THE HELTON-HOWE TRACE FORMULA FOR THE FOCK SPACE

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Abstract. The famous Helton-Howe trace formula was originally established for antisymmetric sums of Toeplitz operators on the Bergman space of the unit ball. This formula has been extended to weighted Bergman spaces and to the Hardy space in [20], and more recently it even has been extended to the Drury-Arveson space [24]. In this paper we show that an analogue of the Helton-Howe trace formula also holds in an unlikely place, the Fock space! We further show that our techniques not only work for the ordinary trace tr, as in the case of the Helton-Howe trace formula, but also for the Dixmier trace.

1. Introduction

Given any bounded operators $A_1, \dots, A_k$ on a Hilbert space $\mathcal{H}$, one has the antisymmetric sum

$$[A_1, \dots, A_k] = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)}.$$

This was first introduced by Helton and Howe in [16]. Later Douglas and Voiculescu showed that there is a deep connection between antisymmetric sums and Fredholm index [9]. These two papers are now considered as some of the early examples of non-commutative geometry [5]. In recent years, the study of the Arveson-Douglas conjecture [1,2,8] has brought renewed interest in antisymmetric sums, with significant results [12,22,23].

As it turns out, operators on reproducing-kernel Hilbert spaces provide some of the particularly interesting examples of antisymmetric sums.

Let $\mathbf{B}$ be the unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. As usual, we write $L^2_a(\mathbf{B})$ for the Bergman space. Given an $f \in L^\infty(\mathbf{B})$, we have the familiar Toeplitz operator defined by the formula

$$T_f^{\mathrm{Berg}} h = P^{\mathrm{Berg}}(fh), \quad h \in L^2_a(\mathbf{B}),$$

where $P^{\mathrm{Berg}} : L^2(\mathbf{B}) \to L^2_a(\mathbf{B})$ is the orthogonal projection. We use the superscript "Berg" to distinguish T_f^{Berg} from Toeplitz operators to be defined on other spaces. We recall the following classic result of Helton and Howe:

Theorem 1.1. [16, Theorem 7.2] *Let $f_1, f_2, \dots, f_{2n}$ be C^∞-functions on an open set containing $\overline{\mathbf{B}}$. Then the antisymmetric sum $[T_{f_1}^{\mathrm{Berg}}, T_{f_2}^{\mathrm{Berg}}, \dots, T_{f_{2n}}^{\mathrm{Berg}}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[T_{f_1}^{\mathrm{Berg}}, T_{f_2}^{\mathrm{Berg}}, \dots, T_{f_{2n}}^{\mathrm{Berg}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}. \tag{1.1}$$

In [20], this formula was extended to a family of reproducing-kernel Hilbert spaces $\mathcal{H}^{(t)}$ on the ball, $-1 \le t < \infty$. Recall $\mathcal{H}^{(t)}$ is the Hilbert space of analytic functions on $\mathbf{B}$ that has the

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function

$$K_z^{(t)}(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1+t}}$$

as its reproducing kernel. It is well known that for each $-1 < t < \infty$, $\mathcal{H}^{(t)}$ is the weighted Bergman space $L^2_a(\mathbf{B}, dv_t)$, where

$$dv_t(w) = a_t(1 - |w|^2)^t dv(w).$$

Here, a_t is chosen so that $v_t(\mathbf{B}) = 1$. For each $-1 < t < \infty$, we have the orthogonal projection $P^{(t)} : L^2(\mathbf{B}, dv_t) \to L^2_a(\mathbf{B}, dv_t)$. Accordingly, we have the Toeplitz operator

$$T_f^{(t)} h = P^{(t)}(fh), \quad h \in L^2_a(\mathbf{B}, dv_t),$$

on the weighted Bergman space $L^2_a(\mathbf{B}, dv_t)$. Note that $\mathcal{H}^{(0)}$ is the unweighted Bergman space $L^2_a(\mathbf{B})$, and we have $T_f^{(0)} = T_f^{\mathrm{Berg}}$.

In the case $t = -1$, $\mathcal{H}^{(-1)}$ is none other than the Hardy space $H^2(S)$, which can be simply described as the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$, where $S = \{z \in \mathbf{C}^n : |z| = 1\}$ and $d\sigma$ is the spherical measure on S with the normalization $\sigma(S) = 1$. Let $P^{(-1)} : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. Accordingly, we have the Toeplitz operator

$$T_f^{(-1)} h = P^{(-1)}(fh), \quad h \in H^2(S),$$

on the Hardy space $H^2(S)$.

Generalizing Theorem 1.1, Tang, Wang and Zheng proved the following:

Theorem 1.2. [20] *Let $f_1, f_2, \dots, f_{2n}$ be C^2-functions on an open set containing $\overline{\mathbf{B}}$. Then for every $-1 \le t < \infty$, the antisymmetric sum $[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \dots, T_{f_{2n}}^{(t)}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[T_{f_1}^{(t)}, T_{f_2}^{(t)}, \dots, T_{f_{2n}}^{(t)}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}. \tag{1.2}$$

More recently, an analogue of (1.1) and (1.2) was proved on the Drury-Arveson space:

Theorem 1.3. [24] *Let $f_1, \dots, f_{2n}, g_1, \dots, g_{2n} \in H_s^\infty$ for some $s > 1$. Then on the Drury-Arveson space H_n^2, the antisymmetric sum $[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}]$ is in the trace class, and we have the trace formula*

$$\mathrm{tr}[M_{f_1}^* M_{g_1}, M_{f_2}^* M_{g_2}, \dots, M_{f_{2n}}^* M_{g_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} d\bar{f}_1 g_1 \wedge d\bar{f}_2 g_2 \wedge \cdots \wedge d\bar{f}_{2n} g_{2n}. \tag{1.3}$$

The purpose of the paper is to prove that an analogue of (1.1), (1.2) and (1.3) also holds on the Fock space!

Let $d\mu$ denote the Gaussian measure on $\mathbf{C}^n$. More precisely, we write

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z),$$

where dV is the standard volume measure on $\mathbf{C}^n$. Recall that the Fock space $H^2(\mathbf{C}^n, d\mu)$ is the norm closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(\mathbf{C}^n, d\mu)$. Let $P : L^2(\mathbf{C}^n, d\mu) \to H^2(\mathbf{C}^n, d\mu)$ be the orthogonal projection. Given an $f \in L^\infty(\mathbf{C}^n, d\mu)$, we define the Toeplitz operator

$$T_f h = P(fh), \quad h \in H^2(\mathbf{C}^n, d\mu).$$

The function f is commonly referred to as the symbol of the Toeplitz operator T_f.

It is well known that the n-variable Fock space $H^2(\mathbf{C}^n, d\mu)$ can be viewed as the tensor product of n copies of the one-variable Fock space. In fact, this tensor-product structure can be visualized rather vividly: Let $h_1, \dots, h_n$ be functions in $H^2(\mathbf{C}^n, d\mu)$ such that h_j depends only on the j-th variable z_j, $j = 1, \dots, n$. Then the function $h_1 \cdots h_n$ also belongs to $H^2(\mathbf{C}^n, d\mu)$, and we have

$$\|h_1 \cdots h_n\| = \|h_1\| \cdots \|h_n\|.$$

This clearly is a structure that is fundamentally incompatible with the structure of the ball. Given that, how can one hope to prove an analogue of (1.1), (1.2) and (1.3) on the Fock space $H^2(\mathbf{C}^n, d\mu)$?

To solve this problem, we first note that if we apply Stoke's theorem to the right-hand side of (1.1), then the Helton-Howe trace formula has the following alternative form:

Theorem 1.4. [16] *Let $f_1, f_2, \dots, f_{2n}$ be C^∞-functions on an open set containing $\overline{\mathbf{B}}$. Then the antisymmetric sum $[T^{\mathrm{Berg}}_{f_1}, T^{\mathrm{Berg}}_{f_2}, \dots, T^{\mathrm{Berg}}_{f_{2n}}]$ is in the trace class. Moreover,*

$$\mathrm{tr}[T^{\mathrm{Berg}}_{f_1}, T^{\mathrm{Berg}}_{f_2}, \dots, T^{\mathrm{Berg}}_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_S f_1 df_2 \wedge \cdots \wedge df_{2n}. \tag{1.4}$$

That is, the right-hand side of (1.4) is an integral on the unit sphere S. This leads to the thought that if we only consider Toeplitz operators with symbol functions that are *radially constant*, then we might be able to prove an analogue of (1.4) on the Fock space. As it turns out, this does lead to a trace formula. The fact that we can do this seems to suggest that even though the ball $\mathbf{B}$ is not quite structurally compatible with the Fock space, the sphere S can somehow be reasonably represented in $H^2(\mathbf{C}^n, d\mu)$.

To prove our trace formula, we will take advantage of a strategy that has been tested in [24]. That is, we will divide the proof of our analogue of the Helton-Howe trace formula for $H^2(\mathbf{C}^n, d\mu)$ into two parts. The first part is a trace-norm bound for antisymmetric sums, and the second part is to prove the trace formula in the special case where the symbol functions are just polynomials. The trace-norm bound then allows us to extend the trace formula to more general symbol functions.

Let us introduce the symbol functions that we will consider. First of all, if f is a function on the unit sphere S, we define the function $\tilde{f}$ by the formula

$$\tilde{f}(z) = f\left(\frac{z}{|z|}\right), \quad z \in \mathbf{C}^n \backslash \{0\}.$$

This notation will be used throughout the paper. Given a $0 < \delta < 1$, define

$$\Omega_\delta = \{z \in \mathbf{C}^n : 1 - \delta < |z| < 1 + \delta\}.$$

We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the natural way. Thus Ω_δ is also considered as an open set in $\mathbf{R}^{2n}$. Denote $\partial_j = \partial/\partial x_j$, $j = 1, \dots, 2n$. As usual, for $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbf{Z}^{2n}_+$, we write

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_{2n}^{\alpha_{2n}}.$$

For each $g \in C^2(\Omega_\delta)$, we define

$$\|g\|_{*,\delta} = \sum_{|\alpha| \le 2} \sup_{1-(\delta/2) \le |x| \le 1+(\delta/2)} |(\partial^\alpha g)(x)|. \tag{1.5}$$

We remind the reader that $\partial^0 g$ means the function g itself. With $\|\cdot\|_{*,\delta}$ so defined, we can state our trace-norm bound:

Theorem 1.5. *Given any $0 < \delta < 1$, there is a constant $0 < C = C(n, \delta) < \infty$ such that for all $f_1, f_2, \ldots, f_{2n} \in C^2(\Omega_\delta)$, the antisymmetric sum $[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n}}]$ on the Fock space $H^2(\mathbf{C}^n, d\mu)$ satisfies the trace-norm bound*

$$\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n}}]\|_1 \le C\|f_1\|_{*,\delta}\|f_2\|_{*,\delta} \cdots \|f_{2n}\|_{*,\delta}.$$

Using Theorem 1.5, we can prove the following analogue of trace formula (1.4) for the Fock space:

Theorem 1.6. *Let $f_1, f_2, \ldots, f_{2n} \in C^2(\Omega_\delta)$ for some $0 < \delta < 1$. Then on the Fock space $H^2(\mathbf{C}^n, d\mu)$, we have the trace formula*

$$\mathrm{tr}[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n}}] = \frac{n!}{(2\pi i)^n} \int_S f_1 df_2 \wedge \cdots \wedge df_{2n}. \tag{1.6}$$

As it turns out, our strategy of combining a norm bound with a polynomial version of the desired formula is a rather powerful one. It works not only for the ordinary Hilbert-space trace tr, as demonstrated by Theorem 1.6 above, but also for Dixmier trace.

Recall from [5,7,18] that Dixmier traces Tr_ω are defined on the Lorentz ideal $\mathcal{C}_1^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_1^+ < \infty\}$, where

$$\|A\|_1^+ = \sup_{k \ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1} + 2^{-1} + \cdots + k^{-1}}.$$

In recent years, Dixmier trace has been extensively studied in the context of operator theory on reproducing-kernel Hilbert spaces [10,11,17,21]. In [10], a formula for Dixmier trace was proved on the Bergman space $L_a^2(\mathbf{B})$. We will show that an analogue of this also holds on the Fock space $H^2(\mathbf{C}^n, d\mu)$.

Let $C^1(\overline{\mathbf{B}})$ be the collection of functions that are C^1 on an open set containing $\overline{\mathbf{B}}$. As defined in [10], the *boundary Poisson bracket* for $f, g \in C^1(\overline{\mathbf{B}})$ is given by the formula

$$\{f, g\}_b = \sum_{j=1}^n (\partial_j^b f \bar{\partial}_j^b g - \partial_j^b g \bar{\partial}_j^b f),$$

where $\partial_j^b = \partial_j - \bar{z}_j R$, $\bar{\partial}_j^b = \bar{\partial}_j - z_j \bar{R}$, and R is the radial derivative. We recall the following result of Engliš, Guo and Zhang:

Theorem 1.7. [10, Theorem 4.4] *Let $f_1, g_1, \ldots, f_n, g_n \in C^1(\overline{\mathbf{B}})$. Then the product of commutators*

$$[T_{f_1}^{\mathrm{Berg}}, T_{g_1}^{\mathrm{Berg}}] \cdots [T_{f_n}^{\mathrm{Berg}}, T_{g_n}^{\mathrm{Berg}}]$$

belongs to the ideal $\mathcal{C}_1^+$, and its Dixmier trace is given by the formula

$$\mathrm{Tr}_\omega([T_{f_1}^{\mathrm{Berg}}, T_{g_1}^{\mathrm{Berg}}] \cdots [T_{f_n}^{\mathrm{Berg}}, T_{g_n}^{\mathrm{Berg}}]) = \frac{1}{n!} \int_S \{f_1, g_1\}_b \cdots \{f_n, g_n\}_b d\sigma. \tag{1.7}$$

As the second part of our paper, we will prove an analogue of (1.7) for the Fock space $H^2(\mathbf{C}^n, d\mu)$. Similar to the case of the ordinary trace, our proof of the Fock-space analogue of (1.7) begins with a bound on the Lorentz norm $\| \cdot \|_1^+$.

Let $\mathrm{Lip}(S)$ be the collection of Lipschitz functions on the sphere S. For each $f \in \mathrm{Lip}(S)$, we write $L(f)$ for its Lipschitz constant.

Theorem 1.8. *Let $f_1, f_2, \dots, f_{2n} \in \mathrm{Lip}(S)$. Then on the Fock space $H^2(\mathbf{C}^n, d\mu)$, the product of commutators*

$$[T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]$$

belongs to $\mathcal{C}_1^+$. Moreover, there is a $0 < C = C(n) < \infty$ such that

$$\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]\|_1^+ \le C L(f_1)L(f_2) \cdots L(f_{2n-1})L(f_{2n})$$

for all $f_1, f_2, \dots, f_{2n} \in \mathrm{Lip}(S)$.

From Theorem 1.8 we can deduce the following analogue of (1.7) for the Fock space:

Theorem 1.9. *Let $f_1, f_2, \dots, f_{2n} \in C^1(\Omega_\delta)$ for some $0 < \delta < 1$. Then on the Fock space $H^2(\mathbf{C}^n, d\mu)$, for every Dixmier trace Tr_ω we have*

$$\mathrm{Tr}_\omega([T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]) = \frac{1}{n!} \int_S \{f_1, f_2\}_b \cdots \{f_{2n-1}, f_{2n}\}_b d\sigma. \tag{1.8}$$

The rest of the paper consists of the proofs of these results. Specifically, we prove Theorems 1.5 and 1.6 in Sections 2-7, and we prove Theorems 1.8 and 1.9 in Sections 8-9. Such an arrangement reflects the fact that the proofs of Theorems 1.5 and 1.6 deal with Schatten classes and the ordinary trace tr, whereas the proofs of Theorems 1.8 and 1.9 deal with Lorentz ideals and Dixmier trace. The point we want makes is that the same strategy works for the proofs of both (1.6) and (1.8).

We start the proofs by collecting a few basic facts in Section 2. Section 3 deals with the Schatten-class memberships of integral operators A_G introduced in Definition 3.1.

Then, using the Schatten-class memberships of A_G, we prove Theorem 1.5 in Section 4. There are two main ideas behind the proof. The first is that, by the first order Taylor expansion, symbol functions can be "extracted from commutation" modulo appropriate Schatten classes. The second idea is to take advantage of the fact that the real dimension of the sphere S is only $2n - 1$. Therefore in a Taylor expansion for $\tilde{f}$, there are only $2n - 1$ first order terms. This and the fact an antisymmetric sum involves $2n$ such $\tilde{f}$'s result in a crucial cancellation, which is the key step in the proof of Theorem 1.5.

We then turn to the proof of Theorem 1.6. Our approach involves "collapsing" products of Toeplitz operators, which we deal with in Section 5. Section 6 is the most crucial one in the paper. First, by careful analysis using asymptotic expansions, we prove formula (6.3). This formula enables us to exploit an old idea of Coburn [3], which was already used in [24], to introduce the unitary operator $U : L_a^2(\mathbf{B}) \to H^2(\mathbf{C}^n, d\mu)$ defined by (6.11). The main result of Section 6 is that for $p_1, p_2, \dots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, we have

$$U^*[T_{\tilde{p}_1}, T_{\tilde{p}_2}, \dots, T_{\tilde{p}_{2n}}]U = [T_{p_1}^{\mathrm{Berg}}, T_{p_2}^{\mathrm{Berg}}, \dots, T_{p_{2n}}^{\mathrm{Berg}}] + Y, \tag{1.9}$$

where Y is in the trace class with zero trace. This and Theorem 1.4 immediately give us the polynomial version of (1.6). Then, by approximation using Theorem 1.5, in Section 7 we derive the general version of (1.6) from the polynomial version.

In Section 8 we first prove analogues of several results in Section 3 for Lorentz ideals. We then deduce Theorem 1.8 from these Lorentz-ideal results. The reader will notice that the proof of Theorem 1.8 is significantly easier than the proof of Theorem 1.5.

Next we show that, similar to (1.9), for $p_1, p_2, \dots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ we have

$$U^*[T_{\tilde{p}_1}, T_{\tilde{p}_2}] \cdots [T_{\tilde{p}_{2n-1}}, T_{\tilde{p}_{2n}}]U = [T_{p_1}^{\mathrm{Berg}}, T_{p_2}^{\mathrm{Berg}}] \cdots [T_{p_{2n-1}}^{\mathrm{Berg}}, T_{p_{2n}}^{\mathrm{Berg}}] + G,$$

where G is in the trace class. This and Theorem 1.7 immediately give us the polynomial version of (1.8). Finally, in Section 9 we derive the general version of (1.8) from the polynomial version by approximation.

2. Preliminaries

Here we collect a number of facts that will be needed in the subsequent proofs.

As one might expect, Schatten classes will be heavily involved in the proofs of Theorems 1.5 and 1.6. For each $1 \le p < \infty$, let $\mathcal{C}_p$ denote the Schatten p-class. In other words, $\mathcal{C}_p = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p < \infty\}$, where $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}$.

Lemma 2.1. [12, Lemma 2.9] *Let $A \in \mathcal{C}_{p_1}$ and $B \in \mathcal{C}_{p_2}$, where $p_1, p_2 \in [1,\infty)$. If $p_1p_2/(p_1+p_2) \ge 1$, then $AB \in \mathcal{C}_{p_1p_2/(p_1+p_2)}$ with*

$$\|AB\|_{p_1p_2/(p_1+p_2)} \le \|A\|_{p_1}\|B\|_{p_2}.$$

If $p_1p_2/(p_1+p_2) < 1$, then $AB \in \mathcal{C}_1$.

By Lemma 2.1 and an obvious induction, we have

Corollary 2.2. *Let $p_1, \dots, p_k \in [1,\infty)$ be such that $(1/p_1) + \cdots + (1/p_k) \ge 1$. If operators $A_1, \dots, A_k$ are such that $A_j \in \mathcal{C}_{p_j}$ for every $j \in \{1, \dots, k\}$, then the product $A_1 \cdots A_k$ is in the trace class with*

$$\|A_1 \cdots A_k\|_1 \le \|A_1\|_{p_1} \cdots \|A_k\|_{p_k}.$$

We write $z_1, \dots, z_n$ for the coordinate functions on $\mathbf{C}^n$ or on $\mathbf{B}$.

Lemma 2.3. [24, Lemma 2.6] *On the Bergman space $L^2_a(\mathbf{B})$, we have $[T^{\mathrm{Berg}}_{z_i}, (T^{\mathrm{Berg}}_{z_j})^*] \in \mathcal{C}_p$ for all $p > n$ and $i, j \in \{1, \dots, n\}$.*

We will need the following property of the orthogonal projection $P : L^2(\mathbf{C}^n, d\mu) \to H^2(\mathbf{C}^n, d\mu)$:

Lemma 2.4. *Let ψ be a bounded measurable function on $\mathbf{C}^n$, and suppose that $\psi = 0$ on $\{z \in \mathbf{C}^n : |z| \ge R\}$ for some $0 < R < \infty$. Then the operator $M_\psi P$ is in the trace class.*

Proof. Let $\{e_\alpha : \alpha \in \mathbf{Z}^n_+\}$ be the standard orthonormal basis for the Fock space $H^2(\mathbf{C}^n, d\mu)$. It is well known that

$$e_\alpha(z) = (\alpha!)^{-1/2} z^\alpha, \quad \alpha \in \mathbf{Z}^n_+.$$

Then

$$M_\psi P = M_\psi \sum_{\alpha \in \mathbf{Z}^n_+} e_\alpha \otimes e_\alpha = \sum_{\alpha \in \mathbf{Z}^n_+} (\psi e_\alpha) \otimes e_\alpha.$$

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^n_+$, we have

$$\begin{aligned}\|\psi e_\alpha\|^2 &= \int |\psi(z)|^2 |e_\alpha(z)|^2 d\mu(z)\\ &\le \|\psi\|^2_\infty \prod_{j=1}^n \frac{1}{\alpha_j!\pi} \int_{|z_j| \le R} |z_j|^{2\alpha_j} e^{-|z_j|^2} dA(z_j) \le \|\psi\|^2_\infty \frac{R^{2|\alpha|}}{\alpha!}.\end{aligned}$$

Thus

$$\sum_{\alpha \in \mathbf{Z}^n_+} \|\psi e_\alpha\| \le \|\psi\|_\infty \sum_{\alpha \in \mathbf{Z}^n_+} \frac{R^{|\alpha|}}{\sqrt{\alpha!}} = \|\psi\|_\infty \left(\sum_{k=0}^\infty \frac{R^k}{\sqrt{k!}} \right)^n < \infty.$$

Therefore $M_\psi P$ is in the trace class. □

3. Integral operators

For convenience, we will write

$$K(z,w) = e^{\langle z,w\rangle},$$

$z, w \in \mathbf{C}^n$, which is the reproducing kernel for the Fock space $H^2(\mathbf{C}^n, d\mu)$.

Definition 3.1. For a Borel function $G(z,w)$ on $\mathbf{C}^n \times \mathbf{C}^n$, A_G denotes the operator defined by the formula

$$(A_G f)(z) = \int G(z,w)K(z,w)f(w)d\mu(w).$$

Lemma 3.2. *We have $\|A_G\| \le 2^{2n}\|G\|_\infty$ for every bounded Borel function $G(z,w)$ on $\mathbf{C}^n \times \mathbf{C}^n$.*

Proof. Define $h(w) = e^{|w|^2/2}$, $w \in \mathbf{C}^n$. Note that $|K(z,w)| = e^{\mathrm{Re}\langle z,w\rangle}$. Thus

$$\begin{aligned}\int |G(z,w)K(z,w)|h(w)d\mu(w) &\le \|G\|_\infty \int |K(z,w)|e^{|w|^2/2}d\mu(w)\\ &= \|G\|_\infty e^{|z|^2/2}\frac{1}{\pi^n}\int e^{-|z-w|^2/2}dV(w) = 2^{2n}\|G\|_\infty h(z).\end{aligned}$$

Similarly,

$$\int |G(z,w)K(z,w)|h(z)d\mu(z) \le 2^{2n}\|G\|_\infty h(w).$$

Applying the Schur test, we obtain the bound $\|A_G\| \le 2^{2n}\|G\|_\infty$. □

Lemma 3.3. *Let $2 \le p < \infty$. There is a constant $0 < C_{3.3} = C_{3.3}(n,p) < \infty$ such that*

$$\|A_G\|_p^p \le C_{3.3}\iint |G(z,w)|^p|K(z,w)|^2 d\mu(w)d\mu(z)$$

for every Borel function $G(z,w)$ on $\mathbf{C}^n \times \mathbf{C}^n$. In particular, if

$$\iint |G(z,w)|^p|K(z,w)|^2 d\mu(w)d\mu(z) < \infty$$

for some $2 \le p < \infty$, then $A_G \in \mathcal{C}_p$.

Proof. The case $p = 2$ is trivial. The case $2 < p < \infty$ is obtained by the standard interpolation between the operator norm (Lemma 3.2) and the Hilbert-Schmidt norm. □

Note that Lemma 3.3 holds only for $p \ge 2$. This causes many of the proofs below to be divided into the cases $n \ge 2$ and $n = 1$. In other words, this causes inconveniences.

Lemma 3.4. *If $a > 2n$, then*

$$\iint \left|\frac{z}{|z|} - \frac{w}{|w|}\right|^a |K(z,w)|^2 d\mu(w)d\mu(z) < \infty.$$

Proof. Denote $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$. Then

$$\begin{aligned}\int_{\mathbf{B}}\left\{\int_{\mathbf{C}^n}\left|\frac{z}{|z|} - \frac{w}{|w|}\right|^a |K(z,w)|^2 d\mu(w)\right\}d\mu(z) &\le 2^a\int_{\mathbf{B}}\left\{\int_{\mathbf{C}^n}|K(z,w)|^2 d\mu(w)\right\}d\mu(z)\\ &= 2^a V(\mathbf{B})/\pi^n.\end{aligned} \tag{3.1}$$

On the other hand, for $z, w \in \mathbf{C}^n \backslash \{0\}$, we have

$$\frac{z}{|z|} - \frac{w}{|w|} = \frac{1}{|z|}(z-w) + \left(\frac{1}{|z|} - \frac{1}{|w|}\right) w = \frac{1}{|z|}(z-w) + \frac{1}{|z|}(|w| - |z|)\frac{w}{|w|}.$$

Hence

$$\left|\frac{z}{|z|} - \frac{w}{|w|}\right| \le \frac{2}{|z|}|z-w|. \tag{3.2}$$

Consequently,

$$\begin{aligned}\int_{\mathbf{C}^n\backslash\mathbf{B}} &\left\{ \int_{\mathbf{C}^n} \left|\frac{z}{|z|} - \frac{w}{|w|}\right|^a |K(z,w)|^2 d\mu(w)\right\} d\mu(z) \\ &\le \frac{1}{\pi^{2n}} \int_{\mathbf{C}^n\backslash\mathbf{B}} \frac{2^a}{|z|^a}\left\{ \int_{\mathbf{C}^n} |z-w|^a e^{-|z-w|^2} dV(w)\right\} dV(z) \\ &= \frac{2^a}{\pi^{2n}} \int_{\mathbf{C}^n\backslash\mathbf{B}} \frac{1}{|z|^a} dV(z) \int_{\mathbf{C}^n} |\zeta|^a e^{-|\zeta|^2} dV(\zeta). \end{aligned} \tag{3.3}$$

For $a > 2n$, this is finite. Combining (3.1) and (3.3), the lemma is proved. □

Lemma 3.5. *Consider the case $n = 1$ and define the function $\varphi(z) = (|z| + 1)^{-1}$, $z \in \mathbf{C}$. Then the operator PM_φ on $L^2(\mathbf{C}, d\mu)$ has the property that $PM_\varphi \in \mathcal{C}_p$ for every $p > 2$.*

Proof. For the one-variable Fock space $H^2(\mathbf{C}, d\mu)$, we have the standard orthonormal basis $\{e_k : k \in \mathbf{Z}_+\}$, where

$$e_k(z) = (k!)^{-1/2} z^k, \quad k \in \mathbf{Z}_+.$$

Note that $\langle \varphi^2 e_k, e_j\rangle = 0$ for all $j \ne k$ in $\mathbf{Z}_+$. Therefore

$$(PM_\varphi(PM_\varphi)^*)^{1/2} = (PM_{\varphi^2}P)^{1/2} = \sum_{k=0}^{\infty} \langle \varphi^2 e_k, e_k\rangle^{1/2} e_k \otimes e_k.$$

For each $k \ge 1$, we have

$$\langle \varphi^2 e_k, e_k\rangle = \frac{1}{k!}\int \frac{|z|^{2k}}{(|z|+1)^2} d\mu(z) \le \frac{1}{k!}\int |z|^{2(k-1)} d\mu(z) = \frac{1}{k}.$$

This obviously means that $(PM_\varphi)^* \in \mathcal{C}_p$ for every $p > 2$. Consequently, $PM_\varphi \in \mathcal{C}_p$ for every $p > 2$. □

Lemma 3.6. (1) *For each $p > 2n$, there is a $0 < C_{3.6.1} = C_{3.6.1}(n,p) < \infty$ such that if the inequality*

$$|G(z,w)| \le \left|\frac{z}{|z|} - \frac{w}{|w|}\right|$$

holds for all $z, w \in \mathbf{C}^n \backslash \{0\}$, then $\|A_G\|_p \le C_{3.6.1}$.
(2) *Suppose that $n \ge 2$. For each $p > n$, there is a $0 < C_{3.6.2} = C_{3.6.2}(n,p) < \infty$ such that if the inequality*

$$|G(z,w)| \le \left|\frac{z}{|z|} - \frac{w}{|w|}\right|^2$$

holds for all $z, w \in \mathbf{C}^n \backslash \{0\}$, then $\|A_G\|_p \le C_{3.6.2}$.

(3) *Suppose that* $n = 1$. *For each* $p > 1$, *there is a* $0 < C_{3.6.3} = C_{3.6.3}(p) < \infty$ *such that if the inequality*

$$|G(z,w)| \le \left|\frac{z}{|z|} - \frac{w}{|w|}\right|^2 \tag{3.4}$$

holds for all $z, w \in \mathbf{C}\backslash\{0\}$, *then* $\|PA_G\|_p \le C_{3.6.3}$.

Proof. Obviously, (1) and (2) follow immediately from Lemmas 3.3 and 3.4.

Let us consider (3). In this case, we define the function

$$H(z,w) = (|z|+1)G(z,w), \quad z, w \in \mathbf{C}.$$

Then $PA_G = PM_\varphi A_H$. Lemma 3.5 tells us that $PM_\varphi \in \mathcal{C}_p$ for every $p > 2$. Thus it suffices to show that for every $p > 2$, there a constant $0 < C(p) < \infty$ such that $\|A_H\|_p \le C(p)$. This essentially repeats the argument in the proof of Lemma 3.4.

Write $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. By (3.4) and (3.2), we have

$$|H(z,w)| \le \frac{8}{|z|}|z-w|^2 \quad \text{for} \quad z \in \mathbf{C}\backslash\mathbf{D} \quad \text{and} \quad w \in \mathbf{C}\backslash\{0\}.$$

Therefore for any $p > 2$,

$$\begin{aligned}
&\int_{\mathbf{C}\backslash\mathbf{D}} \left\{ \int_{\mathbf{C}} |H(z,w)|^p |K(z,w)|^2 d\mu(w) \right\} d\mu(z) \\
&\qquad \le \frac{1}{\pi^2} \int_{\mathbf{C}\backslash\mathbf{D}} \frac{8^p}{|z|^p} \left\{ \int_{\mathbf{C}} |z-w|^{2p} e^{-|z-w|^2} dA(w) \right\} dA(z) \\
&\qquad = \frac{8^p}{\pi^2} \int_{\mathbf{C}\backslash\mathbf{D}} \frac{1}{|z|^p} dA(z) \int_{\mathbf{C}} |\zeta|^{2p} e^{-|\zeta|^2} dA(\zeta) < \infty.
\end{aligned} \tag{3.5}$$

On the other hand, by the definition of $H(z,w)$ and (3.4),

$$\begin{aligned}
\int_{\mathbf{D}} \left\{ \int_{\mathbf{C}} |H(z,w)|^p |K(z,w)|^2 d\mu(w) \right\} d\mu(z) &\le 8^p \int_{\mathbf{D}} \left\{ \int_{\mathbf{C}} |K(z,w)|^2 d\mu(w) \right\} d\mu(z) \\
&= 8^p A(\mathbf{D})/\pi.
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) with Lemma 3.3, we obtain the desired bound $\|A_H\|_p \le C(p)$ for every $p > 2$. □

Recall that we write $S = \{z \in \mathbf{C}^n : |z| = 1\}$, the unit sphere in $\mathbf{C}^n$. If $f \in \mathrm{Lip}(S)$, we write $L(f)$ for its Lipschitz constant.

Lemma 3.7. *For each* $p > 2n$, *there is a* $0 < C_{3.7} = C_{3.7}(n,p) < \infty$ *such that the inequality* $\|[M_{\tilde{f}}, P]\|_p \le C_{3.7} L(f)$ *holds for every* $f \in \mathrm{Lip}(S)$.

Proof. For each $f \in \mathrm{Lip}(S)$, it is obvious that $[M_{\tilde{f}}, P] = A_G$, where

$$G(z,w) = \tilde{f}(z) - \tilde{f}(w) = f\left(\frac{z}{|z|}\right) - f\left(\frac{w}{|w|}\right).$$

Consequently,

$$|G(z,w)| \le L(f)\left|\frac{z}{|z|} - \frac{w}{|w|}\right|.$$

Thus the desired conclusion follows from Lemma 3.6(1). □

Lemma 3.8. (1) *Suppose that $n \geq 2$. For each $p > n$, there is a $0 < C_{3.8.1} = C_{3.8.1}(n,p) < \infty$ such that the inequality $\|[M_{\tilde{g}}, [M_{\tilde{f}}, P]]\|_p \leq C_{3.8.1} L(f)L(g)$ holds for all $f, g \in \mathrm{Lip}(S)$.*
(2) *Suppose that $n = 1$. For each $p > 1$, there is a $0 < C_{3.8.2} = C_{3.8.2}(p) < \infty$ such that the inequality $\|P[M_{\tilde{g}}, [M_{\tilde{f}}, P]]\|_p \leq C_{3.8.2} L(f)L(g)$ holds for all $f, g \in \mathrm{Lip}(S)$.*

Proof. (1) Given $f, g \in \mathrm{Lip}(S)$, we have $[M_{\tilde{g}}, [M_{\tilde{f}}, P]] = A_G$, where

$$G(z,w) = (\tilde{f}(z) - \tilde{f}(w))(\tilde{g}(z) - \tilde{g}(w)) = \left(f\left(\frac{z}{|z|}\right) - f\left(\frac{w}{|w|}\right)\right)\left(g\left(\frac{z}{|z|}\right) - g\left(\frac{w}{|w|}\right)\right).$$

Consequently,

$$|G(z,w)| \leq L(f)L(g)\left|\frac{z}{|z|} - \frac{w}{|w|}\right|^2.$$

Thus the desired conclusion follows from Lemma 3.6(2).

(2) We can write $P[M_{\tilde{g}}, [M_{\tilde{f}}, P]] = PA_G$, where G has the same formula as above. Thus, in this case, the desired conclusion follows from Lemma 3.6(3). □

4. Proof of Theorem 1.5

It will be convenient to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the natural way. That is, we identify

$$(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \quad \text{with} \quad (x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n}),$$

$x_1, x_2, \ldots, x_{2n-1}, x_{2n} \in \mathbf{R}$. Accordingly, the unit sphere S in $\mathbf{C}^n$ is identified with

$$\{(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in \mathbf{R}^{2n} : x_1^2 + x_2^2 + \cdots + x_{2n-1}^2 + x_{2n}^2 = 1\}.$$

For each $1 \leq j \leq 2n$, we define the function

$$\epsilon_j(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = x_j, \quad (x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in \mathbf{R}^{2n}.$$

Under our identification, each ϵ_j is also considered as a function on $\mathbf{C}^n$.

Denote the vector $(0, \ldots, 0, i)$ in $\mathbf{C}^n$ by ι. Under the above convention, ι is identified with the vector $(0, 0, \ldots, 0, 1)$ in $\mathbf{R}^{2n}$.

We fix a C^∞-function ξ on $[0, \infty)$ satisfying the following conditions:
(a) $0 \leq \xi \leq 1$ on $[0, \infty)$.
(b) $\xi = 1$ on $[0, 1/4]$.
(c) $\xi = 0$ on $[1/3, \infty)$.
We then define the function

$$\eta(z) = \xi(|\iota - z|), \quad z \in \mathbf{C}^n.$$

Then η is a C^∞-function on $\mathbf{C}^n$. Moreover, according to the preceding paragraphs, η is also considered as a C^∞-function on $\mathbf{R}^{2n}$.

We will write $x = (x', x_{2n})$ for a general $x \in \mathbf{R}^{2n}$, where $x' \in \mathbf{R}^{2n-1}$ and $x_{2n} \in \mathbf{R}$. Under the identification of $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, we define

$$\Sigma = \{(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in S : \sqrt{x_1^2 + x_2^2 + \cdots + x_{2n-1}^2} < 1/2 \text{ and } x_{2n} > 0\}.$$

Thus a general point $x \in \Sigma$ can be represented in the form $x = (x', x_{2n})$, where $x' \in \mathbf{R}^{2n-1}$ with $|x'| < 1/2$ and $x_{2n} = (1 - |x'|^2)^{1/2}$. By condition (c) above, we have $\eta = 0$ on $S\backslash\Sigma$. Therefore there is a function $\psi \in C_c^\infty(\mathbf{R}^{2n})$ such that

$$\psi(x) = \frac{\eta(x)}{\sqrt{1 - |x'|^2}} \quad \text{for} \;\; x = (x', x_{2n}) \in \Sigma.$$

Let us also define

$$E = \{(x_1, x_2, \dots, x_{2n-1}, x_{2n}) \in S : \sqrt{x_1^2 + x_2^2 + \cdots + x_{2n-1}^2} < 1/8 \text{ and } x_{2n} > 0\}.$$

It is easy to see that $|\iota - x| < 1/4$ for every $x \in E$. Thus, by condition (b) above,

$$\psi(x) = \frac{1}{\sqrt{1 - |x'|^2}} \quad \text{for} \;\; x = (x', x_{2n}) \in E. \tag{4.1}$$

For convenience, we will abbreviate

$$\partial_j = \frac{\partial}{\partial x_j}, \quad j = 1, \dots, 2n.$$

Furthermore, for $1 \le j \le 2n - 1$, we define

$$(D_j f)(x) = (\partial_j f)(x) - \epsilon_j(x)\psi(x)(\partial_{2n} f)(x).$$

Lemma 4.1. *Given any $0 < \delta < 1$, there is a $0 < C_{4.1} = C_{4.1}(\delta, n) < \infty$ such that if $f \in C^2(\Omega_\delta)$ and $x, y \in E$, then*

$$\left| f(y) - f(x) - \sum_{j=1}^{2n-1} (D_j f)(x)(\epsilon_j(y) - \epsilon_j(x)) \right| \le C_{4.1} \|f\|_{*,\delta} |y - x|^2.$$

Proof. Given $x, y \in E$, we use the representations $x = (x', x_{2n})$ and $y = (y', y_{2n})$ mentioned above. By the fundamental theorem of calculus and straightforward differentiation,

$$\begin{aligned}
f(y) - f(x) &= f(y', y_{2n}) - f(x', x_{2n}) \\
&= \int_0^1 \frac{d}{dt} f(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2}) dt \\
&= \int_0^1 \sum_{j=1}^{2n-1} (\partial_j f)(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2})(\epsilon_j(y) - \epsilon_j(x)) dt \\
&- \int_0^1 (\partial_{2n} f)(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2}) \\
&\times \frac{\sum_{j=1}^{2n-1} (\epsilon_j(x) + t(\epsilon_j(y) - \epsilon_j(x)))(\epsilon_j(y) - \epsilon_j(x))}{\sqrt{1 - |x' + t(y' - x')|^2}} dt \\
&= \int_0^1 \sum_{j=1}^{2n-1} (D_j f)(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2})(\epsilon_j(y) - \epsilon_j(x)) dt,
\end{aligned}$$

where for the last $=$ we apply (4.1) and the fact that

$$\epsilon_j(x) + t(\epsilon_j(y) - \epsilon_j(x)) = \epsilon_j(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2})$$

when $1 \le j \le 2n-1$. Continuing with the above, we obtain

$$f(y) - f(x) = \sum_{j=1}^{2n-1} (D_j f)(x)(\epsilon_j(y) - \epsilon_j(x)) + \int_0^1 \sum_{j=1}^{2n-1} H_j(y', x', t)(\epsilon_j(y) - \epsilon_j(x))dt,$$

where, for each $1 \le j \le 2n-1$,

$$H_j(y', x', t) = (D_j f)(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2}) - (D_j f)(x', \sqrt{1 - |x'|^2}).$$

It is easy to see that for each $1 \le j \le 2n-1$,

$$\begin{aligned} |H_j(x', y', t)| &\le C_2 \|f\|_{*,\delta} |(x' + t(y' - x'), \sqrt{1 - |x' + t(y' - x')|^2}) - (x', \sqrt{1 - |x'|^2})| \\ &\le C_3 \|f\|_{*,\delta} |y' - x'| \le C_3 \|f\|_{*,\delta} |y - x|. \end{aligned}$$

This completes the proof. □

We now define the function

$$\gamma(z) = \xi(3|\iota - z|), \quad z \in \mathbf{C}^n.$$

By condition (c) for ξ, we have $\gamma = 0$ on $S \backslash E$. Moreover, by condition (b) for ξ, we have $\gamma = 1$ on the set

$$\mathcal{N} = S \cap B(\iota, 1/12).$$

With this function γ, we now define the operators

$$\Lambda_j = M_{\tilde{\gamma}} [M_{\tilde{\epsilon}_j}, P] M_{\tilde{\gamma}} \tag{4.2}$$

$j = 1, \dots, 2n-1$. By Lemma 3.7, we have $\Lambda_j \in \mathcal{C}_p$ for every $p > 2n$, $j = 1, \dots, 2n-1$.

Lemma 4.2. (1) *Suppose that $n \ge 2$. For each $p > n$, there is a $0 < C_{4.2.1} = C_{4.2.1}(p, n) < \infty$ such that $\|[\Lambda_j, M_{\tilde{f}}]\|_p \le C_{4.2.1} L(f)$ for all $j = 1, \dots, 2n-1$ and $f \in \mathrm{Lip}(S)$.*
(2) *Suppose that $n = 1$. For each $p > 1$, there is a $0 < C_{4.2.2} = C_{4.2.2}(p) < \infty$ such that $\|P[\Lambda_1, M_{\tilde{f}}]\|_p \le C_{4.2.2} L(f)$ for all $f \in \mathrm{Lip}(S)$.*

Proof. By (4.2), (1) is an immediate consequence of Lemma 3.8(1). Also by (4.2), (2) follows from Lemmas 3.8(2) and 3.7. □

Lemma 4.3. (1) *Suppose that $n \ge 2$. Given any $p > n$ and $0 < \delta < 1$, there is a $0 < C_{4.3.1} = C_{4.3.1}(p, \delta, n) < \infty$ such that for every $f \in C^2(\Omega_\delta)$,*

$$\left\| M_{\tilde{\gamma}} [M_{\tilde{f}}, P] M_{\tilde{\gamma}} - \sum_{j=1}^{2n-1} \Lambda_j M_{\widetilde{D_j f}} \right\|_p \le C_{4.3.1} \|f\|_{*,\delta}.$$

(2) *Suppose that $n = 1$. Given any $p > 1$ and $0 < \delta < 1$, there is a $0 < C_{4.3.2} = C_{4.3.2}(p, \delta) < \infty$ such that for every $f \in C^2(\Omega_\delta)$,*

$$\begin{aligned} &\|P(M_{\tilde{\gamma}} [M_{\tilde{f}}, P] M_{\tilde{\gamma}} - \Lambda_1 M_{\widetilde{D_1 f}})\|_p \le C_{4.3.2} \|f\|_{*,\delta} \quad \text{and} \\ &\|(M_{\tilde{\gamma}} [M_{\tilde{f}}, P] M_{\tilde{\gamma}} - \Lambda_1 M_{\widetilde{D_1 f}}) P\|_p \le C_{4.3.2} \|f\|_{*,\delta}. \end{aligned}$$

Proof. By the notation in Section 3, we have

$$M_{\tilde{\gamma}} [M_{\tilde{f}}, P] M_{\tilde{\gamma}} - \sum_{j=1}^{2n-1} \Lambda_j M_{\widetilde{D_j f}} = A_G,$$

where

$$G(z,w)=\gamma\left(\frac{z}{|z|}\right)\left\{f\left(\frac{z}{|z|}\right)-f\left(\frac{w}{|w|}\right)\right.$$
$$\left.-\sum_{j=1}^{2n-1}\left(\epsilon_j\left(\frac{z}{|z|}\right)-\epsilon_j\left(\frac{w}{|w|}\right)\right)(D_jf)\left(\frac{w}{|w|}\right)\right\}\gamma\left(\frac{w}{|w|}\right).$$

By the property of γ mentioned above, we have $G(z,w)\neq 0$ only if $z/|z|\in E$ and $w/|w|\in E$. Therefore by Lemma 4.1,

$$|G(z,w)|\le C_{4.1}\|f\|_{*,\delta}\left|\frac{z}{|z|}-\frac{w}{|w|}\right|^2\quad\text{for all}\quad z,w\in\mathbf{C}^n\backslash\{0\}.$$

Thus the desired conclusions follow from Lemma 3.6. □

As usual, for $f\in L^\infty(\mathbf{C}^n,d\mu)$, we identify the Toeplitz operator T_f on the Fock space $H^2(\mathbf{C}^n,d\mu)$ with the operator PM_fP on $L^2(\mathbf{C}^n,d\mu)$. Thus it makes sense to write T_fM_g or M_gT_f for $f,g\in L^\infty(\mathbf{C}^n,d\mu)$. Also, for $f,g\in L^\infty(\mathbf{C}^n,d\mu)$, easy algebra shows that

$$[T_f,T_g]=P[[M_f,P],[M_g,P]]P. \tag{4.3}$$

Proposition 4.4. *Given any* $0<\delta<1$, *there is a* $0<C_{4.4}=C_{4.4}(\delta,n)<\infty$ *such that*

$$\|[T_{\tilde f_1},T_{\tilde f_2},\dots,T_{\tilde f_{2n-1}},T_{\tilde f_{2n}}]M_{\tilde\gamma^{4n}}\|_1\le C_{4.4}\|f_1\|_{*,\delta}\|f_2\|_{*,\delta}\cdots\|f_{2n-1}\|_{*,\delta}\|f_{2n}\|_{*,\delta}$$

for all $f_1,f_2,\dots,f_{2n-1},f_{2n}\in C^2(\Omega_\delta)$.

Proof. Let $f_1,f_2,\dots,f_{2n-1},f_{2n}\in C^2(\Omega_\delta)$. In this proof, the notation $A\sim_1 B$ means

$$\|A-B\|_1\le C\|f_1\|_{*,\delta}\|f_2\|_{*,\delta}\cdots\|f_{2n-1}\|_{*,\delta}\|f_{2n}\|_{*,\delta}$$

with constant C independent of $f_1,f_2,\dots,f_{2n-1},f_{2n}\in C^2(\Omega_\delta)$.

We will now apply the Schatten-norm bounds provided by Lemmas 3.7, 3.8, 4.2 and 4.3. Keep in mind that if $f\in C^2(\Omega_\delta)$, then $\partial_jf\in\mathrm{Lip}(S)$ with $L(\partial_jf)\le C\|f\|_{*,\delta}$, $j=1,\dots,2n$.

Starting with (4.3), we have

$$\begin{aligned}
&[T_{\tilde f_1},T_{\tilde f_2}]\cdots[T_{\tilde f_{2n-1}},T_{\tilde f_{2n}}]M_{\tilde\gamma^{4n}}\\
&\quad=P[[M_{\tilde f_1},P],[M_{\tilde f_2},P]]P\cdots P[[M_{\tilde f_{2n-1}},P],[M_{\tilde f_{2n}},P]]PM_{\tilde\gamma^{4n}}\\
&\quad\sim_1 P[M_{\tilde\gamma}[M_{\tilde f_1},P]M_{\tilde\gamma},M_{\tilde\gamma}[M_{\tilde f_2},P]M_{\tilde\gamma}]P\cdots P[M_{\tilde\gamma}[M_{\tilde f_{2n-1}},P]M_{\tilde\gamma},M_{\tilde\gamma}[M_{\tilde f_{2n}},P]M_{\tilde\gamma}]P\\
&\quad\sim_1 P\left[\sum_{j=1}^{2n-1}\Lambda_jM_{\widetilde{D_jf_1}},\sum_{j=1}^{2n-1}\Lambda_jM_{\widetilde{D_jf_2}}\right]P\cdots P\left[\sum_{j=1}^{2n-1}\Lambda_jM_{\widetilde{D_jf_{2n-1}}},\sum_{j=1}^{2n-1}\Lambda_jM_{\widetilde{D_jf_{2n}}}\right]P\\
&\quad=\sum_{j_1,j_2,\dots,j_{2n}=1}^{2n-1}P[\Lambda_{j_1}M_{\widetilde{D_{j_1}f_1}},\Lambda_{j_2}M_{\widetilde{D_{j_2}f_2}}]P\cdots P[\Lambda_{j_{2n-1}}M_{\widetilde{D_{j_{2n-1}}f_{2n-1}}},\Lambda_{j_{2n}}M_{\widetilde{D_{j_{2n}}f_{2n}}}]P\\
&\quad\sim_1\sum_{j_1,j_2,\dots,j_{2n}=1}^{2n-1}P[\Lambda_{j_1},\Lambda_{j_2}]P\cdots P[\Lambda_{j_{2n-1}},\Lambda_{j_{2n}}]PM_{\widetilde{D_{j_1}f_1}\widetilde{D_{j_2}f_2}\cdots\widetilde{D_{j_{2n-1}}f_{2n-1}}\widetilde{D_{j_{2n}}f_{2n}}},
\end{aligned}$$

where the first $\sim_1$ follows from Lemmas 3.7 and 3.8, for the second $\sim_1$ we apply Lemmas 4.3 and 3.7, and the third $\sim_1$ is a consequence of Lemmas 3.7 and 4.2. Recall identity (4.9) from [23]. That is,

$$[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \dots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}] = \frac{1}{2^n} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma)[T_{\tilde{f}_{\sigma(1)}}, T_{\tilde{f}_{\sigma(2)}}] \cdots [T_{\tilde{f}_{\sigma(2n-1)}}, T_{\tilde{f}_{\sigma(2n)}}].$$

Therefore it follows from the above that

$$\begin{aligned}[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \dots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]M_{\tilde{\gamma}^{4n}} \sim_1 \frac{1}{2^n} &\sum_{j_1, j_2, \dots, j_{2n}=1}^{2n-1} P[\Lambda_{j_1}, \Lambda_{j_2}]P \cdots P[\Lambda_{j_{2n-1}}, \Lambda_{j_{2n}}]P \\ &\times \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) M_{\widetilde{D_{j_1} f_{\sigma(1)}} \widetilde{D_{j_2} f_{\sigma(2)}} \cdots \widetilde{D_{j_{2n-1}} f_{\sigma(2n-1)}} \widetilde{D_{j_{2n}} f_{\sigma(2n)}}}. \end{aligned} \tag{4.4}$$

Given any $j_1, j_2, \dots, j_{2n} \in \{1, 2, \dots, 2n-1\}$, there are $k \neq \ell$ in the set $\{1, 2, \dots, 2n-1, 2n\}$ such that $j_k = j_\ell$, i.e., $D_{j_k} = D_{j_\ell}$. Hence for each tuple $j_1, j_2, \dots, j_{2n} \in \{1, 2, \dots, 2n-1\}$, we have

$$\sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \widetilde{D_{j_1} f_{\sigma(1)}} \widetilde{D_{j_2} f_{\sigma(2)}} \cdots \widetilde{D_{j_{2n-1}} f_{\sigma(2n-1)}} \widetilde{D_{j_{2n}} f_{\sigma(2n)}} = 0.$$

Combining this with (4.4), we conclude that

$$[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \dots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]M_{\tilde{\gamma}^{4n}} \sim_1 0.$$

This completes the proof. $\square$

Let $\mathcal{U}$ denote the group of unitary transformations on $\mathbf{C}^n$. It is well known that for each $L \in \mathcal{U}$, the formula

$$(W_L f)(z) = f(Lz)$$

defines a unitary operator on both $H^2(\mathbf{C}^n, d\mu)$ and $L^2(\mathbf{C}^n, d\mu)$. We have

$$W_L^* T_\varphi W_L = T_{\varphi \circ L^*} \quad \text{for all} \quad L \in \mathcal{U} \quad \text{and} \quad \varphi \in L^\infty(\mathbf{C}^n, d\mu).$$

If h is a function on S, then $\tilde{h} \circ L = \widetilde{h \circ L}$, $L \in \mathcal{U}$. It is also easy to see that there is a $0 < C = C(n, \delta) < \infty$ such that $\|f \circ L\|_{*,\delta} \leq C\|f\|_{*,\delta}$ for all $f \in C^2(\Omega_\delta)$ and $L \in \mathcal{U}$.

Proof of Theorem 1.5. Recall that $\gamma = 1$ on the open subset $\mathcal{N} = S \cap B(\iota, 1/12)$ of the unit sphere S. There is a finite subset F of $\mathcal{U}$ such that $S = \cup_{L \in F} L\mathcal{N}$. Hence

$$\sum_{L \in F} (\gamma \circ L^*)^{4n} \geq 1 \quad \text{on} \quad S.$$

Consequently,

$$\sum_{L \in F} (\tilde{\gamma} \circ L^*)^{4n} = \sum_{L \in F} (\widetilde{\gamma \circ L^*})^{4n} \geq 1 \quad \text{on} \quad \mathbf{C}^n \backslash \{0\}. \tag{4.5}$$

Let $f_1, f_2, \ldots, f_{2n-1}, f_{2n} \in C^2(\Omega_\delta)$. For each $L \in F$, it follows from Proposition 4.4 that

$$\begin{aligned}
&\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]M_{(\tilde{\gamma}\circ L^*)^{4n}}\|_1 \\
&\qquad = \|W_L^*[T_{\tilde{f}_1\circ L}, T_{\tilde{f}_2\circ L}, \ldots, T_{\tilde{f}_{2n-1}\circ L}, T_{\tilde{f}_{2n}\circ L}]M_{\tilde{\gamma}^{4n}}W_L\|_1 \\
&\qquad = \|[T_{\widetilde{f_1\circ L}}, T_{\widetilde{f_2\circ L}}, \ldots, T_{\widetilde{f_{2n-1}\circ L}}, T_{\widetilde{f_{2n}\circ L}}]M_{\tilde{\gamma}^{4n}}\|_1 \\
&\qquad \le C_{4.4}\|f_1 \circ L\|_{*,\delta}\|f_2 \circ L\|_{*,\delta} \cdots \|f_{2n-1} \circ L\|_{*,\delta}\|f_{2n} \circ L\|_{*,\delta} \\
&\qquad \le C_1\|f_1\|_{*,\delta}\|f_2\|_{*,\delta} \cdots \|f_{2n-1}\|_{*,\delta}\|f_{2n}\|_{*,\delta}.
\end{aligned}$$

Combining this with (4.5), we find that

$$\begin{aligned}
\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]\|_1 &\le \left\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]\sum_{L\in F} M_{(\tilde{\gamma}\circ L^*)^{4n}}\right\|_1 \\
&\le \mathrm{card}(F)C_1\|f_1\|_{*,\delta}\|f_2\|_{*,\delta} \cdots \|f_{2n-1}\|_{*,\delta}\|f_{2n}\|_{*,\delta}.
\end{aligned}$$

This completes the proof. □

5. Semi-commutators

For each $0 < R < \infty$, we define the function

$$h_R(z) = \chi_{[R,\infty)}(|z|), \quad z \in \mathbf{C}^n.$$

Lemma 5.1. *For all $2n < p < \infty$ and $f \in \mathrm{Lip}(S)$, we have*

$$\lim_{R\to\infty} \|[T_{h_R}, T_{\tilde{f}}]\|_p = 0. \tag{5.1}$$

Proof. By (4.3),

$$[T_{h_R}, T_{\tilde{f}}] = P[[M_{h_R}, P], [M_{\tilde{f}}, P]]P.$$

By Lemma 3.7, we have $[M_{\tilde{f}}, P] \in \mathcal{C}_p$ for every $p > 2n$. Combining this with the obvious fact that $M_{h_R} \to 0$ strongly as $R \to \infty$, (5.1) follows. □

Lemma 5.2. *For all $f, g \in \mathrm{Lip}(S)$ and $p > n$, we have $T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}} \in \mathcal{C}_p$. Consequently, $[T_{\tilde{f}}, T_{\tilde{g}}] \in \mathcal{C}_p$.*

Proof. Since

$$T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}} = PM_{\tilde{f}}(1-P)M_{\tilde{g}}P = [P, M_{\tilde{f}}](1-P)[M_{\tilde{g}}, P], \tag{5.2}$$

the desired conclusion again follows from Lemma 3.7. □

Lemma 5.3. (1) *Suppose that $n \ge 2$. For all $f, g, \psi \in \mathrm{Lip}(S)$ and $p > 2n/3$, we have $[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, T_{\tilde{\psi}}] \in \mathcal{C}_p$.*
(2) *Suppose that $n = 1$. For all $f, g, \psi \in \mathrm{Lip}(S)$, we have $[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, T_{\tilde{\psi}}] \in \mathcal{C}_1$.*

Proof. Continuing with (5.2), we have

$$\begin{aligned}
[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, T_{\tilde{\psi}}] &= P[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, M_{\tilde{\psi}}]P \\
&= P[[P, M_{\tilde{f}}], M_{\tilde{\psi}}](1-P)[M_{\tilde{g}}, P]P - P[P, M_{\tilde{f}}][P, M_{\tilde{\psi}}][M_{\tilde{g}}, P]P \\
&\quad + P[P, M_{\tilde{f}}](1-P)[[M_{\tilde{g}}, P], M_{\tilde{\psi}}]P.
\end{aligned}$$

Thus the desired Schatten-class membership for $[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, T_{\tilde{\psi}}]$ follows from Lemmas 3.7 and 3.8. □

Let $\mathcal{R}$ be the algebra generated by the Toeplitz operators $\{T_{\tilde{f}} : f \in \mathrm{Lip}(S)\}$ on $H^2(\mathbf{C}^n, d\mu)$.

Lemma 5.4. *Let $A, B, A_2, \ldots, A_{2n-1}, A_{2n} \in \mathcal{R}$ and let $f, g \in \mathrm{Lip}(S)$. Then the antisymmetric sum*

$$[A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_2, \ldots, A_{2n-1}, A_{2n}] \tag{5.3}$$

is in the trace class with zero trace.

Proof. First of all, if G is a trace-class operator, then for all bounded operators $T_2, \ldots, T_{2n-1}$, T_{2n}, the antisymmetric sum

$$[G, T_2, \ldots, T_{2n-1}, T_{2n}]$$

is in the trace class with zero trace. With that in mind, for each $0 < R < \infty$, we take the decomposition

$$[A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_2, \ldots, A_{2n-1}, A_{2n}] = X_R + Y_R,$$

where

$$X_R = [T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_2, \ldots, A_{2n-1}, A_{2n}] \quad \text{and}$$
$$Y_R = [T_{1-h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_2, \ldots, A_{2n-1}, A_{2n}].$$

Since $1 - h_R(z) = \chi_{[0,R)}(|z|)$, by Lemma 2.4, the Toeplitz operator T_{1-h_R} is in the trace class. Therefore Y_R is in the trace class with zero trace. Thus it suffices to show that

$$\lim_{R\to\infty} \|X_R\|_1 = 0. \tag{5.4}$$

To prove this, we first consider the case where $n \geq 2$. For each $R > 0$ we define $A_{1,R} = T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B$ and $A_{j,R} = A_j$ for every $2 \leq j \leq 2n$. Then

$$X_R = [A_{1,R}, A_{2,R}, \ldots, A_{2n-1,R}, A_{2n,R}].$$

Recalling [23, (4.9)], we have

$$X_R = \frac{1}{2^n}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma)[A_{\sigma(1),R}, A_{\sigma(2),R}]\cdots[A_{\sigma(2n-1),R}, A_{\sigma(2n),R}].$$

Thus (5.4) will follow if we can show that

$$\lim_{R\to\infty} \|[A_{\sigma(1),R}, A_{\sigma(2),R}]\cdots[A_{\sigma(2n-1),R}, A_{\sigma(2n),R}]\|_1 = 0 \tag{5.5}$$

for every $\sigma \in S_{2n}$. Let a $\sigma \in S_{2n}$ be given. Then there is a $\nu \in \{1, \ldots, n\}$ such that $1 \in \{\sigma(2\nu - 1), \sigma(2\nu)\}$. We now take a $p > n$ and a $q > 2n/3$ such that

$$\frac{n-1}{p} + \frac{1}{q} \geq 1.$$

Then

$$\begin{aligned}
&\|[A_{\sigma(1),R}, A_{\sigma(2),R}]\cdots[A_{\sigma(2n-1),R}, A_{\sigma(2n),R}]\|_1 \\
&\qquad \leq \|[A_{\sigma(2\nu-1),R}, A_{\sigma(2\nu),R}]\|_q \prod_{i\in\{1,\ldots,n\}\backslash\{\nu\}} \|[A_{\sigma(2i-1),R}, A_{\sigma(2i),R}]\|_p.
\end{aligned}$$

By the definitions of $A_{j,R}$ and ν, for every $i \in \{1, \ldots, n\}\backslash\{\nu\}$ we have

$$\|[A_{\sigma(2i-1),R}, A_{\sigma(2i),R}]\|_p = \|[A_{\sigma(2i-1)}, A_{\sigma(2i)}]\|_p < \infty,$$

where the $<$ follows from the membership $A_{\sigma(2i-1)}, A_{\sigma(2i)} \in \mathcal{R}$ and Lemma 5.2. Therefore (5.5) will be proved for the given $\sigma \in S_{2n}$ if we can show that

$$\lim_{R\to\infty} \|[A_{\sigma(2\nu-1),R}, A_{\sigma(2\nu),R}]\|_q = 0.$$

By the definitions of $A_{j,R}$ and ν, we also have

$$[A_{\sigma(2\nu-1),R}, A_{\sigma(2\nu),R}] = \pm[T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_d]$$

with some $d \in \{2, \ldots, 2n-1, 2n\}$. Thus the proof will be complete if we can show that

$$\lim_{R\to\infty} \|[T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_d]\|_q = 0.$$

For each $R > 0$, we have

$$[T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_d] = Z_{1,R} + Z_{2,R} + Z_{3,R} + Z_{4,R},$$

where

$$\begin{aligned}
Z_{1,R} &= [T_{h_R}, A_d]A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, \\
Z_{2,R} &= T_{h_R}[A, A_d](T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, \\
Z_{3,R} &= T_{h_R}A[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, A_d]B \quad \text{and} \\
Z_{4,R} &= T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})[B, A_d].
\end{aligned}$$

Thus it suffices to prove that

$$\lim_{R\to\infty} \|Z_{i,R}\|_q = 0 \tag{5.6}$$

for $i = 1, 2, 3, 4$. To prove this, first note that $T_{h_R} \to 0$ strongly as $R \to \infty$. It follows from Lemma 5.2 that $[A, A_d](T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}) \in \mathcal{C}_q$ and that $(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})[B, A_d] \in \mathcal{C}_q$. Hence (5.6) holds in the cases $i = 2$ and $i = 4$. By Lemma 5.3, $[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, A_d] \in \mathcal{C}_q$. Therefore (5.6) also holds in the case $i = 3$.

To prove the case $i = 1$ for (5.6), we use the condition $q > 2n/3$. This allows us to pick an $s > 2n$ and a $t > n$ such that

$$\frac{1}{s} + \frac{1}{t} \geq \frac{1}{q}. \tag{5.7}$$

It follows from Lemma 5.1 that

$$\lim_{R\to\infty} \|[T_{h_R}, A_d]\|_s = 0.$$

By Lemma 5.2, we have $T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}} \in \mathcal{C}_t$. By (5.7), this proves (5.6) for the case $i = 1$. Thus (5.4) is proved in the case $n \geq 2$.

Now suppose that $n = 1$. Then

$$\begin{aligned}
X_R &= [T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B, A_2] \\
&= [T_{h_R}, A_2]A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B + T_{h_R}[A, A_2](T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B \\
&\quad + T_{h_R}A[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, A_2]B + T_{h_R}A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})[B, A_2].
\end{aligned}$$

Applying Lemmas 5.1, 5.2 and 5.3 in the case $n = 1$ and using the fact that $T_{h_R} \to 0$ strongly as $R \to \infty$, we again obtain (5.4). $\square$

Proposition 5.5. *Consider any $f_{i,j} \in \mathrm{Lip}(S)$, where $1 \leq i \leq 2n$ and $1 \leq j \leq m$, $m \in \mathbf{N}$. For each $1 \leq i \leq 2n$, define*

$$A_i = T_{\tilde{f}_{i,1}} \cdots T_{\tilde{f}_{i,m}} \quad \text{and} \quad f_i = f_{i,1} \cdots f_{i,m}.$$

Then the difference

$$[A_1, A_2, \ldots, A_{2n-1}, A_{2n}] - [T_{\tilde{f}_1}, T_{\tilde{f}_2}, \ldots, T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}] \tag{5.8}$$

is in the trace class with zero trace.

Proof. It is easy to see that (5.8) is a linear combination of terms of the form (5.3). Therefore by Lemma 5.4, (5.8) is in the trace class with zero trace. $\square$

6. A unitary operator between two spaces

For each $j = 1, \dots, n$, we define the complex coordinate function

$$c_j(z) = z_j \quad \text{for} \quad z = (z_1, \dots, z_n). \tag{6.1}$$

We will regard c_j both as a function on $\mathbf{C}^n$ and as a function on $\mathbf{B}$. In terms of the functions $\epsilon_1, \dots, \epsilon_{2n}$ defined in Section 4, we have

$$c_j = \epsilon_{2j-1} + i\epsilon_{2j}$$

for every $1 \le j \le n$. For each $1 \le j \le n$, let s_j denote the element in $\mathbf{Z}_+^n$ whose j-th component is 1 and whose other components are 0. For $\alpha \in \mathbf{Z}_+^n$ and $1 \le i \le n$, we write α_i for the i-th component of α. That is, $\alpha = (\alpha_1, \dots, \alpha_n)$.

Let us first recall how the Toeplitz operator $T_{c_j}^{\mathrm{Berg}}$ acts on the Bergman space $L_a^2(\mathbf{B})$. Let $\{u_\alpha : \alpha \in \mathbf{Z}_+^n\}$ be the standard orthonormal basis for $L_a^2(\mathbf{B})$. It is well known that

$$u_\alpha(z) = ((|\alpha| + n)!/\alpha!n!)^{1/2} z^\alpha, \quad \alpha \in \mathbf{Z}_+^n.$$

For all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$, easy computation gives us the formula

$$T_{c_j}^{\mathrm{Berg}} u_\alpha = \{\|z^{\alpha+s_j}\|_{L_a^2(\mathbf{B})}/\|z^\alpha\|_{L_a^2(\mathbf{B})}\} u_{\alpha+s_j} = \left(\frac{\alpha_j + 1}{|\alpha| + 1 + n}\right)^{1/2} u_{\alpha+s_j}. \tag{6.2}$$

Next we will compare this with the action of $T_{\tilde{c}_j}$ on the Fock space $H^2(\mathbf{C}^n, d\mu)$, $j = 1, \dots, n$. For this we need a more precise form of Stirling's asymptotic expansion.

Definition 6.1. A sequence $a = \{a_k\}_{k=0}^\infty$ of real numbers is in the class $\mathcal{S}$ if there is a constant $0 < C = C(a) < \infty$ such that

$$|a_k| \le \frac{C}{k+1} \quad \text{and} \quad |a_k - a_{k+1}| \le \frac{C}{(k+1)^2}$$

for every $k \ge 0$.

Our next lemma is trivial, but we state it any way:

Lemma 6.2. *Given any sequence $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ in $\mathcal{S}$, there is a sequence $\{c_k\}_{k=0}^\infty$ in $\mathcal{S}$ such that*

$$(1 + a_k)(1 + b_k) = 1 + c_k$$

for every $k \in \mathbf{Z}_+$.

Proof. Given any $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ in $\mathcal{S}$, we define

$$c_k = a_k + b_k + a_k b_k,$$

$k \in \mathbf{Z}_+$. Then the sequence $\{c_k\}_{k=0}^\infty$ obviously has all the desired properties. $\square$

Lemma 6.3. *Given any $0 < t < \infty$, there exist an $a(t) \in \mathbf{R}$ and a sequence $\{a_k(t)\}_{k=0}^\infty$ in $\mathcal{S}$ such that*

$$\prod_{j=0}^{k} (t + j) = (t + k)^{t+k+(1/2)} e^{-k} e^{a(t)+a_k(t)}$$

for every $k \in \mathbf{Z}_+$.

Proof. We have the identity

$$\frac{1}{2}\{f(1)+f(0)\} = \int_0^1 f(x)dx - \frac{1}{2}\int_0^1 (x^2-x)f''(x)dx$$

for any C^2-function f on any neighborhood of $[0,1]$. From this it follows that

$$\begin{aligned}\sum_{j=0}^{k}\log(t+j) &= \frac{1}{2}\{\log t+\log(t+k)\} + \int_0^k \log(t+x)dx + \frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \frac{x^2-x}{(t+j+x)^2}dx \\ &= \frac{1}{2}\{\log t+\log(t+k)\} + (k+t)\log(t+k) - k - t\log t \\ &+ \frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \frac{x^2-x}{(t+j+x)^2}dx\end{aligned}$$

for every $k \in \mathbf{N}$. Exponentiating both sides, we obtain

$$\begin{aligned}\prod_{j=0}^{k}(t+j) &= (t+k)^{t+k+(1/2)}e^{-k}t^{-t}\sqrt{t}\exp\left(\frac{1}{2}\sum_{j=0}^{k-1}\int_0^1 \frac{x^2-x}{(t+j+x)^2}dx\right) \\ &= (t+k)^{t+k+(1/2)}e^{-k}e^{a(t)+a_k(t)},\end{aligned}$$

where

$$\begin{aligned}a(t) &= -t\log t + \frac{1}{2}\log t + \frac{1}{2}\sum_{j=0}^{\infty}\int_0^1 \frac{x^2-x}{(t+j+x)^2}dx \quad \text{and} \\ a_k(t) &= \frac{1}{2}\sum_{j=k}^{\infty}\int_0^1 \frac{x-x^2}{(t+j+x)^2}dx, \quad k \geq 1.\end{aligned}$$

There is an obvious choice of $a_0(t)$ such that the desired identity also holds in the case $k=0$. It is easy to see that the sequence $\{a_k(t)\}_{k=0}^{\infty}$ so defined is in $\mathcal{S}$. $\square$

Lemma 6.4. *There is a sequence $\{\gamma_k\}_{k=0}^{\infty}$ in $\mathcal{S}$ such that*

$$\left(1-\frac{1}{2k}\right)^k = e^{-1/2}(1+\gamma_k) \quad \textit{for every } \ k \in \mathbf{N}.$$

Proof. Consider any $0 < \delta < y \le 1$. By the fundamental theorem of calculus, we have

$$\begin{aligned}(1-(\delta/2))^{1/\delta} - (1-(y/2))^{1/y} &= -\int_\delta^y \frac{d}{dx}(1-(x/2))^{1/x}dx \\ &= -\int_\delta^y (1-(x/2))^{1/x}\frac{d}{dx}\frac{\log(1-(x/2))}{x}dx \\ &= \int_\delta^y (1-(x/2))^{1/x}\frac{\frac{x/2}{1-(x/2)} + \log(1-(x/2))}{x^2}dx \\ &= \int_\delta^y (1-(x/2))^{1/x}\frac{1}{x^2}\sum_{j=1}^\infty \left(1-\frac{1}{j}\right)\left(\frac{x}{2}\right)^j dx \\ &= \int_\delta^y (1-(x/2))^{1/x}\sum_{j=2}^\infty \left(1-\frac{1}{j}\right)\frac{x^{j-2}}{2^j}dx.\end{aligned}$$

Letting δ descend to 0, we obtain the identity

$$e^{-1/2} - \left(1-\frac{1}{2k}\right)^k = \int_0^{1/k}(1-(x/2))^{1/x}\sum_{j=2}^\infty\left(1-\frac{1}{j}\right)\frac{x^{j-2}}{2^j}dx$$

for every $k \in \mathbf{N}$. If we define $\gamma_0 = 0$ and

$$\gamma_k = -e^{1/2}\int_0^{1/k}(1-(x/2))^{1/x}\sum_{j=2}^\infty\left(1-\frac{1}{j}\right)\frac{x^{j-2}}{2^j}dx, \quad k \in \mathbf{N},$$

then it is easy to see that the sequence $\{\gamma_k\}_{k=0}^\infty$ has the desired properties. □

Let $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ be the standard orthonormal basis for the Fock space $H^2(\mathbf{C}^n, d\mu)$. It is well known that

$$e_\alpha(z) = (\alpha!)^{-1/2}z^\alpha, \quad \alpha \in \mathbf{Z}_+^n.$$

Lemma 6.5. *There exists a sequence $\{d_k\}_{k=0}^\infty$ in $\mathcal{S}$ such that*

$$T_{\tilde{c}_j}e_\alpha = \left(\frac{\alpha_j+1}{|\alpha|+1+n}\right)^{1/2}(1+d_{|\alpha|})e_{\alpha+s_j} \tag{6.3}$$

for all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$.

Proof. It is easy to see that $\langle T_{\tilde{c}_j}e_\alpha, e_\beta\rangle = 0$ for every $\beta \ne \alpha + s_j$. Therefore for all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$, we have

$$T_{\tilde{c}_j}e_\alpha = \langle T_{\tilde{c}_j}e_\alpha, e_{\alpha+s_j}\rangle e_{\alpha+s_j}. \tag{6.4}$$

Let $d\sigma$ denote the spherical measure on S with normalization $\sigma(S) = 1$. Then for all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$, we have

$$\begin{aligned}\langle T_{\tilde{c}_j}e_\alpha, e_{\alpha+s_j}\rangle &= \frac{1}{\sqrt{\alpha!(\alpha+s_j)!}}\int \frac{z_j}{|z|}z^\alpha\overline{z^{\alpha+s_j}}d\mu(z) \\ &= \frac{1}{\sqrt{\alpha!(\alpha+s_j)!}}\cdot\frac{1}{\pi^n}\int\frac{|z^{\alpha+s_j}|^2}{|z|}e^{-|z|^2}dV(z) \\ &= \frac{A_n}{\sqrt{\alpha!(\alpha+s_j)!}}\int_S|\xi^{\alpha+s_j}|^2d\sigma(\xi)\int_0^\infty e^{-r^2}r^{2n-2+2(|\alpha|+1)}dr. \end{aligned}\tag{6.5}$$

We know from [19, Proposition 1.4.9] that

$$\int_S |\xi^{\alpha+s_j}|^2 d\sigma(\xi) = \frac{(n-1)!(\alpha+s_j)!}{(n-1+|\alpha+s_j|)!} = \frac{(n-1)!(\alpha+s_j)!}{(n+|\alpha|)!},$$

and it is elementary that

$$\int_0^\infty e^{-r^2} r^{2k} dr = \frac{\sqrt{\pi}}{2}\prod_{j=0}^{k-1}\left(j+\frac{1}{2}\right) \quad \text{for every} \quad k \in \mathbf{N}.$$

Substituting these in (6.5), we find that

$$\begin{aligned} \langle T_{\tilde{c}_j} e_\alpha, e_{\alpha+s_j}\rangle &= \frac{A'_n}{\sqrt{\alpha!(\alpha+s_j)!}}\frac{(\alpha+s_j)!}{(n+|\alpha|)!}\prod_{j=0}^{n-1+|\alpha|}\left(j+\frac{1}{2}\right) \\ &= A'_n\sqrt{\alpha_j+1}\frac{1}{(n+|\alpha|)!}\prod_{j=0}^{n-1+|\alpha|}\left(j+\frac{1}{2}\right). \end{aligned} \tag{6.6}$$

It follows from Lemma 6.3 that

$$\begin{aligned} &\frac{1}{(n+|\alpha|)!}\prod_{j=0}^{n-1+|\alpha|}\left(j+\frac{1}{2}\right) \\ &\quad = \frac{(n+|\alpha|-(1/2))^{n+|\alpha|}e^{-(n-1)-|\alpha|}\exp(a(1/2)+a_{n-1+|\alpha|}(1/2))}{(n+|\alpha|)^{n+|\alpha|+(1/2)}e^{-(n-1)-|\alpha|}\exp(a(1)+a_{n-1+|\alpha|}(1))} \\ &\quad = \frac{e^{a(1/2)-a(1)}}{\sqrt{n+|\alpha|}}\left(\frac{n+|\alpha|-(1/2)}{n+|\alpha|}\right)^{n+|\alpha|}\exp(a_{n-1+|\alpha|}(1/2)-a_{n-1+|\alpha|}(1)). \end{aligned} \tag{6.7}$$

Lemma 6.3 tells us that the sequences $\{a_k(1/2)\}_{k=0}^\infty$ and $\{a_k(1)\}_{k=0}^\infty$ are in $\mathcal{S}$. It is easy to see that there is a sequence $\{b_k^{(1)}\}_{k=0}^\infty$ in $\mathcal{S}$ such that

$$\exp(a_{n-1+|\alpha|}(1/2)-a_{n-1+|\alpha|}(1)) = 1+b_{|\alpha|}^{(1)} \tag{6.8}$$

for every $\alpha \in \mathbf{Z}_+^n$. By Lemma 6.4, there is a sequence $\{b_k^{(2)}\}_{k=0}^\infty$ in $\mathcal{S}$ such that

$$\left(\frac{n+|\alpha|-(1/2)}{n+|\alpha|}\right)^{n+|\alpha|} = e^{-1/2}(1+b_{|\alpha|}^{(2)}) \tag{6.9}$$

for every $\alpha \in \mathbf{Z}_+^n$. Finally, it is elementary that there is a sequence $\{b_k^{(3)}\}_{k=0}^\infty$ in $\mathcal{S}$ such that

$$\frac{1}{\sqrt{n+|\alpha|}} = \frac{1+b_{|\alpha|}^{(3)}}{\sqrt{|\alpha|+1+n}} \tag{6.10}$$

for every $\alpha \in \mathbf{Z}_+^n$.

Now, substituting (6.8), (6.9) and (6.10) in (6.7), we obtain the equality

$$\frac{1}{(n+|\alpha|)!}\prod_{j=0}^{n-1+|\alpha|}\left(j+\frac{1}{2}\right) = \frac{e^{a(1/2)-a(1)-(1/2)}}{\sqrt{|\alpha|+1+n}}\prod_{i=1}^{3}(1+b_{|\alpha|}^{(i)}) = \frac{e^{a(1/2)-a(1)-(1/2)}}{\sqrt{|\alpha|+1+n}}(1+d_{|\alpha|})$$

for every $\alpha \in \mathbf{Z}_+^n$, where, by Lemma 6.2, $\{d_k\}_{k=0}^\infty$ is a sequence in $\mathcal{S}$. Substituting this in (6.6), we find that

$$\langle T_{\tilde{c}_j} e_\alpha, e_{\alpha+s_j}\rangle = A_n'' \left(\frac{\alpha_j + 1}{|\alpha| + 1 + n}\right)^{1/2} (1 + d_{|\alpha|})$$

for all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$. Recalling (6.4), the proof will be complete if we can show that $A_n'' = 1$.

To prove that $A_n'' = 1$, we note that $\|T_{\tilde{c}_j}\| \le 1$ and that, by (6.5), $\langle T_{\tilde{c}_j} e_\alpha, e_{\alpha+s_j}\rangle \ge 0$ for all $1 \le j \le n$ and $\alpha \in \mathbf{Z}_+^n$. Thus the conclusion $A_n'' = 1$ will follow if we can show that $\|T_{\tilde{c}_j}\|_{\mathrm{ess}} \ge 1$ for a $j \in \{1, \dots, n\}$, where $\|T_{\tilde{c}_j}\|_{\mathrm{ess}}$ denotes the essential norm of $T_{\tilde{c}_j}$.

For this we use the normalized reproducing kernel $k_w(z) = e^{-|w|^2/2} e^{\langle z, w\rangle}$ for the Fock space $H^2(\mathbf{C}^n, d\mu)$. Take, for example, $j = 1$. Then take the vector $u = (1, 0, \dots, 0)$ in $\mathbf{C}^n$. It is elementary to verify that

$$\lim_{r\to\infty} \langle T_{\tilde{c}_1} k_{ru}, k_{ru}\rangle = 1.$$

Since $k_{ru} \to 0$ weakly as $r \to \infty$, this implies $\|T_{\tilde{c}_1}\|_{\mathrm{ess}} \ge 1$. This completes the proof. □

Let $U : L_a^2(\mathbf{B}) \to H^2(\mathbf{C}^n, d\mu)$ be the unitary operator such that

$$U u_\alpha = e_\alpha \quad \text{for every} \quad \alpha \in \mathbf{Z}_+^n. \tag{6.11}$$

Furthermore, we define the diagonal operator D on $L_a^2(\mathbf{B})$ by the formula

$$D = \sum_{\alpha \in \mathbf{Z}_+^n} d_{|\alpha|} u_\alpha \otimes u_\alpha, \tag{6.12}$$

where $\{d_k\}_{k=0}^\infty$ is the sequence provided by Lemma 6.5. It follows from (6.2) and (6.3) that

$$U^* T_{\tilde{c}_j} U = T_{c_j}^{\mathrm{Berg}} (1 + D) \tag{6.13}$$

for every $1 \le j \le n$.

For each $k \in \mathbf{N}$, we define

$$E_k = \sum_{|\alpha| < k} d_{|\alpha|} u_\alpha \otimes u_\alpha \quad \text{and} \quad F_k = \sum_{|\alpha| \ge k} d_{|\alpha|} u_\alpha \otimes u_\alpha. \tag{6.14}$$

We have $E_k + F_k = D$ with $\mathrm{rank}(E_k) < \infty$, $k \in \mathbf{N}$.

Lemmas 6.6-6.12 below deal with operators on $L_a^2(\mathbf{B})$ and are similar to the corresponding ones in [24, Section 10]. The main reason why we go over the details here again is that the operator D here is slightly different from the D in [24, Section 10]. In particular, we want to make explicit how the two conditions in Definition 6.1 are used in the proofs.

Lemma 6.6. *For each $p > n$, we have*

$$\lim_{k\to\infty} \|F_k\|_p = 0.$$

Proof. Write m_n for the n-dimensional Lebesgue measure on $\mathbf{R}^n$. By the membership $\{d_k\}_{k=0}^\infty \in \mathcal{S}$, we have

$$\begin{aligned}\|F_k\|_p^p &= \sum_{|\alpha|\ge k}(d_{|\alpha|})^p \le \sum_{|\alpha|\ge k}\frac{C^p}{(|\alpha|+1)^p}\\ &\le C_1\int_{(x_1^2+\cdots+x_n^2)^{1/2}\ge k/\sqrt{n}}\frac{dm_n(x_1,\dots,x_n)}{(x_1^2+\cdots+x_n^2)^{p/2}} = C_2\int_{k/\sqrt{n}}^\infty \frac{r^{n-1}dr}{r^p}.\end{aligned}$$

For $p > n$, the right-hand side tends to 0 as $k \to \infty$. $\square$

Lemma 6.7. (1) *Suppose that* $n \ge 2$. *For* $t > n-1$ *and* $1 \le j \le n$, *we have*

$$\lim_{k\to\infty}\|[T_{c_j}^{\mathrm{Berg}}, F_k]\|_t = 0.$$

(2) *Suppose that* $n = 1$. *We have*

$$\lim_{k\to\infty}\|[T_{c_1}^{\mathrm{Berg}}, F_k]\|_1 = 0.$$

Proof. By (6.14), for any $1 \le j \le n$ we have

$$[T_{c_j}^{\mathrm{Berg}}, F_k] = \sum_{|\alpha|\ge k} d_{|\alpha|}[T_{c_j}^{\mathrm{Berg}}, u_\alpha \otimes u_\alpha]. \tag{6.15}$$

For $\alpha \in \mathbf{Z}_+^n$, we have $(T_{c_j}^{\mathrm{Berg}})^* u_\alpha = 0$ if $\alpha_j = 0$ and $(T_{c_j}^{\mathrm{Berg}})^* u_\alpha = \alpha_j^{1/2}(|\alpha|+n)^{-1/2}u_{\alpha - s_j}$ if $\alpha_j \ge 1$. Combining this fact with (6.2) and with (6.15), we find that $[T_{c_j}^{\mathrm{Berg}}, F_k] = A_k - B_k$, where

$$A_k = \sum_{|\alpha|\ge k}(d_{|\alpha|} - d_{|\alpha+s_j|})\left(\frac{\alpha_j+1}{|\alpha|+1+n}\right)^{1/2} u_{\alpha+s_j}\otimes u_\alpha \quad \text{and}$$

$$B_k = \sum_{|\alpha|=k-1} d_{|\alpha+s_j|}\left(\frac{\alpha_j+1}{|\alpha|+1+n}\right)^{1/2} u_{\alpha+s_j}\otimes u_\alpha.$$

Consider the case where $n \ge 2$. Given $t > n-1$, it suffices to show that $\|B_k\|_t \to 0$ and $\|A_k\|_t \to 0$ as $k \to \infty$.

Note that

$$\mathrm{rank}(B_k) = \mathrm{card}\{\alpha \in \mathbf{Z}_+^n : |\alpha| = k-1\} = \frac{(k-1+n-1)!}{(k-1)!(n-1)!} = O(k^{n-1}).$$

By Lemma 6.5 and Definition 6.1, we have

$$d_{|\alpha+s_j|} = O(k^{-1}) \quad \text{when} \quad |\alpha| = k-1.$$

Since $t > n-1$, we have $\|B_k\|_t \to 0$ as $k \to \infty$. Also by Lemma 6.5 and Definition 6.1,

$$|d_{|\alpha|} - d_{|\alpha+s_j|}| \le \frac{C}{(|\alpha|+1)^2}.$$

Therefore

$$\|A_k\|_t^t \le C_1\sum_{|\alpha|\ge k}\frac{1}{(|\alpha|+1)^{2t}} \le C_2\int_{k/\sqrt{n}}^\infty \frac{r^{n-1}dr}{r^{2t}}.$$

Since $t > n-1$ and $n \ge 2$, we have $2t > 2n-2 \ge n$. Thus $\|A_k\|_t \to 0$ as $k \to \infty$.

Now consider the case $n = 1$. From the above its is clear that $\|B_k\|_1 \to 0$ and $\|A_k\|_1 \to 0$ as $k \to \infty$. This completes the proof. □

Definition 6.8. Let $\mathcal{A}$ denote the unital algebra generated by the operators

$$D, \quad T_{c_1}^{\text{Berg}}, \dots, T_{c_n}^{\text{Berg}}, \quad (T_{c_1}^{\text{Berg}})^*, \dots, (T_{c_n}^{\text{Berg}})^*.$$

Lemma 6.9. (1) *Suppose that $n \geq 2$. Let $t > n - 1$. Then for every $A \in \mathcal{A}$ we have*

$$\lim_{k\to\infty} \|[A, F_k]\|_t = 0. \tag{6.16}$$

(2) *Suppose that $n = 1$. Then for every $A \in \mathcal{A}$ we have*

$$\lim_{k\to\infty} \|[A, F_k]\|_1 = 0.$$

Proof. (1) Given $t > n-1$, (6.16) is a consequence of the following three statements:
(a) $\lim_{k\to\infty} \|[T_{c_j}^{\text{Berg}}, F_k]\|_t = 0$ for every $1 \leq j \leq n$.
(b) $\lim_{k\to\infty} \|[(T_{c_j}^{\text{Berg}})^*, F_k]\|_t = 0$ for every $1 \leq j \leq n$.
(c) $[D, F_k] = 0$ for every $k \in \mathbf{N}$.
Obviously, (a) just restates Lemma 6.7(1). Since F_k is self-adjoint, (b) follows from (a). By (6.12) and (6.14), (c) is obvious.

(2) When $n = 1$, the desired limit follows from the following three statements:
(d) $\lim_{k\to\infty} \|[T_{c_1}^{\text{Berg}}, F_k]\|_1 = 0$.
(e) $\lim_{k\to\infty} \|[(T_{c_1}^{\text{Berg}})^*, F_k]\|_1 = 0$.
(f) $[D, F_k] = 0$ for every $k \in \mathbf{N}$.
Again, (d) just restates Lemma 6.7(2). Since F_k is self-adjoint, (e) follows from (d). Finally, (f) is obvious from (6.12) and (6.14). □

Lemma 6.10. *For any $A, B \in \mathcal{A}$, we have $[A, B] \in \mathcal{C}_p$ for every $p > n$.*

Proof, This follows from Lemma 2.3 and the fact that $D \in \mathcal{C}_p$ for every $p > n$. □

Lemma 6.11. (1) *Suppose that $n \geq 2$. Then for all $t > n - 1$ and $A, B, C \in \mathcal{A}$, we have*

$$\lim_{k\to\infty} \|[A, BF_kC]\|_t = 0.$$

(2) *Suppose that $n = 1$. Then for all $A, B, C \in \mathcal{A}$, we have*

$$\lim_{k\to\infty} \|[A, BF_kC]\|_1 = 0.$$

Proof. (1) Applying Lemma 6.9(1), what remains to be shown is that

$$\lim_{k\to\infty} \|[A, B]F_k\|_t = 0 \quad \text{and} \quad \lim_{k\to\infty} \|F_k[A, C]\|_t = 0.$$

Set $p = 2t$. Then $p > 2n - 2 \geq n$. By Lemma 2.1, we have

$$\|[A, B]F_k\|_t \leq \|[A, B]\|_p \|F_k\|_p.$$

Thus it follows from Lemmas 6.6 and 6.10 that $\|[A, B]F_k\|_t \to 0$ as $k \to \infty$. A similar argument shows that $\|F_k[A, C]\|_t \to 0$ as $k \to \infty$.

(2) Applying Lemma 6.9(2), what remains to be shown is that

$$\lim_{k\to\infty} \|[A, B]F_k\|_1 = 0 \quad \text{and} \quad \lim_{k\to\infty} \|F_k[A, C]\|_1 = 0.$$

We have $\|[A, B]F_k\|_1 \leq \|[A, B]\|_2\|F_k\|_2$. Since this is the case $n = 1$, it follows from Lemmas 6.6 and 6.10 that $\|[A, B]F_k\|_1 \to 0$ as $k \to \infty$. A similar argument shows that $\|F_k[A, C]\|_1 \to 0$ as $k \to \infty$. $\square$

Lemma 6.12. *Let $A, B, A_2, A_3, \ldots, A_{2n} \in \mathcal{A}$. Then the antisymmetric sum*

$$[ADB, A_2, A_3, \ldots, A_{2n}]$$

is in the trace class with zero trace.

Proof. Since $D = E_k + F_k$, $\operatorname{rank}(E_k) < \infty$, and consequently

$$\operatorname{tr}[AE_kB, A_2, A_3, \ldots, A_{2n}] = 0,$$

it suffices to show that

$$\lim_{k\to\infty} \|[AF_kB, A_2, A_3, \ldots, A_{2n}]\|_1 = 0. \tag{6.17}$$

First of all, in the case $n = 1$, (6.17) just restates Lemma 6.11(2). Now suppose that $n \geq 2$. Then, using Lemmas 6.10 and 6.11(1), the proof for (6.17) is the same as the argument in the proof of [24, Lemma 10.7], which we will not repeat here. $\square$

Proposition 6.13. *Given arbitrary $p_1, p_2, \ldots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, there is a $Y \in \mathcal{C}_1$ with* $\operatorname{tr}(Y) = 0$ *such that*

$$U^*[T_{\tilde{p}_1}, T_{\tilde{p}_2}, \ldots, T_{\tilde{p}_{2n}}]U = [T^{\mathrm{Berg}}_{p_1}, T^{\mathrm{Berg}}_{p_2}, \ldots, T^{\mathrm{Berg}}_{p_{2n}}] + Y.$$

Proof. Some helpful notation will ease the proof. First of all, for any $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n_+$, we write

$$c^a = c_1^{a_1} \cdots c_n^{a_n},$$

where $c_1, \ldots, c_n$ are the complex coordinate functions defined by (6.1). By the linearity for each slot of an antisymmetric sum, the proposition will be proved if we can show that for arbitrary $\alpha_1, \ldots, \alpha_{2n}, \beta_1, \ldots, \beta_{2n} \in \mathbf{Z}^n_+$, there is a $Z \in \mathcal{C}_1$ with $\operatorname{tr}(Z) = 0$ such that

$$U^*[T_{\widetilde{\bar{c}^{\alpha_1}c^{\beta_1}}}, \ldots, T_{\widetilde{\bar{c}^{\alpha_{2n}}c^{\beta_{2n}}}}]U = [T^{\mathrm{Berg}}_{\bar{c}^{\alpha_1}c^{\beta_1}}, \ldots, T^{\mathrm{Berg}}_{\bar{c}^{\alpha_{2n}}c^{\beta_{2n}}}] + Z. \tag{6.18}$$

To prove this, we need more notation. Consider the operator tuples

$$T = (T_{\tilde{c}_1}, \ldots, T_{\tilde{c}_n}) \quad \text{and} \quad B = (T^{\mathrm{Berg}}_{c_1}, \ldots, T^{\mathrm{Berg}}_{c_n}).$$

By the usual multi-index notation, we write

$$T^a = T^{a_1}_{\tilde{c}_1} \cdots T^{a_n}_{\tilde{c}_n} \quad \text{and} \quad B^a = (T^{\mathrm{Berg}}_{c_1})^{a_1} \cdots (T^{\mathrm{Berg}}_{c_n})^{a_n}$$

for any $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n_+$. By Proposition 5.5, the difference

$$[T_{\widetilde{\bar{c}^{\alpha_1}c^{\beta_1}}}, \ldots, T_{\widetilde{\bar{c}^{\alpha_{2n}}c^{\beta_{2n}}}}] - [(T^{\alpha_1})^*T^{\beta_1}, \ldots, (T^{\alpha_{2n}})^*T^{\beta_{2n}}]$$

is in the trace class with zero trace. On the other hand, we obviously have

$$[T^{\mathrm{Berg}}_{\bar{c}^{\alpha_1}c^{\beta_1}}, \ldots, T^{\mathrm{Berg}}_{\bar{c}^{\alpha_{2n}}c^{\beta_{2n}}}] = [(B^{\alpha_1})^*B^{\beta_1}, \ldots, (B^{\alpha_{2n}})^*B^{\beta_{2n}}].$$

Therefore (6.18) will follow if we can prove that

$$U^*[(T^{\alpha_1})^*T^{\beta_1}, \ldots, (T^{\alpha_{2n}})^*T^{\beta_{2n}}]U = [(B^{\alpha_1})^*B^{\beta_1}, \ldots, (B^{\alpha_{2n}})^*B^{\beta_{2n}}] + X,$$

where X is in the trace class with zero trace. But this just repeats the argument in the proof of [24, Proposition 10.8].

Indeed for any $a = (a_1, \dots, a_n) \in \mathbf{Z}_+^n$, it follows from (6.13) that

$$\begin{aligned} U^*T^aU &= U^*T_{\tilde{c}_1}^{a_1} \cdots T_{\tilde{c}_n}^{a_n}U = \{T_{c_1}^{\mathrm{Berg}}(1+D)\}^{a_1} \cdots \{T_{c_n}^{\mathrm{Berg}}(1+D)\}^{a_n} \\ &= B^a + \sum_{\nu=1}^{2^{|a|}-1} X_\nu D Y_\nu, \end{aligned}$$

where $X_1, Y_1, \dots, X_{2^{|a|}-1}, Y_{2^{|a|}-1} \in \mathcal{A}$. Therefore, given any $\alpha_1, \dots, \alpha_{2n}, \beta_1, \dots, \beta_{2n} \in \mathbf{Z}_+^n$, there exist an $m \in \mathbf{N}$ and $A_{j,i}, C_{j,i} \in \mathcal{A}$, $1 \le j \le 2n$ and $1 \le i \le m$, such that

$$U^*(T^{\alpha_j})^*T^{\beta_j}U = (B^{\alpha_j})^*B^{\beta_j} + \sum_{i=1}^m A_{j,i}DC_{j,i}$$

for every $1 \le j \le 2n$. Combining this identity with the linearity for each slot in an antisymmetric sum, we have

$$\begin{aligned} &U^*[(T^{\alpha_1})^*T^{\beta_1}, \dots, (T^{\alpha_{2n}})^*T^{\beta_{2n}}]U \\ &\quad = [(B^{\alpha_1})^*B^{\beta_1}, \dots, (B^{\alpha_{2n}})^*B^{\beta_{2n}}] + \sum_{r=1}^{(m+1)^{2n}-1} [X_{1,r}, X_{2,r}, \dots, X_{2n,r}], \end{aligned}$$

where the operators $X_{j,r}$ satisfy the following two conditions:

(1) $X_{j,r} \in \mathcal{A}$ for all $1 \le j \le 2n$ and $1 \le r \le (m+1)^{2n} - 1$.

(2) For each $1 \le r \le (m+1)^{2n} - 1$, there is a $j(r) \in \{1, \dots, 2n\}$ such that $X_{j(r),r} = ADC$ with $A, C \in \mathcal{A}$.

By these two conditions and Lemma 6.12, for each $1 \le r \le (m+1)^{2n} - 1$, the antisymmetric sum $[X_{1,r}, X_{2,r}, \dots, X_{2n,r}]$ is in the trace class with zero trace. This completes the proof. □

Proposition 6.14. *For arbitrary $p_1, p_2, \dots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ we have*

$$\mathrm{tr}[T_{\tilde{p}_1}, T_{\tilde{p}_2}, \dots, T_{\tilde{p}_{2n}}] = \frac{n!}{(2\pi i)^n} \int_S p_1 dp_2 \wedge \cdots \wedge dp_{2n}.$$

Proof. This follows immediately from Proposition 6.13 and Theorem 1.4. □

7. Approximation

Again, we remind the reader that we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the natural way. To complete the proof of Theorem 1.6, we need the following:

Lemma 7.1. *Let $f \in C^2(\Omega_\delta)$ for some $0 < \delta < 1$. Then there is a sequence of polynomials $\{p_k\}$ in $\mathbf{C}[x_1, x_2, \dots, x_{2n-1}, x_{2n}]$ such that*

$$\lim_{k\to\infty} \|f - p_k\|_{*,\delta} = 0.$$

Proof. Given a $0 < \delta < 1$, define the open subset

$$\Xi_\delta = \{x \in \mathbf{R}^{2n} : 1 - (2\delta/3) < |x| < 1 + (2\delta/3)\}$$

of Ω_δ. Recalling the definition of $\|\cdot\|_{*,\delta}$, for any $g \in C^2(\Omega_\delta)$, the value of $\|g\|_{*,\delta}$ only depends on the restriction of g to Ξ_δ. Therefore, to prove the lemma, it suffices to consider an $f \in C_c^2(\mathbf{R}^{2n})$.

Given such an f, define

$$f_r(x) = \frac{r^{2n}}{\pi^n} \int f(x+y)e^{-r^2|y|^2} dV(y) = \frac{r^{2n}}{\pi^n} \int f(y)e^{-r^2|x-y|^2} dV(y), \quad r > 0.$$

It is obvious that
$$\lim_{r\to\infty}\|f-f_r\|_{*,\delta}=0.$$
Thus it suffices to show that for every $r>0$, there is a sequence $\{p_k\}$ of polynomials in the variables $x_1,x_2,\dots,x_{2n-1},x_{2n}$ such that
$$\lim_{k\to\infty}\|f_r-p_k\|_{*,\delta}=0. \tag{7.1}$$
Let an $r>0$ be given. Since $f\in C_c^2(\mathbf{R}^{2n})$, i.e., since f has a compact support, we have
$$f_r(x)=\frac{r^{2n}}{\pi^n}\sum_{j=0}^{\infty}\frac{(-r^2)^j}{j!}\int f(y)\left(|x-y|^2\right)^j dV(y).$$
Thus (7.1) holds for the sequence of polynomials
$$p_k(x)=\frac{r^{2n}}{\pi^n}\sum_{j=0}^{k}\frac{(-r^2)^j}{j!}\int f(y)\left(|x-y|^2\right)^j dV(y),\quad k\in\mathbf{N}.$$
This completes the proof. □

We are now ready to prove our trace formula.

Proof of Theorem 1.6. Let $f_1,f_2,\dots,f_{2n}\in C^2(\Omega_\delta)$ for some $0<\delta<1$. By Lemma 7.1, there are sequences of polynomials $\{p_{j,k}\}$ in $\mathbf{C}[z_1,\bar z_1,\dots,z_n,\bar z_n]$, $1\le j\le 2n$, such that
$$\lim_{k\to\infty}\|f_j-p_{j,k}\|_{*,\delta}=0 \tag{7.2}$$
for $1\le j\le 2n$. By Theorem 1.5 and the linearity of antisymmetric sums, (7.2) implies
$$\lim_{k\to\infty}\|[T_{\tilde f_1},T_{\tilde f_2},\dots,T_{\tilde f_{2n}}]-[T_{\tilde p_{1,k}},T_{\tilde p_{2,k}},\dots,T_{\tilde p_{2n,k}}]\|_1=0.$$
Therefore
$$\operatorname{tr}[T_{\tilde f_1},T_{\tilde f_2},\dots,T_{\tilde f_{2n}}]=\lim_{k\to\infty}\operatorname{tr}[T_{\tilde p_{1,k}},T_{\tilde p_{2,k}},\dots,T_{\tilde p_{2n,k}}]. \tag{7.3}$$
Obviously, (7.2) also implies
$$\lim_{k\to\infty}\frac{n!}{(2\pi i)^n}\int_S p_{1,k}dp_{2,k}\wedge\cdots\wedge dp_{2n,k}=\frac{n!}{(2\pi i)^n}\int_S f_1df_2\wedge\cdots\wedge df_{2n}. \tag{7.4}$$
For each $k\in\mathbf{N}$, Proposition 6.14 gives us the identity
$$\operatorname{tr}[T_{\tilde p_{1,k}},T_{\tilde p_{2,k}},\dots,T_{\tilde p_{2n,k}}]=\frac{n!}{(2\pi i)^n}\int_S p_{1,k}dp_{2,k}\wedge\cdots\wedge dp_{2n,k}. \tag{7.5}$$
Combining (7.3), (7.4) and (7.5), we obtain (1.6). This completes the proof. □

8. Lorentz ideals

Recall that, given a $1\le p<\infty$, the formula
$$\|A\|_p^+=\sup_{k\ge1}\frac{s_1(A)+s_2(A)+\cdots+s_k(A)}{1^{-1/p}+2^{-1/p}+\cdots+k^{-1/p}}$$
defines a symmetric norm for operators [15, Section III.14], where $s_1(A),\dots,s_k(A),\dots$ are the s-numbers of A. On any separable Hilbert space $\mathcal{H}$, the set
$$\mathcal{C}_p^+=\{A\in\mathcal{B}(\mathcal{H}):\|A\|_p^+<\infty\}$$

is a norm ideal [15, Section III.2]. This $\mathcal{C}_p^+$ is commonly called a Lorentz ideal. It is well known that $\mathcal{C}_p^+$ contains the Schatten class $\mathcal{C}_p$ and that $\mathcal{C}_p^+ \neq \mathcal{C}_p$. An interesting property of $\mathcal{C}_p^+$ is that it is not separable with respect to the norm $\|\cdot\|_p^+$.

Lemma 8.1. [14, Lemma 5.6] *Suppose that $1 < p < \infty$. For a bounded operator A, define*

$$F_p(A) = \sup_{k \geq 1} k^{1/p} s_k(A).$$

Then

$$\frac{p-1}{p} F_p(A) \leq \|A\|_p^+ \leq F_p(A).$$

Lemma 8.2. *Let $p_1, p_2 \in (1, \infty)$ be such that $p_1 p_2/(p_1 + p_2) \geq 1$. Then*

$$\|AB\|^+_{p_1 p_2/(p_1+p_2)} \leq \frac{p_1 p_2}{(p_1 - 1)(p_2 - 1)} \|A\|^+_{p_1} \|B\|^+_{p_2} \tag{8.1}$$

for all bounded operators A and B.

Proof. By (7.9) on page 63 in [15], for every $k \in \mathbf{N}$ we have the inequality

$$s_1(AB) + \cdots + s_k(AB) \leq s_1(A)s_1(B) + \cdots + s_k(A)s_k(B).$$

Combining this with Lemma 8.1, we find that

$$\sum_{j=1}^k s_j(AB) \leq \frac{p_1 p_2 \|A\|^+_{p_1} \|B\|^+_{p_2}}{(p_1 - 1)(p_2 - 1)} \sum_{j=1}^k \frac{1}{j^{1/p_1} j^{1/p_2}} = \frac{p_1 p_2 \|A\|^+_{p_1} \|B\|^+_{p_2}}{(p_1 - 1)(p_2 - 1)} \sum_{j=1}^k \frac{1}{j^{(p_1+p_2)/p_1 p_2}}$$

for every $k \in \mathbf{N}$. This obviously implies (8.1). □

Corollary 8.3. *Let m be an integer greater than or equal to 2. Then there is a $0 < C = C(m) < \infty$ such that*

$$\|A_1 \cdots A_m\|_1^+ \leq C \|A_1\|_m^+ \cdots \|A_m\|_m^+$$

for all bounded operators $A_1, \ldots, A_m$.

Proof. This is an obvious consequence of Lemma 8.2. □

Recall Definition 3.1 for the operator A_G.

Lemma 8.4. (1) *Suppose that $n \geq 2$. There is a $0 < C_{8.4.1} = C_{8.4.1}(n) < \infty$ such that if the inequality*

$$|G(z,w)| \leq \left| \frac{z}{|z|} - \frac{w}{|w|} \right| \tag{8.2}$$

holds for all $z, w \in \mathbf{C}^n \backslash \{0\}$, then $\|A_G\|^+_{2n} \leq C_{8.4.1}$.
(2) *Suppose that $n = 1$. There is a $0 < C_{8.4.2} < \infty$ such that if the inequality*

$$|G(z,w)| \leq \left| \frac{z}{|z|} - \frac{w}{|w|} \right| \tag{8.3}$$

holds for all $z, w \in \mathbf{C} \backslash \{0\}$, then $\|PA_G\|^+_2 \leq C_{8.4.2}$.

Proof. (1) For any bounded operator X and any $t > 0$, define

$$N_X(t) = \mathrm{card}\{j \in \mathbf{N} : s_j(X) > t\}.$$

We know from [13, (7.1)] that

$$N_{X+Y}(t) \leq N_X(t/2) + N_Y(t/2) \tag{8.4}$$

for all bounded operators X, Y and $t > 0$. Given any $0 < t < \infty$, we define

$$G_1^{(t)}(z,w) = \begin{cases} G(z,w) & \text{if} \quad |G(z,w)| \geq t \\ \\ 0 & \text{if} \quad |G(z,w)| < t \end{cases}$$

and $G_2^{(t)}(z,w) = G(z,w) - G_1^{(t)}(z,w)$. By Lemma 3.2, $\|A_{G_2^{(t)}}\| \leq 2^{2n}t$. Consequently, $N_{A_{G_2^{(t)}}}(2^{2n}t) = 0$. Since $A_G = A_{G_1^{(t)}} + A_{G_2^{(t)}}$, applying (8.4), we have

$$\begin{aligned} N_{A_G}(2^{2n+1}t) &\leq N_{A_{G_1^{(t)}}}(2^{2n}t) + N_{A_{G_2^{(t)}}}(2^{2n}t) = N_{A_{G_1^{(t)}}}(2^{2n}t) \\ &\leq \frac{1}{(2^{2n}t)^2}\|A_{G_1^{(t)}}\|_2^2 = \frac{1}{(2^{2n}t)^2}\iint |G_1^{(t)}(z,w)|^2|K(z,w)|^2 d\mu(w)d\mu(z). \end{aligned} \tag{8.5}$$

By (8.2) and (3.2), if $z, w \in \mathbf{C}^n\backslash\{0\}$ are such that $|G(z,w)| \geq t$, then $|z - w| \geq t|z|/2$. Hence it follows from (8.5) and (3.2) that

$$\begin{aligned} N_{A_G}(2^{2n+1}t) &\leq \frac{4}{(2^{2n}t)^2}\iint_{|z-w|\geq t|z|/2} \frac{|z-w|^2}{|z|^2}|K(z,w)|^2 d\mu(w)d\mu(z) \\ &= \frac{4}{\pi^{2n}(2^{2n}t)^2}\int_{\mathbf{C}^n} \frac{1}{|z|^2}\left(\int_{|\zeta|\geq t|z|/2} |\zeta|^2 e^{-|\zeta|^2}dV(\zeta)\right)dV(z). \end{aligned}$$

Obviously, there is a $0 < C_1 < \infty$ such that

$$\int_{|\zeta|\geq a} |\zeta|^2 e^{-|\zeta|^2}dV(\zeta) \leq C_1 e^{-(1/2)a^2}$$

for every $a > 0$. Therefore for every $t > 0$ we have

$$\begin{aligned} N_{A_G}(2^{2n+1}t) &\leq \frac{4C_1}{\pi^{2n}(2^{2n}t)^2}\int_{\mathbf{C}^n} \frac{1}{|z|^2}e^{-(1/2)(t|z|/2)^2}dV(z) \\ &= \frac{4C_1}{\pi^{2n}2^{4n}t^{2n}}\int_{\mathbf{C}^n} \frac{1}{|z|^2}e^{-(1/8)|z|^2}dV(z) = \frac{C_2}{t^{2n}}, \end{aligned}$$

where the last step requires the condition $n \geq 2$. Equivalently,

$$N_{A_G}(t) \leq C_3 t^{-2n} \quad \text{for every} \quad t > 0,$$

where $C_3 = 2^{2n(2n+1)}C_2$.

For each $k \in \mathbf{N}$, let $t_k \in (0,\infty)$ be such that $C_3 t_k^{-2n} = k$. Then $N_{A_G}(t_k) \leq C_3 t_k^{-2n} = k$, and consequently

$$s_{k+1}(A_G) \leq t_k = (C_3/k)^{1/2n} \leq \frac{2C_3^{1/2n}}{(k+1)^{1/2n}}.$$

We have $s_1(A_G) \leq \|A_G\| \leq 2^{2n+1}$ by Lemma 3.2 . Set $C_{8.4.1} = \max\{2^{2n+1}, 2C_3^{1/2n}\}$. Then $s_k(A_G) \leq C_{8.4.1}k^{-1/2n}$ for every $k \in \mathbf{N}$. Hence $\|A_G\|_{2n}^+ \leq C_{8.4.1}$.

(2) Using the techniques in Section 3, we define

$$H(z,w) = (|z| + 1)G(z,w).$$

Then $PA_G = PM_\varphi A_H$, where $\varphi(z) = (|z| + 1)^{-1}$. Recall from the proof of Lemma 3.5 that $\langle \varphi^2 e_k, e_k\rangle \leq k^{-1}$ for every $k \in \mathbf{N}$. Since $\langle \varphi^2 e_j, e_k\rangle = 0$ for all $j \neq k$ in $\mathbf{N}$, this means $PM_\varphi \in \mathcal{C}_2^+$.

Thus it suffices to show that there is a constant $0 < C < \infty$ such that $\|A_H\| \le C$ whenever G satisfies (8.3).

By (8.3) and (3.2), we have $|H(z,w)| \le 4|z-w|$ if $|z| \ge 1$. On the other hand, if $|z| < 1$, then by (8.3) we have $|H(z,w)| \le 4$. It follows from these bounds that $\|A_H\| \le \|L\|$, where L is the operator defined by the formula

$$(Lf)(z) = \int_{\mathbf{C}} (4 + 4|z-w|)|K(z,w)|f(w)d\mu(w).$$

Thus it suffices to show that L is a bounded operator on $L^2(\mathbf{C}, d\mu)$. But the boundedness of L follows from an application of the Schur test with the familiar test function $h(w) = e^{|w|^2/2}$. This completes the proof. □

Lemma 8.5. (1) *Suppose that $n \ge 2$. Then the inequality $\|[M_{\tilde{f}}, P]\|_{2n}^+ \le C_{8.4.1}L(f)$ holds for every $f \in \mathrm{Lip}(S)$, where $C_{8.4.1}$ the constant provided by Lemma* 8.4(1).
(2) *Suppose that $n = 1$. Then the inequality $\|P[M_{\tilde{f}}, P]\|_2^+ \le C_{8.4.2}L(f)$ holds for every $f \in \mathrm{Lip}(S)$, where $C_{8.4.2}$ the constant provided by Lemma* 8.4(2).

Proof. As we saw in the proof of Lemma 3.7, if $f \in \mathrm{Lip}(S)$, then $[M_{\tilde{f}}, P] = A_G$, where G satisfies the condition

$$|G(z,w)| \le L(f)\left|\frac{z}{|z|} - \frac{w}{|w|}\right|.$$

Applying Lemma 8.4, we obtain the desired conclusions. □

Proof of Theorem 1.8. By (4.3), we have

$$[T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}] = P[[M_{\tilde{f}_1}, P], [M_{\tilde{f}_2}, P]]P \cdots P[[M_{\tilde{f}_{2n-1}}, P], [M_{\tilde{f}_{2n}}, P]]P.$$

Thus the desired conclusion follows from Lemma 8.5 and Corollary 8.3. □

Recall from Section 5 that $\mathcal{R}$ denotes the algebra generated by the Toeplitz operators $\{T_{\tilde{f}} : f \in \mathrm{Lip}(S)\}$ on $H^2(\mathbf{C}^n, d\mu)$.

Lemma 8.6. *Let $A_1, \dots, A_{2n} \in \mathcal{R}$. Suppose that there is a $j \in \{1, \dots, 2n\}$ such that $A_j = A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B$, where $A, B \in \mathcal{R}$ and $f, g \in \mathrm{Lip}(S)$. Then the product of commutators*

$$[A_1, A_2] \cdots [A_{2n-1}, A_{2n}]$$

is in the trace class.

Proof. It suffices to consider the case where $j = 2\nu - 1$ for some $\nu \in \{1, \dots, n\}$. Then

$$[A_{2\nu-1}, A_{2\nu}] = [A, A_{2\nu}](T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})B + A[T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}}, A_{2\nu}]B + A(T_{\tilde{f}\tilde{g}} - T_{\tilde{f}}T_{\tilde{g}})[B, A_{2\nu}].$$

Applying Lemmas 5.2 and 5.3 in the above, we have $[A_{2\nu-1}, A_{2\nu}] \in \mathcal{C}_p$ for $p > 2n/3$ if $n \ge 2$ and $[A_{2\nu-1}, A_{2\nu}] \in \mathcal{C}_1$ if $n = 1$. On the other hand, for any $i \in \{1, \dots, n\} \backslash \{\nu\}$, by Lemma 5.2 we have $[A_{2i-1}, A_{2i}] \in \mathcal{C}_p$ for every $p > n$. Combining these Schatten-class memberships with Corollary 2.2, we see that $[A_1, A_2] \cdots [A_{2n-1}, A_{2n}] \in \mathcal{C}_1$. □

Proposition 8.7. *Consider any $f_{i,j} \in \mathrm{Lip}(S)$, where $1 \le i \le 2n$ and $1 \le j \le m$, $m \in \mathbf{N}$. For each $1 \le i \le 2n$, define*

$$A_i = T_{\tilde{f}_{i,1}} \cdots T_{\tilde{f}_{i,m}} \quad \text{and} \quad f_i = f_{i,1} \cdots f_{i,m}.$$

Then the difference

$$[A_1, A_2] \cdots [A_{2n-1}, A_{2n}] - [T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]$$

is in the trace class.

Proof. This is obtained from Lemma 8.6 by using the obvious linearity. □

We recall Definition 6.8 for the algebra $\mathcal{A}$.

Lemma 8.8. *Let $A_1, \ldots, A_{2n} \in \mathcal{A}$. If there is a $j \in \{1, \ldots, 2n\}$ such that $A_j = ADB$, where $A, B \in \mathcal{A}$, then the product of commutators*

$$[A_1, A_2] \cdots [A_{2n-1}, A_{2n}]$$

is in the trace class.

Proof. As in the proof of Lemma 8.6, it suffices to consider the case where $j = 2\nu - 1$ for some $\nu \in \{1, \ldots, n\}$. Then Lemma 6.11 tells us that $[A_{2\nu-1}, A_{2\nu}] \in \mathcal{C}_t$ for every $t > n - 1$ in the case $n \geq 2$, and that $[A_{2\nu-1}, A_{2\nu}] \in \mathcal{C}_1$ in the case $n = 1$. On the other hand, for any $i \in \{1, \ldots, n\} \backslash \{\nu\}$, by Lemma 6.10 we have $[A_{2i-1}, A_{2i}] \in \mathcal{C}_p$ for every $p > n$. Combining these Schatten-class memberships with Corollary 2.2, we see that $[A_1, A_2] \cdots [A_{2n-1}, A_{2n}] \in \mathcal{C}_1$. □

Proposition 8.9. *Given arbitrary $p_1, p_2, \ldots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, there is a trace-class operator G such that*

$$U^*[T_{\tilde{p}_1}, T_{\tilde{p}_2}] \cdots [T_{\tilde{p}_{2n-1}}, T_{\tilde{p}_{2n}}]U = [T^{\text{Berg}}_{p_1}, T^{\text{Berg}}_{p_2}] \cdots [T^{\text{Berg}}_{p_{2n-1}}, T^{\text{Berg}}_{p_{2n}}] + G.$$

Proof. First of all, we recall the notation c^a, $a \in \mathbf{Z}^n_+$, from the proof of Proposition 6.13. Furthermore, T, B, T^a and B^a will be the same as in the proof of Proposition 6.13, $a \in \mathbf{Z}^n_+$.

By the linearity involved, the proposition will be proved if we can show that given any $\alpha_1, \ldots, \alpha_{2n}, \beta_1, \ldots, \beta_{2n} \in \mathbf{Z}^n_+$, there is a $Z \in \mathcal{C}_1$ such that

$$\begin{aligned} U^*[T_{\widetilde{\bar{c}^{\alpha_1} c^{\beta_1}}}, T_{\widetilde{\bar{c}^{\alpha_2} c^{\beta_2}}}] \cdots [T_{\widetilde{\bar{c}^{\alpha_{2n-1}} c^{\beta_{2n-1}}}}, T_{\widetilde{\bar{c}^{\alpha_{2n}} c^{\beta_{2n}}}}]U \\ = [T^{\text{Berg}}_{\bar{c}^{\alpha_1} c^{\beta_1}}, T^{\text{Berg}}_{\bar{c}^{\alpha_2} c^{\beta_2}}] \cdots [T^{\text{Berg}}_{\bar{c}^{\alpha_{2n-1}} c^{\beta_{2n-1}}}, T^{\text{Berg}}_{\bar{c}^{\alpha_{2n}} c^{\beta_{2n}}}] + Z. \end{aligned} \tag{8.6}$$

By Proposition 8.7, the difference

$$\begin{aligned} [T_{\widetilde{\bar{c}^{\alpha_1} c^{\beta_1}}}, T_{\widetilde{\bar{c}^{\alpha_2} c^{\beta_2}}}] \cdots [T_{\widetilde{\bar{c}^{\alpha_{2n-1}} c^{\beta_{2n-1}}}}, T_{\widetilde{\bar{c}^{\alpha_{2n}} c^{\beta_{2n}}}}] \\ - [(T^{\alpha_1})^* T^{\beta_1}, (T^{\alpha_2})^* T^{\beta_2}] \cdots [(T^{\alpha_{2n-1}})^* T^{\beta_{2n-1}}, (T^{\alpha_{2n}})^* T^{\beta_{2n}}] \end{aligned}$$

is in the trace class. On the other hand, we know that

$$\begin{aligned} [T^{\text{Berg}}_{\bar{c}^{\alpha_1} c^{\beta_1}}, T^{\text{Berg}}_{\bar{c}^{\alpha_2} c^{\beta_2}}] \cdots [T^{\text{Berg}}_{\bar{c}^{\alpha_{2n-1}} c^{\beta_{2n-1}}}, T^{\text{Berg}}_{\bar{c}^{\alpha_{2n}} c^{\beta_{2n}}}] \\ = [(B^{\alpha_1})^* B^{\beta_1}, (B^{\alpha_2})^* B^{\beta_2}] \cdots [(B^{\alpha_{2n-1}})^* B^{\beta_{2n-1}}, (B^{\alpha_{2n}})^* B^{\beta_{2n}}]. \end{aligned}$$

Therefore (8.6) will follow if we can prove that

$$\begin{aligned} U^*[(T^{\alpha_1})^* T^{\beta_1}, (T^{\alpha_2})^* T^{\beta_2}] \cdots [(T^{\alpha_{2n-1}})^* T^{\beta_{2n-1}}, (T^{\alpha_{2n}})^* T^{\beta_{2n}}]U \\ = [(B^{\alpha_1})^* B^{\beta_1}, (B^{\alpha_2})^* B^{\beta_2}] \cdots [(B^{\alpha_{2n-1}})^* B^{\beta_{2n-1}}, (B^{\alpha_{2n}})^* B^{\beta_{2n}}] + X, \end{aligned}$$

where X is in the trace class.

As we saw in the proof of Proposition 6.13, for the given $\alpha_1, \ldots, \alpha_{2n}, \beta_1, \ldots, \beta_{2n} \in \mathbf{Z}^n_+$, there exist an $m \in \mathbf{N}$ and $A_{j,i}, C_{j,i} \in \mathcal{A}$, $1 \leq j \leq 2n$ and $1 \leq i \leq m$, such that

$$U^*(T^{\alpha_j})^* T^{\beta_j} U = (B^{\alpha_j})^* B^{\beta_j} + \sum_{i=1}^{m} A_{j,i} D C_{j,i}$$

for every $1 \le j \le 2n$. Therefore

$$U^*[(T^{\alpha_1})^*T^{\beta_1},(T^{\alpha_2})^*T^{\beta_2}]\cdots[(T^{\alpha_{2n-1}})^*T^{\beta_{2n-1}},(T^{\alpha_{2n}})^*T^{\beta_{2n}}]U$$
$$= [(B^{\alpha_1})^*B^{\beta_1},(B^{\alpha_2})^*B^{\beta_2}]\cdots[(B^{\alpha_{2n-1}})^*B^{\beta_{2n-1}},(B^{\alpha_{2n}})^*B^{\beta_{2n}}]$$
$$+\sum_{r=1}^{(m+1)^{2n}-1}[X_{1,r},X_{2,r}]\cdots[X_{2n-1,r},X_{2n,r}],$$

where the operators $X_{j,r}$ satisfy the following two conditions:

(1) $X_{j,r} \in \mathcal{A}$ for all $1 \le j \le 2n$ and $1 \le r \le (m+1)^{2n}-1$.

(2) For each $1 \le r \le (m+1)^{2n}-1$, there is a $j(r) \in \{1,\dots,2n\}$ such that $X_{j(r),r} = ADC$ with $A, C \in \mathcal{A}$.

By these two conditions and Lemma 8.8, for each $1 \le r \le (m+1)^{2n}-1$, the product of commutators $[X_{1,r},X_{2,r}]\cdots[X_{2n-1,r},X_{2n,r}]$ is in the trace class. This completes the proof. □

9. Dixmier trace

Before we prove Theorem 1.9, a general discussion of Dixmier trace is in order. For this, we cite [5,7,18] as general references. To define Dixmier trace, one starts with a Banach limit ω on $\ell^\infty(\mathbf{N})$. But in addition to the properties that Banach limits [6, Section III.7] possess in general, ω is required to have the following "doubling" property:

(DB) For each $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, $\omega(\{a_k\}_{k\in\mathbf{N}}) = \omega(\{a_1,a_1,a_2,a_2,\dots,a_k,a_k,\dots\})$.

Such an ω can be easily constructed. For example, one can start with the doubling operator $E : \ell^\infty(\mathbf{N}) \to \ell^\infty(\mathbf{N})$. That is,

$$E\{a_1,a_2,\dots,a_k,\dots\} = \{a_1,a_1,a_2,a_2,\dots,a_k,a_k,\dots\}$$

for $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$. Take any Banach limits L_1 and L_2, distinct or identical. Then an elementary exercise shows that the formula

$$\omega(a) = L_2\left(\left\{\frac{1}{k}\sum_{j=1}^{k}L_1(E^j a)\right\}_{k\in\mathbf{N}}\right),$$

$a \in \ell^\infty(\mathbf{N})$, defines a Banach limit that has the doubling property (DB).

With such an ω, for any *positive* operator $A \in \mathcal{C}_1^+$, its Dixmier trace is defined to be

$$\mathrm{Tr}_\omega(A) = \omega\left(\left\{\frac{1}{\log(k+1)}\sum_{j=1}^{k}s_j(A)\right\}_{k\in\mathbf{N}}\right).$$

The doubling property of ω ensures the additivity $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A)+\mathrm{Tr}_\omega(B)$ for positive operators $A, B \in \mathcal{C}_1^+$. Thus Tr_ω naturally extends to a linear functional on $\mathcal{C}_1^+$. This definition guarantees that Tr_ω is unitary invariant. For any $T \in \mathcal{C}_1^+$ and any unitary operator W, WT is unitarily equivalent to TW. Therefore $\mathrm{Tr}_\omega(WT) = \mathrm{Tr}_\omega(TW)$. From this it follows that $\mathrm{Tr}_\omega(XT) = \mathrm{Tr}_\omega(TX)$ for every $T \in \mathcal{C}_1^+$ and every bounded operator X, which is what one expects of a trace.

It is well known that if A is a trace-class operator, then $\mathrm{Tr}_\omega(A) = 0$ for every Dixmier trace Tr_ω. This fact was heavily involved in previous works [4,10,11,21] on Dixmier trace. Our proof of Theorem 1.9 also relies on this fact. Thus from Proposition 8.9 and Theorem 1.7 we immediately obtain

Corollary 9.1. *Let* $p_1, p_2, \ldots, p_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. *Then on the Fock space we have*

$$\mathrm{Tr}_\omega([T_{\tilde{p}_1}, T_{\tilde{p}_2}] \cdots [T_{\tilde{p}_{2n-1}}, T_{\tilde{p}_{2n}}]) = \frac{1}{n!} \int_S \{p_1, p_2\}_b \cdots \{p_{2n-1}, p_{2n}\}_b d\sigma$$

for every Dixmier trace Tr_ω.

Mimicking (1.5), for any $f \in C^1(\Omega_\delta)$ we now define

$$\|f\|_{\#,\delta} = \sum_{|\alpha| \le 1} \sup_{1-(\delta/2) \le |x| \le 1+(\delta/2)} |(\partial^\alpha f)(x)|.$$

Lemma 9.2. *Let* $f \in C^1(\Omega_\delta)$ *for some* $0 < \delta < 1$. *Then there is a sequence of polynomials* $\{p_k\}$ *in* $\mathbf{C}[x_1, x_2, \ldots, x_{2n-1}, x_{2n}]$ *such that*

$$\lim_{k \to \infty} \|f - p_k\|_{\#,\delta} = 0.$$

The proof of Lemma 9.2 is the same as that of Lemma 7.1 and will not be repeated.

Proof of Theorem 1.9. Since $\| \cdot \|_{\#,\delta}$ involves all the first-order partial derivatives, it is an immediate consequence of Theorem 1.8 that there is a $0 < C < \infty$ such that

$$\|[T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]\|_1^+ \le C\|f_1\|_{\#,\delta} \cdots \|f_{2n}\|_{\#,\delta} \tag{9.1}$$

for all $f_1, \ldots, f_{2n} \in C^1(\Omega_\delta)$.

Suppose that we are given $f_1, f_2, \ldots, f_{2n} \in C^1(\Omega_\delta)$. Then by Lemma 9.2, there are sequences of polynomials $\{p_{j,k}\}$ in $\mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, $1 \le j \le 2n$, such that

$$\lim_{k \to \infty} \|f_j - p_{j,k}\|_{\#,\delta} = 0 \tag{9.2}$$

for $1 \le j \le 2n$. By (9.1) and the obvious linearity, (9.2) implies

$$\lim_{k \to \infty} \|[T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}] - [T_{\tilde{p}_{1,k}}, T_{\tilde{p}_{2,k}}] \cdots [T_{\tilde{p}_{2n-1,k}}, T_{\tilde{p}_{2n,k}}]\|_1^+ = 0.$$

Therefore for any Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega([T_{\tilde{f}_1}, T_{\tilde{f}_2}] \cdots [T_{\tilde{f}_{2n-1}}, T_{\tilde{f}_{2n}}]) = \lim_{k \to \infty} \mathrm{Tr}_\omega([T_{\tilde{p}_{1,k}}, T_{\tilde{p}_{2,k}}] \cdots [T_{\tilde{p}_{2n-1,k}}, T_{\tilde{p}_{2n,k}}]). \tag{9.3}$$

Obviously, (9.2) also implies

$$\lim_{k \to \infty} \frac{1}{n!} \int_S \{p_{1,k}, p_{2,k}\}_b \cdots \{p_{2n-1,k}, p_{2n,k}\}_b d\sigma = \frac{1}{n!} \int_S \{f_1, f_2\}_b \cdots \{f_{2n-1}, f_{2n}\}_b d\sigma. \tag{9.4}$$

For each $k \in \mathbf{N}$, Corollary 9.1 gives us the identity

$$\mathrm{Tr}_\omega([T_{\tilde{p}_{1,k}}, T_{\tilde{p}_{2,k}}] \cdots [T_{\tilde{p}_{2n-1,k}}, T_{\tilde{p}_{2n,k}}]) = \frac{1}{n!} \int_S \{p_{1,k}, p_{2,k}\}_b \cdots \{p_{2n-1,k}, p_{2n,k}\}_b d\sigma. \tag{9.5}$$

Combining (9.3), (9.4) and (9.5), we obtain (1.8). This completes the proof. $\square$

Data availability

No data was used for the research described in the article.

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