PORTFOLIO THEORY WITH MULTI-INDEX MODELS

Fall 2000

For illustrative purposes, consider a two-index model. The results clearly generalize

Assume the following return generating process:

$$R_{it} = a_i + b_{i1} R_{mt} + b_{i2} I_{It} + e_i$$

Where:

(1) R_{mt} **is return on market.**

(2) I_I **is an inflation index.**

Assume investor in question cares about inflation and that the market cares about inflation.

<u>Implications</u>:

(1) **Sensitivity to inflation should be priced and we should observe an APT model like:**

$$\bar{R}_i = R_{Ft} + b_{i1} l_m + b_{i2} l_I$$

(2) **Investors should be willing to trade off inflation hedging for expected return.**

3.1 Multiple Factors: An N-Fund Theorem

Figure 2 shows how the simple two fund theorem of Figure 1 changes if there are multiple sources of priced risk. (This section is a graphical version of Fama's 1996 analysis. Most of the theory was first worked out by Merton 1969, 1971a, 1971b.)

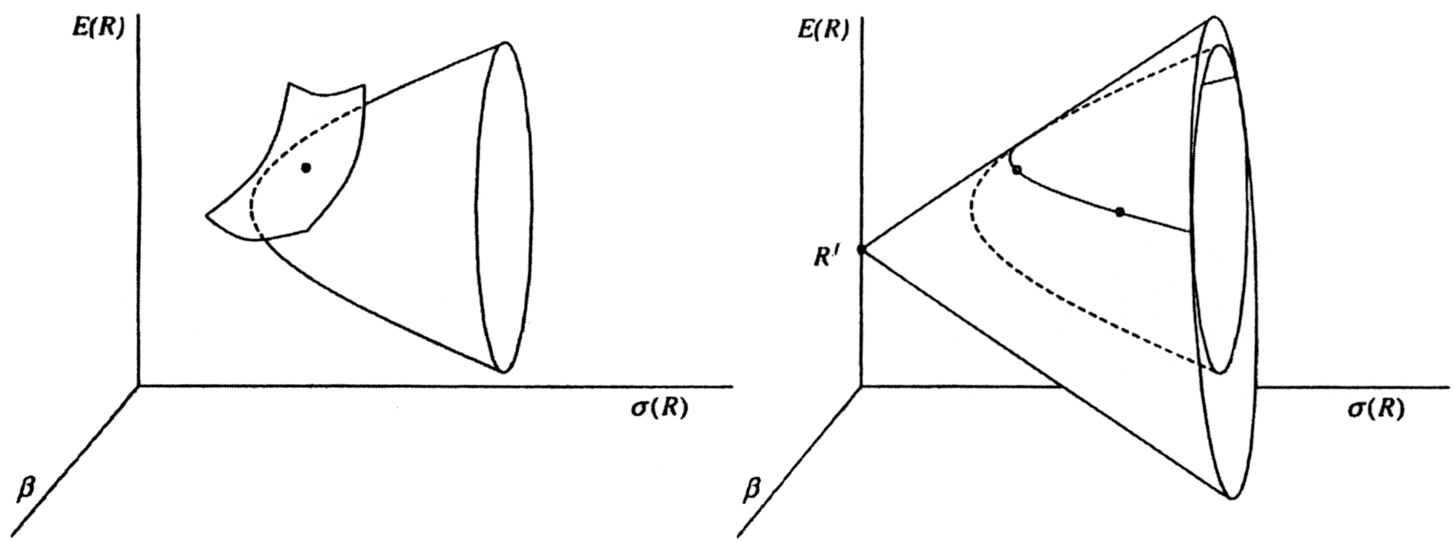

Figure 2. Portfolio theory in a multifactor world. The left hand panel shows an indifference surface and optimal portfolio in the case with no riskfree rate. The dot marks the optimal portfolio where the indifference sheet touches the multifactor efficient frontier. The right hand panel shows the set of multifactor efficient portfolios with a riskfree rate. The two cone-shaped surfaces intersect on the marked line. The two dots are the market portfolio and an additional multifactor-efficient portfolio; all multifactor-efficient portfolios on the outer cone can be reached by combinations of the risk free rate, the market, and the extra multifactor-efficient portfolio.

If the investor does not care about b_1 per se, but does care about b_2, the choice set is three-dimensional. It can be represented with these three axies:

$$\bar{\mathrm{R}}_{\mathrm{p}},\ s_{\mathrm{p}}^{2},\ b_{2}$$

Separation Theorem.

(1) The efficient frontier can be obtained using three efficient portfolios.

If the investor cares about the b_1, then we have a four-fund theorem and the efficient frontier becomes a combination of:

(1) The riskless asset.

(2) Two portfolios that have a beta of one and minimum risk, e.g., factor replication portfolios.

(3) A special portfolio that maximizes a for any residual risk.

When we have a multi-index model, it is often sensible to assume that residuals are un-correlated. In this case, an extremely easy version of simple rules exists and the optimal proportion in any asset is proportional to:

$$Z_i = \frac{a_i}{s^2_{e_i}}$$

Now assume the investor does not care about inflation, but the market does. Then the market's equilibrium model is the capm model and the investor does not pay a cost in expected return by adjusting sensitivity to it.