SIMPLIFY INPUTS --

SINGLE INDEX CASE

September 2000

A little later we will look at the solution after we have put some more structure on the problem. This will give us greater insight.

inputs		N = 10	N = 100	N = 200
$\bar{R}_i$	N	10	100	200
s_i	N	10	100	200
r_{ik} or s_{ik}	$\frac{N(N-1)}{2}$	$\frac{90}{2}$	$\frac{9{,}900}{2}$	$\frac{39{,}800}{2}$

Analysts may be able to estimate $\bar{R}_i$ and s_i, but covariance comes from models. Thus, index models were developed to estimate covariances. Many more uses have been developed for them and they have become very important.

Index Models

A. **Single index = splits return into unique and systematic.**

Unique Part

$$R_{it} = a_i + b_i R_{mt} + e_{it}$$

Systematic Part

Where

(1) $E(e_i) = 0$

(2) $E(e_i e_j) = 0$ **all i and j**

(3) $E(R_m e_i) = 0$

(4) a_i and b_i are constants

1. Expected Value

$$\overline{R}_i = E(R_i) = E(a_i + b_i R_{mt} + e_{it})$$

$$= E(a_i) + E(b_i R_{mt}) + E(e_{it})$$

$$= a_i + b_i \overline{R}_m$$

2. Variance

$$s_i^2 = \mathrm{E}\left(\mathrm{R}_{it} - \overline{\mathrm{R}}_i\right) = \mathrm{E}\left[\left(a_i + b_i \mathrm{R}_{mt} + e_{it}\right) - \left(a_i + b_i \overline{\mathrm{R}}_m\right)\right]^2$$

$$= \mathrm{E}\left[b_i\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right) + e_{it}\right]^2$$

$$s_i^2 = \mathrm{E}\left[b_i^2\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right)^2 + 2b_i e_{it}\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right) + e_{it}^2\right]$$

$$= b_i^2 \mathrm{E}\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right)^2 + \mathrm{E}\left(e_i\right)^2$$

3. Covariance

$$s_{ij} = \mathrm{E}\left[\left(\mathrm{R}_{it} - \overline{\mathrm{R}}_{i}\right)\left(\mathrm{R}_{jt} - \overline{\mathrm{R}}_{j}\right)\right]$$

$$= \mathrm{E}\left[\left(a_{i} + b_{i}\mathrm{R}_{mt} + e_{it}\right) - \left(a_{i} + b_{i}\overline{\mathrm{R}}_{mt}\right)\right]$$

$$\left[\left(a_{j} + b_{j}\mathrm{R}_{mt} + e_{jt}\right) - \left(a_{j} + b_{j}\overline{\mathrm{R}}_{mt}\right)\right]$$

$$= \mathrm{E}\left[\left(b_{i}\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_{m}\right) + e_{it}\right)\left(b_{j}\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_{m}\right) + e_{jt}\right)\right]$$

$$= b_i b_j \mathrm{E}\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right)^2 + b_j \mathrm{E}\left[\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right) e_{it}\right]$$

$$+ b_i \mathrm{E}\left[\left(\mathrm{R}_{mt} - \overline{\mathrm{R}}_m\right) e_{jt}\right] + \mathrm{E}\left(e_i e_j\right)$$

$$= b_i b_j s^2_{\mathrm{R}_m} + \mathrm{E}\left(e_i e_j\right)$$

Observe

Date	Return GM	Return Market
Dec	11	5
Nov	3	4
Oct	7	7
Sept	0	-2
Aug	6	4
July	9	6
	36	24

Good Fairy says Beta = 1

Return Gm	(constant) + a_i	Beta (return market) $b_i R_{mt}$ +	e_{it}
11	2	5	4
3	2	4	-3
7	2	7	-2
0	2	-2	0
6	2	4	0
9	2	6	+1
36	12	24	0

Did the Good Fairy <u>lie</u>???

Assume $b = 1.5$

Return GM	a_i	$b_i R_m$	$+e_i$
11	0	7.5	3.5
3	0	6	-3
7	0	10.5	-3.5
0	0	-3	3
6	0	6	0
9	0	9	0
36	0	36	0

Covariance with market

$$\underline{b = 1}$$

1 * 4 = 4

0 * -3 = 0

3 * -2 = -6

-6 * 0 = 0

0 * 0 = 0

2 * 1 = 2

0

$$\underline{b = 1.5}$$

1 * 3.5 = 3.5

0 * -3 = 0

3 * -3.5 = -10.5

-6 * 3 = -18

0 * 0 = 0

2 * 0 = 0

-25

$$b_i = \frac{\operatorname{cov}\left(R_i R_m\right)}{\operatorname{Var}\left(R_m\right)}$$

<u>Note</u>:

(1). Mean return unchanged

$$\bar{R}_i = a_i + b_i \bar{R}_m$$

$$6 = 2 + 1 \cdot 4$$

(2). Standard deviation unchanged

$$s_i^2 = b_i^2 s_m^2 + s_{e_i}^2$$

$$13\tfrac{1}{3} = 1^2 \cdot 8\tfrac{1}{3} + 5$$

(3). Only change is covariance

$$E\left(e_i e_j\right) = 0$$

True <u>only</u> by assumption.

Economic content is only reason securities move together is common response to market movements.

Consider Ford

Month	Return	a_i	$b_i R_m$	e_i
Dec	9.5	2	7.5	0
Nov	7.5	2	6	-.5
Oct	9.5	2	10.5	-3
Sept	-2	2	-3	-1
Aug	11	2	6	+3
July	12.5	2	9	1.5
	48	12	36	0

$$\frac{\text{cov}(R_i R_j)}{}$$

5 * 1.5 = 7.5

-3 * -.5 = +1.5

1 * 1.5 = 1.5

-6 * -10 = 60

0 * 3 = 0

3 * 4.5 = 13.5

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$$\frac{E(e_i e_j)}{}$$

4 * 0 = 0

-3 * -.5 = +1.5

-2 * -3 = 6

0 * -1 = 0

0 * 3 = 0

1 * 1.5 = 1.5

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$$\text{cov}(R_i R_j) = b_i b_j s_m^2 + E(e_i e_j)$$

$$= 1 \cdot (1.5)(8\,{}^1\!/\!_3) + 1.5$$

$$14 = 12\,{}^1\!/\!_2 + 1.5$$

Adjusting Beta

			observe		
True	.6	.8	1.0	1.2	1.4
1.2			.2	.6	.2
1.0		.2	.6	.2	
.8	.2	.6	.2		

Extremely high betas likely upper
Tail of a true lower beta thus

(1) Improve if adjust to mean

(2) Improve if use information about company

Some Adjustment Techniques

I. Blumes

Companies	1985-1990	1990-1995
1	b_{11}	b_{21}
2	b_{12}	b_{22}
3	b_{13}	b_{23}

Get adjustment from first to second period

$$b_{i2} = a + b b_{i1} + e_i$$

This shows normal adjustment to forecast for 1995-2000.

$$b_{adj} = \hat{a} + \hat{b} b_{i2}$$

II. Vasichek

$$b_{adj} = \frac{s^{2}_{b_{i1}}}{s^{2}_{b_{1}} + s^{2}_{b_{i1}}} \bar{b}_{1} \frac{s^{2}_{\bar{b}_{1}}}{s^{2}_{\bar{b}_{1}} + s^{2}_{b_{i1}}} b_{i1}$$

Puts more weight on mean if imperfect estimates of Beta.

III. Fundamentals

$b_{adj} = b_{hist}$ **+ fundamentals**

- **a. Dividend Payment**
- **b. Growth**

+ firm variables