# TRACE INVARIANTS ASSOCIATED WITH QUOTIENT MODULES OF THE HARDY MODULE

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**Abstract.** We consider the quotient module $\mathcal{Q}$ of the Hardy module $H^2(S)$ defined by an analytic set $\tilde{M}$ satisfying certain conditions. A representation for the orthogonal projection $Q : L^2(S, d\sigma) \to \mathcal{Q}$ was derived in [26], which allowed us to prove the geometric Arveson-Douglas conjecture for $\mathcal{Q}$. In this paper we derive a new representation for $Q$, which makes it possible for us to take the next step: We show that for $f, g \in \text{Lip}(S)$, the double commutator $[M_f, [M_g, Q]]$ is in the Schatten class $\mathcal{C}_p$ for $p > \dim_{\mathbf{C}} \tilde{M}$. This Schatten-class membership leads to a number of results for trace invariants on $\mathcal{Q}$ and $H^2(S)$. In addition, we report an unexpected discovery: if $\dim_{\mathbf{C}} \tilde{M} = 1$, then $\mathcal{Q}$ is 1-essentially normal. This is a stronger result than the prediction of the Arveson-Douglas conjecture.

## 1. Introduction

This paper is a continuation of the work in [26]. As such, we will follow the notations in [26]. To discuss what we will do in this paper, let us first recall the setting in [26].

Denote $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$ and $S = \{z \in \mathbf{C}^n : |z| = 1\}$ as usual. Let $H^2(S)$ be the Hardy space on $S$. Consider an analytic subset $\tilde{M}$ of an open neighborhood of $\overline{\mathbf{B}}$ with $1 \le d \le n-1$, where $d = \dim_{\mathbf{C}} \tilde{M}$. We assume that $\tilde{M}$ has no singular points on $S$ and that $\tilde{M}$ intersects $S$ transversely. Denote $M = \mathbf{B} \cap \tilde{M}$. Then we have a submodule

$$\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$$

of $H^2(S)$. The corresponding quotient module is

$$\mathcal{Q} = H^2(S) \ominus \mathcal{R}.$$

Both $\mathcal{R}$ and $\mathcal{Q}$ are the focus of the Arveson-Douglas conjecture [1,2,10], which commands intense current research interest [4,11,13-15,19,23,25].

In [26], the geometric Arveson-Douglas conjecture was proved for $\mathcal{Q}$. That is, we showed that $\mathcal{Q}$ is $p$-essentially normal for $p > d$. Central to this essential normality is the orthogonal projection

$$Q : H^2(S) \to \mathcal{Q}.$$

It was shown that if $f$ is a Lipschitz function on $S$, then the commutator $[M_f, Q]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$ [26, Proposition 8.3], which implies that the Geometric Arveson-Douglas conjecture holds for the quotient module $\mathcal{Q}$.

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For the Bergman space $L_a^2(\mathbf{B})$, the analogous result was proved earlier in [25]. Taking [25] as a guide, we see that more can be done in the Hardy-space case. Specifically, here we are referring to [25, Proposition 5.11], which says that, in the Bergman-space case, the double commutator $[M_f, [M_g, Q]]$ improves the Schatten-class membership of the single commutator $[M_g, Q]$. This improved Schatten-class membership has implications in terms of trace invariants [25, Theorem 1.8] in the Bergman-space case.

Thus an obvious question for the Hardy-space case is, what about the Schatten-class membership of the double commutator $[M_f, [M_g, Q]]$? This is a question that we did not address in [26], for two reasons. First, much of [26] was devoted to the proof of [26, Theorem 1.3], a somewhat unexpected compactness criterion on $\mathcal{Q}$, which certainly deserved priority. Second, and more important, the handling of the double commutator $[M_f, [M_g, Q]]$ in the Hardy-space case involves a non-trivial hurdle that does not exist in the Bergman-space case. Let us explain where the difficulty lies.

In [26], the proof of the Schatten-class membership for $[M_f, Q]$ relied on the inequality

$$cQ \le T_\mu \le CQ, \tag{1.1}$$

where $0 < c \le C < \infty$. Here, $\mu$ is an explicitly given measure on $M$, and

$$T_\mu = \int_M K_w \otimes K_w d\mu(w), \tag{1.2}$$

where

$$K_w(\zeta) = \frac{1}{(1 - \langle \zeta, w\rangle)^n}, \tag{1.3}$$

which is the reproducing kernel for the Hardy space $H^2(S)$. (1.1) and (1.2) give us a reasonably good handle on $Q$, which is why we were able to do what we did in [26].

In the case of the Bergman space $L_a^2(\mathbf{B})$, the analogues of (1.1) and (1.2) also hold [14,25]. But the difference is that in the Bergman-space case, the reproducing kernel is

$$K_w^{\mathrm{Berg}}(\zeta) = \frac{1}{(1 - \langle \zeta, w\rangle)^{n+1}}. \tag{1.4}$$

Because the power on the right-hand side of (1.4) is $n+1$, we were able to deal with the double commutator $[M_f, [M_g, Q]]$ in [25]. Because the power on the right-hand side of (1.3) is $n$, we were only able to obtain the desired Schatten-class membership for the single commutator $[M_f, Q]$ in [26].

Let us explain the difficulty in more detail. After taking the commutator of $T_\mu$ once, the requisite estimates will "consume" 1 unit in the power of the reproducing kernel. In the case of (1.3), $n-1$ will coincide with $d$ when $\tilde{M}$ has the maximum dimension $n-1$. That is why, in the Hardy-space case, (1.1) and (1.2) only allow us to handle the single commutator $[M_f, T_\mu]$. In contrast, in the Bergman-space case, by taking commutator once, the same consumption of 1 unit of power only reduces the power in (1.4) to $n+1-1=n$, which is still greater than any $1 \le d \le n-1$, and which allows us to take commutator one more time.

Thus we see precisely what the difficulty is: If we want to show that in the Hardy-space case, the double commutator $[M_f, [M_g, Q]]$ also has an improved Schatten-class membership over the single commutator $[M_g, Q]$, we will have to somehow "raise the power in (1.3)". This means that we are looking for an operator inequality of the form

$$cQ \le T' \le CQ, \tag{1.5}$$

where $T'$ is constructed from "kernels with power greater than $n$".

We are pleased to report that we have managed to find such a $T'$, although it is quite technical to describe it. Therefore we are able to obtain the desired Schatten-class membership for $[M_f, [M_g, Q]]$ in the Hardy-space case. Consequently, the trace results in [25] can be proved in the Hardy-space case. We state our results below.

**Theorem 1.1.** *For any Lipschitz functions $f, g$ on $S$, the double commutator $[M_f, [M_g, Q]]$ is in the Schatten class $\mathcal{C}_p$ for every $p > d$.*

We write $\sigma$ for the spherical measure on $S$ with the normalization $\sigma(S) = 1$. Let $P : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. Recall that for $f \in L^\infty(S, d\sigma)$, the Toeplitz operator $T_f$ is defined by the formula

$$T_f = PM_f|H^2(S).$$

Let $R$ be the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{R}$. Then $P = R + Q$. Given any $f \in L^\infty(S, d\sigma)$, we define the operators

$$R_f = RM_f|\mathcal{R} \quad \text{and} \quad Q_f = QM_f|\mathcal{Q}.$$

We think of $R_f$ and $Q_f$ as "Toeplitz operators" for the submodule $\mathcal{R}$ and the quotient module $\mathcal{Q}$ respectively.

Given operators $A_1, \dots, A_k$ on a Hilbert space $\mathcal{H}$, one has the antisymmetric sum

$$[A_1, \dots, A_k] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)}.$$

This was first introduced by Helton and Howe in [21], and has since become an important part of operator theory [8,12]. As it turns out, operators of the types $T_f$, $R_f$ and $Q_f$ make particularly interesting antisymmetric sums. In fact, Theorem 1.1 leads to

**Theorem 1.2.** *Let $\ell > d$. Then for any Lipschitz functions $f_1, f_2, \dots, f_{2\ell}$ on $S$, the antisymmetric sum*

$$[Q_{f_1}, Q_{f_2}, \dots, Q_{f_{2\ell}}]$$

*is in the trace class with zero trace.*

This is the Hardy-space analogue of [25, Theorem 1.8]. But, as we explained above, the proof in the Hardy-space case requires extra efforts. These extra efforts, however, are well justified by the implications of Theorem 1.2, of which we present two below.

Let $\mathcal{T}(C(S))$ be the $C^*$-algebra generated by the Toeplitz operators $\{T_f : f \in C(S)\}$. It is well known that we have the exact sequence

$$\{0\} \to \mathcal{K}(H^2(S)) \to \mathcal{T}(C(S)) \to C(S) \to \{0\} \tag{1.6}$$

of $C^*$-algebras. By the BDF theory, this represents an element in the extension group $\mathrm{Ext}(S)$ [5,9]. Let $\mathcal{TR}(C(S))$ be the $C^*$-algebra generated by the operators $\{R_f : f \in C(S)\}$ on $\mathcal{R}$. Then it is easy to show that we also have the exact sequence

$$\{0\} \to \mathcal{K}(\mathcal{R}) \to \mathcal{TR}(C(S)) \to C(S) \to \{0\}, \tag{1.7}$$

which also represents an element in $\mathrm{Ext}(S)$. From Theorem 1.2 we will deduce

**Theorem 1.3.** *Exact sequences* (1.6) *and* (1.7) *represent the same element in* $\mathrm{Ext}(S)$.

The second implication of Theorem 1.2 is more straightforward:

**Theorem 1.4.** *For any Lipschitz functions $f_1, f_2, \ldots, f_{2n}$ on $S$, the difference*

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] - [R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}], \tag{1.8}$$

*as an operator on $H^2(S)$, is in the trace class with zero trace.*

**Remark 1.** It should be pointed out that the fact that (1.8) is in the trace class actually follows from the work in [26]. What is new for this paper is to show that the trace of (1.8) is zero, and Theorem 1.2 is an indispensable step in this endeavor.

**Remark 2.** Because we only assume that $f_1, f_2, \ldots, f_{2n} \in \mathrm{Lip}(S)$, we do not know if the individual antisymmetric sums

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}], \quad [R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}]$$

are in the trace class. Regardless, Theorem 1.4 tells us that the difference of the two is in the trace class with zero trace.

Next we switch gears and take another look at the Arveson-Douglas conjecture [1,2,10] itself. Write $\zeta_1, \ldots, \zeta_n$ for the coordinate functions on $\mathbf{C}^n$. In our setting, the prediction of the geometric Arveson-Douglas conjecture is that for all $i, j \in \{1, \ldots, n\}$, the commutator $[Q_{\zeta_i}, Q^*_{\zeta_j}]$ is in the Schatten class $\mathcal{C}_p$ for $p > d$, which we proved in [26]. A short way of saying this is that the quotient module $\mathcal{Q}$ is $p$-essentially normal for $p > d$. What has so far eluded all investigators is the fact that the case $d = 1$ is special. In this case, as an unexpected discovery, we report a result that is stronger than the original prediction of the Arveson-Douglas conjecture:

**Theorem 1.5.** *In the case $d = 1$, the quotient module $\mathcal{Q}$ is 1-essentially normal. That is, if $d = 1$, then for every pair of $i, j \in \{1, \ldots, n\}$, the commutator $[Q_{\zeta_i}, Q^*_{\zeta_j}]$ belongs to the trace class $\mathcal{C}_1$.*

The reader will see that the proof of Theorem 1.5 requires a technique that is distinctly different from the techniques in the proofs of Theorems 1.1-1.4.

The rest of the paper is organized as follows. In Section 2 we first record the precise definitions of $\tilde{M}$, $M$, $\mathcal{R}$, $\mathcal{Q}$ etc, and then we collect a number of results that will be needed in the subsequent sections. Section 3 is the key to the proofs of Theorems 1.1-1.4, in which we prove a version of Hardy's inequality on $M$. In Section 4 we introduce the kernel $K_{w,u}$ associated with derivatives in the tangential directions of $M$. Such a $K_{w,u}$ serves our purpose: it has a power $n+1$ in the denominator. Using these kernels, we then construct the $T'$ promised above. The conclusion of Section 4 is that (1.5) indeed holds, which is proved using the inequalities in Section 3.

Section 5 contains estimates for operators that are discrete sums constructed from $K_{w,u}$ over lattices in $\mathbf{B}$. With these estimates established, we prove Theorem 1.1 in Section 6. The proofs of Theorems 1.2, 1.3 and 1.4 are then presented in Sections 7, 8 and 9 respectively.

After that, we turn to the proof of Theorem 1.5. Central to the proof of Theorem 1.5 is the idea of *range space*, which will be the subject of Section 10. With the preparations in Section 10, we prove Theorem 1.5 in Section 11. To conclude the paper, in Section 12 we discuss two examples and an open problem that are related to Theorem 1.5.

## 2. Preliminaries

In this section we present the precise definitions of the analytic sets, submodules and quotient modules, etc, that we consider in this paper. We also collect a number of known results that will be needed in the subsequent sections.

**Definition 2.1.** [6] Let $\Omega$ be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of $\Omega$ if for each point $a \in \Omega$ there are a neighborhood $U$ of $a$ and functions $f_1, \cdots, f_N$ analytic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood $U$ of $a$ in $\Omega$ such that $A \cap U$ is a complex submanifold of $\Omega$. A point $a \in A$ is called a *singular point* of $A$ if it is not regular.

**Assumption 2.2.** Let $\tilde{M}$ be an analytic subset in an open neighborhood of the closed ball $\overline{\mathbf{B}}$. Furthermore, $\tilde{M}$ satisfies the following conditions:

(1) $\tilde{M}$ intersects $\partial\mathbf{B}$ transversely.

(2) $\tilde{M}$ has no singular points on $\partial\mathbf{B}$.

(3) $\tilde{M}$ is of pure dimension $d$, where $1 \leq d \leq n-1$.

We emphasize that Assumption 2.2 will always be in force for the rest of the paper. Given such an $\tilde{M}$, we fix $M$, $\mathcal{R}$, $R$, $\mathcal{Q}$ and $Q$ as follows.

**Notation 2.3.** (a) Let $M = \tilde{M} \cap \mathbf{B}$.

(b) Denote $\mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}$.

(c) Denote $\mathcal{Q} = H^2(S) \ominus \mathcal{R}$.

(d) Let $R$ be the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{R}$.
(e) Let $Q$ be the orthogonal projection from $L^2(S, d\sigma)$ onto $\mathcal{Q}$.

By Assumption 2.2, there is an $s \in (0, 1)$ such that

$$\mathcal{M} = \{z \in \tilde{M} : 1 - s < |z| < 1 + s\} \tag{2.1}$$

is a complex manifold of complex dimension $d$ and of finite volume. We take the value $s \in (0, 1)$ so small that the closure of $\mathcal{M}$ is contained in the regular part of $\tilde{M}$. Thus

$$K = \{z \in \tilde{M} : 1 - (s/2) \le |z| \le 1\}$$

is a compact subset of the complex manifold $\mathcal{M}$.

**Definition 2.4.** (a) We define the measure $v_M$ on $M = \tilde{M} \cap \mathbf{B}$ by the formula $v_M(E) = v_{\mathcal{M}}(E \cap \mathcal{M})$ for Borel sets $E \subset M$, where $v_{\mathcal{M}}$ is the natural volume measure on $\mathcal{M}$.
(b) We define the measure $\mu$ on $M$ by the formula

$$d\mu(w) = (1 - |w|^2)^{n-1-d} dv_M(w).$$

We further extend $\mu$ to a measure on $\mathbf{B}$ by setting $\mu(\mathbf{B} \backslash M) = 0$.

With the measure $\mu$ in Definition 2.4(b), we define the Toeplitz operator $T_\mu$ on the Hardy space $H^2(S)$ by the formula

$$(T_\mu f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^n} d\mu(w),$$

$f \in H^2(S)$. It is straightforward to verify that we can also write $T_\mu$ as

$$T_\mu = \int K_w \otimes K_w d\mu(w), \tag{2.2}$$

where $K_w(z) = (1 - \langle z, w\rangle)^{-n}$, the reproducing kernel for $H^2(S)$. Moreover,

$$\langle T_\mu f, f\rangle = \int |f(w)|^2 d\mu(w)$$

for each $f \in H^2(S)$. If we consider each $K_w$ as a vector in $L^2(S, d\sigma)$, then (2.2) automatically extends $T_\mu$ to an operator on $L^2(S, d\sigma)$.

**Theorem 2.5.** [26, Theorem 3.5] *There are scalars $0 < c \le C < \infty$ such that the operator inequality*

$$cQ \le T_\mu \le CQ$$

*holds on $L^2(S, d\sigma)$.*

As usual, we write $\beta$ for the Bergman metric on unit ball $\mathbf{B}$. For $z \in \mathbf{B}$ and $r > 0$, we denote $D(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}$.

**Definition 2.6.** (i) Let $a$ be a positive number. A subset $\Gamma$ of $\mathbf{B}$ is said to be $a$-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements $z$, $w$ in $\Gamma$.
(ii) A subset $\Gamma$ of $\mathbf{B}$ is simply said to be separated if it is $a$-separated for some $a > 0$.

**Lemma 2.7.** [22, Lemma 2.1] *For any pair of* $0 < a \leq R < \infty$, *there is a natural number* $m = m(a,R)$ *such that every* $a$*-separated set* $\Gamma$ *in* $\mathbf{B}$ *admits a partition* $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ *with the property that each* $\Gamma_j$ *is* $R$*-separated,* $j = 1, \ldots, m$.

**Proposition 2.8.** [26, Proposition 2.14] *For each* $r \geq 1$, *there exist* $0 < c_{2.8}(r) \leq C_{2.8}(r) < \infty$ *such that for every* $z \in M \cap K$, *we have*

$$c_{2.8}(r)(1-|z|^2)^{d+1} \leq v_M(D(z,r)) \leq C_{2.8}(r)(1-|z|^2)^{d+1}.$$

**Lemma 2.9.** [26, Lemma 8.2] *Given any* $\epsilon > 0$, *there is a constant* $0 < C_{2.9} = C_{2.9}(\epsilon) < \infty$ *such that the following holds true: Let* $\Gamma$ *be a* 1*-separated set in* $M \cap K$ *and let* $\{e_z : z \in \Gamma\}$ *be an orthonormal set in a Hilbert space* $\mathcal{H}$. *Then the operator*

$$T = \sum_{z,w\in\Gamma} \frac{(1-|z|^2)^{(d+\epsilon)/2}(1-|w|^2)^{(d+\epsilon)/2}}{|1-\langle z,w\rangle|^{d+\epsilon}} e_z \otimes e_w$$

*satisfies the estimate* $\|T\| \leq C_{2.9}$.

For each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k\geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}}$$

defines a symmetric norm for operators. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [18]. The relation between these ideals and the Schatten classes is well known: For all $1 \leq p < p' < \infty$, we have $\mathcal{C}_p \subset \mathcal{C}_p^+ \subset \mathcal{C}_{p'}$. Thus, for any operator $A$ and for any given $1 \leq t < \infty$, the statement that $A \in \mathcal{C}_p^+$ for every $p > t$ is equivalent to the statement that $A \in \mathcal{C}_p$ for every $p > t$.

The reason why the $\mathcal{C}_p^+$'s are the preferred ideals in the study of the Arveson-Douglas conjecture is that norm estimates in these ideals are particularly easy:

**Lemma 2.10.** [25, Lemma 2.9] *Given any positive numbers* $0 < a \leq b < \infty$, *there is a constant* $0 < B(a,b) < \infty$ *such that the following holds true: Let* $\mathcal{H}$ *be a Hilbert space, and suppose that* $F_0, F_1, \ldots, F_k, \ldots$ *are operators on* $\mathcal{H}$ *such that the following two conditions are satisfied for every* $k$:
(1) $\|F_k\| \leq 2^{-ak}$,
(2) $\text{rank}(F_k) \leq 2^{bk}$.
*Then the operator* $F = \sum_{k=0}^{\infty} F_k$ *satisfies the estimate* $\|F\|_{b/a}^+ \leq B(a,b)$. *In particular,* $F \in \mathcal{C}_{b/a}^+$.

**Lemma 2.11** [25, Lemma 2.8] *Let $T$ be an operator in the weak operator closure of a set of operators $\{T_\alpha : \alpha \in I\}$. Suppose that $T_\alpha \in \mathcal{C}_p^+$ for every $\alpha \in I$ and that*

$$\sup_{\alpha \in I} \|T_\alpha\|_p^+ \le C < \infty.$$

*Then $T \in \mathcal{C}_p^+$ and $\|T\|_p^+ \le C$.*

**Proposition 2.12.** [26, Proposition 8.4] *For any Lipschitz function $f$ on $S$, the commutator $[M_f, Q]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

We conclude the preliminaries with some standard estimates on $S$.

**Lemma 2.13.** (a) *Given any $\eta > 0$, there is a $0 < C_{2.13}^{(1)} = C_{2.13}^{(1)}(\eta) < \infty$ such that*

$$\int \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n+\eta}|1 - \langle \zeta, w\rangle|^{n+\eta}} \le \frac{C_{2.13}^{(1)}}{|1 - \langle z, w\rangle|^{n+\eta}}\left(\frac{1}{(1-|z|^2)^\eta} + \frac{1}{(1-|w|^2)^\eta}\right)$$

*for all $z, w \in \mathbf{B}$.*
(b) *Given any $\epsilon > 0$, there is a $0 < C_{2.13}^{(2)} = C_{2.13}^{(2)}(\epsilon) < \infty$ such that*

$$\int \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n}|1 - \langle \zeta, w\rangle|^{n}} \le \frac{C_{2.13}^{(2)}}{|1 - \langle z, w\rangle|^{n}}\left(\frac{1}{(1-|z|^2)^\epsilon} + \frac{1}{(1-|w|^2)^\epsilon}\right)$$

*for all $z, w \in \mathbf{B}$.*

*Proof.* Recall from [24, Proposition 5.1.2] that the triangle inequality

$$|1 - \langle z, w\rangle|^{1/2} \le |1 - \langle z, \zeta\rangle|^{1/2} + |1 - \langle \zeta, w\rangle|^{1/2}$$

holds for all $z, w \in \mathbf{B}$ and $\zeta \in S$. Thus, given any $z, w \in \mathbf{B}$, if we define

$$A = \{\zeta \in S : |1 - \langle z, \zeta\rangle| \ge (1/4)|1 - \langle z, w\rangle|\} \quad \text{and}$$
$$B = \{\zeta \in S : |1 - \langle \zeta, w\rangle| \ge (1/4)|1 - \langle z, w\rangle|\},$$

then $A \cup B = S$. Hence

$$\int \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n+\eta}|1 - \langle \zeta, w\rangle|^{n+\eta}} \le \frac{4^{n+\eta}}{|1 - \langle z, w\rangle|^{n+\eta}} \int_A \frac{d\sigma(\zeta)}{|1 - \langle \zeta, w\rangle|^{n+\eta}} + \frac{4^{n+\eta}}{|1 - \langle z, w\rangle|^{n+\eta}} \int_B \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n+\eta}}.$$

Now an application of [24, Proposition 1.4.10] completes the proof for (a). The fact $A \cup B = S$ also leads to

$$\int \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n}|1 - \langle \zeta, w\rangle|^{n}} \le \frac{4^n}{|1 - \langle z, w\rangle|^{n}}\left(\int_A \frac{d\sigma(\zeta)}{|1 - \langle \zeta, w\rangle|^{n}} + \int_B \frac{d\sigma(\zeta)}{|1 - \langle \zeta, z\rangle|^{n}}\right),$$

and another application of [24, Proposition 1.4.10] completes the proof for (b). □

## 3. Integral inequalities on $M$

We begin with a classic inequality of Hardy: For $-1 < \alpha < \infty$, $1 \le p < \infty$ and $g \in C_c[0,\infty)$, we have

$$\int_0^\infty \left| \int_x^\infty \frac{1}{t} g(t)dt \right|^p x^\alpha dx \le \left( \frac{p}{\alpha+1} \right)^p \int_0^\infty |g(x)|^p x^\alpha dx.$$

See [3,20], or Google. From this it is easy to deduce that

$$\int_0^\infty |f(x)|^p x^\alpha dx \le \left( \frac{p}{\alpha+1} \right)^p \int_0^\infty |xf'(x)|^p x^\alpha dx \tag{3.1}$$

for every $f \in C_c^\infty[0,\infty)$. We need an analogue of this on $M$ in the case $p = 2$.

For each $z \in \mathcal{M}$, let $T_z$ be the tangent space to $\mathcal{M}$ at the point $z$, which naturally is a subspace of $\mathbf{C}^n$. Under the usual identification of $\mathbf{C}$ with $\mathbf{R}^2$ we can also view $T_z$ as a subspace of $\mathbf{R}^{2n}$ of real dimension $2d$, equipped with the real inner product. Thus if $z \in \mathcal{M}$ and $h$ is a real-valued $C^\infty$-function on an open neighborhood $U$ of $z$ in $\mathbf{C}^n \cong \mathbf{R}^{2n}$, then we define $(\nabla_{\mathcal{M}} h)(z)$ as the orthogonal projection of the real vector $(\nabla h)(z)$ onto the real subspace $T_z$. If $h$ is complex-valued, we can write $h = h_1 + ih_2$, where $h_1$ and $h_2$ are real-valued. In this case, we define $(\nabla_{\mathcal{M}} h)(z) = (\nabla_{\mathcal{M}} h_1)(z) + i(\nabla_{\mathcal{M}} h_2)(z)$. This defines the operation $\nabla_{\mathcal{M}}$. We think of $\nabla_{\mathcal{M}}$ as the gradient in the directions tangent to $\mathcal{M}$.

For each $0 < t < 1$, we define

$$M^{(t)} = \{z \in M : 1 - |z|^2 < t\} \quad \text{and} \quad N^{(t)} = \{z \in M : 1 - |z|^2 \ge t\}. \tag{3.2}$$

**Lemma 3.1.** *There are constants $0 < a < b < 1$ and $0 < C_{3.1} < \infty$ such that if $f$ is any $C^\infty$ function on an open set containing $\overline{\mathbf{B}}$, then*

$$\int_M |f(w)|^2 d\mu(w) \le C_{3.1} \int_{M^{(b)}} |(\nabla_{\mathcal{M}} f)(w)|^2 (1 - |w|^2)^2 d\mu(w) + C_{3.1} \int_{N^{(a)}} |f(w)|^2 d\mu(w).$$

*Proof.* We begin with a $0 < b < 1$ such that $M^{(b)} \subset \mathcal{M}$. Consider the function $r(w) = 1 - |w|^2$. Since $\mathcal{M}$ intersects $S$ transversely, the vector $\nabla_{\mathcal{M}} r$ does not vanish near $\mathcal{M} \cap S$. Thus we can use $r$ as one of the real coordinates on $\mathcal{M}$ near $S$. More precisely, if $\zeta \in \mathcal{M} \cap S$, then $\zeta$ has an open neighborhood $U_\zeta$ in $\mathcal{M}$ that has the following properties:

(1) $U_\zeta = G((-c,c) \times V)$, where $0 < c < b$, $V$ is a bounded open set in $\mathbf{R}^{2d-1}$ and $G : (-c,c) \times V \to \mathbf{C}^n$ is a one-to-one $C^\infty$ map.

(2) there are $0 < \delta < C < \infty$ such that $DG$, the derivative of $G$, satisfies the matrix inequality $\delta \le (DG)^*(x,y)(DG)(x,y) \le C$ for all $x \in (-c,c)$ and $y \in V$.

(3) If $w = G(x,y)$ for some $x \in (-c,c)$ and $y \in V$, then $x = 1 - |w|^2$. Equivalently, for each $w \in U_\zeta$, there is a unique $y_w \in V$ such that $w = G(1 - |w|^2, y_w)$.

Obviously, (3) implies $U_\varsigma \cap M = G((0,c)\times V) \subset M^{(b)}$.

Once we have this $c$, by the standard technique of using a smooth cutoff function, we can apply (3.1) with $p=2$ on the interval $[0,c]$. That is, there are $C_1$ and $C_2$ such that

$$\int_0^c |h(x)|^2 x^{n-1-d}dx \le C_1\int_0^c |xh'(x)|^2 x^{n-1-d}dx + C_2\int_{c/2}^c |h(x)|^2x^{n-1-d}dx \tag{3.3}$$

for every $h \in C^\infty[0,c]$. By the definition of $\mu$ and property (3) above,

$$\begin{aligned}\int_{U_\varsigma\cap M}|f(w)|^2d\mu(w) &= \int_{U_\varsigma\cap M}|f(w)|^2(1-|w|^2)^{n-1-d}dv_M(w)\\ &= \int_V\int_0^c |f(G(x,y))|^2x^{n-1-d}J(x,y)dxdy\\ &\le C_3\int_V\int_0^c |f(G(x,y))|^2x^{n-1-d}dxdy\\ &\le C_4\int_V\int_0^c \left|x\frac{d}{dx}f(G(x,y))\right|^2 x^{n-1-d}dxdy\\ &+ C_5\int_V\int_{c/2}^c |f(G(x,y))|^2x^{n-1-d}dxdy,\end{aligned}\tag{3.4}$$

where the last $\le$ follows from (3.3). By the chain rule for differentiation,

$$\frac{d}{dx}f(G(x,y)) = \langle(\nabla f)(G(x,y)), \tau(x,y)\rangle,$$

where $\tau(x,y)$ is a (real) tangent vector to $\mathcal{M}$ at the point $G(x,y)$. Moreover, (2) implies the bound $|\tau(x,y)| \le C^{1/2}$. Hence $|df(G(x,y))/dx| \le C^{1/2}|(\nabla_{\mathcal{M}}f)(G(x,y))|$. Thus

$$\begin{aligned}\int_V\int_0^c\left|x\frac{d}{dx}f(G(x,y))\right|^2x^{n-1-d}dxdy &\le C\int_V\int_0^c |(\nabla_{\mathcal{M}}f)(G(x,y))|^2x^{2+n-1-d}dxdy\\ &\le C_6\int_V\int_0^c |(\nabla_{\mathcal{M}}f)(G(x,y))|^2x^{n+1-d}J(x,y)dxdy\\ &= C_6\int_{U_\varsigma\cap M}|(\nabla_{\mathcal{M}}f)(w)|^2(1-|w|^2)^{n+1-d}dv_M(w)\\ &= C_6\int_{U_\varsigma\cap M}|(\nabla_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2d\mu(w),\end{aligned}$$

where the third step uses property (3). Combining this with (3.4), we find that

$$\begin{aligned}\int_{U_\varsigma\cap M}|f(w)|^2d\mu(w) &\le C_7\int_{U_\varsigma\cap M}|(\nabla_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2d\mu(w)\\ &+ C_8\int_{N^{(c/2)}}|f(w)|^2d\mu(w).\end{aligned}\tag{3.5}$$

Since $\mathcal{M} \cap S$ is compact, there are $\zeta_1, \dots, \zeta_k \in \mathcal{M} \cap S$ such that the corresponding open sets $U_{\zeta_1}, \dots, U_{\zeta_k}$ have the property $U_{\zeta_1} \cup \cdots \cup U_{\zeta_k} \supset \mathcal{M} \cap S = \{w \in \mathcal{M} : 1 - |w|^2 = 0\}$. Thus $U_{\zeta_1} \cup \cdots \cup U_{\zeta_k} \supset \{w \in \mathcal{M} : -\rho < 1 - |w|^2 < \rho\}$ for some $0 < \rho < 1$. Consequently, $U_{\zeta_1} \cup \cdots \cup U_{\zeta_k} \supset M^{(\rho)}$. Combining this containment with (3.5), the lemma follows. □

As usual, we write $\partial = (\partial_1, \dots, \partial_n)$, the analytic gradient on $\mathbf{C}^n$. For any $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{C}^n$, we denote

$$\partial_\xi = \xi_1 \partial_1 + \cdots + \xi_n \partial_n.$$

Let $z \in \mathcal{M}$, and let $f$ be an analytic function on an open neighborhood $W$ of $z$ in $\mathbf{C}^n$. Now we view the tangent space $T_z$ naturally as a complex subspace of $\mathbf{C}^n$. Let $\{u_1, \dots, u_d\}$ be any orthonormal basis for $T_z$. We define

$$|(\partial_{\mathcal{M}} f)(z)|^2 = |(\partial_{u_1} f)(z)|^2 + \cdots + |(\partial_{u_d} f)(z)|^2.$$

It is easy to see that this value is independent of the choice of the orthonormal basis $\{u_1, \dots, u_d\}$ for $T_z$, which justifies the notation $|(\partial_{\mathcal{M}} f)(z)|^2$.

Let $z \in \mathcal{M}$ and let $h$ be a $C^\infty$-function on an open neighborhood $U$ of $z$ in $\mathbf{C}^n \cong \mathbf{R}^{2n}$. By our definition of $\nabla_{\mathcal{M}}$, there is a $C^1$ map $\gamma : (-c, c) \to \mathcal{M}$ such that $\gamma(0) = z$, $|\gamma'(0)| = 1$ and

$$|(\nabla_{\mathcal{M}} h)(z)| \le \sqrt{2} \left| \frac{d}{dt} h(\gamma(t)) \Big|_{t=0} \right|.$$

From this fact we see that if $z \in \mathcal{M}$ and if $f$ is an analytic function on an open neighborhood $W$ of $z$ in $\mathbf{C}^n$, then

$$|(\nabla_{\mathcal{M}} f)(z)|^2 \le 2 |(\partial_{\mathcal{M}} f)(z)|^2.$$

Combining this inequality with Lemma 3.1, we obtain

**Corollary 3.2.** *If $f$ is an analytic function on an open set containing $\overline{\mathbf{B}}$, then*

$$\int_M |f(w)|^2 d\mu(w) \le 2C_{3.1} \int_{M^{(b)}} |(\partial_{\mathcal{M}} f)(w)|^2 (1 - |w|^2)^2 d\mu(w) + C_{3.1} \int_{N^{(a)}} |f(w)|^2 d\mu(w),$$

*where $0 < a < b < 1$ and $0 < C_{3.1} < \infty$ are the constants given in Lemma* 3.1.

**Lemma 3.3.** *Define the measure $\nu$ by the formula*

$$d\nu(w) = (1 - |w|^2)^2 d\mu(w). \tag{3.6}$$

*Then $\nu$ is a Carleson measure for the weighted Bergman space $L^2_{a,1} = L^2_a(\mathbf{B}, (1-|z|^2)dv(z))$.*

*Proof.* For each pair of $\zeta \in S$ and $r > 0$, define $Q(\zeta, r) = \{z \in \mathbf{B} : |1 - \langle z, \zeta \rangle| < r\}$. By the well-known [7, Theorem 1], to show that $\nu$ is a Carleson measure for $L^2_{a,1}$, it suffices to find a $C_1$ such that

$$\nu(Q(\zeta, r)) \le C_1 r^{n+2} \tag{3.7}$$

for all $\zeta \in S$ and $r > 0$. We have

$$\nu(Q(\zeta,r)) = \int_{Q(\zeta,r)\cap M} (1-|w|^2)^2 d\mu(w) \le \sup_{w\in Q(\zeta,r)} (1-|w|^2)^2 \mu(Q(\zeta,r)) \le 4r^2\mu(Q(\zeta,r)).$$

It was shown in the proof of [26, Proposition 2.13] that

$$\mu(Q(\zeta,r)) \le Cr^n$$

for all $\zeta \in S$ and $r > 0$. Thus (3.7) indeed holds. □

**Proposition 3.4.** *There is a constant* $0 < C_{3.4} < \infty$ *such that*

$$\int_M |(\partial_j f)(w)|^2(1-|w|^2)^2 d\mu(w) \le C_{3.4}\|f\|^2$$

*for every* $f \in H^2(S)$ *and every* $j \in \{1,\dots,n\}$.

*Proof.* By Lemma 3.3, we have

$$\int_M |(\partial_j f)(w)|^2(1-|w|^2)^2 d\mu(w) \le C \int_{\mathbf{B}} |(\partial_j f)(z)|^2(1-|z|^2)dv(z)$$

for $f \in H^2(S)$ and $j \in \{1,\dots,n\}$. But it is well known that

$$\int_{\mathbf{B}} |(\partial_j f)(z)|^2(1-|z|^2)dv(z) \le C_1\|f\|^2$$

for $f \in H^2(S)$ and $j \in \{1,\dots,n\}$. This completes the proof. □

## 4. Modifying the reproducing kernel

We will now try to modify the Hardy-space reproducing kernel $K_w(\zeta)$ to suit our need. For each pair of $w \in \mathbf{B}$ and $u \in \mathbf{C}^n$, we define

$$K_{w,u}(\zeta) = \frac{n\langle \zeta, u\rangle}{(1-\langle \zeta, w\rangle)^{n+1}}.$$

For $w \in \mathbf{B}$, $u \in \mathbf{C}^n$ and $f \in H^2(S)$, it is easy to see that

$$\langle f, K_{w,u}\rangle = \frac{d}{dt}\langle f, K_{w+tu}\rangle\big|_{t=0} = \frac{d}{dt} f(w+tu)\big|_{t=0} = (\partial_u f)(w). \tag{4.1}$$

That is, $K_{w,u}$ is the reproducing kernel for the directional derivative $\partial_u$.

**Lemma 4.1.** *Let* $w \in M^{(b)}$. *If* $u \in T_w$, *then* $K_{w,u} \in \mathcal{Q}$.

*Proof.* Since $u \in T_w$, there is a smooth path $\gamma : (-c,c) \to M^{(b)}$ such that $\gamma(0) = w$ and $\gamma'(0) = u$. Thus

$$K_{w,u} = \frac{d}{dt} K_{\gamma(t)}\big|_{t=0}.$$

Let $f \in \mathcal{R}$. Since the range of $\gamma$ is contained in $M$, we have $\langle f, K_{\gamma(t)}\rangle = 0$ for every $t \in (-c, c)$. Therefore

$$\langle f, K_{w,u}\rangle = \frac{d}{dt}\langle f, K_{\gamma(t)}\rangle\big|_{t=0} = \frac{d}{dt}0\big|_{t=0} = 0.$$

This shows that $K_{w,u} \perp \mathcal{R}$. That is, $K_{w,u} \in \mathcal{Q}$. □

Since the regular part of $\tilde{M}$ is a complex manifold, there are local frames. That is, if $z$ is a regular point of $\tilde{M}$, then there exist an open neighborhood $W_z$ of $z$ in $\tilde{M}$ and continuous maps $u_{z,1}, \dots, u_{z,d} : W_z \to \mathbf{C}^n$ such that for each $w \in W_z$, $\{u_{z,1}(w), \dots, u_{z,d}(w)\}$ is an orthonormal basis for $T_w$.

By the choice of $s \in (0, 1)$ in (2.1), the closure of $\mathcal{M}$ is contained in the regular part of $\tilde{M}$. Therefore the subset $M^{(b)}$ of $\mathcal{M} \cap M$ is covered by a finite number of $W_z$'s described in the above paragraph. Consequently, there is a finite subset $F$ of $\tilde{M}$ and a corresponding family of Borel subsets $\{V_z : z \in F\}$ of $\mathcal{M}$ such that the following hold true:

(1) $V_z \subset W_z \cap M^{(b)}$ for every $z \in F$.

(2) $V_z \cap V_{z'} = \emptyset$ for all $z \neq z'$ in $F$.

(3) $\cup_{z\in F} V_z = M^{(b)}$.

**Definition 4.2.** (a) With the finite set $F$ and the family $\{V_z : z \in F\}$ described above, we define the operator

$$Y = \sum_{z\in F}\sum_{i=1}^{d}\int_{V_z} K_{w,u_{z,i}(w)} \otimes K_{w,u_{z,i}(w)}(1 - |w|^2)^2 d\mu(w).$$

(b) With the number $a$ given in Lemma 3.1, we define

$$Z = \int_{N^{(a)}} K_w \otimes K_w d\mu(w).$$

Obviously, both $Y$ and $Z$ are positive operators. We have

$$\mathrm{tr}(Z) = \int_{N^{(a)}} \langle K_w, K_w\rangle d\mu(w) = \int_{N^{(a)}} \frac{d\mu(w)}{(1 - |w|^2)^n} < \infty$$

(cf. (3.2)). In other words, $Z$ is in the trace class.

**Proposition 4.3.** *There are* $0 < c \leq C < \infty$ *such that*

$$cQ \leq Y + Z \leq CQ.$$

*Proof.* We first show that $Y$ is a bounded operator on $H^2(S)$. By (4.1), for $f \in H^2(S)$,

$$\begin{aligned}\langle Yf, f\rangle &= \sum_{z\in F}\sum_{i=1}^{d}\int_{V_z} |\langle f, K_{w,u_{z,i}(w)}\rangle|^2(1-|w|^2)^2 d\mu(w)\\ &= \sum_{z\in F}\sum_{i=1}^{d}\int_{V_z} |(\partial_{u_{z,i}(w)}f)(w)|^2(1-|w|^2)^2 d\mu(w)\\ &\le \sum_{z\in F}\sum_{i=1}^{d}\int_{V_z}\sum_{j=1}^{n} |(\partial_j f)(w)|^2(1-|w|^2)^2 d\mu(w)\\ &\le \mathrm{card}(F)d\sum_{j=1}^{n}\int_M |(\partial_j f)(w)|^2(1-|w|^2)^2 d\mu(w).\end{aligned}$$

From Proposition 3.4 we now obtain $\langle Yf, f\rangle \le \mathrm{card}(F)dnC_{3.4}\|f\|^2$, $f \in H^2(S)$. Thus $Y$ is indeed a bounded operator on $H^2(S)$. By Lemma 4.1, the range of $Y$ is contained in $\mathcal{Q}$. The range of $Z$ is, of course, also contained in $\mathcal{Q}$. Therefore we have the upper bound $Y + Z \le CQ$ for some $0 < C < \infty$.

To prove the lower bound, we use the fact that for each pair of $z \in F$ and $w \in V_z$, $\{u_{z,1}(w), \dots, u_{z,d}(w)\}$ is an orthonormal basis for $T_w$. Therefore, if $f \in H^2(S)$, then

$$\sum_{i=1}^{d} |(\partial_{u_{z,i}(w)}f)(w)|^2 = |(\partial_{\mathcal{M}}f)(w)|^2$$

for each pair of $z \in F$ and $w \in V_z$. Consequently,

$$\begin{aligned}\langle Yf, f\rangle &= \sum_{z\in F}\sum_{i=1}^{d}\int_{V_z} |(\partial_{u_{z,i}(w)}f)(w)|^2(1-|w|^2)^2 d\mu(w)\\ &= \sum_{z\in F}\int_{V_z} |(\partial_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2 d\mu(w) = \int_{M^{(b)}} |(\partial_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2 d\mu(w),\end{aligned}$$

where the last $=$ follows from properties (2) and (3) of the family $\{V_z : z \in F\}$. Hence

$$\langle (Y+Z)f, f\rangle = \int_{M^{(b)}} |(\partial_{\mathcal{M}}f)(w)|^2(1-|w|^2)^2 d\mu(w) + \int_{N^{(a)}} |f(w)|^2 d\mu(w),$$

$f \in H^2(S)$. Applying Corollary 3.2, we obtain the inequality

$$2C_{3.1}\langle (Y+Z)f, f\rangle \ge \int_M |f(w)|^2 d\mu(w) = \langle T_\mu f, f\rangle.$$

Thus the desired lower bound for $Y + Z$ follows from the lower bound in Theorem 2.5. □

## 5. Estimates of various norms

In this section we establish bounds for operators that are discrete sums constructed from the $K_{w,u}$ introduced in Section 4. For this purpose, a simplification of notation will be beneficial. For each pair of $z \in \mathbf{B}$ and $u \in \mathbf{C}^n$, we write

$$k_{z,u} = (1-|z|^2)^{1+(n/2)} K_{z,u}. \tag{5.1}$$

**Lemma 5.1.** *There is a constant $0 < C_{5.1} < \infty$ such that the following bound holds: Let $\Gamma$ be any 1-separated set contained in $M$. For each $z \in \Gamma$, let $u(z) \in \mathbf{C}^n$ be such that $|u(z)| \leq 1$. Let $\{c_z : z \in \Gamma\}$ be any bounded set of complex coefficients. Then*

$$\left\| \sum_{z \in \Gamma} c_z k_{z,u(z)} \otimes e_z \right\| \leq C_{5.1} \sup_{z \in \Gamma} |c_z|,$$

*where $\{e_z : z \in \Gamma\}$ is any orthonormal set.*

*Proof.* If $\Gamma$ is 1-separated, then for each $\delta > 0$, $\mathrm{card}(\Gamma \cap N^{(\delta)})$ is bounded by a constant determined solely by $\delta$. Therefore we only need to consider the case where $\Gamma \subset M^{(\delta)}$ for a sufficiently small $\delta > 0$. That is, we only need to consider those $\Gamma$ to which Lemma 2.9 is applicable.

With such a $\Gamma$, write

$$A = \sum_{z \in \Gamma} c_z k_{z,u(z)} \otimes e_z.$$

Then

$$A^*A = \sum_{w,z \in \Gamma} a(z,w) e_z \otimes e_w,$$

where

$$a(z,w) = \bar{c}_z c_w \langle k_{w,u(w)}, k_{z,u(z)} \rangle,$$

$z, w \in \Gamma$. By (4.1), we have

$$\begin{aligned}\langle K_{w,u(w)}, K_{z,u(z)} \rangle &= (\partial_{u(z)} K_{w,u(w)})(z) \\ &= \frac{n(n+1)\langle z, u(w)\rangle \langle u(z), w\rangle}{(1-\langle z,w\rangle)^{n+2}} + \frac{n\langle u(z), u(w)\rangle}{(1-\langle z,w\rangle)^{n+1}}.\end{aligned}$$

Thus there is a constant $C_1$ such that

$$|a(z,w)| \leq C_1 c^2 \left( \frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle z,w\rangle|} \right)^{n+2}$$

for $z, w \in \Gamma$, where $c = \sup_{z \in \Gamma} |c_z|$. Combining this with Lemma 2.9, the desired conclusion follows. □

Given any integer $k \geq 0$, we define

$$M_k = \{z \in M : 1 - 2^{-2k} \leq |z| < 1 - 2^{-2(k+1)}\} \tag{5.2}$$

as in [25,26]. If $f \in \mathrm{Lip}(S)$, we write $L(f)$ for its Lipschitz constant.

**Lemma 5.2.** *There is a constant $0 < C_{5.2} < \infty$ such that the following bound holds: Let $k \geq 0$, and let $\Gamma$ be any 1-separated set contained in $M_k$. For each $z \in \Gamma$, let $u(z) \in \mathbf{C}^n$ be such that $|u(z)| \leq 1$. Let $\{c_z : z \in \Gamma\}$ be any set of complex coefficients, and let $f \in \mathrm{Lip}(S)$. Then*

$$\Big\| \sum_{z \in \Gamma} c_z (f - f(z/|z|)) k_{z,u(z)} \otimes e_z \Big\| \leq C_{5.2} \sup_{z \in \Gamma} |c_z| L(f) 2^{-k},$$

*where $\{e_z : z \in \Gamma\}$ is any orthonormal set.*

*Proof.* As in the previous lemma, we only need to consider $k$ large enough so that Lemma 2.9 can be applied to 1-separated $\Gamma \subset M_k$. This time, we write

$$B = \sum_{z \in \Gamma} c_z (f - f(z/|z|)) k_{z,u(z)} \otimes e_z.$$

Then

$$B^* B = \sum_{w,z \in \Gamma} b(z,w) e_z \otimes e_w,$$

where

$$b(z,w) = \bar{c}_z c_w \langle (f - f(w/|w|)) k_{w,u(w)}, (f - f(z/|z|)) k_{z,u(z)} \rangle,$$

$z, w \in \Gamma$. It is easy to see that

$$|f(\zeta) - f(z/|z|)| \leq 2L(f) |1 - \langle \zeta, z \rangle|^{1/2} \tag{5.3}$$

for $\zeta \in S$. Hence, writing $c = \sup_{z \in \Gamma} |c_z|$, we have

$$\begin{aligned} |b(z,w)| &\leq 4L^2(f) c^2 \{(1 - |z|^2)(1 - |w|^2)\}^{1+(n/2)} \int \frac{n^2 d\sigma(\zeta)}{|1 - \langle \zeta, w \rangle|^{n+(1/2)} |1 - \langle \zeta, z \rangle|^{n+(1/2)}} \\ &\leq 4n^2 L^2(f) c^2 C \frac{\{(1 - |z|^2)(1 - |w|^2)\}^{1+(n/2)}}{|1 - \langle z, w \rangle|^{n+(1/2)}} \left( \frac{1}{(1 - |z|^2)^{1/2}} + \frac{1}{(1 - |w|^2)^{1/2}} \right), \end{aligned}$$

where the second $\leq$ follows from Lemma 2.13(a). For $z, w \in \Gamma \subset M_k$, we have

$$2^{-2(k+1)} \leq 1 - |z|^2 \leq 2^{-2k+1} \quad \text{and} \quad 2^{-2(k+1)} \leq 1 - |w|^2 \leq 2^{-2k+1}. \tag{5.4}$$

Therefore

$$|b(z,w)| \leq 4(1 + 8^{1/4}) n^2 C L^2(f) c^2 2^{-2k+1} \left( \frac{(1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2}}{|1 - \langle z, w \rangle|} \right)^{n+(1/2)},$$

$z, w \in \Gamma$. Thus it follows from Lemma 2.9 that $\|B^*B\| \le C_1 L^2(f)c^2 2^{-2k}$. Since $\|B^*B\| = \|B\|^2$, the lemma is proved. $\square$

**Lemma 5.3.** *Let any $0 < \epsilon < 1$ be given. There is a constant $0 < C_{5.3} = C_{5.3}(\epsilon) < \infty$ such that the following bound holds: Let $k \ge 0$, and let $\Gamma$ be any 1-separated set contained in $M_k$. For each $z \in \Gamma$, let $u(z) \in \mathbf{C}^n$ be such that $|u(z)| \le 1$. Let $\{c_z : z \in \Gamma\}$ be any set of complex coefficients, and let $f, g \in \mathrm{Lip}(S)$. Then*

$$\left\| \sum_{z\in\Gamma} c_z (f - f(z/|z|))(g - g(z/|z|)) k_{z,u(z)} \otimes e_z \right\| \le C_{5.3} \sup_{z\in\Gamma} |c_z| L(f) L(g) 2^{-2k(1-\epsilon)},$$

*where $\{e_z : z \in \Gamma\}$ is any orthonormal set.*

*Proof.* As before, we only need to consider $k$ such that Lemma 2.9 can be applied to 1-separated $\Gamma \subset M_k$. Now we write

$$H = \sum_{z\in\Gamma} c_z (f - f(z/|z|))(g - g(z/|z|)) k_{z,u(z)} \otimes e_z.$$

Then

$$H^*H = \sum_{w,z\in\Gamma} h(z, w) e_z \otimes e_w,$$

where

$$h(z, w) = \bar{c}_z c_w \langle (f - f(w/|w|))(g - g(w/|w|)) k_{w,u(w)}, (f - f(z/|z|))(g - g(z/|z|)) k_{z,u(z)} \rangle,$$

$z, w \in \Gamma$. Again, we write $c = \sup_{z\in\Gamma} |c_z|$. This time, (5.3) and (5.4) lead to

$$\begin{aligned}
|h(z, w)| &\le 16 L^2(f) L^2(g) c^2 \{(1 - |z|^2)(1 - |w|^2)\}^{1+(n/2)} \int \frac{n^2 d\sigma(\zeta)}{|1 - \langle \zeta, w\rangle|^n |1 - \langle \zeta, z\rangle|^n} \\
&\le 16 n^2 L^2(f) L^2(g) c^2 C(\epsilon) \frac{\{(1 - |z|^2)(1 - |w|^2)\}^{1+(n/2)}}{|1 - \langle z, w\rangle|^n} \left( \frac{1}{(1 - |z|^2)^{2\epsilon}} + \frac{1}{(1 - |w|^2)^{2\epsilon}} \right) \\
&\le 16 n^2 L^2(f) L^2(g) c^2 C(\epsilon) \left( \frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|1 - \langle z, w\rangle|} \right)^n \{(1 - |z|^2)(1 - |w|^2)\}^{1-\epsilon}(1 + 8^\epsilon) \\
&\le C_1 L^2(f) L^2(g) c^2 \left( \frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|1 - \langle z, w\rangle|} \right)^n 2^{-4k(1-\epsilon)},
\end{aligned}$$

where the second $\le$ follows from Lemma 2.13(b). Applying Lemma 2.9 again, we have

$$\|H^*H\| \le C_2 L^2(f) L^2(g) c^2 2^{-4k(1-\epsilon)}.$$

Since $\|H^*H\| = \|H\|^2$, the proof is complete. $\square$

**Proposition 5.4.** (a) *Let any $0 < \epsilon < 1$ be given. There is a constant $0 < C_{5.4.1} = C_{5.4.1}(\epsilon) < \infty$ such that the following bound holds: Let $k \ge 0$, and let $\Gamma$ be any 1-separated* *set contained in $M_k$. For each $z \in \Gamma$, let $u(z) \in \mathbf{C}^n$ be such that $|u(z)| \leq 1$. Given a set of complex coefficients $\{c_z : z \in \Gamma\}$, define the operator*

$$T = \sum_{z\in\Gamma} c_z k_{z,u(z)} \otimes k_{z,u(z)}. \tag{5.5}$$

*Then for $f, g \in \mathrm{Lip}(S)$, we have*

$$\|[M_f,[M_g,T]]\| \leq C_{5.4.1} \sup_{z\in\Gamma} |c_z| L(f)L(g)2^{-2k(1-\epsilon)}.$$

(b) *There is a constant $0 < C_{5.4.2} < \infty$ such that for every $T$ given by* (5.5) *and every $f \in \mathrm{Lip}(S)$, we have*

$$\|[M_f,T]\| \leq C_{5.4.2} \sup_{z\in\Gamma} |c_z| L(f)2^{-k}.$$

*Proof.* (a) It is easy to see that $[M_f,[M_g,T]] = A - B - C + D$, where

$$A = \sum_{z\in\Gamma} c_z\{(f - f(z/|z|))(g - g(z/|z|))k_{z,u(z)}\} \otimes k_{z,u(z)},$$
$$B = \sum_{z\in\Gamma} c_z\{(g - g(z/|z|))k_{z,u(z)}\} \otimes \{(\bar{f} - \bar{f}(z/|z|))k_{z,u(z)}\},$$
$$C = \sum_{z\in\Gamma} c_z\{(f - f(z/|z|))k_{z,u(z)}\} \otimes \{(\bar{g} - \bar{g}(z/|z|))k_{z,u(z)}\} \quad \text{and}$$
$$D = \sum_{z\in\Gamma} c_z k_{z,u(z)} \otimes \{(\bar{f} - \bar{f}(z/|z|))(\bar{g} - \bar{g}(z/|z|))k_{z,u(z)}\}.$$

Taking any orthonormal set $\{e_z : z \in \Gamma\}$, we have the factorization $A = A_1A_2$, where

$$A_1 = \sum_{z\in\Gamma} c_z\{(f - f(z/|z|))(g - g(z/|z|))k_{z,u(z)}\} \otimes e_z \quad \text{and} \quad A_2 = \sum_{z\in\Gamma} e_z \otimes k_{z,u(z)}.$$

Writing $c = \sup_{z\in\Gamma} |c_z|$, we have $\|A_1\| \leq C_{5.3}L(f)L(g)c2^{-2k(1-\epsilon)}$ by Lemma 5.3 and $\|A_2\| \leq C_{5.1}$ by Lemma 5.1. Hence

$$\|A\| \leq \|A_1\|\|A_2\| \leq C_{5.1}C_{5.3}L(f)L(g)c2^{-2k(1-\epsilon)}.$$

For $B$, we have the factorization $B = B_1B_2$, where

$$B_1 = \sum_{z\in\Gamma}\{(g - g(z/|z|))k_{z,u(z)}\} \otimes e_z \quad \text{and} \quad B_2 = \sum_{z\in\Gamma} c_z e_z \otimes \{(\bar{f} - \bar{f}(z/|z|))k_{z,u(z)}\}.$$

Now Lemma 5.2 gives us $\|B_1\| \leq C_{5.2}L(g)2^{-k}$ and $\|B_2\| \leq C_{5.2}cL(f)2^{-k}$. Therefore

$$\|B\| \leq \|B_1\|\|B_2\| \leq C_{5.2}^2 L(f)L(g)c2^{-2k}.$$

Since $C$ is just another $B$ and $D^*$ another $A$, this completes the proof for (a).

(b) For the single commutator, we have $[M_f, T] = U - V$, where

$$U = \sum_{z\in\Gamma} c_z\{(f - f(z/|z|))k_{z,u(z)}\} \otimes k_{z,u(z)} \quad \text{and}$$
$$V = \sum_{z\in\Gamma} c_z k_{z,u(z)} \otimes \{(\bar{f} - \bar{f}(z/|z|))k_{z,u(z)}\}.$$

This time, we have the factorization $V = A_2^* B_2$, where $A_2$ and $B_2$ are the same as in (a). Therefore it follows from Lemmas 5.1 and 5.2 that

$$\|V\| \le \|A_2^*\|\|B_2\| \le C_{5.1}C_{5.2}cL(f)2^{-k}.$$

The proof is complete upon the observation that, this time, $U^*$ is just another $V$. $\square$

**Proposition 5.5.** (a) *Let any $p > d$ be given. There is a constant $0 < C_{5.5.1} = C_{5.5.1}(p) < \infty$ such that the following bound holds: Let $\Gamma$ be any 1-separated set contained in $M$. For each $z \in \Gamma$, let $u(z) \in \mathbf{C}^n$ be such that $|u(z)| \le 1$. Given a set of complex coefficients $\{c_z : z \in \Gamma\}$, define the operator*

$$W = \sum_{z\in\Gamma} c_z k_{z,u(z)} \otimes k_{z,u(z)}.$$

*Then for $f, g \in \mathrm{Lip}(S)$, we have*

$$\|[M_f, [M_g, W]]\|_p^+ \le C_{5.5.1} \sup_{z\in\Gamma} |c_z| L(f)L(g).$$

(b) *There is a constant $0 < C_{5.5.2} < \infty$ such that for every $W$ given as above and every $f \in \mathrm{Lip}(S)$, we have*

$$\|[M_f, W]\|_{2d}^+ \le C_{5.5.2} \sup_{z\in\Gamma} |c_z| L(f).$$

*Proof.* (a) Given any $p > d$, we pick an $\epsilon > 0$ such that

$$(1-\epsilon)p > d.$$

Given a 1-separated set $\Gamma$ in $M$, we define $\Gamma_k = \Gamma \cap M_k$ for each $k \ge 0$. It is known that

$$\text{(5.6)} \qquad \mathrm{card}(\Gamma_k) \le C2^{2dk} \quad \text{for every} \quad k \ge 0.$$

See [25, page 1080]. For each $k \ge 0$, we define

$$W_k = \sum_{z\in\Gamma_k} c_z k_{z,u(z)} \otimes k_{z,u(z)}.$$

For $f, g \in \mathrm{Lip}(S)$, Proposition 5.4(a) provides the bound

$$\|[M_f, [M_g, W_k]]\| \le C_{5.4.1} c L(f) L(g) 2^{-2k(1-\epsilon)} \tag{5.7}$$

for every $k \ge 0$, where $c = \sup_{z \in \Gamma} |c_z|$. Obviously, (5.6) implies that

$$\mathrm{rank}([M_f, [M_g, W_k]]) \le 4C 2^{2dk}, \tag{5.8}$$

$k \ge 0$. Since

$$[M_f, [M_g, W]] = \sum_{k=0}^{\infty} [M_f, [M_g, W_k]],$$

from (5.7), (5.8) and Lemma 2.10 we obtain

$$\|[M_f, [M_g, W]]\|^+_{d/(1-\epsilon)} \le B(2(1-\epsilon), 2d)(1 + 4C) C_{5.4.1} c L(f) L(g).$$

Since $d/(1-\epsilon) < p$, this proves (a).

(b) Let $W_k$, $k \ge 0$, be the same as in part (a). For $f \in \mathrm{Lip}(S)$, Proposition 5.4(b) provides the bound

$$\|[M_f, W_k]\| \le C_{5.4.2} c L(f) 2^{-k} \tag{5.9}$$

for every $k \ge 0$. By (5.6), we have

$$\mathrm{rank}([M_f, W_k]) \le 2C 2^{2dk}, \tag{5.10}$$

$k \ge 0$. Since

$$[M_f, W] = \sum_{k=0}^{\infty} [M_f, W_k],$$

from (5.9), (5.10) and Lemma 2.10 we obtain

$$\|[M_f, W]\|^+_{2d} \le B(1, 2d)(1 + 2C) C_{5.4.2} c L(f).$$

This completes the proof. □

## 6. Double commutators

Let $\mathcal{X}$ denote the collection of operators of the form

$$X = \int_V K_{w,u(w)} \otimes K_{w,u(w)} (1 - |w|^2)^2 d\mu(w), \tag{6.1}$$

where $V$ is any Borel subset of $M^{(b)}$ (cf. Lemma 3.1) and $u : V \to \mathbf{C}^n$ is any continuous map satisfying the condition $|u(w)| \le 1$ for every $w \in V$. By the argument in the proof of the upper bound in Proposition 4.3, each $X \in \mathcal{X}$ is a bounded operator on $H^2(S)$. By (5.1) and Definition 2.4, we can rewrite (6.1) as

$$X = \int_V k_{w,u(w)} \otimes k_{w,u(w)} \frac{dv_M(w)}{(1-|w|^2)^{d+1}}. \tag{6.2}$$

The following is the main technical result of the section:

**Proposition 6.1.** (a) *Given any $p > d$, there is a $0 < C_{6.1.1} = C_{6.1.1}(p) < \infty$ such that*

$$\|[M_f, [M_g, X]]\|_p^+ \le C_{6.1.1} L(f) L(g)$$

*for all $X \in \mathcal{X}$ and $f, g \in \mathrm{Lip}(S)$.*
(b) *There is a $0 < C_{6.1.2} < \infty$ such that*

$$\|[M_f, X]\|_{2d}^+ \le C_{6.1.2} L(f)$$

*for all $X \in \mathcal{X}$ and $f \in \mathrm{Lip}(S)$.*

The basic idea for the proof of Proposition 6.1 is to bring the estimates for discrete sums in Section 5 and Lemma 2.11 to bear. This involves a standard covering scheme using balls with respect to Bergman metric.

Let a Borel subset $V$ of $M^{(b)}$ be given as in (6.1) and (6.2), and suppose that $V \neq \emptyset$. We choose a subset $\mathcal{L}_0$ of $V$ that is maximal with respect to the property

$$D(z,1) \cap D(w,1) = \emptyset \quad \text{for all} \quad z \neq w \in \mathcal{L}_0.$$

This maximality implies that $\cup_{z\in\mathcal{L}_0} D(z,2) \supset V$. By a standard construction, we obtain a family of Borel sets $\{\Delta_z : z \in \mathcal{L}_0\}$ satisfying the following conditions:

(1) $D(z,1) \cap V \subset \Delta_z \subset D(z,2)$ for every $z \in \mathcal{L}_0$.
(2) $\Delta_z \cap \Delta_w = \emptyset$ for $z \neq w$ in $\mathcal{L}_0$.
(3) $\cup_{z\in\mathcal{L}_0} \Delta_z = V$.

For each $z \in \mathcal{L}_0$, define

$$\alpha_z = \int_{\Delta_z} \frac{dv_M(w)}{(1-|w|^2)^{d+1}}.$$

By (1) and Proposition 2.8, we have

$$\alpha_z \le C_{6.3} \quad \text{for every} \quad z \in \mathcal{L}_0. \tag{6.3}$$

We then define $\mathcal{L} = \{z \in \mathcal{L}_0 : \alpha_z > 0\}$.

For each $z \in \mathcal{L}$, we define the measure

$$d\nu_z(w) = \frac{\chi_{\Delta_z}(w) dv_M(w)}{\alpha_z(1-|w|^2)^{d+1}}.$$

We can rewrite (6.2) as

$$X = \sum_{z\in\mathcal{L}} \alpha_z X_z, \quad \text{where} \;\; X_z = \int_{\Delta_z} k_{w,u(w)} \otimes k_{w,u(w)} d\nu_z(w).$$

For each $z \in \mathcal{L}$, $\nu_z$ is a probability measure concentrated on $\Delta_z$. Therefore $\nu_z$ can be approximated in the weak-* topology by measures of the form $(1/k)\sum_{j=1}^{k}\delta_{w_j}$ with $w_1, \dots, w_k \in \Delta_z$. Consequently, each $X_z$ is in the weak closure of operators of the form

$$\frac{1}{k}\sum_{j=1}^{k} k_{w_j,u(w_j)} \otimes k_{w_j,u(w_j)}, \quad \text{where} \;\; w_1, \dots, w_k \in \Delta_z.$$

Summarizing the above, we have established the following:

**Lemma 6.2.** *The operator $X$ given by* (6.2) *is in the weak closure of convex combinations of operators of the form*

$$\Sigma = \sum_{z\in G} \alpha_z k_{w(z),u(w(z))} \otimes k_{w(z),u(w(z))}, \tag{6.4}$$

*where $G$ is any finite subset of $\mathcal{L}$ and $w(z) \in \Delta_z$ for every $z \in G$.*

*Proof of Proposition* 6.1. Let $p > d$ be given. By Lemmas 6.2 and 2.11, to prove (a), it suffices to find a $0 < C_{6.1.1} < \infty$ such that

$$\|[M_f, [M_g, \Sigma]]\|_p^+ \le C_{6.1.1} L(f) L(g) \tag{6.5}$$

for every $\Sigma$ given by (6.4) and every pair of $f, g \in \mathrm{Lip}(S)$. Similarly, to prove (b), it suffices to find a $0 < C_{6.1.2} < \infty$ such that

$$\|[M_f, \Sigma]\|_{2d}^+ \le C_{6.1.2} L(f) \tag{6.6}$$

for every $\Sigma$ given by (6.4) and every $f \in \mathrm{Lip}(S)$.

We begin with Lemma 2.7, which tells us that the 1-separated set $\mathcal{L}$ admits a partition

$$\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_\ell$$

such that for each $j \in \{1, \dots, \ell\}$, the set $\mathcal{L}_j$ has the property that $\beta(z, z') > 6$ for all $z \ne z'$ in $\mathcal{L}_j$. We emphasize that this $\ell$ is completely determined by the numbers $n$, 1 and 6.

For the $G \subset \mathcal{L}$ in (6.4), define $G_j = G \cap \mathcal{L}_j$, $1 \le j \le \ell$. Consider any $G_j$. For $z \in G_j$, since $w(z) \in \Delta_z \subset D(z, 2)$, we have $\beta(z, w(z)) < 2$. Thus for any $z \ne z'$ in $G_j$,

$$6 < \beta(z, z') \le \beta(z, w(z)) + \beta(w(z), w(z')) + \beta(z', w(z')) < 4 + \beta(w(z), w(z')).$$

This shows that for every $j \in \{1, \dots, \ell\}$, the set $\{w(z) : z \in G_j\}$ is 1-separated. For the $\Sigma$ given by (6.4), we have $\Sigma = \Sigma_1 + \cdots + \Sigma_\ell$, where

$$\Sigma_j = \sum_{z \in G_j} \alpha_z k_{w(z),u(w(z))} \otimes k_{w(z),u(w(z))},$$

$1 \le j \le \ell$. Since $\{w(z) : z \in G_j\}$ is 1-separated, it follows from Proposition 5.5(a) and (6.3) that

$$\|[M_f, [M_g, \Sigma_j]]\|_p^+ \le C_{5.5.1} C_{6.3} L(f) L(g),$$

$1 \le j \le \ell$. Thus (6.5) holds for the constant $C_{6.1.1} = \ell C_{5.5.1} C_{6.3}$. Similarly, it follows from Proposition 5.5(b) and (6.3) that

$$\|[M_f, \Sigma_j]\|_{2d}^+ \le C_{5.5.2} C_{6.3} L(f),$$

$1 \le j \le \ell$. That is, (6.6) holds for $C_{6.1.2} = \ell C_{5.5.2} C_{6.3}$. This completes the proof. □

**Corollary 6.3.** *The operator $Y$ given in Definition* 4.2 *has the following properties*:
(a) *If $p > d$, then $[M_f, [M_g, Y]] \in \mathcal{C}_p^+$ for all $f, g \in \mathrm{Lip}(S)$.*
(b) *For $f \in \mathrm{Lip}(S)$, we have $[M_f, Y] \in \mathcal{C}_{2d}^+$ with $\|[M_f, Y]\|_{2d}^+ \le C_{6.3} L(f)$.*

*Proof.* Obviously, $Y$ is a linear combination of operators in $\mathcal{X}$. Therefore (a), (b) follow from the corresponding parts in Proposition 6.1. □

*Proof of Theorem* 1.1. It follows from Proposition 4.3 that the spectrum of $Y + Z$ is contained in $\{0\} \cup [c, C]$, and that the spectral projection of $Y + Z$ corresponding to the interval $[c, C]$ equals $Q$. Thus

$$Q = \frac{1}{2\pi i} \int_\gamma (\lambda - Y - Z)^{-1} d\lambda, \tag{6.7}$$

where $\gamma$ is a simple Jordan curve in $\mathbf{C} \backslash (\{0\} \cup [c, C])$ whose winding number about 0 is 0 and whose winding number about every $x \in [c, C]$ is 1. For any $f, g \in \mathrm{Lip}(S)$, we have

$$[M_f, [M_g, Q]] = \frac{1}{2\pi i} \int_\gamma [M_f, [M_g, (\lambda - Y - Z)^{-1}]] d\lambda = \frac{1}{2\pi i} \int_\gamma \{F(\lambda) + G(\lambda) + H(\lambda)\} d\lambda,$$

where

$$\begin{aligned} F(\lambda) &= (\lambda - Y - Z)^{-1} [M_f, Y + Z] (\lambda - Y - Z)^{-1} [M_g, Y + Z] (\lambda - Y - Z)^{-1}, \\ G(\lambda) &= (\lambda - Y - Z)^{-1} [M_f, [M_g, Y + Z]] (\lambda - Y - Z)^{-1} \quad \text{and} \\ H(\lambda) &= (\lambda - Y - Z)^{-1} [M_g, Y + Z] (\lambda - Y - Z)^{-1} [M_f, Y + Z] (\lambda - Y - Z)^{-1}. \end{aligned}$$

Recall that the operator $Z$ is in the trace class. Thus from Corollary 6.3 and the above identities we conclude that $[M_f, [M_g, Q]] \in \mathcal{C}_p^+$ for every $p > d$. We know that $\mathcal{C}_t^+ \subset \mathcal{C}_{t'}$ for all $1 \le t < t' < \infty$. Therefore $[M_f, [M_g, Q]]$ is in the Schatten class $\mathcal{C}_p$ for every $p > d$. □

**Theorem 6.4.** *For $f \in \mathrm{Lip}(S)$, we have $[M_f, Q] \in \mathcal{C}_{2d}^+$ with $\|[M_f, Q]\|_{2d}^+ \le C_{6.4} L(f)$.*

*Proof.* This follows immediately from Corollary 6.3(b) and (6.7). □

One might characterize Theorem 6.4 as a "marginal" improvement of Proposition 2.12, but we want to emphasize that the membership $[M_f, Q] \in \mathcal{C}^+_{2d}$ cannot be proved using the $T_\mu$ given by (2.2) alone. This is again due to the power in the reproducing kernel $K_w$: Since the power in $K_w$ is $n$, Proposition 2.12 is the best that we can prove if we do not look beyond operators constructed from $K_w$.

## 7. Antisymmetric sums on $\mathcal{Q}$

We will now derive Theorem 1.2 from Theorem 1.1. This involves a classic vanishing principle for trace due to Helton and Howe:

**Lemma 7.1.** [21, Lemma 1.3] *Suppose that $X$ is a self-adjoint operator and $C$ is a compact operator. If $[X, C]$ is in the trace class, then* $\mathrm{tr}[X, C] = 0$.

**Proposition 7.2.** *For $f, g \in \mathrm{Lip}(S)$, we have $[Q_f, Q_g] \in \mathcal{C}^+_d$.*

*Proof.* We have

$$\begin{aligned}[Q_f, Q_g] &= QM_g(1 - Q)M_fQ - QM_f(1 - Q)M_gQ \\ &= [Q, M_g](1 - Q)[M_f, Q] - [Q, M_f](1 - Q)[M_g, Q]. \end{aligned} \tag{7.1}$$

Now an application of Theorem 6.4 completes the proof. □

**Proposition 7.3.** (a) *If $d \geq 2$, then for $f, g, h \in \mathrm{Lip}(S)$ we have $[Q_h, [Q_f, Q_g]] \in \mathcal{C}_p$ for every $p > 2d/3$.*
(b) *If $d = 1$, then $[Q_h, [Q_f, Q_g]]$ is in the trace class for all $f, g, h \in \mathrm{Lip}(S)$.*

*Proof.* (a) Continuing with (7.1), we have

$$\begin{aligned}[M_h, [Q_f, Q_g]] = {} & [M_h, [Q, M_g]](1 - Q)[M_f, Q] - [Q, M_g][M_h, Q][M_f, Q] \\ & + [Q, M_g](1 - Q)[M_h, [M_f, Q]] - [M_h, [Q, M_f]](1 - Q)[M_g, Q] \\ & + [Q, M_f][M_h, Q][M_g, Q] - [Q, M_f](1 - Q)[M_h, [M_g, Q]]. \end{aligned} \tag{7.2}$$

Thus it follows from Theorem 1.1 and Proposition 2.12 that $[M_h, [Q_f, Q_g]] \in \mathcal{C}_p$ for every $p > 2d/3$. Consequently, $[Q_h, [Q_f, Q_g]] = Q[M_h, [Q_f, Q_g]]Q \in \mathcal{C}_p$ for $p > 2d/3$.

(b) In the case $d = 1$, from (7.2), Theorem 1.1 and Proposition 2.12 we see that $[M_h, [Q_f, Q_g]]$ is in the trace class. Consequently, so is $[Q_h, [Q_f, Q_g]] = Q[M_h, [Q_f, Q_g]]Q$. □

**Proposition 7.4.** *Let $\nu \geq d$. Then for all $f, g, f_1, f_2, \ldots, f_{2\nu} \in \mathrm{Lip}(S)$, the operator*

$$[Q_f, Q_g[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2\nu}}]]$$

*is in the trace class with zero trace.*

*Proof.* For convenience, denote

$$W = [Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2\nu}}].$$

For each $1 \le j \le \nu$, let $\tau_j : \{1,2,\dots,2\nu\} \to \{1,2,\dots,2\nu\}$ be the transposition such that $\tau_j(2j-1) = 2j$, $\tau_j(2j) = 2j-1$ and $\tau_j(k) = k$ for every $k \in \{1,2,\dots,2\nu\}\backslash\{2j-1,2j\}$. Let $T_{2\nu}$ be the subgroup of $S_{2\nu}$ generated by $\tau_1,\dots,\tau_\nu$. Then there is a subset $G_{2\nu}$ of $S_{2\nu}$ such that $S_{2\nu} = \cup_{\lambda\in G_{2\nu}}\lambda T_{2\nu}$ and such that $\lambda T_{2\nu} \cap \lambda' T_{2\nu} = \emptyset$ for all $\lambda \ne \lambda'$ in $G_{2\nu}$. Therefore

$$W = \sum_{\lambda\in G_{2\nu}} \mathrm{sgn}(\lambda)[Q_{f_{\lambda(1)}}, Q_{f_{\lambda(2)}}]\cdots[Q_{f_{\lambda(2\nu-1)}}, Q_{f_{\lambda(2\nu)}}]. \tag{7.3}$$

Since $\nu \ge d$, it follows from Proposition 7.2 that $W \in \mathcal{C}_p$ for every $p > 1$. Proposition 7.2 also says that $[Q_f, Q_g] \in \mathcal{C}_{d+\epsilon}$ if $\epsilon > 0$. Hence $[Q_f, Q_g]W \in \mathcal{C}_1$.

Next we show that $[Q_f, W] \in \mathcal{C}_1$. If $d = 1$, this follows from (7.3) and Proposition 7.3(b). Suppose that $d \ge 2$. Then Proposition 7.3(a) tells us that $[Q_f, [Q_{f_{\lambda(2i-1)}}, Q_{f_{\lambda(2i)}}]] \in \mathcal{C}_p$ for every $p > 2d/3$, where $1 \le i \le \nu$ and $\lambda \in G_{2\nu}$. Since $2d/3 < d$ and since for every $j \ne i$ we have $[Q_{f_{\lambda(2j-1)}}, Q_{f_{\lambda(2j)}}] \in \mathcal{C}_{d+\epsilon}$ for every $\epsilon > 0$, it follows that $[Q_f, W] \in \mathcal{C}_1$.

From the last two paragraphs we obtain the membership $[Q_f, Q_gW] \in \mathcal{C}_1$. Similarly, $[(Q_f)^*, Q_gW] = [Q_{\bar f}, Q_gW] \in \mathcal{C}_1$ since $\bar f$ is also in $\mathrm{Lip}(S)$. Since $W$ is compact, it follows from Lemma 7.1 that

$$\mathrm{tr}[Q_f + (Q_f)^*, Q_gW] = 0 = \mathrm{tr}[Q_f - (Q_f)^*, Q_gW].$$

From this we obtain $\mathrm{tr}[Q_f, Q_gW] = 0$ as promised. □

*Proof of Theorem* 1.2. Write $\ell = \nu + 1$. Then the condition $\ell > d$ translates to $\nu \ge d$. Since $2\nu + 2$ is even, for any $f_1, f_2, \dots, f_{2\nu+2} \in \mathrm{Lip}(S)$, [21, Proposition 1.1] tells us that the antisymmetric sum

$$[Q_{f_1}, Q_{f_2}, \dots, Q_{f_{2\nu+1}}, Q_{f_{2\nu+2}}]$$

is a linear combination of terms of the form

$$[Q_{f_{\xi(1)}}, Q_{f_{\xi(2)}}[Q_{f_{\xi(3)}}, Q_{f_{\xi(4)}}, \cdots, Q_{f_{\xi(2\nu+1)}}, Q_{f_{\xi(2\nu+2)}}]],$$

where $\xi$ runs over a certain subset of the symmetric group $S_{2\nu+2}$. Thus Theorem 1.2 follows from Proposition 7.4. □

## 8. Exact sequence and the associated index

Our goal for this section is to prove Theorem 1.3. Let $(T_1, \dots, T_n)$ be an $n$-tuple of bounded operators on a separable Hilbert space $\mathcal{H}$. We will say that the tuple $(T_1, \dots, T_n)$ is *essentially spherical* if the operators

$$1 - (T_1^*T_1 + \cdots + T_n^*T_n), \quad [T_i, T_j], \quad [T_i, T_j^*],$$

$i, j \in \{1, \dots, n\}$, are all compact. In addition to Theorem 1.2, for the proof of Theorem 1.3 we need an index formula due to Douglas and Voiculescu for certain essentially spherical tuples. Our main reference for this section will be [12].

On the Grassmann algebra $\wedge\mathbf{C}^n$, let $a_1, \dots, a_n$ be the representation of the canonical anticommutation relations (CARs). That is, $a_i h = e_i \wedge h$ for $i = 1, \dots, n$, where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis for $\mathbf{C}^n$. Accordingly, we have the operators

$$d' = T_1 \otimes a_1 + \cdots + T_n \otimes a_n \quad \text{and} \quad d'' = T_1^* \otimes a_1^* + \cdots + T_n^* \otimes a_n^*$$

on $\mathcal{H} \otimes \wedge\mathbf{C}^n$. Let $\eta : \wedge^{\mathrm{o}}\mathbf{C}^n \to \wedge^{\mathrm{e}}\mathbf{C}^n$ be a unitary transformation. Then define the operator

$$A = (1 \otimes \eta)(d' + d'')\big|(\mathcal{H} \otimes \wedge^{\mathrm{e}}\mathbf{C}^n) \tag{8.1}$$

on $\mathcal{H} \otimes \wedge^{\mathrm{e}}\mathbf{C}^n$. One can think of $A$ as a $2^{n-1} \times 2^{n-1}$ matrix with entries in $\mathcal{B}(\mathcal{H})$.

**Proposition 8.1.** [12, Proposition 2] *Suppose that $T_1, \dots, T_n$ satisfy the conditions*

$$[T_i, T_j] \in \mathcal{C}_n, \quad [T_i, T_j^*] \in \mathcal{C}_n \tag{8.2}$$

*for all $i, j \in \{1, \dots, n\}$ and*

$$1 - (T_1^* T_1 + \cdots + T_n^* T_n) \in \mathcal{C}_n. \tag{8.3}$$

*Then for the operator $A$ defined by* (8.1), *we have*

$$\operatorname{index}(A) = \operatorname{tr}[T_1, T_1^*, \dots, T_n, T_n^*]. \tag{8.4}$$

Note that (8.2) and (8.3) are actually very demanding conditions. For example, for the Toeplitz operators $T_{\zeta_1}, \dots, T_{\zeta_n}$ on $H^2(S)$, we only have $[T_{\zeta_i}, T_{\zeta_j}^*] \in \mathcal{C}_n^+$. In other words, the essentially spherical tuple $(T_{\zeta_1}, \dots, T_{\zeta_n})$ on $H^2(S)$ does not satisfy the conditions required for applying index formula (8.4). It is also known that the essentially spherical tuple $(R_{\zeta_1}, \dots, R_{\zeta_n})$ on the submodule $\mathcal{R}$ does not satisfy the conditions required for applying index formula (8.4) [17, Theorem 1.1].

But it is a different story on the quotient module $\mathcal{Q}$. It follows from Proposition 2.12 that for all $i, j \in \{1, \dots, n\}$ and for every $p > d$, we have $Q_{\zeta_i} Q_{\bar{\zeta}_j} - Q_{\zeta_i \bar{\zeta}_j} \in \mathcal{C}_p$. In particular, $Q_{\zeta_i} Q_{\bar{\zeta}_j} - Q_{\zeta_i \bar{\zeta}_j} \in \mathcal{C}_n$, $i, j \in \{1, \dots, n\}$. In other words, the tuple $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$ on the quotient module $\mathcal{Q}$ satisfies conditions (8.2) and (8.3).

**Proposition 8.2.** *Let $A_{\mathcal{Q}}$ denote the operator defined by* (8.1) *in the case where $\mathcal{H} = \mathcal{Q}$ and $(T_1, \dots, T_n) = (Q_{\zeta_1}, \dots, Q_{\zeta_n})$. Then* $\operatorname{index}(A_{\mathcal{Q}}) = 0$.

*Proof.* As we explained above, Proposition 8.1 is applicable to the tuple $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$. Therefore

$$\operatorname{index}(A_{\mathcal{Q}}) = \operatorname{tr}[Q_{\zeta_1}, Q_{\bar{\zeta}_1}, \dots, Q_{\zeta_n}, Q_{\bar{\zeta}_n}],$$

which is 0 according to Theorem 1.2. □

**Remark.** It is easy to see that the "symbol" of $A_{\mathcal{Q}}$ is a $2^{n-1} \times 2^{n-1}$ matrix $\varphi$ with entries in $C(X)$, where $X = \tilde{M} \cap S$. Thus the conclusion $\operatorname{index}(A_{\mathcal{Q}}) = 0$ in Proposition 8.2 can also be obtained if somehow we can show that $\varphi$ represents the 0 element in the $K_1$-group of $C(X)$. For some simple examples of $M$, we can indeed directly verify that $[\varphi] = 0$ in $K_1(C(X))$. But, at least for now, we do not have a topological argument for $[\varphi] = 0$ for the general $M$ considered in this paper. Even if one manages to find such a general topological argument, it does not detract from the value of proving $\text{index}(A_{\mathcal{Q}}) = 0$ by an analytical method, namely trace computation.

*Proof of Theorem* 1.3. Let $A_{\text{Har}}$ be the operator defined by (8.1) in the case where $\mathcal{H} = H^2(S)$ and $(T_1, \ldots, T_n) = (T_{\zeta_1}, \ldots, T_{\zeta_n})$. Also, we let $A_{\mathcal{R}}$ denote the operator defined by (8.1) in the case where $\mathcal{H} = \mathcal{R}$ and $(T_1, \ldots, T_n) = (R_{\zeta_1}, \ldots, R_{\zeta_n})$. It is known from the BDF theory that

$$\text{index}(A_{\text{Har}}) \quad \text{and} \quad \text{index}(A_{\mathcal{R}})$$

respectively determine the elements in $\text{Ext}(S)$ represented by exact sequences (1.6) and (1.7). See the discussion on page 107 in [12], and see [5,9].

Since $P = R + Q$, by the essential normality of either $\mathcal{Q}$ or $\mathcal{R}$, we have

$$A_{\text{Har}} = (A_{\mathcal{R}} \oplus A_{\mathcal{Q}}) + K,$$

where $K$ is a compact operator on $H^2(S) \otimes \wedge^{\text{e}}\mathbf{C}^n$. Therefore

$$\text{index}(A_{\text{Har}}) = \text{index}(A_{\mathcal{R}} \oplus A_{\mathcal{Q}}) = \text{index}(A_{\mathcal{R}}) + \text{index}(A_{\mathcal{Q}}) = \text{index}(A_{\mathcal{R}}),$$

where the last $=$ follows from Proposition 8.2. This, as we explained above, means that exact sequences (1.6) and (1.7) represent the same element in $\text{Ext}(S)$. □

## 9. Difference of antisymmetric sums

Our goal for this section is to prove Theorem 1.4. We begin with the part of Theorem 1.4 that only requires Proposition 2.12:

**Proposition 9.1.** *For any Lipschitz functions $f_1, f_2, \ldots, f_{2n}$ on $S$, the difference*

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] - [R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}], \tag{9.1}$$

*as an operator on $H^2(S)$, is in the trace class.*

*Proof.* As it was the case for (7.3), there is a subset $G_{2n}$ of $S_{2n}$ such that

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] = \sum_{\lambda \in G_{2n}} \text{sgn}(\lambda)[T_{f_{\lambda(1)}}, T_{f_{\lambda(2)}}] \cdots [T_{f_{\lambda(2n-1)}}, T_{f_{\lambda(2n)}}] \quad \text{and}$$
$$[R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}] = \sum_{\lambda \in G_{2n}} \text{sgn}(\lambda)[R_{f_{\lambda(1)}}, R_{f_{\lambda(2)}}] \cdots [R_{f_{\lambda(2n-1)}}, R_{f_{\lambda(2n)}}].$$

Thus it suffices to show that for $f_1, f_2, \ldots, f_{2n} \in \text{Lip}(S)$, we have

$$[T_{f_1}, T_{f_2}] \cdots [T_{f_{2n-1}} T_{f_{2n}}] - [R_{f_1}, R_{f_2}] \cdots [R_{f_{2n-1}} R_{f_{2n}}] \in \mathcal{C}_1. \tag{9.2}$$

The key to this is Proposition 2.12: if $f \in \mathrm{Lip}(S)$, then $[M_f, Q] \in \mathcal{C}_p$ for $p > 2d$. Using this fact, the proof of (9.2) is the same as the proof of [25, Lemma 4.2]. $\square$

What remains for the proof of Theorem 1.4 is to show that the trace of (9.1) is zero. This trace computation can be further divided into two parts: Proposition 9.3 and "the rest of the argument". The proof of Proposition 9.3 is basically a repeat of the proof of Theorem 1.2, and therefore relies on Theorem 1.1. In other words, Proposition 9.3 relies on the work in Sections 3-6. By contrast, "the rest of the argument" only involves previous techniques, although these previous techniques are themselves recent developments in [16].

To show that (9.1) has zero trace, we follow the approach in [16]. For each natural number $m \in \mathbf{N}$, we define the subset

$$\Omega^{(m)} = \{z \in M : 1 - 2^{-2m} \leq |z| < 1\}$$

of $M$. We recall the measure $\mu$ from Definition 2.4. For each $m \in \mathbf{N}$, by restricting $\mu$ to $\Omega^{(m)}$ and $M \backslash \Omega^{(m)}$ we obtain two measures. That is, we define the measures $\mu^{(m)}$ and $\lambda^{(m)}$ by the formulas

$$\mu^{(m)}(E) = \mu(E \cap \Omega^{(m)}) \quad \text{and} \quad \lambda^{(m)}(E) = \mu(E \cap \{M \backslash \Omega^{(m)}\})$$

for Borel sets $E$. We have, of course, $\mu = \mu^{(m)} + \lambda^{(m)}$ for each $m$. The measures $\mu^{(m)}$ and $\lambda^{(m)}$ give rise to Toeplitz operators $T_{\mu^{(m)}}$ and $T_{\lambda^{(m)}}$. More precisely, we have

$$(T_{\mu^{(m)}} f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^n} d\mu^{(m)}(w) \quad \text{and}$$
$$(T_{\lambda^{(m)}} f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^n} d\lambda^{(m)}(w)$$

for $f \in H^2(S)$.

By adapting the work in [16, Section 3] to the Hardy-space setting, we obtain

$$\lim_{m \to \infty} \|[T_{\mu^{(m)}}, M_f]\|_p = 0 \quad \text{for } f \in \mathrm{Lip}(S) \text{ and } p > 2d.$$

By smooth functional calculus (see the proof of [16, Proposition 4.1]) and the relation $\mu = \mu^{(m)} + \lambda^{(m)}$, the above leads to

$$\lim_{m \to \infty} \|[\varphi(T_\mu), M_f] - [\varphi(T_{\lambda^{(m)}}), M_f]\|_p = 0 \tag{9.3}$$

for $\varphi \in C_c^\infty(\mathbf{R})$, $f \in \mathrm{Lip}(S)$ and $p > 2d$.

Recall from Theorem 2.5 that $cQ \leq T_\mu \leq CQ$ for some $0 < c \leq C < \infty$. By [16, Lemma 4.2] and the construction preceding it, there is an $h \in C_c^\infty(\mathbf{R})$ that has the following properties:

(1) $0 \leq h \leq 1$ on $\mathbf{R}$.

(2) $h = 0$ on $(-\infty, c/3] \cup [C + 2, \infty)$.

(3) $h(T_\mu) = Q$.
(4) $(1-h^2)^{1/2}h \in C_c^\infty(\mathbf{R})$.
With this $h$ we define

$$A_m = h(T_{\lambda^{(m)}}), \tag{9.4}$$

$m \in \mathbf{N}$. Since the measure $\lambda^{(m)}$ is concentrated on $M\backslash\Omega^{(m)}$, the Toeplitz operator $T_{\lambda^{(m)}}$ is compact. Since $h = 0$ on $(-\infty, c/3]$, we conclude that $\text{rank}(A_m) < \infty$ for every $m \in \mathbf{N}$.

We have $0 \le T_{\lambda^{(m)}} \le T_\mu \le CQ$. Since $h$ vanishes on a neighborhood of 0, $h(T_{\lambda^{(m)}})$ is the limit in operator norm of operators of the form $T_{\lambda^{(m)}}q(T_{\lambda^{(m)}})$, where $q$ are polynomials. Hence for each $m \in \mathbf{N}$, the range of $A_m$ is contained in the quotient module $\mathcal{Q}$. Thus (9.4) defines a sequence of finite-rank positive contractions $\{A_m\}$ satisfying the operator inequality

$$0 \le A_m \le Q \tag{9.5}$$

for every $m \in \mathbf{N}$.

By (9.5), it is straightforward to verify that the operator

$$Q^{(m)} = \begin{bmatrix} A_m^2 & (1-A_m^2)^{1/2}A_m \\ & \\ (1-A_m^2)^{1/2}A_m & Q - A_m^2 \end{bmatrix}$$

is an orthogonal projection on $L^2(S, d\sigma) \oplus L^2(S, d\sigma)$. On this space we further define

$$R' = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}, \quad Q' = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad P' = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q'' = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}.$$

Since the ranges of $R'$ and $Q^{(m)}$ are orthogonal to each other, we also have the orthogonal projection

$$P^{(m)} = R' + Q^{(m)}$$

on $L^2(S, d\sigma) \oplus L^2(S, d\sigma)$, $m \in \mathbf{N}$. For $f \in \text{Lip}(S)$, we write

$$D_f = \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix}.$$

With $P'$, $R'$, $P^{(m)}$, $Q^{(m)}$ and $D_f$, we define the "Toeplitz operators"

$$T'_f = P'D_fP', \quad R'_f = R'D_fR', \quad T_f^{(m)} = P^{(m)}D_fP^{(m)} \quad \text{and} \quad Q_f^{(m)} = Q^{(m)}D_fQ^{(m)},$$

$m \in \mathbf{N}$, as in [16].

**Proposition 9.2.** *For any $f_1, f_2, \dots, f_{2n} \in \text{Lip}(S)$, we have*

$$\lim_{m\to\infty} \|[T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] - [T_{f_1}^{(m)}, T_{f_2}^{(m)}, \dots, T_{f_{2n}}^{(m)}]\|_1 = 0.$$

*Proof.* This is deduced from (9.3) the same way [16, Proposition 6.1] is deduced from [16, Proposition 4.1]. □

**Proposition 9.3.** *Let $\nu \geq d$. Then for all $f_1, f_2, \ldots, f_{2\nu+1}, f_{2\nu+2} \in \mathrm{Lip}(S)$ and $m \in \mathbf{N}$, the antisymmetric sum*

$$[Q^{(m)}_{f_1}, Q^{(m)}_{f_2}, \ldots, Q^{(m)}_{f_{2\nu+1}}, Q^{(m)}_{f_{2\nu+2}}]$$

*is in the trace class with zero trace.*

*Proof.* Since $\mathrm{rank}(A_m) < \infty$, we have $Q^{(m)} = Q'' + L_m$, where $L_m$ is a finite-rank operator. Thus it follows from Theorem 1.1 that for $f, g \in \mathrm{Lip}(S)$,

$$[D_f, [D_g, Q^{(m)}]] \in \mathcal{C}_p \quad \text{if} \;\; p > d.$$

Similarly, it follows from Proposition 2.12 that for $f \in \mathrm{Lip}(S)$,

$$[D_f, Q^{(m)}] \in \mathcal{C}_p \quad \text{if} \;\; p > 2d.$$

Once these two Schatten-class memberships are established, the rest of the proof is a repeat of the work in Section 7. □

**Proposition 9.4.** *For all $f_1, f_2, \ldots, f_{2n} \in \mathrm{Lip}(S)$ and $m \in \mathbf{N}$, the difference*

$$[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \ldots, T^{(m)}_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \ldots, R'_{f_{2n}}]$$

*is in the trace class with zero trace.*

*Proof.* Given any $m \in \mathbf{N}$, we let $\mathcal{Z}^{(m)}$ denote the collection of $(2n+1)$-tuples $(X_0, \ldots, X_{2n})$ satisfying the following two conditions:

(1) For each $j \in \{0, 1, \ldots, 2n\}$, $X_j$ is either $R'$ or $Q^{(m)}$.

(2) For each $(X_0, \ldots, X_{2n})$, there is at least one $i \in \{0, 1, \ldots, 2n\}$ such that $X_i = R'$ and at least one $j \in \{0, 1, \ldots, 2n\}$ such that $X_j = Q^{(m)}$.

Then from the relation $P^{(m)} = R' + Q^{(m)}$ we obtain

$$[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \ldots, T^{(m)}_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \ldots, R'_{f_{2n}}] = [Q^{(m)}_{f_1}, Q^{(m)}_{f_2}, \ldots, Q^{(m)}_{f_{2n}}] + G_m,$$

where

$$G_m = \sum_{(X_0, X_1, \ldots, X_{2n}) \in \mathcal{Z}^{(m)}} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) X_0 D_{f_{\sigma(1)}} X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n}.$$

Proposition 9.3 tells us that $[Q^{(m)}_{f_1}, Q^{(m)}_{f_2}, \ldots, Q^{(m)}_{f_{2n}}]$ is in the trace class with zero trace. By the argument for [16, Lemma 6.7(a)], $G_m$ is a finite-rank operator. Thus it suffices to show that $\mathrm{tr}(G_m) = 0$. But this is just a repeat of the argument on pages 24-26 in [16]. □

*Proof of Theorem* 1.4. Since $P' = P \oplus 0$ and $R' = R \oplus 0$, we have

$$\begin{aligned}
&[T'_{f_1}, T'_{f_2}, \ldots, T'_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \ldots, R'_{f_{2n}}] \\
&\qquad = \{[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] - [R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}]\} \oplus 0.
\end{aligned}$$

Thus, by Proposition 9.1, it suffices to show that

$$\mathrm{tr}([T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \dots, R'_{f_{2n}}]) = 0. \tag{9.6}$$

Applying Proposition 9.4, for each $m \in \mathbf{N}$ we have

$$\begin{aligned}
&\mathrm{tr}([T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \dots, R'_{f_{2n}}]) \\
&\qquad = \mathrm{tr}([T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] - [T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}]) \\
&\qquad + \mathrm{tr}([T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] - [R'_{f_1}, R'_{f_2}, \dots, R'_{f_{2n}}]) \\
&\qquad = \mathrm{tr}([T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] - [T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}]) \\
&\qquad + 0.
\end{aligned}$$

Combining this with Proposition 9.2, (9.6) follows. □

## 10. The range space

The idea of range space was first introduced in [14] for quotient modules of the Bergman module. This turns out to be the only technique we have at the moment that proves the 1-essential normality in Theorem 1.5. The reader will see that the technique of range space is distinctly different from the techniques we used in the previous sections.

First, we remind the reader that the measure $d\mu$ is given in Definition 2.4(b). Accordingly, we have the Hilbert space $L^2(\mu) = L^2(M, d\mu)$ of measurable functions on $M$ that are square-integrable with respect to $d\mu$.

We will now introduce the restriction operator from $\mathbf{B}$ to $M$. Ordinarily, the restriction operator should be denoted by the symbol $R$, which is common in the literature. But for us, $R$ denotes the orthogonal projection from $L^2(S, d\sigma)$ to the submodule $\mathcal{R}$. To avoid confusion, we'd better choose a different symbol for the restriction operator.

Let $f \in \mathcal{Q}$. Since $f$ is an analytic function on $\mathbf{B}$, we define $Jf$ to be the restriction of this analytic function to the subset $M$ of $\mathbf{B}$. By (2.2) we have

$$\int_M |(Jf)(w)|^2 d\mu(w) = \int_M |f(w)|^2 d\mu(w) = \langle T_\mu f, f\rangle \tag{10.1}$$

for every $f \in \mathcal{Q}$. Thus, by the upper bound in Theorem 2.5, $J$ is a bounded operator that maps $\mathcal{Q}$ into $L^2(\mu)$. By the lower bound in Theorem 2.5 and (10.1), we have

$$\|Jf\|^2 \geq c\|f\|^2 \quad \text{for every} \quad f \in \mathcal{Q}. \tag{10.2}$$

Therefore the range of $J$ is a closed linear subspace of $L^2(\mu)$.

**Definition 10.1.** (a) Write $\mathcal{P}$ for the range of the restriction operator $J$ introduced above. (b) Let $E$ denote the orthogonal projection from $L^2(\mu)$ onto $\mathcal{P}$.

Obviously, (10.1) is equivalent to the statement that

$$J^*Jf = T_\mu f \quad \text{for every} \quad f \in \mathcal{Q}. \tag{10.3}$$

Moreover, (10.2) says that $J$ is an invertible operator from $\mathcal{Q}$ to $\mathcal{P}$.

If $f \in \mathcal{R}$, then its restriction to $M$ is the zero function. Since $H^2(S) = \mathcal{R} \oplus \mathcal{Q}$, we see that the range space $\mathcal{P}$ is actually the collection of the restrictions of all $f \in H^2(S)$ to $M$.

**Lemma 10.2.** [26, Lemma 2.10] *Given any $a > 0$ and $\kappa > -1$, there is a $0 < C_{10.2} < \infty$ such that*

$$\int_M \frac{(1-|z|^2)^a(1-|w|^2)^\kappa}{|1-\langle w,z\rangle|^{d+1+a+\kappa}} dv_M(w) \le C_{10.2}$$

*for every $z \in M$.*

Moreover, it is known that if $\kappa > -1$, then

$$\int_M (1-|w|^2)^\kappa dv_M(w) < \infty$$

[26, page 15]. This finiteness is due to the fact that we can use the function $1 - |w|^2$ as one of the $2d$ real coordinates on $M$ for $w \in M$ near $S$.

We define the operator $\hat{T}_\mu$ on $L^2(\mu)$ by the formula

$$(\hat{T}_\mu\varphi)(\zeta) = \int_M \varphi(w)K_w(\zeta)d\mu(w), \quad \zeta \in M,$$

$\varphi \in L^2(\mu)$.

**Proposition 10.3.** (a) *$\hat{T}_\mu$ is a bounded operator on $L^2(\mu)$.*
(b) *$\hat{T}_\mu$ maps $L^2(\mu)$ into $\mathcal{P}$.*
(c) *Let $\tilde{T}_\mu$ denote the restriction of $\hat{T}_\mu$ to the subspace $\mathcal{P}$. Then $\tilde{T}_\mu = JJ^*$. In particular, $\tilde{T}_\mu$ is invertible on $\mathcal{P}$.*
(d) *With respect to the orthogonal decomposition $L^2(\mu) = \mathcal{P} \oplus \mathcal{P}^\perp$, we have $\hat{T}_\mu = \tilde{T}_\mu \oplus 0$.*

*Proof.* (a) The boundedness of $\hat{T}_\mu$ follows from the Rudin-Forelli estimate in Lemma 10.2 by a standard Schur-test argument. See the proof of Lemma 10.8 below for a more general version of this boundedness.

(b) Once we know that $\hat{T}_\mu$ is bounded, the inclusion $\hat{T}_\mu L^2(\mu) \subset \mathcal{P}$ becomes obvious.

(c) For each $f \in \mathcal{Q}$, it is easy to see that $\tilde{T}_\mu Jf = JT_\mu f$. Combining this with (10.3), we have $\tilde{T}_\mu Jf = JT_\mu f = JJ^*Jf$. Since $J\mathcal{Q} = \mathcal{P}$, this implies $\tilde{T}_\mu = JJ^*$. Since $J : \mathcal{Q} \to \mathcal{P}$ and $J^* : \mathcal{P} \to \mathcal{Q}$ are invertible, so is $\tilde{T}_\mu$.

(d) This follows from (b) and the obvious fact that $\hat{T}_\mu$ is self-adjoint. □

**Definition 10.4.** For $f \in L^\infty(\mu)$, $\hat{M}_f$ denotes the operator of multiplication by the function $f$ on $L^2(\mu)$.

As usual, we write $\zeta_1, \dots, \zeta_n$ for the coordinate functions on $\mathbf{B}$.

**Proposition 10.5.** *For each $j \in \{1, \dots, n\}$, $\mathcal{P}$ is an invariant subspace for $\hat{M}_{\zeta_j}$.*

*Proof.* Let $f \in \mathcal{Q}$. Then $Q_{\zeta_j} f = \zeta_j f - g_j$ for some $g_j \in \mathcal{R}$. Since $g_j = 0$ on $M$, we have

$$JQ_{\zeta_j} f = \zeta_j f\big|M - g_j\big|M = \zeta_j f\big|M = \zeta_j Jf = \hat{M}_{\zeta_j} Jf. \tag{10.4}$$

That is, for each $f \in \mathcal{Q}$, we have $\hat{M}_{\zeta_j} Jf \in J\mathcal{Q} = \mathcal{P}$, which proves the proposition. □

Proposition 10.5 makes it possible for us to introduce

**Definition 10.6.** For each $j \in \{1, \dots, n\}$, let $M_{\zeta_j}$ denote the restriction of the operator $\hat{M}_{\zeta_j}$ to the invariant subspace $\mathcal{P}$.

Thus we can restate (10.4) in the form

**Corollary 10.7.** *We have $JQ_{\zeta_j} = M_{\zeta_j} J$ for every $j \in \{1, \dots, n\}$.*

**Lemma 10.8.** *Let $G(\zeta, w)$ be a Borel function on $M \times M$. Consider the operator on $L^2(\mu)$ given by the formula*

$$(A_G\varphi)(\zeta) = \int_M \varphi(w)G(\zeta, w)K_w(\zeta)d\mu(w), \tag{10.5}$$

*$\varphi \in L^2(\mu)$. If $G$ satisfies the condition*

$$\iint |G(\zeta, w)|^p |K_w(\zeta)|^2 d\mu(w)d\mu(\zeta) < \infty$$

*for some $2 \le p < \infty$, then $A_G$ belongs to the Schatten class $\mathcal{C}_p$.*

*Proof.* This follows from a standard interpolation between the Hilbert-Schmidt norm and the operator norm. One end of this interpolation, the case $p = 2$, is obvious. Thus we only need to show that in the case where $G$ is bounded, we have $\|A_G\| \le C_1\|G\|_\infty$. But the case of operator norm follows from Lemma 10.2 and the Schur test, as follows.

Consider the function $h(w) = (1 - |w|^2)^{-1/2}$ on $M$. Recalling Definition 2.4, we have

$$\begin{aligned}\int_M h(w)|G(\zeta, w)K_w(\zeta)|d\mu(w) &\le C\|G\|_\infty \int_M \frac{(1 - |w|^2)^{-1/2}}{|1 - \langle\zeta, w\rangle|^{d+1}} dv_M(w) \\ &= C\|G\|_\infty \int_M \frac{(1 - |w|^2)^{-1/2}}{|1 - \langle\zeta, w\rangle|^{d+1+(1/2)-(1/2)}} dv_M(w) \le C_1\|G\|_\infty h(\zeta),\end{aligned}$$

where the last step is an application of Lemma 10.2. A similar argument shows that

$$\int_M h(\zeta)|G(\zeta, w)K_w(\zeta)|d\mu(\zeta) \le C_1\|G\|_\infty h(w).$$

Thus the Schur test gives us $\|A_G\| \le C_1\|G\|_\infty$ as promised. This completes the proof. □

**Proposition 10.9.** *For every pair of $j \in \{1, \dots, n\}$ and $p > 2d$, we have $[\hat{M}_{\zeta_j}, \hat{T}_\mu] \in \mathcal{C}_p$ and $[\hat{M}_{\zeta_j}, E] \in \mathcal{C}_p$.*

*Proof.* It follows from parts (d) and (c) in Proposition 10.3 that there is an $h \in C_c^\infty(\mathbf{R})$ such that $E = h(\hat{T}_\mu)$. Thus by the standard smooth functional calculus, it suffices to show that $[\hat{M}_{\zeta_j}, \hat{T}_\mu] \in \mathcal{C}_p$ for every pair of $j \in \{1, \dots, n\}$ and $p > 2d$.

Note that for any $j \in \{1, \dots, n\}$, if we define $G_j(\zeta, w) = \zeta_j - w_j$, then $[\hat{M}_{\zeta_j}, \hat{T}_\mu] = A_{G_j}$, where $A_{G_j}$ is given by (10.5). Thus, by Lemma 10.8, it suffices to show that

$$\iint |\zeta_j - w_j|^p |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) < \infty$$

for every $p > 2d$. We have $|\zeta_j - w_j| \le \sqrt{2}|1 - \langle \zeta, w \rangle|^{1/2}$. Given any $p > 2d$, we write it in the form $p = 2d + 2r$ with some $r > 0$. Then

$$|\zeta_j - w_j|^p |K_w(\zeta)|^2 \le C|1 - \langle \zeta, w \rangle|^{d+r} |K_w(\zeta)|^2.$$

Recalling Definition 2.4, we now have

$$\iint |\zeta_j - w_j|^p |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) \le C_1 \iint \frac{|1 - \langle \zeta, w \rangle|^{d+r}}{|1 - \langle \zeta, w \rangle|^{2d+2}} dv_M(w) dv_M(\zeta).$$

Since $r > 0$, Lemma 10.2 tells us that this is finite. This completes the proof. □

**Proposition 10.10.** *For every $p > d$, the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on the range space $\mathcal{P}$ is $p$-essentially normal. In other words, for all $i, j \in \{1, \dots, n\}$ and $p > d$, the commutator $[M_{\zeta_i}, M_{\zeta_j}^*]$ on $\mathcal{P}$ is in the Schatten class $\mathcal{C}_p$.*

*Proof.* This follows from Proposition 10.9 by the easy identity

$$[M_{\zeta_i}, M_{\zeta_j}^*] = [E, \hat{M}_{\zeta_i}](E - 1)[E, \hat{M}_{\zeta_j}]^*, \tag{10.6}$$

$i, j \in \{1, \dots, n\}$. □

Let

$$J^* = U|J^*| \tag{10.7}$$

be the polar decomposition of the operator $J^*$. We know that $J^* : \mathcal{P} \to \mathcal{Q}$ is invertible. Therefore the $U$ above is a unitary operator. Also, by Proposition 10.3(c), we have $|J^*| = (JJ^*)^{1/2} = \tilde{T}_\mu^{1/2}$. Combining this with Corollary 10.7, we find that

$$Q_{\zeta_j} = J^{-1} M_{\zeta_j} J = U\tilde{T}_\mu^{-1/2} M_{\zeta_j} \tilde{T}_\mu^{1/2} U^* = UM_{\zeta_j}U^* + K_j \tag{10.8}$$

for each $j \in \{1, \dots, n\}$, where

$$K_j = U\tilde{T}_\mu^{-1/2}[M_{\zeta_j}, \tilde{T}_\mu^{1/2}]U^*. \tag{10.9}$$

We alert the reader that (10.8) and (10.9) are important identities.

**Lemma 10.11.** *The operator $K_j$ given by* (10.9) *belongs to $\mathcal{C}_p$ for every $p > 2d$.*

*Proof.* It suffices to show that $[M_{\zeta_j}, \tilde{T}_\mu^{1/2}] \in \mathcal{C}_p$ for $p > 2d$. By Proposition 10.3(c), the spectrum of $\tilde{T}_\mu$ is contained in $[c, C]$ for some $0 < c < C < \infty$. Therefore there is an $h \in C_c^\infty(\mathbf{R})$ such that $\tilde{T}_\mu^{1/2} = h(\tilde{T}_\mu)$. By the standard smooth functional calculus, this reduces the proof of the lemma to that of the membership $[M_{\zeta_j}, \tilde{T}_\mu] \in \mathcal{C}_p$ for $p > 2d$. Since $[M_{\zeta_j}, \tilde{T}_\mu] = E[\hat{M}_{\zeta_j}, \hat{T}_\mu]E$, Proposition 10.9 gives us the desired conclusion. □

The above naturally leads to the notion of "essential joint subnormality":

**Definition 10.12.** (a) Let $(S_1, \dots, S_n)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$. We say that $(S_1, \dots, S_n)$ is *jointly subnormal* if it extends to a commuting tuple of normal operators. That is, if there exist a Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$ and a commuting tuple of normal operators $(N_1, \dots, N_n)$ on $\hat{\mathcal{H}}$ such that $\mathcal{H}$ is an invariant subspace for each $N_j$ and such that $S_j = N_j|\mathcal{H}$ for every $j \in \{1, \dots, n\}$.
(b) Let $(T_1, \dots, T_n)$ be a tuple of operators on a Hilbert space $\mathcal{H}$, which may or may not be commuting. Let $1 \le p < \infty$. We say that $(T_1, \dots, T_n)$ is *$p$-essentially jointly subnormal* if there is a commuting, jointly subnormal tuple $(S_1, \dots, S_n)$ on $\mathcal{H}$ such that $T_j - S_j \in \mathcal{C}_p$ for every $j \in \{1, \dots, n\}$.

Obviously, the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on the range space $\mathcal{P}$ is jointly subnormal, as are the tuple $(T_{\zeta_1}, \dots, T_{\zeta_n})$ on $H^2(S)$ and the tuple $(R_{\zeta_1}, \dots, R_{\zeta_n})$ on the submodule $\mathcal{R}$. Moreover, the property of being jointly subnormal is preserved under unitary equivalence.

For a well-known example of a tuple that is essentially jointly subnormal but not jointly subnormal, we mention the multiplication operators $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on the Drury-Arveson space $H_n^2$.

Since $\mathcal{Q}$ is a quotient module, in general we do not know if the tuple $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$ is jointly subnormal. Thus it is significant that from (10.8) and Lemma 10.11 we obtain

**Theorem 10.13.** *The commuting tuple $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$ on $\mathcal{Q}$ is $p$-essentially jointly subnormal for every $p > 2d$.*

## 11. The case $d = 1$

With the preparations in Section 10, we are now ready to specialize to the case $d = 1$ and prove Theorem 1.5. We begin with a special property of $M$ in the case $d = 1$.

**Proposition 11.1.** *Suppose that $d = 1$. Then there is a $0 < C < \infty$ such that*

$$|\zeta - w| \le C|1 - \langle \zeta, w \rangle| \tag{11.1}$$

*for all $\zeta, w \in M$.*

*Proof.* First, recall the set $M^{(t)}$ from (3.2). For any fixed $0 < t < 1$, if we have $w \in M \backslash M^{(t)}$, then (11.1) obviously holds. Thus we only need to consider the case where $w \in M^{(t)}$ for a fixed $t \in (0, 1)$ that is sufficiently close to 0. Similarly, for any fixed $\delta > 0$, if $\zeta, w \in M$

satisfy the condition $|\zeta - w| \geq \delta$, then (11.1) obviously holds. Thus we only need to consider $\zeta, w \in M$ satisfying the condition $|\zeta - w| \leq \delta$ for some given $\delta > 0$.

Then note that the Möbius transforms of the unit disc give us the inequality

$$|a - b| \leq |1 - a\bar{b}| \tag{11.2}$$

for all complex numbers $a, b$ with $|a| < 1$ and $|b| < 1$.

For each $z \in K$ (cf. Section 2), let $T_z$ be the tangent space to the complex manifold $\mathcal{M}$ at $z$, and let $p_z$ be the orthogonal projection of $z$ on $T_z$. By Assumption 2.2(1), if $z \in \tilde{M} \cap S$, then $p_z \neq 0$. Reducing the value of $s \in (0, 1)$ in the definition of $K$ if necessary, we may assume that $p_z \neq 0$ for every $z \in K$. Thus under the condition $d = 1$, for each $z \in K$, the modified tangent space $T_z^{\text{mod}}$ (cf. [26, Definition 2.6]) is just $\mathbf{C}z$.

Let $P_z : \mathbf{C}^n \to \mathbf{C}z$ be the orthogonal projection, $z \in K$. By [26, Lemma 2.7], $P_z : \mathcal{M} \to \mathbf{C}z$ is biholomorphic near $z$. Moreover, by the analysis on pages 7-9 in [26], there are constants $0 < C < \infty$ and $\epsilon > 0$ such that

$$|z - \xi| \leq C|P_z(z - \xi)| \quad \text{if} \;\; z \in K, \;\; \xi \in M \;\; \text{and} \;\; |z - \xi| \leq \epsilon.$$

Thus for every pair of $\zeta \in K$ and $w \in M$ satisfying the condition $|\zeta - w| \leq \epsilon$, since $P_\zeta \zeta = \zeta$ and since $\zeta$ and $P_\zeta w$ both belong to the subspace $\mathbf{C}\zeta$ of complex dimension 1, we have

$$|\zeta - w| \leq C|\zeta - P_\zeta w| \leq C|1 - \langle \zeta, P_\zeta w\rangle| = C|1 - \langle \zeta, w\rangle|,$$

where the second $\leq$ follows from (11.2). As we explained in the first paragraph, this proves the proposition. □

Let $\text{Lip}(M)$ denote the collection of Lipschitz functions on $M$.

**Proposition 11.2.** *Suppose that $d = 1$. Then for every $f \in \text{Lip}(M)$, the commutators $[\hat{M}_f, \hat{T}_\mu]$ and $[\hat{M}_f, E]$ are in the Hilbert-Schmidt class $\mathcal{C}_2$. Moreover, we have $[M_{\zeta_j}, \tilde{T}_\mu] \in \mathcal{C}_2$ for every $j \in \{1, \dots, n\}$.*

*Proof.* Obviously, $[M_{\zeta_j}, \tilde{T}_\mu] = E[\hat{M}_{\zeta_j}, \hat{T}_\mu]E$. As we mentioned in the proof of Proposition 10.9, there is an $h \in C_c^\infty(\mathbf{R})$ such that $E = h(\hat{T}_\mu)$. Thus by the standard smooth functional calculus, we only need to show that $[\hat{M}_f, \hat{T}_\mu] \in \mathcal{C}_2$ for every $f \in \text{Lip}(M)$.

Given any $f \in \text{Lip}(M)$, we have $[\hat{M}_f, \hat{T}_\mu] = A_G$, where $G(\zeta, w) = f(\zeta) - f(w)$ and $A_G$ is given by (10.5). Thus it suffices to show that

$$\iint |f(\zeta) - f(w)|^2 |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) < \infty$$

under the condition $d = 1$. But when $d = 1$, it follows from the Lipschitz condition for $f$ and Proposition 11.1 that

$$|f(\zeta) - f(w)|^2 |K_w(\zeta)|^2 \leq C_1 |\zeta - w|^2 |K_w(\zeta)|^2 \leq C_2 |1 - \langle \zeta, w\rangle|^2 |K_w(\zeta)|^2$$

for $w, \zeta \in M$. Recalling Definition 2.4, this inequality leads to

$$\iint |f(\zeta) - f(w)|^2 |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) \leq C_3 \iint \frac{1}{|1 - \langle \zeta, w \rangle|^2} dv_M(w) dv_M(\zeta).$$

Applying Lemma 10.2 in the case $d = 1$, the above is finite. This completes the proof. □

**Proposition 11.3.** *Suppose that $d = 1$. Then the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on the range space $\mathcal{P}$ is 1-essentially normal. That is, for all $i, j \in \{1, \dots, n\}$, the commutator $[M_{\zeta_i}, M_{\zeta_j}^*]$ on $\mathcal{P}$ is in the trace class $\mathcal{C}_1$.*

*Proof.* This follows immediately from Proposition 11.2 and identity (10.6). □

We now define the function

$$\rho(\zeta) = 1 - |\zeta|^2, \quad \zeta \in \mathbf{B}.$$

**Lemma 11.4.** *Suppose that $d = 1$. Let $\epsilon > 0$. If $\delta > 0$ satisfies the condition*

$$(2 + \epsilon)((1/2) - \delta) > 1,$$

*then the operator $\hat{M}_{\rho^{(1/2)-\delta}} E$ is in the Schatten class $\mathcal{C}_{2+\epsilon}$.*

*Proof.* It follows from Proposition 10.3 that $E = \hat{T}_\mu(\tilde{T}_\mu^{-1} \oplus 0)$. Therefore it suffices to show that $\hat{M}_{\rho^{(1/2)-\delta}} \hat{T}_\mu \in \mathcal{C}_{2+\epsilon}$. Denote $t = (2 + \epsilon)((1/2) - \delta)$.

By (10.5), we have $\hat{M}_{\rho^{(1/2)-\delta}} \hat{T}_\mu = A_G$ with $G(\zeta, w) = \rho^{(1/2)-\delta}(\zeta)$. Thus, applying Lemma 10.8, the membership $\hat{M}_{\rho^{(1/2)-\delta}} \hat{T}_\mu \in \mathcal{C}_{2+\epsilon}$ will follow if we can show that

$$\iint |G(\zeta, w)|^{2+\epsilon} |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) < \infty.$$

That is, we have reduced the proof of the lemma to that of the inequality

$$\iint (1 - |\zeta|^2)^t |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) < \infty.$$

Specializing Definition 2.4 to the case $d = 1$, we have

$$\iint (1 - |\zeta|^2)^t |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) \leq C \iint \frac{(1 - |\zeta|^2)^t}{|1 - \langle \zeta, w \rangle|^4} dv_M(w) dv_M(\zeta).$$

Since $t > 1$, we deduce from Lemma 10.2 that the above is finite. □

**Lemma 11.5.** *Given any $f, g \in \mathrm{Lip}(M)$, define the operator $T$ on $L^2(\mu)$ by the formula*

$$(T\varphi)(\zeta) = \int_M \varphi(w)(f(\zeta) - f(w))(g(\zeta) - g(w)) K_w(\zeta) d\mu(w), \tag{11.3}$$

$\varphi \in L^2(\mu)$. *If* $d = 1$, *then* $\hat{M}_{\rho^{-(1/2)+\delta}} T \hat{M}_{\rho^{-(1/2)+\delta}} \in \mathcal{C}_2$ *for every* $\delta > 0$.

*Proof.* Given any $\delta > 0$, it suffices to show that

$$\iint \{\rho^{-(1/2)+\delta}(\zeta)\rho^{-(1/2)+\delta}(w)\}^2 |(f(\zeta) - f(w))(g(\zeta) - g(w))|^2 |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta) < \infty.$$

By the Lipschitz condition for $f$, $g$ and Proposition 11.1, the left-hand side does not exceed

$$C_1 \iint \{(1 - |\zeta|^2)(1 - |w|^2)\}^{-1+2\delta} |1 - \langle \zeta, w\rangle|^4 |K_w(\zeta)|^2 d\mu(w) d\mu(\zeta).$$

Applying Definition 2.4 in the case $d = 1$, the above is dominated by

$$C_2 \iint \{(1 - |\zeta|^2)(1 - |w|^2)\}^{-1+2\delta} dv_M(w) dv_M(\zeta).$$

Since $\delta > 0$, this is finite. This completes the proof. □

**Lemma 11.6.** *Suppose that* $d = 1$. *Then for all* $f, g \in \mathrm{Lip}(M)$, *the operator*

$$E[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu]]E$$

*is in the trace class.*

*Proof.* Note that the double commutator $[\hat{M}_f, [\hat{M}_g, \hat{T}_\mu]]$ is none other than the operator $T$ defined by (11.3). Thus our task is to show that $ETE \in \mathcal{C}_1$.

Given a small $\epsilon > 0$, we will show that there is a factorization $ETE = A_1 A_2 A_3$ such that $A_1, A_3 \in \mathcal{C}_{2+\epsilon}$ and $A_2 \in \mathcal{C}_2$. This, of course, will imply that $ETE \in \mathcal{C}_1$.

To obtain the desired factorization, we pick a $\delta > 0$ such that $(2 + \epsilon)((1/2) - \delta) > 1$. We then define

$$A_1 = E\hat{M}_{\rho^{(1/2)-\delta}}, \quad A_2 = \hat{M}_{\rho^{-(1/2)+\delta}} T \hat{M}_{\rho^{-(1/2)+\delta}} \quad \text{and} \quad A_3 = \hat{M}_{\rho^{(1/2)-\delta}} E.$$

For these operators, we have $A_1, A_3 \in \mathcal{C}_{2+\epsilon}$ by Lemma 11.4 and $A_2 \in \mathcal{C}_2$ by Lemma 11.5. Since these three operators do give us the factorization $ETE = A_1 A_2 A_3$, the proof is complete. □

**Lemma 11.7.** *Suppose that* $d = 1$. *Then for each pair of* $i, j \in \{1, \ldots, n\}$, *the operator* $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}_\mu]]$ *is in the trace class.*

*Proof.* It is easy to see that

$$\begin{aligned}[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}_\mu]] &= E[\hat{M}_{\zeta_i}, E[\hat{M}_{\bar{\zeta}_j}, \hat{T}_\mu]E]E \\ &= E\{[\hat{M}_{\zeta_i}, E][\hat{M}_{\bar{\zeta}_j}, \hat{T}_\mu]E + E[\hat{M}_{\zeta_i}, [\hat{M}_{\bar{\zeta}_j}, \hat{T}_\mu]]E + E[\hat{M}_{\bar{\zeta}_j}, \hat{T}_\mu][\hat{M}_{\zeta_i}, E]\}E.\end{aligned}$$

Applying Proposition 11.2 and Lemma 11.6 to the appropriate terms inside the $\{\quad\}$, we obtain the membership $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}_\mu]] \in \mathcal{C}_1$. □

**Lemma 11.8.** *Suppose that $d = 1$. Then for each pair of $i, j \in \{1, \dots, n\}$, the operator $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}_\mu]]$ is in the trace class.*

*Proof.* As we mentioned in the proof of Lemma 10.11, the spectrum of $\tilde{T}_\mu$ is contained in $[c, C]$, where $0 < c < C < \infty$. Consider $H_+ = \{\lambda \in \mathbf{C} : \mathrm{Re}(\lambda) > 0\}$, the right half-plane. Let $\gamma$ be a simple Jordan curve in $H_+\backslash[c, C]$ whose winding number about every $x \in [c, C]$ is 1. Taking advantage of the fact that the square-root function $\lambda^{1/2}$ is analytic on $H_+$, from the Riesz functional calculus we obtain

$$\tilde{T}^{1/2}_\mu = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}(\lambda - \tilde{T}_\mu)^{-1} d\lambda.$$

Similar to the argument in the proof of Theorem 1.1, the above formula leads to

$$[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}_\mu]] = \frac{1}{2\pi i}\int_\gamma \lambda^{1/2}\{A(\lambda) + B(\lambda) + C(\lambda)\} d\lambda,$$

where

$$\begin{aligned} A(\lambda) &= (\lambda - \tilde{T}_\mu)^{-1}[M_{\zeta_i}, \tilde{T}_\mu](\lambda - \tilde{T}_\mu)^{-1}[M^*_{\zeta_j}, \tilde{T}_\mu](\lambda - \tilde{T}_\mu)^{-1}, \\ B(\lambda) &= (\lambda - \tilde{T}_\mu)^{-1}[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}_\mu]](\lambda - \tilde{T}_\mu)^{-1} \quad \text{and} \\ C(\lambda) &= (\lambda - \tilde{T}_\mu)^{-1}[M^*_{\zeta_j}, \tilde{T}_\mu](\lambda - \tilde{T}_\mu)^{-1}[M_{\zeta_i}, \tilde{T}_\mu](\lambda - \tilde{T}_\mu)^{-1}. \end{aligned}$$

Applying Lemma 11.7 to $B(\lambda)$ and Proposition 11.2 to $A(\lambda)$ and $C(\lambda)$, we obtain the membership $[M_{\zeta_i}, [M^*_{\zeta_j}, \tilde{T}^{1/2}_\mu]] \in \mathcal{C}_1$. □

**Lemma 11.9.** *Suppose that $d = 1$. Then for each pair of $i, j \in \{1, \dots, n\}$, the operator $[M_{\zeta_i}, [M_{\zeta_j}, \tilde{T}^{1/2}_\mu]]$ is in the trace class.*

*Proof.* It suffices to observe that if we replace $M^*_{\zeta_j}$ by $M_{\zeta_j}$ in Lemmas 11.7 and 11.8, then the proofs still work. □

**Lemma 11.10.** *Suppose that $d = 1$. Then for every $j \in \{1, \dots, n\}$ we have $[M_{\zeta_j}, \tilde{T}^{1/2}_\mu] \in \mathcal{C}_2$ and $K_j \in \mathcal{C}_2$, where, as we recall, $K_j$ is given by* (10.9).

*Proof.* The membership $[M_{\zeta_j}, \tilde{T}^{1/2}_\mu] \in \mathcal{C}_2$ is obtained from Proposition 11.2 by applying smooth functional calculus, or the Riesz functional calculus used above. □

**Lemma 11.11.** *Suppose that $d = 1$. Then for every pair of $i, j \in \{1, \dots, n\}$, we have $[UM_{\zeta_i}U^*, K^*_j] \in \mathcal{C}_1$ and $[UM_{\zeta_i}U^*, K_j] \in \mathcal{C}_1$, where $U$ is given in* (10.7).

*Proof.* By (10.9), we have

$$\begin{aligned} [UM_{\zeta_i}U^*, K^*_j] &= U[M_{\zeta_i}, [\tilde{T}^{1/2}_\mu, M^*_{\zeta_j}]\tilde{T}^{-1/2}_\mu]U^* \\ &= U\{[M_{\zeta_i}, [\tilde{T}^{1/2}_\mu, M^*_{\zeta_j}]]\tilde{T}^{-1/2}_\mu + [\tilde{T}^{1/2}_\mu, M^*_{\zeta_j}]\tilde{T}^{-1/2}_\mu[\tilde{T}^{1/2}_\mu, M_{\zeta_i}]\tilde{T}^{-1/2}_\mu\}U^*. \end{aligned}$$

Thus the membership $[UM_{\zeta_i}U^*, K^*_j] \in \mathcal{C}_1$ follows from Lemmas 11.8 and 11.10. Similarly, the membership $[UM_{\zeta_i}U^*, K_j] \in \mathcal{C}_1$ follows from Lemmas 11.9 and 11.10. □

With the above preparation we can now accomplish the main goal of the section.

*Proof of Theorem* 1.5. By (10.8), for any $i, j \in \{1, \dots, n\}$, we have

$$\begin{aligned}[Q_{\zeta_i}, Q^*_{\zeta_j}] &= [UM_{\zeta_i}U^* + K_i, UM^*_{\zeta_j}U^* + K^*_j] \\ &= U[M_{\zeta_i}, M^*_{\zeta_j}]U^* + [UM_{\zeta_i}U^*, K^*_j] + [K_i, UM^*_{\zeta_j}U^*] + [K_i, K^*_j].\end{aligned}$$

By Proposition 11.3 and Lemma 11.10, the first and the last term above are in the trace class. By Lemma 11.11, the two middle terms are also in the trace class. Hence $[Q_{\zeta_i}, Q^*_{\zeta_j}]$ is in the trace class as promised. □

Once we have established the 1-essential normality of the tuple $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$ in the case $d = 1$, we can begin to consider the trace of commutators of polynomials in $Q_{\zeta_1}, \dots, Q_{\zeta_n}, Q^*_{\zeta_1}, \dots, Q^*_{\zeta_n}$. Let $k, m \in \mathbf{N}$ and consider any $S_i, T_j \in \{Q_{\zeta_1}, \dots, Q_{\zeta_n}, Q^*_{\zeta_1}, \dots, Q^*_{\zeta_n}\}$, $1 \le i \le k$ and $1 \le j \le m$. Under the assumption $d = 1$, Theorem 1.5 implies

$$[S_1 \cdots S_k, T_1 \cdots T_m] \in \mathcal{C}_1.$$

Moreover, it is an easy consequence of the 1-essential normality of $(Q_{\zeta_1}, \dots, Q_{\zeta_n})$ that

$$\mathrm{tr}[S_1 \cdots S_k, T_1 \cdots T_m] = \mathrm{tr}[S_{\alpha(1)} \cdots S_{\alpha(k)}, T_{\beta(1)} \cdots T_{\beta(m)}]$$

for every permutation $\alpha$ of the set $\{1, \dots, k\}$ and every permutation $\beta$ of the set $\{1, \dots, m\}$. Therefore we only need to consider traces of the form

$$\mathrm{tr}[Q^*_p Q_q, Q^*_r Q_s],$$

where $p, q, r, s \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$. The same is true if we consider the trace of commutators of polynomials of the operators $M_{\zeta_1}, \dots, M_{\zeta_n}, M^*_{\zeta_1}, \dots, M^*_{\zeta_n}$ on $\mathcal{P}$.

**Theorem 11.12.** *When $d = 1$, we have*

$$\mathrm{tr}[Q^*_p Q_q, Q^*_r Q_s] = \mathrm{tr}[M^*_p M_q, M^*_r M_s] \tag{11.4}$$

*for all $p, q, r, s \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$.*

*Proof.* First of all, in addition to the 1-essential normality provided by Theorem 1.5, it follows from (10.8) and Lemmas 11.10 and 11.11 that for $i, j \in \{1, \dots, n\}$, the commutators $[Q_{\zeta_i}, K^*_j]$ and $[Q_{\zeta_i}, K_j]$ are in the trace class. Let $\mathcal{A}$ be the unital algebra generated by $Q_{\zeta_1}, \dots, Q_{\zeta_n}$, $Q^*_{\zeta_1}, \dots, Q^*_{\zeta_n}$, $K_1, \dots, K_n$, $K^*_1, \dots, K^*_n$. Then for any $j \in \{1, \dots, n\}$ and any $A, B, C \in \mathcal{A}$, we have $[AK_jB, C] \in \mathcal{C}_1$. We also have $[AK_jB, C^*] \in \mathcal{C}_1$ since the membership $C \in \mathcal{A}$ implies $C^* \in \mathcal{A}$. Thus it follows from Lemma 7.1 that $\mathrm{tr}[AK_jB, C] = 0$. Similarly, we have $\mathrm{tr}[AK^*_jB, C] = 0$ for all $A, B, C \in \mathcal{A}$ and $j \in \{1, \dots, n\}$.

Given any $p, q, r, s \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, from (10.8) it is easy to deduce that

$$[Q^*_p Q_q, Q^*_r Q_s] = U[M^*_p M_q, M^*_r M_s]U^* + X,$$

where $X$ is a linear combination of commutators of the form $[AK_jB, C]$ and $[AK_j^*B, C]$, where $A, B, C \in \mathcal{A}$ and $j \in \{1, \dots, n\}$. By the preceding paragraph, we have $\mathrm{tr}(X) = 0$. Therefore (11.4) holds. □

Since $M_{\zeta_1}, \dots, M_{\zeta_n}$ are just multiplication operators on $\mathcal{P}$, the hope is that the right-hand side of (11.4) is more commutable. The significance of Theorem 11.12 is that any explicit formula for the right-hand side of (11.4) (say in terms of some integral involving $p$, $q$, $r$, $s$) is an explicit formula for the left-hand side.

## 12. Examples and an open problem

The assumption $d = 1$ is the obvious geometric condition in Theorem 1.5. But what is also involved in Theorem 1.5 is the algebraic notion of "multiplicity" associated with the submodule that gives arise to $\mathcal{Q}$. It is worth reminding the reader that in this paper, we only consider quotient modules of the form

$$\mathcal{Q} = H^2(S) \ominus \mathcal{R}, \quad \text{where } \mathcal{R} = \{f \in H^2(S) : f = 0 \text{ on } M\}. \tag{12.1}$$

This kind of $\mathcal{R}$ has multiplicity 1, because $f$ is only required to vanish on $M$ to the first order. If we increase the multiplicity, then the corresponding quotient module no longer has 1-essential normality, even if the underlying variety still has complex dimension 1. We will see this in the following example.

**Example 12.1.** Consider the case $n = 2$. That is, suppose that $S \subset \mathbf{C}^2$. Let $[\zeta_1^2]$ be the principal submodule of $H^2(S)$ generated by the monomial $\zeta_1^2$. In other words, $[\zeta_1^2]$ is the closure of $\{\zeta_1^2 f : f \in H^2(S)\}$ in $H^2(S)$. Clearly, this submodule $[\zeta_1^2]$ has multiplicity 2 with the underlying variety $\{(0, z) : |z| < 1, z \in \mathbf{C}\}$. Define

$$[\zeta_1^2]^\perp = H^2(S) \ominus [\zeta_1^2],$$

which is the quotient module corresponding to the submodule $[\zeta_1^2]$. Then the module operator $Q_{\zeta_1}$ on $[\zeta_1^2]^\perp$ has the properties that $[Q_{\zeta_1}^*, Q_{\zeta_1}] \in \mathcal{C}_1^+$ and that $[Q_{\zeta_1}^*, Q_{\zeta_1}] \notin \mathcal{C}_1$. To see this, we use the standard orthonormal basis $\{e_{i,j} : i, j \geq 0\}$ for $H^2(S)$. Recall that $e_{i,j}(\zeta) = \{\frac{(i+j+1)!}{i!j!}\}^{1/2}\zeta_1^i\zeta_2^j$ for $i, j \geq 0$. Let $Q : H^2(S) \to [\zeta_1^2]^\perp$ be the orthogonal projection. Then $Q = Q_0 + Q_1$, where

$$Q_i = \sum_{j=0}^{\infty} e_{i,j} \otimes e_{i,j}, \quad i = 0, 1.$$

By straightforward calculation,

$$\begin{aligned}
[Q_{\zeta_1}^*, Q_{\zeta_1}] &= QM_{\bar{\zeta}_1}QM_{\zeta_1}Q - QM_{\zeta_1}QM_{\bar{\zeta}_1}Q \\
&= Q_0M_{\bar{\zeta}_1}Q_1M_{\zeta_1}Q_0 - Q_1M_{\zeta_1}Q_0M_{\bar{\zeta}_1}Q_1 \\
&= \sum_{j=0}^{\infty} |\langle \zeta_1 e_{0,j}, e_{1,j}\rangle|^2 e_{0,j} \otimes e_{0,j} - \sum_{j=0}^{\infty} |\langle \zeta_1 e_{0,j}, e_{1,j}\rangle|^2 e_{1,j} \otimes e_{1,j} \\
&= \sum_{j=0}^{\infty} \frac{1}{j+2} e_{0,j} \otimes e_{0,j} - \sum_{j=0}^{\infty} \frac{1}{j+2} e_{1,j} \otimes e_{1,j}.
\end{aligned}$$

This gives us three pieces of information: that $[Q^*_{\zeta_1}, Q_{\zeta_1}] \notin \mathcal{C}_1$, that $[Q^*_{\zeta_1}, Q_{\zeta_1}] \in \mathcal{C}_1^+$, and that the Dixmier trace of $[Q^*_{\zeta_1}, Q_{\zeta_1}]$ is 0. □

Our next example provides a contrast to Example 12.1.

**Example 12.2.** Again, consider the case $n = 2$. This time, consider the principal submodule $[\zeta_1\zeta_2]$ of $H^2(S)$ generated by the monomial $\zeta_1\zeta_2$. Define $\tilde{V} = \{(0, z) : z \in \mathbf{C}\} \cup \{(z, 0) : z \in \mathbf{C}\}$ and $V = \tilde{V} \cap \mathbf{B}_2$. It is easy to see that

$$[\zeta_1\zeta_2] = \{f \in H^2(S) : f = 0 \text{ on } V\}.$$

Since $\dim_{\mathbf{C}} \tilde{V} = 1$, the quotient module $[\zeta_1\zeta_2]^\perp = H^2(S) \ominus [\zeta_1\zeta_2]$ is 1-essentially normal by Theorem 1.5. Of course, for such a simple quotient module $[\zeta_1\zeta_2]^\perp$, its 1-essential normality can also be verified by hand. In fact, this is one of the examples that led us to the discovery of Theorem 1.5. □

Summarizing the above discussion, the point we want to make is that essential normality depends on the underlying multiplicity as well as on the underlying dimension.

Once we have Theorem 1.5, a natural question is, is there a generalization of it in our setting (12.1) for the case $1 < d \le n - 1$? We end the paper with an open problem that is worth exploring:

**Problem 12.3.** Consider general $\mathcal{Q}$ in (12.1) with $1 < d \le n-1$. For analytic polynomials $p_1, \dots, p_d, q_1, \dots, q_d \in \mathbf{C}[\zeta_1, \dots, \zeta_n]$, is the antisymmetric sum

$$[Q_{p_1}, Q^*_{q_1}, \dots, Q_{p_d}, Q^*_{q_d}]$$

in the trace class? If it is in the trace class, is there a formula for its trace, say in terms of some integral on $M$? In other words, is there an analogue of the Helton-Howe trace formula [21] for the above antisymmetric sum?

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