where U'_i is the partial derivative of U with respect to c_i . Or, equivalently,

$$U'_1 = U'_2(1 + r_2) = \dots = U'_N \prod_{t=1}^{N-1} (1 + r_{t+1}) = \lambda.$$

Note that between any pair of periods whether adjacent or not we continue to have a simple tangency solution of precisely the same kind as in the two-

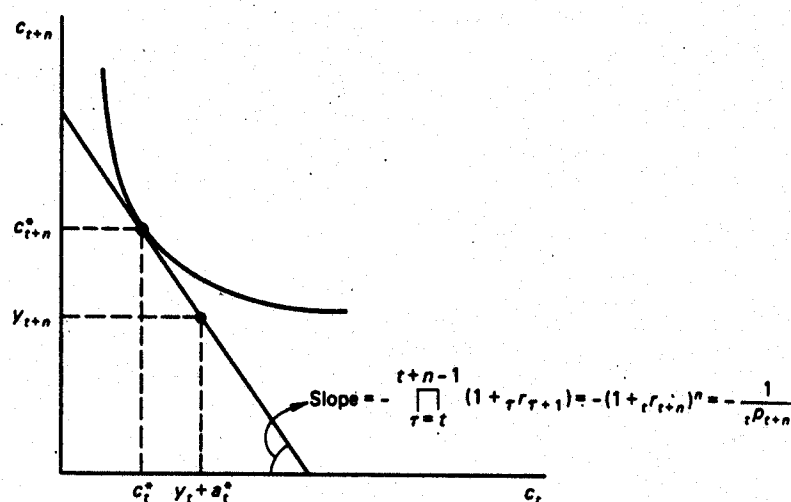


Figure 1.9 Preferred Allocation between Nonadjacent Periods

period case. For example, between periods 1 and 2 we have, after multiplying both sides by -1 ,

$$-\frac{U'_1}{U'_2} = -(1 + r_2);$$

between periods 3 and 7,

$$-\frac{U'_3}{U'_7} = -(1 + r_1)(1 + r_3) \cdots (1 + r_7);$$

and for the general term,

$$-\frac{U'_t}{U'_{t+n}} = -\left(\prod_{\tau=t}^{t+n-1} (1 + r_{\tau+1})\right). \quad (1.29)$$

The term on the left-hand side is the slope of the indifference curve in the $(t, t+n)$ plane, that is, dc_{t+n}/dc_t , and the term on the right represents the slope of the opportunity set, which is, of course, (minus) the force of interest over the indicated n periods. A graphical illustration for this n -period solution is shown in Figure 1.9 with c_t^* and c_{t+n}^* as the optimal total consumption in the two periods when a_t^* is the optimal initial assets for period t .

II.C. Conclusion

We now have in hand most of the essential elements of the model of wealth allocation over time under certainty and perfect capital markets. An illustration of how the abstract model can be specialized and adapted to

help explain important features of observed savings behavior is given in the appendix to the present chapter. The development of the remaining aspects of the model and in particular its extension to allow for durable goods, real investment, and corporations is the task of Chapters 2 and 3.

REFERENCES

Elementary discussions of the economic theory of choice under certainty, such as in Section I, can be found in many intermediate economics texts. An excellent early advanced treatment, still very much worth reading, is that of

1. Hicks, John R., *Value and Capital*, 2d ed. Oxford: Clarendon Press, 1946. ✓

The standard advanced mathematical treatments of the theory are those of

2. Debreu, Gerard, *The Theory of Value*. New York: John Wiley & Sons, Inc., 1959. ✓
3. Samuelson, Paul, *The Foundations of Economic Analysis*. Cambridge, Mass.: Harvard University Press, 1947. ✓

A lucid introduction to the methodology of economics and especially of the positivist position is in

4. Friedman, Milton, "The Methodology of Positive Economics," in *Essays in Positive Economics*. Chicago: The University of Chicago Press, 1956. ✓

The original but still surprisingly up-to-date application of the theory of choice to the allocation of resources over time is that of

5. Fisher, Irving, *The Theory of Interest*. New York: Augustus M. Kelley, Publishers, 1965. Reprinted from the 1930 edition. ✓

A graphical and algebraic exposition of the Fisherian wealth allocation model, touching on many of the same topics covered in Part I, is that of

6. Hirshleifer, Jack, "On the Theory of Optimal Investment Decision," *Journal of Political Economy* (August 1958). ✓

Further extensive exposition, especially of the general equilibrium implications of the model, is given by the same author in his recent treatise,

7. Hirshleifer, Jack, *Investment, Interest and Capital*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1970. ✓

The Fisherian model has provided the micro foundations, along lines illustrated in the Appendix, for much of the modern theory of aggregate consumption and saving by households. Two of the most influential treatments have been those of

8. Modigliani, Franco, and Richard Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in Kenneth Kurihara, Ed., *Post-Keynesian Economics*. New Brunswick, 1954. ✓
9. Friedman, Milton, *A Theory of the Consumption Function*. Princeton, N.J.: Princeton University Press, 1957. ✓

APPENDIX

An Illustrative Application of the Wealth Allocation Model: The Life Cycle of Savings

The specific application of the model of wealth allocation over time that we consider in this appendix is that of savings decisions by individuals. In particular, we try to show how the model, when fleshed out with certain additional specifying assumptions about the structure of tastes and opportunities, can be made to yield propositions about how the level of current savings by an individual varies over his lifetime and how it responds to differences in age, wealth, income, and the rate of interest. Our approach is to start with the simplest possible version and then successively bring in additional complications, pointing up along the way some of the implications of the various models for the field of finance.

I. THE BASIC FRAMEWORK¹

I.A. Smoothing Irregularities in the Income Stream

In Chapter 1, the general model of wealth allocation was used to show, among other things, how an individual could "smooth" his standard of

¹ The classic statement of the "life-cycle" approach that we take here is in F. Modigliani and R. Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in K. Kurihara, ed., *Post-Keynesian Economics*, New Brunswick, 1954.

living over time in the face of an unsmooth income stream, for example, by consuming less than his income in periods when it was abnormally high and carrying over the funds saved to periods when it was abnormally low; or by borrowing, that is, "dissaving" during a period of low income to maintain the standard of living in anticipation of repayment of the loan from the proceeds of higher income receipts in the future. A natural way, therefore, to begin any more detailed study of savings decisions is to seek out the major sources of irregularity in income streams that might call for such smoothing. Under our assumptions of perfect certainty and perfect capital markets—which rule out the need to save for unforeseen contingencies, such as illness or unemployment, or to accumulate liquid assets in advance of purchasing large consumer durables—there are still, typically, two major sources of unevenness in his lifetime income experience that an individual attempts to smooth by his saving plan. First, in most occupations, annual earnings can be expected to rise more or less steadily with age, partly as a reflection of increases in skill and experience and partly because of the continued, projected growth of productivity that steadily raises the level of wages and salaries throughout the economy. Second, most wage and salary earners can confidently look forward to compulsory or voluntary retirement, at which point their nonproperty income ceases entirely.²

I.B. Some Specifications and Simplifications

The major implications for savings behavior of these irregularities in the income stream can be brought out most sharply by further specifying and simplifying the model in Chapter 1. In particular, we assume that:

1. The individual plans to leave no estate to his heirs at the time of his death and, by the same token, will not be the beneficiary of any bequests during his own lifetime. This unrealistic assumption, we hasten to add, is a good deal stronger than is really necessary for our purposes. In principle, the estate motive can readily be incorporated into the analysis simply by adding an additional "consumption" term to the utility function in the year of death. We omit it completely here, however, partly because so little is known about motives for gifts and bequests and partly to emphasize how much can be said about saving behavior even in the absence of estate motive saving.

2. The individual has no "time preference" for either immediate or deferred consumption. More precisely, if A and B are two fixed levels of resources and t and $t + 1$ are any two time periods, then $U(c_t = A,$

² A third source of irregularity, even in a certainty framework, would be changes in family size and hence in per capita income within the family. We do not go into such matters here, however, except in passing and as a qualification to certain results obtained from the single-decision-maker models.

$c_{t+1} = B) = U(c_t = B, c_{t+1} = A)$. In terms of Figure 1.4 or 1.7, this assumption implies that the indifference curves for standards of living are symmetric around a 45° line through the origin. Like the no-bequest assumption, this too is much stronger than is necessary, but it has the virtue of great simplicity, as well as emphasizing that psychological or cultural preferences for deferred gratification are in no sense essential to the explanation of saving behavior.

3. The market rate of interest is zero. This assumption is introduced only temporarily to facilitate the calculation of specific examples. It is dropped after the main features of the simplest models have been fully explored. In the meantime, it may perhaps also serve as a useful reminder that, although interest is indeed the "reward for saving," the absence of any such reward does not imply the absence of any motive for individual saving.

I.C. Consumption, Saving, and Wealth over the Life Cycle

Given these assumptions and assuming further that the individual is entering his t th year in the labor force, has a further life expectancy of $N - t$ years, a further working-life expectancy of $W - t$ years, and will have no wage or salary income during the $N - W$ years of his retirement, we can express his choice problem as

$$\max_{c_t, \dots, c_N} U(c_t, c_{t+1}, \dots, c_N) \quad (\text{A.1})$$

subject to the budget constraint

$$\sum_{r=t}^N c_r = a_t + \sum_{r=t}^W y_r = a_t + (W - t + 1)\bar{y}_t, \quad (\text{A.2})$$

where $\bar{y}_t = \left(\sum_{r=t}^W y_r \right) / (W - t + 1)$ is the average income of the individual over the remainder of the working life as of the start of period t , and a_t is the accumulated value of his financial assets at the start of t .

By virtue of our assumptions of zero time preference and a zero market rate of interest we know that the optimal value of consumption must be the same in every period, that is,

$$c_t^* = c_{t+1}^* = \dots = c_N^*.$$

* Readers for whom this conclusion is not immediately obvious may find it helpful to portray the decision problem graphically for a two-dimensional case. They should quickly see that the assumption of symmetry of the indifference curves implies that the slope of any indifference curve equals -1 at the point where the indifference curve intersects a 45° line through the origin. But -1 is the slope of an opportunity line when the interest rate is zero. Hence, for this rate, the optimal solutions lie along the 45° line, with equal consumptions in both periods.

We may thus rewrite the resource constraint (A.2) as

$$\sum_{r=t}^N c_r^* = [N - t + 1]c_t^* = a_t + [W - t + 1]\bar{y}_t$$

and, by rearranging, obtain

$$c_t^* = \frac{1}{N - t + 1} a_t + \frac{W - t + 1}{N - t + 1} \bar{y}_t \quad (\text{A.3})$$

as an explicit "consumption function" for an individual during the t th year of his participation in the labor force.

The economic meaning of this result can perhaps best be appreciated by following a hypothetical individual through a complete life cycle. For concreteness, suppose that his total work expectancy when he enters the labor force at, say, age 20 is 40 years, after which he anticipates a retirement period of an additional 10 years; that is, at age 20, $t = 1$, $W - t + 1 = 40$, and $N - t + 1 = 50$. Suppose, further, that his average annual earnings when he enters employment are expected to be \$10,000 and that he expects to earn exactly this amount in each of the 40 years of employment. (The assumption of a constant annual income is made at this point solely for the purpose of isolating the pure retirement effect.) Because he starts with no inherited or accumulated wealth, his consumption during his first year of entry, that is, at $t = 1$, as given by Equation (A.3), is

$$c_1^* = \frac{1}{50} 0 + \frac{40 - 0}{50} 10,000 = 8000.$$

Because his income exceeds his consumption, he is saving and thereby accumulating capital, so that his wealth at the start of the following period is \$2000. Hence, his consumption during the next year is

$$c_2^* = \frac{1}{49} 2000 + \frac{40 - 1}{49} 10,000 = \frac{392,000}{49} = 8000.$$

Once again, he consumes less than his income, adding a further increment of \$2000 to his wealth. During each of his years in the labor force thereafter the pattern is repeated. His expected total future income from work falls by one year's earnings of \$10,000 each year; but his accumulated wealth rises by an additional \$2000, and the number of years of remaining life over which he must spread his resources falls by one. These forces just balance to maintain his desired consumption at a level of \$8000.

By the time he retires he will have accumulated a retirement fund of \$80,000 (40 years at \$2000 per year). His consumption during his first year of retirement is then

$$c_{41}^* = \frac{1}{10} 80,000 = 8,000,$$

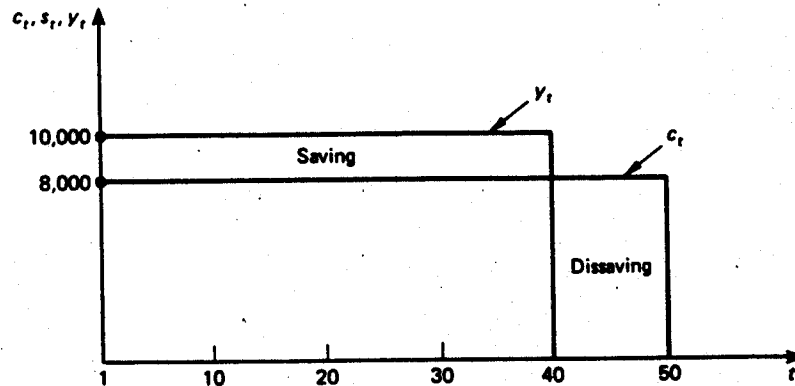


Figure A.1 Life Cycle Profile for the Basic Case

and because his income is zero, this \$8000 of consumption represents net dissaving on his part. The decumulation of capital proceeds steadily thereafter until his life and his resources terminate together. (Needless to say, in the real world, the story does not always have such a neat and happy ending.)

The lifetime profiles of earnings, consumption, saving, and wealth implied by this highly simplified model are shown in Figures A.1 and A.2. Several features of these profiles are worth some comment at this point. Note first the implied difference in "investment objectives" between the young and the old—the former being concerned with the growth of their retirement funds and the latter with their orderly liquidation. Note also the obvious tendency of the process to lead to a substantial concentration in the distribution of wealth. If, for example, there were equal total numbers in each age group and if all entered the labor force at age 20, then despite our extreme egalitarian distribution of wage income among those still

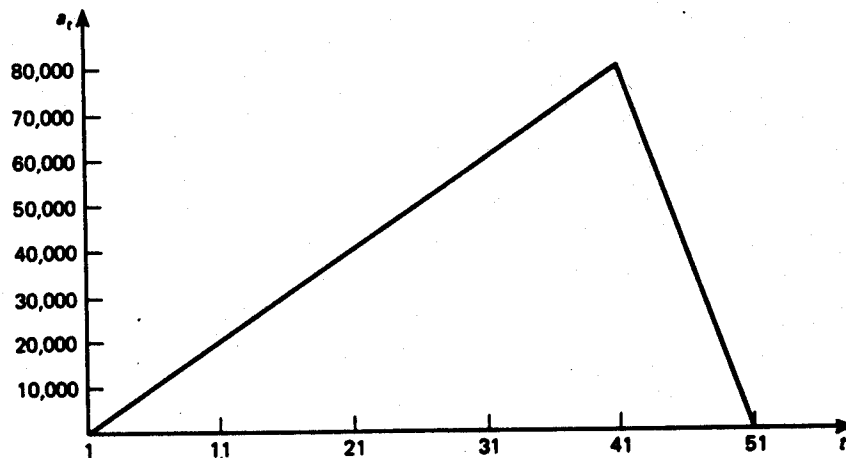


Figure A.2 Asset Profile in the Basic Case

employed, some 55 percent of total wealth would be held by a group amounting to only 40 percent of the total population, that is, the group of relatively elderly persons aged between 50 and 70. Note finally that if the anticipated retirement period is of any sizable length, the volume of wealth necessary to sustain the intergenerational cycle is quite large in relation to total labor income. Assuming again equal numbers in each age group, we find that the ratio of wealth to income is 5 to 1, a figure of about the same order of magnitude as found in modern industrial societies and one that the simple life-cycle model can produce without having to invoke any estate motive for saving.⁴

I.D. Consumption and Saving with a Growing Income

Additional insights into saving behavior are provided when we drop the assumption of a constant annual income during the working years and replace it with one involving the steady rise in income over time anticipated in most occupations. To effect this change, it is convenient to separate current income and average future income in the resources constraint (A.2), rewriting it as

$$\begin{aligned}\sum_{\tau=t}^N c_{\tau}^* &= a_t + \sum_{\tau=t}^W y_{\tau} = a_t + y_t + \sum_{\tau=t+1}^W y_{\tau} \\ &= a_t + y_t + (W - t)\bar{y}_{t+1},\end{aligned}\quad (\text{A.4})$$

where \bar{y}_{t+1} = average annual income from $t + 1$ on. Because the change in assumptions with respect to the income pattern does not affect the optimal consumption pattern, we still have a solution of the form $c_t^* = c_{t+1}^* \cdots = c_N^*$, and repeating the steps in the previous section, we obtain

$$c_t^* = \frac{1}{N - t + 1} a_t + \frac{1}{N - t + 1} y_t + \frac{W - t}{N - t + 1} \bar{y}_{t+1} \quad (\text{A.5})$$

as the explicit consumption function and

$$s_t^* = y_t - c_t^* = \frac{N - t}{N - t + 1} y_t - \frac{1}{N - t + 1} a_t - \frac{W - t}{N - t + 1} \bar{y}_{t+1} \quad (\text{A.6})$$

⁴ Readers interested in comparing these and subsequent profiles with profiles actually compiled from survey data can find examples of the latter in "The 'Permanent Income' and 'Life-Cycle' Hypotheses of Saving Behavior," by F. Modigliani and A. Ando, in *Proceedings of the Conference on Consumption and Saving*, vol. 2, Philadelphia, 1960, and in "The Life Cycle in Income, Saving and Asset Ownership," by H. Lydall, in *Econometrica*, April 1955.

The main qualitative discrepancy between these profiles and ours arises from the fact that the family rather than the individual is really the basic decision unit and the family typically changes in size over the life cycle. Thus even for extreme consumption smoothers, family consumption tends to rise as children enter the family, reach a peak as they reach adolescence, and then decline as they leave home to set up new family units of their own. Hence, in practice, saving and wealth holdings tend to be even more concentrated in the upper age groups than our simple models suggest.

as the explicit savings function in terms of wealth, current income, average future income, and the life-cycle parameters W , N , and t .

To facilitate comparison with the simpler model in the previous section, suppose that N and W are 50 and 40 years, as before, and that the individual's average annual income over his entire working lifetime is still \$10,000. But let his starting income at $t = 1$, that is, at calendar age 20, be only \$5125 with a steady annual increment of \$250 a year thereafter, thus reaching a level of \$14,875 in the last year before retirement, 40 years later. Because the total lifetime level of resources is still \$400,000, the actual amount of consumption is also still \$8000 per year, as in the case considered earlier. The pattern of saving and of wealth holdings, however, is quite different.

During the first year of entry into the labor force, the individual's saving is

$$s_1^* = \frac{49}{50} \cdot 5125 - \frac{1}{50} \cdot 0 - \frac{39}{50} \cdot \frac{394,875}{39} = -2875.$$

He is, in short, actually dissaving initially, and financing this excess of consumption expenditures over current income by borrowing, for example, by tuition loans to attend graduate school. During the next period, his income rises by \$250, his net wealth becomes $-\$2875$, and his saving for the year thus becomes

$$s_2^* = \frac{48}{49} \cdot 5375 - \frac{1}{49} (-2875) - \frac{38}{49} \cdot \frac{389,500}{38} = -2625.$$

It is easy to see that with the numerical values assumed, this dissaving will

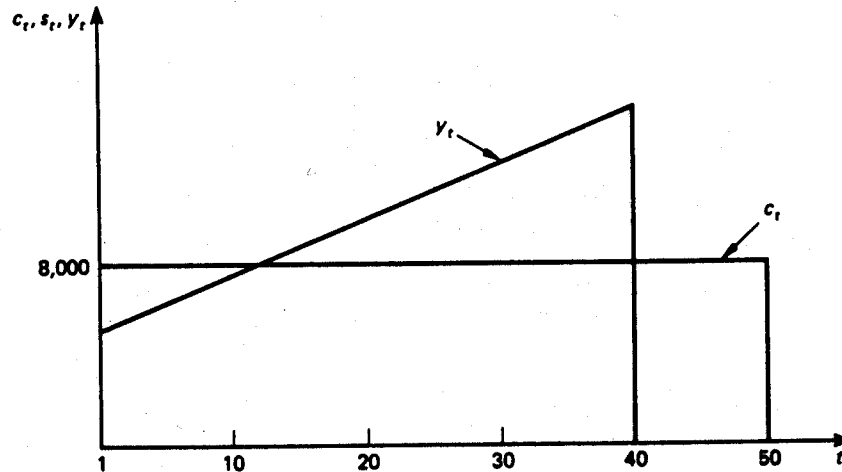


Figure A.3 Savings and Consumption Profiles with Rising Income

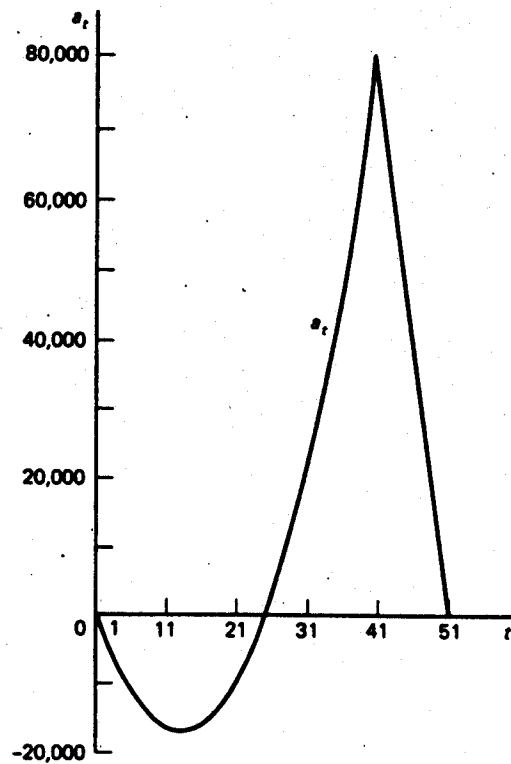


Figure A.4 Asset Profile for the Rising Income Case

continue to diminish steadily by \$250 per year and cease altogether after 12 years. It will take another 13 years until his net wealth reaches zero, that is, until all the debts incurred during the early, low-income years have been repaid. Hence the entire retirement fund of \$80,000 is actually accumulated during the last 16 years of work, with nearly 20 percent of it actually coming during the last 2 years before retirement. The profiles of income, consumption, saving, and wealth for this income growth example are shown graphically in Figures A.3 and A.4.⁵

⁵ Note, incidentally, that by introducing anticipated income growth over the working years, we have in effect provided a rationale for the standard Keynesian consumption function as applied to a cross section of income receivers. By a Keynesian consumption function we mean one of the form $c = a + by$, with $a > 0$ and b , the "marginal propensity to consume," positive, but less than unity. These conditions on a and b imply that c/y , the average propensity to consume, falls with y or, conversely, that s/y rises with y . As can readily be seen from Figure A.3, the average propensity to save out of current income, s^*/y , does indeed rise steadily with income during the working years, although putting the emphasis entirely on income, as is all too often the case in the standard texts, obscures the basic life cycle and hence age-dependent mechanism at work.

I.E. The Effects of Changes in Income and Wealth

The consumption and savings functions in the previous section, with their separate coefficients for current income and expected future income, can also be used to show at least the essential rationale for the distinction between "permanent" and "transitory" changes in income that figures so heavily in recent discussions and controversies over saving behavior.⁶

Consider, for example, an individual somewhere in the working age group, say at age 45, that is, at $t = 26$, for concreteness. Using the previous values of 40 years and 50 years for the work span and life span respectively, his consumption function for the year is

$$\begin{aligned} c_{26}^* &= \frac{1}{50 - 25} a_t + \frac{1}{50 - 25} y_t + \frac{40 - 25 - 1}{50 - 25} \bar{y}_{t+1} \quad (\text{A.7}) \\ &= 0.04a_t + 0.04y_t + 0.56\bar{y}_{t+1}. \end{aligned}$$

Suppose now that the individual received an "unanticipated" raise in salary of \$100 per year.⁷ Then his permanent labor income has risen by \$100 and his current (and future annual) consumption expenditures will rise by $0.04(\$100) + 0.56(\$100) = \$60$. Alternatively and equivalently, of course, we could say that his lifetime level of resources has risen by \$1500 (\$100 in each of the 15 remaining years of employment); and because he spreads his resources equally over his remaining lifetime—in this case, 25 years—his annual consumption rises by $\$1500/25$ or \$60 per year.

Suppose, on the other hand, that the \$100 raise were in the form of a bonus that was regarded as purely transitory, that is, as not expected to be repeated in any subsequent year. Then from Equation (A.7) we see that his current (and future annual) consumption will rise by only \$4. This much lower marginal propensity to consume out of transitory labor income reflects the fact that his lifetime resources have in this case risen by only \$100, with 25 years of life remaining over which consumption is to be spread.

⁶ The classic statement of the permanent income hypothesis is in Milton Friedman, *A Theory of the Consumption Function*. Princeton, N.J.: Princeton University Press, 1957.

⁷ We are, of course, speaking loosely in referring to "unanticipated changes." Given the certainty assumption of the model, we can really speak validly only of cross-sectional differences as between individuals, not of changes for a given individual. In the present context, however, these loose, although expositively convenient, references to changes do no great harm, because in these and similar examples in this appendix our concern is really only with the likely signs and rough relative orders of magnitude of the responses. For insights at this level of generality, the certainty models are "good enough."

II. CONSUMPTION, SAVING, AND THE RATE OF INTEREST⁸

II.A. Some Additional Assumptions: Homogeneity and Its Implications

Having explored some of the basic forces molding consumption and savings patterns over the life cycle, we now go on to enrich the picture further by dropping our simplifying assumption that the rate of interest is zero. As before, our goal is to obtain a simple and transparent consumption function, and the first step in its derivation is again that of introducing additional specifications with respect to the utility function. In particular, we assume that preferences have the following property: If

$$U(c_1^i, c_2^i, \dots, c_N^i) = U(c_1^j, c_2^j, \dots, c_N^j),$$

$$\text{then } U(\lambda c_1^i, \lambda c_2^i, \dots, \lambda c_N^i) = U(\lambda c_1^j, \lambda c_2^j, \dots, \lambda c_N^j);$$

that is, if the decision maker is indifferent when asked to choose between two patterns of lifetime consumption i and j , he would also be indifferent between two patterns that represent a mere proportionate scaling up or down of i and j .

An illustration for a two-period case is provided in Figure A.5. The points x and y are two particular combinations of c_1 and c_2 that have equal utility indexes and hence lie on the same indifference curve. If we draw a straight line, or ray, from the origin through the point x , any point along the ray represents the same pattern of consumptions in the two periods as the point x , although the absolute amounts differ. At the point x' , for example, the total consumption in each period is λ_1 times that of the point x . Our assumption requires that if we now draw the ray through y and find the point $y' = \lambda_1 y$, then y' lies on the same indifference curve as x' . Because what is true for the two points x and y is true for all the points on indifference curve I, our assumption also means that all indifference curves are parallel, radial projections of one another.

The important economic consequences of this property can be seen as soon as we add in the opportunity set. In Figure A.5, for example, the line K_1K_2 represents the opportunities available to an individual, the present value of whose resources is K_1 , at a time when the interest rate is r_2 . Because the indifference curves are radially parallel, we know that the slope of the indifference curve II at the point x' must also equal $-(1 + r_2)$ and hence that the point x' is the chosen combination for this value of the interest rate and a level of initial resources $K'_1 = \lambda_1 K_1$. Thus, under our assumption, all expansion paths of tangency points traced out by varying levels of

⁸ This section is intended mainly for those readers with a special interest in the micro foundations of aggregate savings functions.

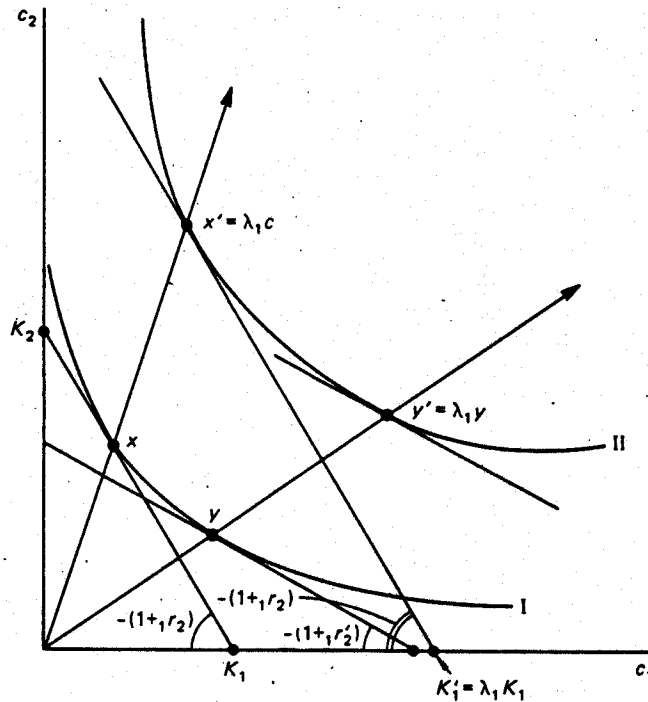


Figure A.5 Homogeneity Postulate

wealth for some given value of r (the analogs of the Engel curves in ordinary consumer demand theory) are straight lines through the origin.

This property, in turn, has some obvious but important implications for the form of the consumption function. The general budget constraint of Equation (1.28) can be written for an individual with a further work expectancy of $W - t$ years and a life expectancy of $N - t$ years as

$$c_t + \sum_{\tau=t+1}^N \frac{c_\tau}{\prod_{\alpha=t+1}^{\tau} (1 + a_{-1}r_\alpha)} = a_t + y_t + \sum_{\tau=t+1}^W \frac{y_\tau}{\prod_{\alpha=t+1}^{\tau} (1 + a_{-1}r_\alpha)} = w_t(r), \quad (\text{A.8})$$

where $w_t(r)$ is the value of lifetime resources as of the start of period t . Because the Engel curves or expansion paths are straight lines through the origin, we have, between consumption in t and any later period $t + \delta$, the proportionality relation

$$c_{t+\delta} = k_{t+\delta}(r)c_t,$$

where the factor of proportionality, which is, of course, the slope of the expansion path, is written as $k_{t+\delta}(r)$ to emphasize its dependence in general on the level of interest rates but independence of the level of wealth.

Substituting into Equation (A.8) and simplifying yields

$$c_t \left[1 + \sum_{\tau=t+1}^N \frac{k_\tau(r)}{\prod_{\alpha=t+1}^{\tau} (1 + a_{-1}r_\alpha)} \right] = w_t(r) \quad (\text{A.9})$$

or, equivalently, in consumption function form

$$c_t = \left[1 + \sum_{\tau=t+1}^N \frac{k_\tau(r)}{\prod_{\alpha=t+1}^{\tau} (1 + a_{-1}r_\alpha)} \right]^{-1} w_t(r) = \gamma_t(r) w_t(r); \quad (\text{A.10})$$

that is, consumption during period t is proportional to lifetime resources at the start of the period—the specific proportion depending on the stage in the life cycle and the level of interest rates.

This property of proportionality or linear homogeneity of the consumption function is one that we have already exploited in the previous examples in Section I, although we arrived at it by a somewhat different route.⁹ It is also a property that in one form or another has come to play a central role in many recent theoretical and empirical studies of savings behavior.

II.B. Interest Rates and Consumption Decisions over the Life Cycle

To bring out more sharply some of the main interactions between interest rates and consumption decisions over the life cycle implicit in a consumption function such as (A.10), it is helpful to specialize the utility function still further. In particular, we assume that the utility function can be approximated by a simple, equally weighted product of the standard of living in each year,

$$U(c_1, c_2, \dots, c_N) = c_1 \cdot c_2 \cdot \dots \cdot c_N. \quad (\text{A.11})$$

Given such a function, which satisfies our assumption of linear, homogeneous expansion paths and which also preserves the symmetry or zero time preference of the earlier examples, and given, for further simplicity, that the rate of interest is the same in all periods, so that the lifetime resources constraint becomes

$$c_t + \sum_{\tau=t+1}^N \frac{c_\tau}{(1+r)^{\tau-t}} = a_t + y_t + \sum_{\tau=t+1}^N \frac{y_\tau}{(1+r)^{\tau-t}} \equiv w_t(r), \quad (\text{A.12})$$

⁹ Our assumption of zero time preference implies in effect that the Engel curve corresponding to a zero rate of interest is a straight line through the origin. And because zero was the value that we specifically assumed for the rate of interest, we got the proportionality properties over the particular range we were considering.

it can readily be shown that the relation between the optimal value of consumption in year t and that in any subsequent period τ is

$$c_{\tau}^* = c_t^* (1 + r)^{\tau-t}; \quad (\text{A.13})$$

that is, for this particular form of the utility function and constraint set

$$k_s(r) = (1 + r)^s.^{10}$$

Substituting once again for the c_{τ} on the left-hand side of Equation (A.9), the terms $(1 + r)^s$ cancel, and we obtain

$$c_t^* + \sum_{\tau=t+1}^N \frac{c_{\tau}^* (1 + r)^{\tau-t}}{(1 + r)^{\tau-t}} = c_t^* (N - t + 1) = w_t(r) \quad (\text{A.14})$$

or in consumption function form

$$c_t^* = \left[\frac{1}{N - t + 1} \right] w_t(r). \quad (\text{A.15})$$

Thus, for the special utility function that we have chosen, the proportionality factor $\gamma_t(r)$ can be stated in terms of the life-cycle parameters N and t as simply $1/[N - t + 1]$, exactly the same factor as in our earlier examples.¹¹

As for the income and asset components of $w_t(r)$, assume for simplicity, as well as to provide a direct basis of comparison with earlier examples, that the rate of labor earnings is a constant amount y during each year of employment. If so,

$$w_t(r) = a_t + y_t + \sum_{\tau=t+1}^W \frac{\bar{y}_{\tau+1}}{(1 + r)^{\tau-t}} = a_t + y_t + A(r; W - t) \bar{y}_{t+1}, \quad (\text{A.16})$$

¹⁰ We leave the proof as an exercise for any student who may feel in need of some further drill.

¹¹ That $\gamma_t(r)$ is independent of r is a consequence of the particular utility function used—Equation (A.11)—and does not rest on our further simplifying assumption of constant interest rates. Equation (A.15) would continue to hold in exactly that form even if interest rates differed from period to period.

Note also that Equation (A.15) is essentially the Friedman form of the consumption function. For empirical testing, however, and especially for facilitating comparison with the Keynesian form, he prefers to state it not in terms of the stock variable $w_t(r)$ but in terms of a flow variable that he dubbed permanent income (see Section I.E.). In principle, at least, the conversion is simple: let $F(r; N - t + 1)$ = the capital recovery factor for $N - t + 1$ years at the rate r —see Equation (1.24)—and let y_t^P = permanent income at the start of t = $w_t(r) \cdot F(r; N - t + 1)$; and let $\alpha_t' = (1/N - t + 1)/F(r; N - t + 1)$. Then the equivalent permanent income form of Equation (A.15) would be $c_t^* = \alpha_t' y_t^P$.

where the term $A(r; W - t)$ denotes the present value of an annuity of \$1 for $W - t$ years at the rate r —see Equation (1.23)—and \bar{y}_{t+1} is the uniform, and hence also average, labor income from $t + 1$ on. Combining Equations (A.15) and (A.16), we thus have as our explicit consumption function in terms of the life-cycle parameters N , W , and t , labor income, financial assets, and the rate of interest

$$c_t^* = \frac{1}{N - t + 1} a_t + \frac{1}{N - t + 1} y_t + \frac{A(r; W - t)}{N - t + 1} \bar{y}_{t+1}. \quad (\text{A.17})$$

Note that for the special case of $r = 0$, Equation (A.17) reduces exactly to our earlier consumption functions, such as (A.5), because $A(0; W - t) = W - t$. For $r > 0$, of course, $A(r; W - t) < W - t$.

It is useful to have variants of Equation (A.17) running in terms of total current income, that is, labor income plus interest income, rather than of labor income alone. In particular, if we define w'_{t-1} as the value of any assets carried over before the addition of the interest earned, so that $w'_{t-1}(1 + r) = a_t$, then we can rewrite Equation (A.17) as

$$c_t^* = \frac{1}{N - t + 1} w'_{t-1} + \frac{1}{N - t + 1} [rw'_{t-1} + y_t] + \frac{A(r; W - t)}{N - t + 1} \bar{y}_{t+1} \quad (\text{A.18})$$

and the corresponding saving function as

$$\begin{aligned} s_t^* &= y_t + rw'_{t-1} - c_t^* = \left(r - \frac{1 + r}{N - t + 1} \right) w'_{t-1} + \frac{N - t}{N - t + 1} y_t \\ &\quad - \frac{A(r; W - t)}{N - t + 1} \bar{y}_{t+1} = - \frac{1}{N - t + 1} w'_{t-1} \\ &\quad + \left[\frac{N - t}{N - t + 1} \right] (y_t + rw'_{t-1}) - \frac{A(r; W - t)}{N - t + 1} \bar{y}_{t+1}. \quad (\text{A.19}) \end{aligned}$$

For purposes of comparison with the results in Section I, suppose again that N and W are 50 and 40 years, respectively, that y_t is \$10,000 in each year of the working life, and that the annual interest rate is 4 percent. Then, during the first year in which the individual enters the labor force, his consumption is

$$c_1^* = \frac{1}{50} 0 + \frac{1}{50} 10,000 + \frac{19.58}{50} 10,000 = \frac{20.58}{50} 10,000 = 4117,$$

and his saving for the year is

$$s_1^* = \left(0.04 - \frac{1.04}{50} \right) 0 + \frac{49}{50} 10,000 - \frac{19.58}{50} 10,000 = \frac{29.42}{50} 10,000 = 5883.$$

The corresponding figures for the zero interest rate model, it will be recalled, were \$8000 for consumption and \$2000 for saving. What has happened, of course, is that the prospect of a 4 percent return on savings has led him to reduce his immediate consumption substantially—the loss in utility due to this reduction being compensated for by the higher level of consumption and utility that this abstinence makes possible in subsequent years. During the next year, for example, his income will rise by the \$235 of interest earned on his fund of \$5883, and his consumption will increase to

$$c_2^* = \frac{(1.04)}{49} 5883 + \frac{1}{49} 10,000 + \frac{19.37}{49} 10,000 = 4281;$$

and his saving to

$$s_2^* = \left(0.04 - \frac{1.04}{49}\right) 5883 + \frac{48}{49} 10,000 - \frac{19.37}{49} 10,000 = 5984.$$

By year $t = 18$, his consumption will have regained the \$8000 level, and in the first year of retirement at $t = 41$ it will have increased to slightly more than \$19,000 for the year. The sum total of his consumption expenditures over his entire lifetime will be more than a million dollars, as compared with only \$400,000 in the no-interest case, the entire difference representing

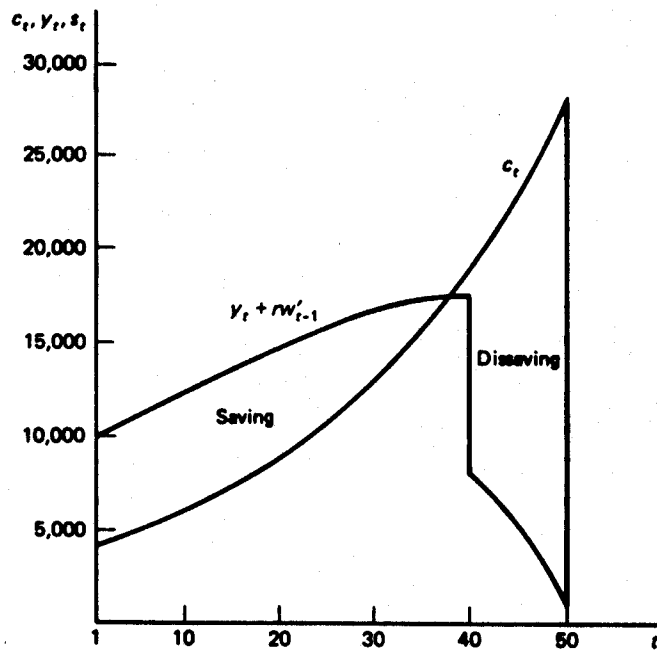


Figure A.6 Savings and Consumption Profiles with a Nonzero Interest Rate

the investment earnings on his accumulated savings. Such is the power of compound interest!

The profiles of earnings, consumption, saving, and wealth are presented graphically in Figures A.6 and A.7.

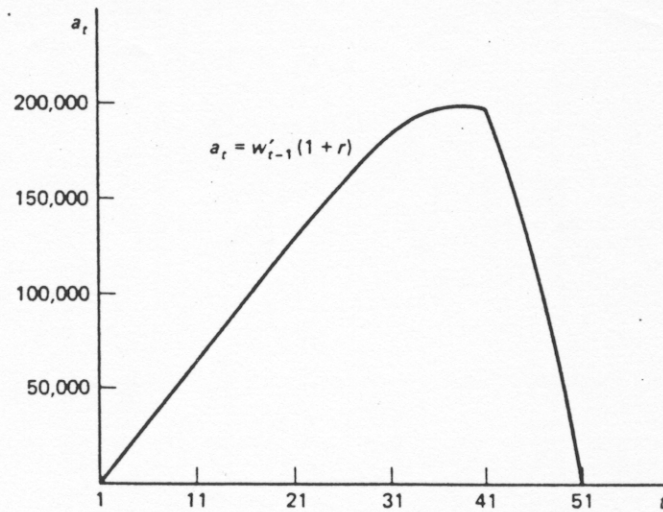


Figure A.7 Asset Profile for the Case of a Nonzero Interest Rate