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Inventory Management Under Uncertainty: a Financial Theory for the Transactions Motive

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This paper examines the effects of risk aversion and output market uncertainty on optimal inventory policy decisions for a transactions demand for inventory using the capital asset pricing theory. The paper shows that (1) the optimal order quantity of the risk-adjusted value-maximizing firm is smaller than that of the expected-profit-maximizing one and (2) the greater the firm's output market uncertainty, the smaller its optimal order quantity, where the output market uncertainty is defined as the relative volatility of the demand for the firm's output.

A classic problem faced by firms is to decide the level of inventory (output) for transactions purposes before the level of demand is known.[1] If actual demand is less than the level of inventory, any items left over may be sold at a loss; if demand exceeds supply, the firm may turn customers away. This problem has been identified as the 'newsboy problem' in the operations research literature (e.g. Atkinson, 1979; Lau, 1980) and as the 'Mills' firm' in the economics literature (e.g. Baron, 1971; Leland, 1972).[2]

Firms in these studies are typically viewed as maximizing the von Neumann–Morgenstern expected utility of profit, where the utility function involved represents either the collective will of shareholders or management's perception of it. This approach, however, ignores the role of financial markets, and individuals are not characterized as holding multiple-asset portfolios. Thus, from the perspective of financial theory, this approach is limited.

Magee (1975) recognizes such a limitation and introduces the maximization of project value in the context of cost-volume-profit (CVP) analysis using the capital asset pricing theory. Magee's study, however, does not properly address the stochastic demand problem because of his rather strong and hard-to-defend assumption of the relationship between the demand for the firm's output and the market return.[3] Constantinides *et al.* (1981) extend the stochastic CVP analysis with a specific linear demand function. Their study, however, differs from ours in its specification of uncertainty.[4]

This paper uses the theory of capital asset pricing (Sharpe, 1964; Lintner, 1965; Mossin, 1966) to value risky cash flows resulting from the inventory investment (as determined by the transactions motive) of the firm facing uncertain demand. In essence, the paper formally establishes two optimal inventory policy implications: (1) the optimal order quantity of the risk-adjusted value-maximizing firm is smaller than that of the expected profit-maximizing one; and (2) the higher the firm's output market uncertainty, the lower its optimal order quantity, where the output market uncertainty is defined as the relative volatility of the firm's demand with respect to the general market movement.

The paper is organized as follows. The next section introduces the model and establishes the relationship between the firm's inventory investment and its operating systematic risk. The third section then presents the model's optimality condition and evaluates its economic rationale under risk neutrality and risk aversion. The comparative-statics property of the decision model also is examined in this section. The final section includes a summary and conclusions.

A FINANCIAL MODEL OF OPTIMAL INVENTORY CONTROL

Basic Model

We assume that the firm places an order (or makes a production decision) of Q units at the beginning of the period. We also assume, initially, that there is no beginning inventory and there are no ordering costs (or fixed costs) at the beginning of the period. (In

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Appendix A we relax these two assumptions.) The random demand (X) and the delivery of product occur at the end of the period.

The product is sold at a substantially lower price (p) than its original one (P) if the company sells the product after a specified demand period. Examples of such products are many. Certain types of inventory are subject to obsolescence, whether it be in technology, in consumer tastes or due to regulations. A change in technology may make an electronic component worthless. A change in style may cause a retailer to sell fashion goods at substantially reduced prices. Other inventories, such as agricultural products, are subject to physical deterioration. With deterioration, inventories will have to be sold at lower prices, other things equal.

Demand is assumed to be a normally distributed random variable with density $f(X)$.[5] If an item is demanded when the product is out of stock, the only implicit cost is assumed to be the profit lost on that item.

We use the following additional notation throughout:

$F(X)$ = the cumulative distribution function of X,

σ = the standard deviation of X,

$E(\ )$ = the expected value operator,

$VAR(\ )$ = the variance operator,

$COV(\ ,\ )$ = the covariance operator,

C = the purchase or production cost per unit ($C > p$),

r_m = the market required rate of return,

r_f = the risk-free rate of return, and

$\lambda = \{E(r_m) - r_f\}/VAR(r_m)$.

According to the usual convention that cash flows are realized at the end of the period, the risk-adjusted net present value of cash flows resulting from an order quantity of Q units, $V(Q)$, is defined as:

$$V(Q) = \{E(Y) - \lambda COV(Y, r_m) - CQ\}/(1 + r_f) \quad (1)$$

where Y is a random cash inflow at the end of the period, defined as:

$$Y = \begin{cases} pQ & \text{if } X < 0 \\ PX + p(Q - X) & \text{if } 0 \le X < Q. \\ PQ & \text{if } X \ge Q \end{cases} \quad (2)$$

The expected value of this cash flow is, therefore,

$$E(Y) = \int_{-\infty}^{0} pQf(X)\,dX + \int_{0}^{Q} \{PX + p(Q - X)\} f(X)\,dX + \int_{Q}^{\infty} PQf(X)\,dX \quad (3)$$

Now, we know the following algebraic identities:[6]

$$\int_a^b f(X)\,dX \equiv F(b) - F(a) \quad (4)$$

and

$$\int_a^b Xf(X)\,dX \equiv \sigma^2\{f(a) - f(b)\} + E(X)\{F(b) - F(a)\} \quad (5)$$

Thus, applying identities (4) and (5) to Eqn (3), and after simplification, we obtain (see Appendix B for derivation):

$$E(Y) = PQ\{1 - F(Q)\} + pQF(Q) + (P - p)[E(X) \times \{F(Q) - F(0)\} - \sigma^2\{f(Q) - f(0)\}] \quad (6)$$

The first term in the right-hand side of Eqn (6) is the expected cash flow when the quantity demanded is greater than the quantity ordered. All other terms on the right-hand side of Eqn (6) jointly capture the expected cash flows when the quantity demanded is less than the quantity ordered. The covariance between the end-of-the-period cash flow and the market return is, by definition:

$$COV(Y, r_m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{Y - E(Y)\} \times \{r_m - E(r_m)\} f(X, r_m)\,dX\,dr_m \quad (7)$$

Equation (7) can be rewritten as follows:[7]

$$COV(Y, r_m) = (COV(X, r_m)/\sigma^2)\int_{-\infty}^{\infty} \{Y - E(Y)\} \times \{X - E(X)\} f(X)\,dX \quad (8)$$

We also have the following algebraic identities:[8]

$$\int_a^b (d^2f(X)/dX^2)\,dX \equiv \{df(X)/dX\}_{X=b} - \{df(X)/dX\}_{X=a} \quad (9)$$

and

$$\int_a^b X^2 f(X)\,dX \equiv \sigma^4 \int_a^b (d^2f(X)/dX^2)\,dX + 2E(X)\int_a^b Xf(X)\,dX + [\sigma^2 - \{E(X)\}^2]\int_a^b f(X)\,dX \quad (10)$$

Thus, applying identities (4), (5), (9) and (10) to Eqn (8) and simplifying, we obtain (see Appendix B for derivation):

$$COV(Y, r_m) = (P - p)\,COV(X, r_m)\{F(Q) - F(0)\} \quad (11)$$

Finally, substituting Eqns (6) and (11) into Eqn (1), the risk-adjusted net present value of cash flows resulting from an order quantity of Q units [$V(Q)$] can be expressed as:

$$V(Q)=\frac{PQ\{1-F(Q)\}+pQF(Q)+(P-p)[\{E(X)-\lambda COV(X,r_m)\}\{F(Q)-F(0)\}-\sigma^2\{f(Q)-f(0)\}]-CQ}{1+r_f} \quad (12)$$

where the numerator of Eqn (12) is the certainty-equivalent cash flow.

Inventory and Risk

Next, we examine the relationship between the firm's inventory investment and its operating systematic risk. First, the operating systematic risk associated with the order quantity of Q units is defined as:

$$\beta_Q=\frac{COV(r,r_m)}{VAR(r_m)} \quad (13)$$

where $r=(Y-CQ)/V(Q)$. Because CQ and $V(Q)$ are not random variables, Eqn (13) can be rewritten as:

$$\beta_Q=\frac{1}{V(Q)}\frac{COV(Y,r_m)}{VAR(r_m)} \quad (14)$$

Substituting Eqn (11) into Eqn (14), β_Q can be rewritten as:

$$\beta_Q=\frac{(P-p)}{V(Q)}\frac{COV(X,r_m)}{VAR(r_m)}\{F(Q)-F(0)\} \quad (15)$$

where $F(Q)$ is the probability of excess inventory.

Now consider a hypothetical firm facing the same output market where delivery (or production) of the output is instantaneous, so that the quantity demanded is always the same as the quantity sold. For such a firm, the net present value of cash flows is defined as:

$$V(X)=[(P-C)E(X)-\lambda COV\{(P-C)X,r_m\}]/(1+r_f) \quad (16)$$

As above, the operating systematic risk of this hypothetical firm is defined as:

$$\beta_I=\frac{(P-C)}{V(X)}\frac{COV(X,r_m)}{VAR(r_m)} \quad (17)$$

Finally, from Eqns (15) and (17), the operating systematic risk of the firm with delivery lag (or production lag), β_Q, can be seen as a function of the operating systematic risk of the firm with instantaneous delivery, β_I:

$$\beta_Q=\beta_I\frac{(P-p)}{(P-C)}\frac{V(X)}{V(Q)}\{F(Q)-F(0)\} \quad (18)$$

Intuitively, therefore, the firm's operating risk associated with its inventory policy can be decomposed into two components: pure output market risk (i.e. the risk inherent in the demand uncertainty, β_I), and that resulting from the need to determine the order quantity (or production level) before the actual demand is revealed (i.e. the inability of the firm to meet the demand instantaneously, $\{(P-p)V(X)\}\{F(Q)-F(0)\}/(P-C)V(Q)$).

INVENTORY POLICY OF THE FIRM UNDER DEMAND UNCERTAINTY

Optimality Condition

Letting $dV(Q)/dQ=0$, and after simplification, we have the following optimality condition:

$$A\{1-F(Q^*)-\lambda_1\}=B\{F(Q^*)-\lambda_1\} \quad (19)$$

where $A=P-C$, $B=C-p$ and $\lambda_1=\lambda COV(X,r_m)f(Q^*)$. Assuming risk-neutrality (i.e. $\lambda_1=0$), Eqn (19) simplifies to:

$$A\{1-F(Q^*)\}=B\{F(Q^*)\} \quad (20)$$

The intuition underlying Eqn (20) is clear. A represents the implicit cost of shortage (i.e. lost profit per unit) and $1-F(Q^*)$ represents the probability of shortage. Thus, the left-hand side of Eqn (20) measures the expected cost of shortage. Similarly, B represents the implicit cost of each unsold unit (i.e. the difference between the product cost and salvage value) and $F(Q^*)$ represents the probability of the excess inventory. Thus, the right-hand side of Eqn (20) measures the expected cost of excess inventory.

If the firm orders too little, the expected cost of shortage will dominate that of excess inventory. However, if the firm orders too much the expected cost of excess inventory will dominate that of shortage. The optimal inventory policy will occur when the firm orders up to the point at which the expected cost of shortage equals that of excess inventory. Similar interpretations apply to Eqn (19), but in this case, both sides of the equation are adjusted by the investors' degree of risk aversion (i.e. λ_1).

The effect of risk aversion on optimal inventory policy becomes clearer if we rewrite Eqns (19) and (20) as follows:

$$F(Q^*)+\lambda_1=\frac{A}{A+B} \quad (19a)$$

$$F(Q^*)=\frac{A}{A+B} \quad (20a)$$

Equation (20a) is the traditional formula for the optimal order quantity. From Eqns (19a) and (20a), the optimal order quantity of the firms owned by risk-averse investors is seen to be smaller than that of those owned by risk-neutral ones, as long as $\lambda_1>0$. (This condition will hold if $COV(X,r_m)>0$, which is a reasonable proposition for most demand situations.[9])

The economic rationale underlying this intuitively appealing result becomes apparent once we consider the tradeoff involved in this decision-making situation: the *potential additional profit* (i.e. the net cash inflows resulting from additional sales, or equivalently, the difference between the selling price and product cost) and the *potential additional cost* (i.e. the net cash outflows resulting from the unsold units, or equivalently, the difference between the production cost and salvage value).

The higher the order quantity, the higher the potential profit, but at the expense of higher potential excess inventory cost, and vice versa. However, for risk-averse owners (i.e. investors) the utility loss resulting from unit monetary loss is greater than the gain of utility resulting from unit monetary gain, while these are the same for risk-neutral owners.[10] Therefore, the expected utility of the risk-averse owners will be maximized at a lower level of order quantity than that of the risk-neutral ones, all things being equal.

Finally, the higher the cost of excess inventory, the smaller the optimal order quantity, and the higher the cost of shortage, the larger the optimal order quantity, regardless of the firm's attitude toward risk.

In addition to the above first-order condition, it is necessary that the second-order condition hold to ensure that the solution is indeed maximum. First,

$$\left.\frac{\mathrm{d}^2V(Q)}{\mathrm{d}Q^2}\right|_{Q=Q^*} = -(P-p)[f(Q^*) - [\{Q^*-E(X)\}/\sigma^2]\{R-F(Q^*)\}] \quad (21)$$

where $R=A/(A+B)$. If $Q^* \leq E(X)$, then $\mathrm{d}^2V(Q)/\mathrm{d}Q^2|_{Q=Q^*}<0$, given $R>F(Q^*)$. We know that $R>F(Q^*)$ from Eqn (19a), given the positive covariance between the market return and the demand for the firm's product. If $Q^*>E(X)$, we have the following property of the normal distribution (see Feller, 1967, p. 175):

$$f(Q)>[\{Q-E(X)\}/\sigma^2]\{1-F(Q)\} \quad (22)$$

Because $R<1$ by assumption (i.e. the salvage value is less than the cost of the product), it also follows that $\mathrm{d}^2V(Q)/\mathrm{d}Q^2|_{Q=Q^*}<0$. This completes the proof that the second-order condition is satisfied.

Comparative Statics

To examine the effect of output market volatility on optimal inventory policy we differentiate Eqn (19a) with respect to the demand beta, β_D, which is defined as $\beta_D=COV(X, r_m)/VAR(r_m)$. After simplification, we have:

$$\frac{\partial Q^*}{\partial \beta_D} = \frac{-\{E(r_m)-r_f\}f(Q^*)}{f(Q^*)-[\{Q^*-E(X)\}/\sigma^2]\{R-F(Q^*)\}} \quad (23)$$

If $Q^* \leq E(X)$, the second term in the denominator of Eqn (23) is negative and, therefore, the denominator will be positive. Since the numerator of Eqn (23) is negative, $\partial Q^*/\partial\beta_D<0$. If $Q^*>E(X)$, from Eqn (22), the denominator of Eqn (23) is negative. Thus, $\partial Q^*/\partial\beta_D<0$.

Therefore, the firm facing higher demand uncertainty will have a lower optimal order quantity.[11] Let us examine the economic rationale underlying this result. Two firms with the same (but not perfectly correlated) distribution of output demand will have the same expected shortage costs; therefore, they have the same expected excess inventory costs. The present value of the expected excess inventory costs will be lower for the riskier firm because of the higher risk premium. Similarly, the present value of the expected shortage costs will be lower for the riskier firm, but to a greater degree than in the case of the excess inventory costs. Therefore, the high-risk firm should choose a lower order quantity.

Similarly, it can be shown that:

$$\frac{\partial Q^*}{\partial R} = \frac{1}{f(Q^*)-[\{Q^*-E(X)\}/\sigma^2]\{R-F(Q^*)\}} > 0 \quad (24)$$

As was shown for Eqn (23), the denominator of Eqn (24) is always positive. Thus, Eqn (24) indicates that there is an inverse association between the level of inventory and the degree of product obsolescence. Therefore, we would expect firms with a lower degree of product obsolescence (i.e. high values of R) to maintain higher inventory levels. The intuition for this result is straightforward. If the product has a lower degree of product obsolescence, the implicit cost of each unsold unit would be lower and, as a result, the firm would be better off with a higher order quantity.

SUMMARY AND CONCLUSIONS

Normative approaches to decision making in the management of inventory are mostly *ad hoc* in nature. It is not surprising, therefore, that these approaches have not been incorporated into the mainstream of finance theory, where net present value (NPV) maximization is the central focus.[12] In this paper we have developed a financial theory for the transactions motive inventory decisions under uncertainty based on the risk-adjusted NPV maximization approach using the CAPM framework.

We have found that (1) the optimal inventory level of the risk-adjusted value-maximizing firm is lower than that of the expected-profit-maximizing one; and (2) the higher the firm's output market uncertainty, the lower its optimal inventory level, where output market uncertainty is defined as the relative volatility of the demand for the firm's output. The first finding implies that the optimal level of inventory suggested by the traditional expected-profit-maximizing inventory model is incorrect if investors (i.e. owners) are not risk neutral. In fact, the inventory level is always higher than that of the risk-adjusted NPV maximizing inventory model, when investors are assumed to be risk averse.

With respect to the second finding, we would expect an inverse association between demand volatility and the level of inventory investment by firms. In this context the following explanation of why firms recently held lower inventories provides some empirical support for our model:

Greater volatility in the business cycle itself has led to increased efforts by business to reduce inventories. Over the past five years, total demand has shown unusual instability with two discrete recessionary episodes separated by a short-lived period of reflation. Under these circumstances, business has experienced extreme difficulty in forecasting the actual level of demand and in gauging the correct level of inventory relative to consumer purchases. Consequently, business has become more cautious in holding large inventories in order to minimize the risk of unanticipated surplus due to increased business-cycle volatility (Jasinowski, 1984).

Such evidence is not conclusive, however, even though consistent with our analysis. Empirical research should await the incorporation of all motives—speculative, precautionary and transactions—into a general theory of inventory management under uncertainty. We hope the current study provides sufficient direction towards establishing such a general theory.

APPENDIX A: OPTIMAL INVENTORY POLICY WITH INITIAL INVENTORY AND ORDERING COSTS

This appendix extends the basic model by relaxing the assumptions of zero initial inventory and zero ordering cost. First, let us assume that the beginning inventory is I. If we redefine Q as the number of units to be available at the end of the period, then the present value of cash flows resulting from the beginning inventory of Q units can be defined as follows:

$$V(Q)=\{E(Y)-\lambda COV(Y,r_m)-C(Q-I)\}/(1+r_f) \quad \text{(A1)}$$

Since CI is a constant, Eqn (19) yields the optimal solution to Eqn (A1) as well as to Eqn (12) provided that $Q^* \geq I$. If $Q^* < I$, the optimal policy is not to order. The decision process can be summarized as follows:

(1) Find Q^* such that

$$F(Q^*)+\lambda COV(X,r_M)f(Q^*)=\frac{A}{A+B}$$

(2) If $Q^* \leq I$, do not order; and
(3) If $Q^* > I$, order $(Q^* - I)$ units.

If we further relax the assumption of zero ordering cost, then $V(Q)$ becomes:

$$V(Q)=\{E(Y)-\lambda COV(Y,r_M)-C(Q-I)-K\}/(1+r_f) \quad \text{(A2)}$$

where K is the ordering cost.

As in Eqn (A1), the optimal inventory policy can be described by Eqn (19) because K is also a constant, provided incurring the ordering cost of K is economically justified. If incurring K is not justified, the optimal policy is not to order any additional units. In that case, the present value of the sales revenue, $S(I)$, is defined as:

$$S(I)=\{E(y)-\lambda COV(y,r_m)\} \quad \text{(A3)}$$

where

$$y=\begin{cases} pI & \text{if } X<0 \\ PX+p(I-X) & \text{if } 0<X<I \\ PI & \text{if } X\geq I \end{cases}$$

If we denote $G(Q)=\{E(Y)-\lambda COV(Y,r_m)\}$, where Y is defined as in Eqn (2), the relationship between Q and $G(Q)-CQ$ can be described as in Fig. A1. In this figure H is the value of Q that maximizes $G(Q)-CQ$ and L is the smallest value of Q for which $G(L)-CL=G(H)-CH-K$. In the case of $I>H$, it is evident that:

$$G(Q)-CQ-K<S(I)-CI,\ \text{for all } Q>I \quad \text{(A4)}$$

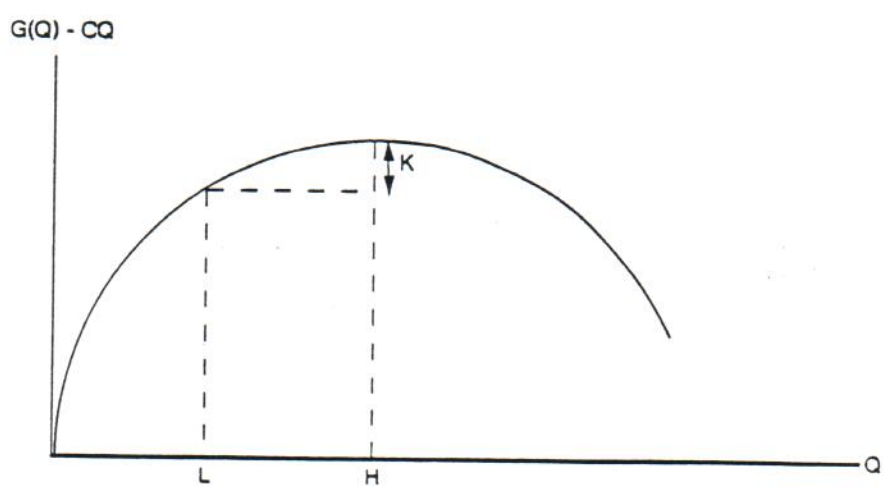

Figure A1. The relationship between Q and $G(Q)-CQ$.

Equation (A4) can be rewritten as:

$$G(Q)-C(Q-I)-K<S(I), \text{ for all } Q>I \quad \text{(A5)}$$

where the left-hand side of the inequality represents the net cash flow if the firm orders up to Q and the right-hand side represents the cash flow if no ordering occurs. Hence, if $I>H$, the optimal policy is not to order.

If $L \le I \le H$, it is evident again from Fig. A1 that Eqn (A5) holds. Thus, no ordering is more profitable than ordering.

Finally, if $I<L$, it follows from Fig. A1 that:

$$\operatorname*{Max}_{Q \ge I} \{G(Q)-CQ-K\} = G(H)-CH-K>S(I)-CI \quad \text{(A6)}$$

or alternatively,

$$\operatorname*{Max}_{Q \ge I} \{G(Q)-C(Q-I)-K\} = G(H)-C(H-I) - K>S(I) \quad \text{(A7)}$$

so that it pays to order. The maximum value is obtained if one orders up to H. Thus, the optimal ordering policy can be summarized as follows:

(1) Find H such that

$$F(H)+\lambda\, COV(X, r_{\mathrm{m}}) f(H) = \frac{A}{A+B}$$

(2) Find the smallest value of L such that

$$G(L)-CL=G(H)-CH-K$$

(3) If $I<L$, order up to H; and
(4) If $I \ge L$, do not order.

APPENDIX B: DERIVATION OF EQNS (6) AND (11)

Equation (3) can be rewritten as follows:

$$E(Y)=pQ\int_{-\infty}^{0} f(X)\,\mathrm{d}X+(P-p)\int_{0}^{Q} Xf(X)\,\mathrm{d}X + pQ\int_{0}^{Q} f(X)\,\mathrm{d}X+PQ\int_{Q}^{\infty} f(X)\,\mathrm{d}X \quad \text{(B1)}$$

Substituting Eqns (4) and (5) into Eqn (B1), we obtain:

$$\begin{aligned} E(Y) &= pQF(0)+(P-p)[\sigma^2\{f(0)-f(Q)\} \\ &\quad +E(X)\{F(Q)-F(0)\}] \\ &\quad +pQ\{F(Q)-F(0)\}+PQ\{1-F(Q)\} \end{aligned} \quad \text{(B2)}$$

Equation (B2) simplifies to:

$$\begin{aligned} E(Y) &= (P-p)[\sigma^2\{f(0)-f(Q)\}+E(X)\{F(Q) \\ &\quad -F(0)\}]+pQF(Q)+PQ\{1-F(Q)\} \end{aligned} \quad (6)$$

Equation (8) can be rewritten as:

$$\begin{aligned} COV(Y, r_{\mathrm{m}}) &= (COV(X, r_{\mathrm{m}})/\sigma^2)\left\{\int_{-\infty}^{\infty} YXf(X)\,\mathrm{d}X \right. \\ &\quad -E(X)\int_{-\infty}^{\infty} Yf(X)\,\mathrm{d}X \\ &\quad -E(Y)\int_{-\infty}^{\infty} Xf(X)\,\mathrm{d}X \\ &\quad \left. +E(X)E(Y)\int_{-\infty}^{\infty} f(X)\,\mathrm{d}X\right\} \end{aligned} \quad \text{(B3)}$$

From Eqn (2), Y is defined as:

$$Y=\begin{cases} pQ & \text{if } X<0 \\ PX+p(Q-X) & \text{if } 0<X<Q \\ PQ & \text{if } X \ge Q \end{cases} \quad (2)$$

Substituting Eqn (2) into Eqn (B3), we obtain:

$$\begin{aligned} COV(Y, r_{\mathrm{m}}) &= (COV(X, r_{\mathrm{m}})/\sigma^2)\left[\int_{-\infty}^{0} pQXf(X)\,\mathrm{d}X \right. \\ &\quad +\int_{0}^{Q}\{PX+p(Q-X)\}Xf(X)\,\mathrm{d}X \\ &\quad +\int_{Q}^{\infty} PQXf(X)\,\mathrm{d}X \\ &\quad -E(X)\left[\int_{-\infty}^{0} pQf(X)\,\mathrm{d}X+\int_{0}^{Q}\{PX \right. \\ &\quad +p(Q-X)\}f(X)\,\mathrm{d}X \\ &\quad \left. +\int_{Q}^{\infty} PQf(X)\,\mathrm{d}X\right] \\ &\quad -E(Y)\int_{-\infty}^{\infty} Xf(X)\,\mathrm{d}X \\ &\quad \left. +E(X)E(Y)\int_{-\infty}^{\infty} f(X)\,\mathrm{d}X\right] \end{aligned} \quad \text{(B4)}$$

Simplifying Eqn (B4), we obtain:

$$\begin{aligned} COV(Y, r_{\mathrm{m}}) &= \{COV(X, r_{\mathrm{m}})/\sigma^2\}\left[pQ \right. \\ &\quad \left\{\int_{-\infty}^{0} Xf(X)\,\mathrm{d}X-E(X)\int_{-\infty}^{0} f(X)\,\mathrm{d}X\right\} \\ &\quad +(P-p)\left\{\int_{0}^{Q} X^2f(X)\,\mathrm{d}X \right. \\ &\quad \left. -E(X)\int_{0}^{Q} Xf(X)\,\mathrm{d}X\right\} \\ &\quad +pQ\left\{\int_{0}^{Q} Xf(X)\,\mathrm{d}X \right. \\ &\quad \left. -E(X)\int_{0}^{Q} f(X)\,\mathrm{d}X\right\} \\ &\quad +PQ\left\{\int_{Q}^{\infty} Xf(X)\,\mathrm{d}X \right. \\ &\quad \left.\left. +E(X)\int_{Q}^{\infty} f(X)\,\mathrm{d}X\right\}\right] \end{aligned} \quad \text{(B5)}$$

Now,

$$\int_a^b (\mathrm{d}^2 f(X)/\mathrm{d}X^2)\,\mathrm{d}X = \{\mathrm{d}f(X)/\mathrm{d}X\}_{X=b} - \{\mathrm{d}f(X)/\mathrm{d}X\}_{X=a} \qquad (9)$$

and

$$\int_a^b X^2 f(X)\,\mathrm{d}X = \sigma^4 \int_a^b \{\mathrm{d}^2 f(X)/\mathrm{d}X^2\}\,\mathrm{d}X + 2E(X)\int_a^b Xf(X)\,\mathrm{d}X + [\sigma^2 - \{E(X)\}^2]\int_a^b f(X)\,\mathrm{d}X \qquad (10)$$

Applying Eqn (10), we have:

$$\int_0^Q X^2 f(X)\,\mathrm{d}X = \sigma^4 \int_0^Q \{\mathrm{d}^2 f(X)/\mathrm{d}X^2\}\,\mathrm{d}X + 2E(X)\int_0^Q Xf(X)\,\mathrm{d}X + [\sigma^2 - \{E(X)\}^2]\int_0^Q f(X)\,\mathrm{d}X \qquad (B6)$$

From Eqn (9):

$$\int_0^Q \{\mathrm{d}^2 f(X)/\mathrm{d}X^2\}\,\mathrm{d}X = \{\mathrm{d}f(X)/\mathrm{d}X\}_{X=Q} - \{\mathrm{d}f(X)/\mathrm{d}X\}_{X=0} \qquad (B7)$$

where

$$\{\mathrm{d}f(X)/\mathrm{d}X\}_{X=Q} = -(1/\sigma^2)\{Q - E(X)\}f(Q)$$

and

$$\{\mathrm{d}f(X)/\mathrm{d}X\}_{X=0} = -(1/\sigma^2)\{-E(X)\}f(0) \qquad (B8)$$

Substituting Eqn (B8) into Eqn (B6) yields:

$$\int_0^Q \{\mathrm{d}^2 f(X)/\mathrm{d}X^2\}\,\mathrm{d}X = -(1/\sigma^2)[\{Q - E(X)\}f(Q) + E(X)f(0)] \qquad (B9)$$

Finally, first applying Eqns (4), (5) and (B9) to Eqn (B6), and subsequently applying Eqns (4), (5), (6) and (B6) to Eqn (B5), and after simplification we obtain:

$$COV(Y, r_m) = (P-p)\,COV(X, r_m)\{F(Q) - F(0)\} \qquad (11)$$

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NOTES

1. Generally, the firm maintains inventory for three different motives: transactions, speculative and precautionary. This paper focuses on the inventory problem, which primarily deals with the transactions motive.
2. There are generally three possible sources of uncertainty in the theory of the firm: future demand for the firm's output, future selling price of the firm's output and production cost. For the theory of the firm facing price uncertainty see Sandmo (1971), and for production cost uncertainty see Adar *et al.* (1977), respectively.
3. Specifically, Magee (1975, p. 262) assumes that $E(r_m|X) = a + bX$, where r_m is the one-period random market return, X is the firm's random demand, E is the expected value operator and a and b are constants.
4. They introduced uncertainty by defining the demand function as $P = g - hX + e$, where P = selling price, X = sales volume, g and h are constants and e is a random variable with an expected value of zero and a variance greater than or equal to zero.

   Three studies (Anvari, 1987; Singhal, 1988; Thorstenson, 1988) that also use the theory of capital asset pricing to analyze stochastic inventory systems were brought to our attention after we had completed the paper. The first work has not been developed to the stage where a comparative-static analysis can be performed; the second relies on a numerical evaluation without the benefit of an analytical framework. The last is primarily a comment on the first.
5. Because of the truncation, the quadratic utility function of investors is assumed for the validity of the Capital Asset Pricing Model.
6. Identity (5) is derived using the following identity, obtained by differentiating $f(X)$:

   $$\mathrm{d}f(X)/\mathrm{d}X \equiv -[\{X - E(X)\}/\sigma^2]\,f(X)$$

   or

   $$Xf(X) \equiv -\sigma^2\{\mathrm{d}f(X)/\mathrm{d}X\} + E(X)\,f(X)$$

   Integrating both sides of the second identity, we have:

   $$\int_a^b Xf(X)\,\mathrm{d}X \equiv -\sigma^2 \int_a^b \mathrm{d}f(X) + E(X)\int_a^b f(X)\,\mathrm{d}X \equiv \sigma^2\{f(a) - f(b)\} + E(X)\{F(b) - F(a)\}$$
7. This is because

   $$\int_{-\infty}^{\infty} \{r_m - E(r_m)\}\,f(r_m/X)\,\mathrm{d}r_m = \{COV(X, r_m)/\sigma^2\}\{X - E(X)\}$$

   (see Mood and Graybill, 1963, p. 202).
8. Identity (10) is derived using the following, obtained by differentiating $\mathrm{d}f(X)/\mathrm{d}X$:

   $$\mathrm{d}^2 f(X)/\mathrm{d}X^2 \equiv -\{f(X)/\sigma^2\} + [\{X - E(X)\}^2/\sigma^4]\,f(X)$$

   or

   $$\sigma^4\{\mathrm{d}^2 f(X)/\mathrm{d}X^2\} \equiv -\sigma^2 f(X) + \{X - E(X)\}^2 f(X)$$

   Integrating both sides of the second identity, we have:

   $$\sigma^4 \int_a^b \{\mathrm{d}^2 f(X)/\mathrm{d}X^2\}\,\mathrm{d}X \equiv -\sigma^2 \int_a^b f(X)\,\mathrm{d}X + \int_a^b \{X - E(X)\}^2 f(X)\,\mathrm{d}X$$

   Rearrangement of the above identity yields:

   $$\int_a^b X^2 f(X)\,\mathrm{d}X \equiv \sigma^4 \int_a^b (\mathrm{d}^2 f(X)/\mathrm{d}X^2)\,\mathrm{d}X + 2E(X)\int_a^b Xf(X)\,\mathrm{d}X + (\sigma^2 - (E(X))^2)\int_a^b f(X)\,\mathrm{d}X$$

where

$$\int_a^b (\mathrm{d}^2 f(X)/\mathrm{d}X^2)\,\mathrm{d}X \equiv [\mathrm{d}f(X)/\mathrm{d}X]_{X=b} - [\mathrm{d}f(X)/\mathrm{d}X]_{X=a}$$

and

$$\mathrm{d}f(X)/\mathrm{d}X \equiv -[\{X - E(X)\}/\sigma^2]\, f(X)$$

9. Risk-averse firms reduce output when there is price or demand uncertainty; this has been repeatedly demonstrated in the economics and operations research literature (see, e.g. Atkinson, 1979; Baron, 1971; Leland, 1972; Sandmo, 1971).
10. Since we are assuming that the firm is maximizing the risk-adjusted present value of cash flows using the CAPM, it is implicit that we are assuming that the firm is maximizing the von Neumann–Morgenstern expected utility of the risk-averse owners (i.e. investors) of the firm.
11. Constantinides *et al.* (1981) made a similar observation. As pointed out in note 3, they used a different specification of uncertainty. They found an inverse relationship between $COV(e, r_m)$ and the optimal output level, assuming that the firm's output always *equaled* actual sales.
12. For a net present value-maximizing framework for inventory analysis under certainty, see Kim and Chung (1985) and Kim *et al.* (1986).

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