SOLVING FOR THE EFFICIENT FRONTIER
Recall $T$ is point on the efficient frontier in most counter-clockwise directions. Equation of line Tangent at $T$ is:

$$\bar{R}_i = R_F + \frac{\bar{R}_T - R_F}{\sigma_T} \sigma_i$$

Moving in most counter-clockwise directions is equivalent to maximizing slope. Thus, the objective is to find portfolio where:

$$\theta = \frac{\bar{R}_T - R_F}{\sigma_T}$$

is maximum.
\[ \Theta = \frac{\sum_{i} X \bar{R}_i - \lambda R_i F}{\left[ \sum_{i} \sigma_i^2 + \sum_{i,j} X_i X_j \sigma_{ij} \right]^{1/2}} \]

Subject to \( \sum_{i} X_i = 1 \)

\[ \Theta = \frac{\sum_{i} X \bar{R}_i - \left[ \sum_{i} X_i \right] R_i F}{\left[ \sum_{i} \sigma_i^2 + \sum_{i,j} X_i X_j \sigma_{ij} \right]^{1/2}} \]

\[ = \frac{\sum_{i} X_i \left( \bar{R}_i - R_i F \right)}{\left[ \sum_{i} \sigma_i^2 + \sum_{i,j} X_i X_j \sigma_{ij} \right]^{1/2}} \]
\( \Theta \) can be written as:

\[
\Theta = \left( \begin{array}{cc}
- \bar{R} & - \bar{R} \\
- \bar{R} & - \bar{R} \\
\end{array} \right) \left( \sigma^2 \right)^{-1/2}
\]

recall how to take a derivative using the chain rule namely:

\[
d\Theta = f(X, Y) = XdY + YdX
\]

and that

\[
dX^N = NX^{N-1}
\]
thus

\[
\frac{d\Theta}{dX_k} = (R_T - R_F)^{-1/2} \left( \sigma^2 \right)^{-3/2} \frac{d\sigma^2}{dX_k} + \frac{d[\bar{R}_T - R_F]}{dX_k} \left( \sigma^2 \right)^{-1/2}
\]

or

\[
\frac{d\Theta}{dX_k} = (R_T - R_F)^{-1/2} \left( \sigma^2 \right)^{-3/2} \left[ 2 \sigma_k^2 + 2 \sum_{k \neq i} X_{k} \sigma_{ik} \right] + (R_T - R_F) \left( \sigma^2 \right)^{-1/2}
\]
\[ \frac{d\Theta}{dX_k} = -\frac{\overline{R}_T - R_F}{\sigma^2} \left[ X_k \sigma_k^2 + \sum_{k \neq i} X_k \sigma_{ik} \right] + \left( \overline{R}_k - R_F \right) \]

\[ \left( \overline{R}_k - R_F \right) = \lambda \left[ X_k \sigma_k^2 + \sum_{k \neq i} X_k \sigma_{ik} \right] \]

where:

\[ \lambda = \frac{\overline{R}_T - R_F}{\sigma^2} \]
First order conditions

\[ \bar{R}_k - R_F = \lambda \left[ X_1 \sigma_{1k} + X_2 \sigma_{2k} + \ldots + X_k \sigma_k^2 + \ldots + X_N \sigma_{Nk} \right] \]

for all \( i \)

where \( \lambda = \frac{\bar{R}_T - R_F}{\sigma_T^2} \)

define \( Z_i = \lambda X_i \) then

\[ \bar{R}_k - R_F = Z_1 \sigma_{ik} + Z_2 \sigma_{2k} + \ldots + Z_k \sigma_k^2 + \ldots + Z_N \sigma_{Nk} \]

all \( k \)
Example 3 securities:

\[ \overline{R}_1 - R_F = Z_1 \sigma_1^2 + Z_2 \sigma_{21} + Z_3 \sigma_{31} \]

\[ \overline{R}_2 - R_F = Z_1 \sigma_{12} + Z_2 \sigma_2^2 + Z_3 \sigma_{32} \]

\[ \overline{R}_3 - R_F = Z_1 \sigma_{13} + Z_2 \sigma_{23}^2 + Z_3 \sigma_3^2 \]

X's are \[ X_i = \frac{Z_i}{\sum_j Z_j} \]
\[ \bar{R} = \begin{pmatrix} \bar{R}_1 \\ \bar{R}_2 \\ \bar{R}_3 \end{pmatrix} \]

\[ \Sigma = \begin{pmatrix} \sigma^2 & \sigma & \sigma \\ \sigma & \sigma^2 & \sigma \\ \sigma & \sigma & \sigma^2 \end{pmatrix} \]

\[ 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

\[ Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \]

\[ \bar{R} - R_{F1} = \Sigma Z \]

and

\[ Z = \Sigma^{-1} \bar{R} + R_{F} \Sigma^{-1} 1 \]
Typical Element

\[ Z_i = C_0 + C_1 R_F \]

Thus
Short Sales Not Allowed

Minimize \( \sigma^2 \)

Subject to:

1. \( \sum X_i = 1 \)

2. \( \sum X_i \bar{R}_i = \bar{R}_p \)

3. \( \sum X_i \geq 0 \)

Quadratic Programming Problem

Since have squared and cross product terms.
Between corner portfolio the portfolio consists of linear combination of corner. At corner securities enter or leave portfolio.