1. Assumptions that residuals are uncorrelated from a single-index model turns out to be inaccurate. This assumption can be improved by assuming additional indexes.

2. Once again multi-index models have developed a life of their own beyond their original purpose of estimating covariances.
Consider two index model:

\[ r_{it} = a_i + b_{il} I_{lt} + b_{12} I_{2t} + e_{it} \]

Where:

1. \( E \begin{pmatrix} I_{lt} & I_{2t} \end{pmatrix} = 0 \)

2. \( E \begin{pmatrix} e_{it} \end{pmatrix} = 0 \)

3. \( E \begin{pmatrix} e_{it} I_{lt} \text{ or } I_{2t} \end{pmatrix} = 0 \)

4. \( E \begin{pmatrix} e_{it} e_{jt} \end{pmatrix} = 0 \)

5. \( a_i, b_{il}, \text{ and } b_{i2} \) are constant.

Only real assumption is \( E \begin{pmatrix} e_{it} e_{jt} \end{pmatrix} = 0 \)
Mean return:

\[
E\left( r_{it} \right) = a_i + b_{i1} \bar{I}_1 + b_{i2} \bar{I}_2
\]

\[
\text{Var}\left( r_{it} \right) = b_{i1}^2 \sigma_{I1}^2 + b_{i2}^2 \sigma_{I2}^2 + \sigma_{\epsilon_i}^2
\]

in other words:

\[
\sigma_{\epsilon_i}^2 = b_{i2}^2 \sigma_{I2}^2 + \sigma_{\epsilon_i}^2
\]

\[
\text{cov}\left( r_{i} r_{j} \right) = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2
\]
Note:

1. if add $E\left(\epsilon_i \epsilon_j\right)$ get historical.

2. assuming $E\left(\epsilon_i \epsilon_j\right) = b_i b_j \sigma^2 + 0$
Generalizing to more than two indexes is straightforward.

**Types of indexes:**

(1). Statistically derived

(2). Portfolios of securities

   A. S&P, H-L, B-M
   B. S&P, H-L, B-M, Bonds

(3). Economic Factors

   A. $\Delta IP$, surprise in inflation
   B. Surprise GNP

(4). Market and Industry

   A. S&P
   B. Industry factors
In bond area:

\[ R_{it} = \bar{R}_i + OAS - D_i \frac{\Delta r}{1 + r} - V_i dV_i + C_i \text{(spread)} + e_i \]
Issues:

(1). Real influences

(2). Parsimonious
Uses:

(1). Covariance structure

(2). Selection of exposure

(3). Return attribution

(4). Portfolio evaluation