ESSENTIAL COMMUTANTS
ON STRONGLY PSEUDO-CONVEX DOMAINS

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Abstract. Consider a bounded strongly pseudo-convex domain Ω with a smooth boundary in $\mathbf{C}^n$. Let $\mathcal{T}$ be the Toeplitz algebra on the Bergman space $L^2_a(\Omega)$. That is, $\mathcal{T}$ is the C^*-algebra generated by the Toeplitz operators $\{T_f : f \in L^\infty(\Omega)\}$. Extending the work [27,28] in the special case of the unit ball, we show that on any such Ω, $\mathcal{T}$ and $\{T_f : f \in \mathrm{VO_{bdd}}\}+\mathcal{K}$ are essential commutants of each other. On a general Ω considered in this paper, the proofs require many new ideas and techniques. These same techniques also enable us to show that for $A \in \mathcal{T}$, if $\langle Ak_z, k_z\rangle \to 0$ as $z \to \partial\Omega$, then A is a compact operator.

1. Introduction

An enduring question in the study of Toeplitz operators is their essential commutativity. In this paper we consider this question on strongly pseudo-convex domains. It will be beneficial to start the paper with a recollection of necessary definitions and background.

Suppose that $\mathcal{Z}$ is a collection of bounded operators on a Hilbert space $\mathcal{H}$. Then its essential commutant is defined to be

$$\mathrm{EssCom}(\mathcal{Z}) = \{A \in \mathcal{B}(\mathcal{H}) : [A, T] \ \text{ is compact for every } \ T \in \mathcal{Z}\}.$$

The study of essential commutants began with the classic papers of Johnson-Parrott [13], Voiculescu [23] and Popa [19]. Ever since, essential-commutant problems have become a mainstay of operator theory and operator algebras. As it turns out, many of the most interesting examples in the study of essential commutants are associated with Toeplitz operators, of various kinds [4,6-9,11,24,25,27,28]. Perhaps, one reason why essential-commutant problems attract attention is that they are generally not easy.

In this paper we consider an arbitrary bounded, strongly pseudo-convex domain Ω with a smooth boundary in $\mathbf{C}^n$. Recall that the Bergman space $L^2_a(\Omega)$ is the collection of analytic functions h on Ω satisfying the condition

$$\int_\Omega |h|^2 dv < \infty,$$

where dv is the volume measure on Ω. Let $P : L^2(\Omega) \to L^2_a(\Omega)$ be the orthogonal projection. For each $f \in L^\infty(\Omega)$, we have the Toeplitz operator T_f defined by the formula

$$T_f h = P(fh), \quad h \in L^2_a(\Omega).$$

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Let $\mathcal{T}$ denote the C^*-algebra generated by $\{T_f : f \in L^\infty(\Omega)\}$. Then $\mathcal{T}$ is called the *Toeplitz algebra* on the Bergman space $L_a^2(\Omega)$. It is well known that $\mathcal{T}$ contains $\mathcal{K}$, the collection of compact operators on $L_a^2(\Omega)$ [22,Theorem 4.1.25]. Obviously, this is a convenient fact for the study of essential commutants.

In the case of the unit ball $\mathbf{B}$ in $\mathbf{C}^n$, the essential commutant problems related to $\mathcal{T}$ were solved in [27,28], with [26,Theorem 1.3] playing a pivotal role. Specifically, in the case of the unit ball, it was shown that $\mathrm{EssCom}(\mathcal{T}) = \{T_f : f \in \mathrm{VO_{bdd}}\} + \mathcal{K}$ in [27] and that $\mathrm{EssCom}\{T_f : f \in \mathrm{VO_{bdd}}\} = \mathcal{T}$ in [28]. Once one knows that, a question naturally presents itself: what happens if one replaces the unit ball $\mathbf{B}$ by a general strongly pseudo-convex domain Ω? Equally naturally, one would expect that the same results hold on a general Ω. But here one immediately runs into two difficulties:

(1) The works in [27,28], particularly in [26], rely heavily on the explicit formula for the Bergman metric β on $\mathbf{B}$. Without such an explicit formula, it is not clear how to redo many of the estimates in [27,28]. By contrast, in the case of a general strongly pseudo-convex domain, we only know the *asymptotics* of the Bergman metric [10,20], but we do not have a formula for it that is explicit enough. In other words, on a general Ω, we do not have good enough a handle on the Bergman metric to do many of the necessary estimates. The same is true if one considers the Kobayashi metric instead of the Bergman metric.

(2) The techniques in [26,27,28] depend heavily on the Möbius transforms on $\mathbf{B}$. But on a general strongly pseudo-convex domain Ω, there is no such thing as Möbius transform. In other words, compared with the unit ball, a general Ω totally lacks global symmetry. Compared with (1), this difficulty is more substantive, but it also makes an exciting challenge: can we prove the results in [27,28] on a domain without symmetry?

We are pleased to report that we have managed to overcome these difficulties. The way we deal with difficulty (1) is to simply introduce a new metric that serves our purpose. Since Ω is a strongly pseudo-convex domain, it has a defining function r, i.e., $\Omega = \{z \in \mathbf{C}^n : r(z) < 0\}$. Then the formula

$$b_{ij}(z) = \partial_i \bar{\partial}_j \log \frac{1}{-r(z)}, \quad 1 \le i, j \le n,$$

for z near $\partial\Omega$ gives us the infinitesimal generator of a metric d on Ω. One might call this d a poor man's imitation of the Bergman metric, but the above formula is explicit enough to allow us to do all the necessary analysis. We will have more to say about this point below, and the precise definition of d will be given at the beginning of Section 2.

Difficulty (2) simply requires new approaches. Examining the involvements of Möbius transforms in [27,28] one by one, we have managed to find a new idea or new technique as a replacement in each case. Thus the results about essential commutants mentioned above can indeed be proved without symmetry.

To state our results, we need the notion of *vanishing oscillation*, which was first introduced in [5,3] for functions on bounded symmetric domains with respect to the Bergman metric. In this paper we need to define functions of vanishing oscillation with respect to the metric d on Ω. Let f be a continuous function on Ω. Then f is said to have vanishing oscillation if

$$\lim_{z \to \partial\Omega} \sup\{|f(z) - f(w)| : d(z,w) \le 1\} = 0.$$

Let VO denote the collection of functions of vanishing oscillation on Ω. Further, define

$$\mathrm{VO}_{\mathrm{bdd}} = \mathrm{VO} \cap L^\infty(\Omega).$$

Our main results are the two theorems below:

Theorem 1.1. *On any bounded, strongly pseudo-convex domain Ω with a smooth boundary in $\mathbf{C}^n$, the following hold true*:
(i) *The Toeplitz algebra $\mathcal{T}$ is the essential commutant of $\{T_f : f \in \mathrm{VO}_{\mathrm{bdd}}\}$.*
(ii) *The essential commutant of $\mathcal{T}$ equals $\{T_f : f \in \mathrm{VO}_{\mathrm{bdd}}\} + \mathcal{K}$.*

Let $\mathcal{Q}$ denote the Calkin algebra $\mathcal{B}(L^2_a(\Omega))/\mathcal{K}$, and let

$$\pi : \mathcal{B}(L^2_a(\Omega)) \to \mathcal{Q}$$

be the quotient homomorphism. Then $\pi(\mathrm{EssCom}(\mathcal{Z})) = \{\pi(\mathcal{Z})\}'$ for every subset $\mathcal{Z} \subset \mathcal{B}(L^2_a(\Omega))$. Obviously, a subset $\mathcal{A}$ of $\mathcal{Q}$ satisfies the double-commutant relation $\mathcal{A} = \mathcal{A}''$ if and only if $\mathcal{A} = \mathcal{G}'$ for some $\mathcal{G} \subset \mathcal{Q}$. Thus Theorem 1.1(i) implies that $\pi(\mathcal{T})$ satisfies the double-commutant relation in $\mathcal{Q}$.

As it turns out, the techniques that allow us to prove Theorem 1.1(i), also give us a classic compactness criterion for $A \in \mathcal{T}$ in terms of its Berezin transform on Ω. Let us write k_z, $z \in \Omega$, for the normalized reproducing kernel for the Bergman space $L^2_a(\Omega)$.

Theorem 1.2. *Consider any bounded, strongly pseudo-convex domain Ω with a smooth boundary in $\mathbf{C}^n$. Let $A \in \mathcal{T}$. If*

$$\lim_{z \to \partial\Omega} \langle Ak_z, k_z \rangle = 0, \tag{1.1}$$

then A is a compact operator on $L^2_a(\Omega)$.

At this point, it is appropriate to briefly recall the long history of this line of investigations. The first result of this genre was due to Axler and Zheng [1], where the domain was the unit disc in $\mathbf{C}$ and A was a finite algebraic combination of Toeplitz operators. Later in [21], Suárez showed that this compactness criterion holds for all $A \in \mathcal{T}$ on the unit ball $\mathbf{B}$ in $\mathbf{C}^n$. The fact that Suárez was able to do this for *arbitrary* $A \in \mathcal{T}$ on the ball, rather than just for finite algebraic combinations of Toeplitz operators, was considered to be a major breakthrough. Consequently, [21] inspired many generalizations [2,12,29], including generalizations on the Fock space. But all these papers depend on the Möbius transforms on the domain in question. In this regard, Theorem 1.2 is the first to remove any and all involvement of Möbius transforms, since in general there aren't any on Ω.

The rest of the paper is taken up by the proofs of these results. Because we have to start from scratch, there are numerous steps involved. We conclude the introduction by an outline of our plan.

First of all, in Section 2 we precisely define the metric d mentioned above. In addition to d, another important quantity for the paper is the "gauge"

$$\rho(z,w) = |z-w|^2 + |\langle z-w, (\bar{\partial}r)(z)\rangle|$$

on Ω. Section 2 contains several fundamental estimates involving d and $\rho(z,w)$. Section 3 brings in another important ingredient for our analysis, the function

$$F(z,w) = |r(z)| + |r(w)| + \rho(z,w),$$

which is a familiar fixture on strongly pseudo-convex domains. The main result of the section is Lemma 3.8, which is a version of the Forelli-Rudin estimates for Ω in which d and F are quantitatively involved.

Sections 4 and 5 are devoted to operators that are discrete sums constructed from the Bergman kernel $K(z,w)$ over d-lattices. The main goal for these two sections is Corollary 5.3, which provides the norm-continuity of such discrete sums under small perturbation of the lattice.

In Section 6, we introduce LOC(A), the class of "localized versions of A" for any bounded operator A on $L^2_a(\Omega)$. Using Lemma 3.8 and Corollary 5.3 mentioned above and doing quite a bit of additional work, we show in Section 6 that LOC$(A) \subset \mathcal{T}$ for every $A \in \mathcal{B}(L^2_a(\Omega))$. This is a major step in the proof of Theorem 1.1(i).

Section 7 is devoted to matters related to functions of vanishing oscillation. In particular, we consider the scalar quantity

$$\text{diff}(f) = \sup\{|f(z) - f(w)| : d(z,w) < 1\},$$

which is another essential ingredient in the proof of Theorem 1.1(i). We show that every operator in EssCom$\{T_f : f \in \text{VO}_{\text{bdd}}\}$ satisfies an "ϵ-δ" condition involving "diff".

In Section 8 we construct approximate partitions of the unity on Ω that satisfy two competing requirements: (1) The "diff" for the partition functions must be small. (2) There is a fixed, finite cap on the overlaps of the sets involved. This construction is based on a suitable analogue of "radial-spherical decomposition" for Ω. As it turns out, the gauge $\rho(z,w)$ plays the role of "spherical coordinates" in our decomposition, whereas the defining function r gives us a convenient "radial coordinate".

With all the above preparation, we prove Theorem 1.1(i) in Section 9. The gist of the proof is that the "ϵ-δ" condition mentioned above characterizes the membership $X \in \mathcal{T}$. The same work also shows that for $A \in \mathcal{T}$, if

$$\lim_{z\to\partial\Omega} \sup\{|\langle Ak_w, k_z\rangle| : d(z,w) < R\} = 0 \tag{1.2}$$

for every given $0 < R < \infty$, then A is a compact operator. This is a major step in the proof of Theorem 1.2. In fact, what remains for the proof of Theorem 1.2 is to show that (1.1) implies (1.2).

Then in Section 10, we turn to the proof of part (ii) in Theorem 1.1. With the work in Section 9, this is now relatively easy. First of all, Theorem 1.1(i) tells us that $\text{EssCom}(\mathcal{T})$ coincides with the essential center of $\mathcal{T}$. That is, $\text{EssCom}(\mathcal{T}) \subset \mathcal{T}$. Then we show that the membership $A \in \text{EssCom}(\mathcal{T})$ implies that the Berezin transform $\tilde{A}$ of A is in VO_{bdd}. Since $A - T_{\tilde{A}} \in \mathcal{T}$, the membership $\tilde{A} \in \text{VO}_{\text{bdd}}$ and the work in Section 9 lead to an easy proof of the fact that $A - T_{\tilde{A}} \in \mathcal{K}$, which proves Theorem 1.1(ii).

Finally, in Section 11 we show that (1.1) indeed implies (1.2). For all previous works involving this step, this was easy, because one could use Möbius transforms. But in our case of a general strongly pseudo-convex domain, this becomes a non-trivial step. Material from Sections 2-4 will be needed for this step.

2. A metric on Ω and related facts

First of all, we cite [15,20] as general references for strongly pseudo-convex domains. Throughout the paper, Ω denotes a bounded, connected, strongly pseudo-convex domain in $\mathbf{C}^n$ with smooth boundary. More precisely, we always assume that Ω is bounded and connected, and that there is a real-valued C^∞ function r defined in an open neighborhood of the closure of Ω such that the following three conditions are satisfied:

(1) $\Omega = \{z \in \mathbf{C}^n : r(z) < 0\}$.

(2) $|(\nabla r)(z)| \neq 0$ for every $z \in \partial\Omega$.

(3) There is a $c > 0$ such that

$$\sum_{i,j=1}^{n} (\partial_i \bar{\partial}_j r)(z)\xi_i\bar{\xi}_j \geq c(|\xi_1|^2 + \cdots + |\xi_n|^2) \tag{2.1}$$

for all $z \in \partial\Omega$ and $\xi_1, \ldots, \xi_n \in \mathbf{C}$.

Such an r is called a defining function for the domain, and will be fixed along with Ω.

It will be convenient to adopt the following convention: We will consider $\mathbf{C}^n$ as a column space whenever an $n \times n$ matrix acts on it. When there is no matrix involved, we will consider $\mathbf{C}^n$ either as a column space or as a row space, whichever is more appropriate.

Let $A(z)$ be the $n \times n$ matrix whose entry in the intersection of i-th column and j-row is $(\partial_i \bar{\partial}_j r)(z)$, $i, j = 1, \ldots, n$. By (2) and (3), there is a $\theta > 0$ such that if $w \in \Omega$ and $r(w) > -3\theta$, then $|(\nabla r)(w)| \neq 0$ and

$$\langle A(w)\xi, \xi\rangle \geq (c/2)|\xi|^2 \tag{2.2}$$

for all $\xi \in \mathbf{C}^n$. Let $\psi : \mathbf{R} \to [0, 1]$ be a C^∞ function such that $\psi = 1$ on $[-\theta, \infty)$ and $\psi = 0$ on $(-\infty, -2\theta]$. Write δ_{ij} for Kronecker's delta. We then define

$$b_{ij}(z) = \psi(r(z))\left(\frac{1}{-r(z)}(\partial_i \bar{\partial}_j r)(z) + \frac{1}{r^2(z)}(\partial_i r)(z)(\bar{\partial}_j r)(z)\right) + (1 - \psi(r(z)))\delta_{ij} \tag{2.3}$$

for $i, j \in \{1, \ldots, n\}$ and $z \in \Omega$. Let $\mathcal{B}(z)$ be the $n \times n$ matrix whose entry in the intersection of i-th column and j-row is $b_{ij}(z)$, $i, j = 1, \ldots, n$. From (2.2) and the definition of ψ we see that the $\mathcal{B}(z)$ is invertible for every $z \in \Omega$. Thus the local Hermitian form

$$H_z(\xi, \eta) = \langle \mathcal{B}(z)\xi, \eta \rangle, \quad \xi, \eta \in T_z\Omega = \mathbf{C}^n,$$

generates a non-degenerate metric d on Ω. That is, for $z, w \in \Omega$,

$$d(z, w) = \inf \int_0^1 \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t) \rangle} dt, \tag{2.4}$$

where the infimum is taken over all C^1 maps $g : [0, 1] \to \Omega$ satisfying the conditions $g(0) = z$ and $g(1) = w$. The definition of ψ ensures that for $i, j \in \{1, \ldots, n\}$,

$$b_{ij}(z) = \partial_i \bar{\partial}_j \log \frac{1}{-r(z)} \quad \text{whenever} \quad -\theta \le r(z) < 0.$$

Denote $\bar{\partial} = (\bar{\partial}_1, \ldots, \bar{\partial}_n)$, which will play a prominent role throughout the paper. Then

$$\langle \mathcal{B}(z)\xi, \xi \rangle = \frac{\langle A(z)\xi, \xi \rangle}{-r(z)} + \left(\frac{|\langle \xi, (\bar{\partial} r)(z) \rangle|}{-r(z)} \right)^2 \quad \text{whenever} \quad -\theta \le r(z) < 0, \tag{2.5}$$

$\xi \in \mathbf{C}^n$. These identities make d an *imitation* of the Bergman metric on Ω. Compared with the real Bergman metric β, our imitation d has the advantage that the explicit formulas above will greatly simplify many of the estimates below.

Lemma 2.1. *There is a $c_{2.1} > 0$ such that $-r(w) \ge -c_{2.1} 2^{-4d(z,w)} r(z)$ for all $z, w \in \Omega$.*

Proof. Consider any $z, w \in \Omega$ such that $-r(z) \le \theta$ and

$$-r(w) \le -(1/2)r(z).$$

Let $g : [a, b] \to \Omega$ be a C^1 map such that $g(a) = z$ and $g(b) = w$. Let a' be the largest number in $[a, b]$ such that $r(g(a')) = r(z)$. Then $-r(g(t)) \le -r(g(a'))$ for $t \in [a', b]$ and

$$\int_{a'}^b \frac{d}{dt} r(g(t)) dt = r(g(b)) - r(g(a')) \ge -(1/2)r(z).$$

Note that

$$\frac{d}{dt} r(g(t)) = 2\mathrm{Re}\langle g'(t), (\bar{\partial} r)(g(t)) \rangle.$$

Therefore the above implies

$$2 \int_{a'}^b |\langle g'(t), (\bar{\partial} r)(g(t)) \rangle| dt \ge \int_{a'}^b \frac{d}{dt} r(g(t)) dt \ge -(1/2)r(z).$$

Since $-r(g(t)) \leq -r(z) \leq \theta$ for every $t \in [a', b]$, by (2.5) we have

$$\int_{a'}^{b} \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle} dt \geq \int_{a'}^{b} \frac{|\langle g'(t), (\bar{\partial} r)(g(t))\rangle|}{-r(g(t))} dt \geq \int_{a'}^{b} \frac{|\langle g'(t), (\bar{\partial} r)(g(t))\rangle|}{-r(z)} dt \geq \frac{1}{4}.$$

From the above we see that if $z, w \in \Omega$ satisfy the conditions $-r(z) \leq \theta$ and

$$-r(w) \leq -2^{-m} r(z) \quad \text{for some} \quad m \in \mathbf{N},$$

then for any C^1 map $g : [0,1] \to \Omega$ with the properties $g(0) = z$ and $g(1) = w$ we have

$$\int_{0}^{1} \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle} dt \geq \frac{m}{4}.$$

Combining this with (2.4), we have $m \leq 4d(z, w)$. This implies the inequality

$$-r(w) > -(2^{-4d(z,w)-1}) r(z)$$

for all $w \in \Omega$ whenever $-r(z) \leq \theta$.

Suppose that $-r(z) > \theta$. If $-r(w) \geq \theta$, then the case is trivial, as the function $-r$ has a maximum on Ω. Suppose that $-r(w) < \theta$. There is a C^1 map $g : [0,1] \to \Omega$ such that $g(0) = z$, $g(1) = w$ and

$$\int_{0}^{1} \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle} dt \leq d(z, w) + (1/4).$$

We have $-r(g(0)) > \theta$ and $-r(g(1)) < \theta$. Thus there is an $a \in [0,1]$ such that $-r(g(a)) = \theta$. Define $z' = g(a)$. By what we proved above, $-r(w) > 2^{-4d(z',w)-1}\theta$. Since $a \in [0,1]$, $z' = g(a)$ and $g(1) = w$, we have $d(z', w) \leq d(z, w) + (1/4)$. Hence

$$-r(w) \geq 2^{-4\{d(z,w)+(1/4)\}-1}\theta = -\{\theta/(-r(z))\}(2^{-4d(z,w)-2}) r(z).$$

Since $-r$ has a maximum on Ω, the lemma also holds in the case $-r(z) > \theta$. $\square$

Lemma 2.2. *There is a constant $0 < C_{2.2} < \infty$ such that*

$$|z - w|^2 + |\langle z - w, (\bar{\partial} r)(z)\rangle| \leq C_{2.2}\{d(z, w) + d^2(z, w)\} 2^{12d(z,w)}(-r(z))$$

for all $z, w \in \Omega$.

Proof. We first show that there is a C such that

$$|z - w| \leq C d(z, w) 2^{4d(z,w)} \sqrt{-r(z)} \tag{2.6}$$

for all $z, w \in \Omega$. By (2.4), for any given $z, w \in \Omega$, there is a C^1 map $g : [0,1] \to \Omega$ such that $g(0) = z$, $g(1) = w$, and

$$\int_{0}^{1} \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle} dt \leq 2d(z, w).$$

There is a $t_0 \in [0,1]$ such that $-r(g(t_0)) \geq -r(g(t))$ for every $t \in [0,1]$. By (2.5) and (2.3), there is a $c_1 > 0$ such that

$$\frac{c_1}{\sqrt{-r(g(t_0))}} \int_0^1 |g'(t)|dt \leq \int_0^1 \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle}dt \leq 2d(z,w).$$

Set $C_1 = 2/c_1$. Since $g(0) = z$ and $g(1) = w$, the above implies

$$|z - w| \leq C_1 d(z,w)\sqrt{-r(g(t_0))}. \tag{2.7}$$

If we write $\zeta = g(t_0)$, then

$$d(z,\zeta) \leq \int_0^{t_0} \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle}dt \leq 2d(z,w).$$

By Lemma 2.1, we have $-r(\zeta) \leq c_{2.1}^{-1}2^{4d(z,\zeta)}(-r(z)) \leq c_{2.1}^{-1}2^{8d(z,w)}(-r(z))$. Combining this with (2.7), (2.6) follows.

The same argument also shows that $d(z, g(t)) \leq 2d(z,w)$ for every $t \in [0,1]$. Therefore

$$|z - g(t)| \leq 2Cd(z,w)2^{8d(z,w)}\sqrt{-r(z)},$$

$t \in [0,1]$. Using Lemma 2.1 and the obvious Lipschitz condition for $\bar{\partial}r$, we have

$$\begin{aligned}
|\langle z - w, (\bar{\partial}r)(z)\rangle| &= |\langle g(1) - g(0), (\bar{\partial}r)(z)\rangle| \leq \int_0^1 |\langle g'(t), (\bar{\partial}r)(z)\rangle|dt \\
&\leq \int_0^1 |\langle g'(t), (\bar{\partial}r)(g(t))\rangle|dt + \int_0^1 |g'(t)||(\bar{\partial}r)(z) - (\bar{\partial}r)(g(t))|dt \\
&\leq \frac{2^{8d(z,w)}}{c_{2.1}}(-r(z)) \int_0^1 \frac{|\langle g'(t), (\bar{\partial}r)(g(t))\rangle|}{-r(g(t))}dt \\
&\quad + 2Cd(z,w)2^{8d(z,w)} \cdot C_2 \left(\frac{2^{8d(z,w)}}{c_{2.1}}\right)^{1/2} (-r(z)) \int_0^1 \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle}dt \\
&\leq C_3(-r(z))\{2^{8d(z,w)}d(z,w) + 2^{12d(z,w)}d^2(z,w)\}.
\end{aligned}$$

Combining this with (2.6), the lemma is proved. □

For $z \in \Omega$ and $a > 0$, define the imitation Bergman metric ball

$$D(z,a) = \{w \in \Omega : d(z,w) < a\}.$$

Definition 2.3. For $\eta \in \mathbf{C}^n \backslash \{0\}$, $a > 0$ and $b > 0$, we let $\mathcal{P}(\eta; a, b)$ be the collection of vectors $u + v$ satisfying the following three conditions:

(1) $u, v \in \mathbf{C}^n$ with $|u| < a$ and $|v| < b$.
(2) $u \perp \eta$.
(3) $v \in \{\xi\eta : \xi \in \mathbf{C}\}$.

Proposition 2.4. *Given any $0 < a < \infty$, there are $0 < c \le C < \infty$ such that*

$$z + \mathcal{P}((\bar{\partial} r)(z); c\sqrt{-r(z)}, -cr(z)) \subset D(z,a) \subset z + \mathcal{P}((\bar{\partial} r)(z); C\sqrt{-r(z)}, -Cr(z))$$

for every $z \in \Omega$ satisfying the condition $-r(z) < \theta$.

Proof. Let $0 < a < \infty$ be given and consider a sufficiently small $c > 0$. Let $u, v \in \mathbf{C}^n$ satisfy the conditions $u \perp (\bar{\partial} r)(z)$, $|u| < c\sqrt{-r(z)}$, $v \in \{\xi(\bar{\partial} r)(z) : \xi \in \mathbf{C}\}$, and $|v| < -cr(z)$. We want to show that $d(z, z+u+v) < a$. To prove this, consider the path

$$g(t) = z + t(u+v),$$

$t \in [0,1]$. Then $g'(t) = u + v$. By the Taylor expansion for $\bar{\partial} r$, we have

$$\begin{aligned} \langle g'(t), (\bar{\partial} r)(g(t)) \rangle &= \langle u+v, (\bar{\partial} r)(z + t(u+v)) \rangle \\ &= \langle u+v, (\bar{\partial} r)(z) + tX(z)(u+v) + o(|u+v|) \rangle \\ &= \langle v, (\bar{\partial} r)(z) \rangle + \langle u+v, tX(z)(u+v) + o(|u+v|) \rangle, \end{aligned}$$

where $X(z)$ is the derivative of $\bar{\partial} r$ at z, which is a linear map from $\mathbf{C}^n$ to $\mathbf{C}^n$. Since $|v| < -cr(z)$ and $|u+v| \le c(-r(z) + \sqrt{-r(z)})$, we see that

$$|\langle g'(t), (\bar{\partial} r)(g(t)) \rangle| \le cM(-r(z))$$

for $t \in [0,1]$. Since r is real-valued, Taylor expansion gives us

$$\begin{aligned} r(g(t)) = r(z + t(u+v)) &= r(z) + 2t\mathrm{Re}\langle u+v, (\bar{\partial} r)(z) \rangle + O(|u+v|^2) \\ &= r(z) + 2t\mathrm{Re}\langle v, (\bar{\partial} r)(z) \rangle + O(-c^2 r(z)). \end{aligned}$$

Since $|v| < -cr(z)$ and since c is small, we obtain

$$\frac{|\langle g'(t), (\bar{\partial} r)(g(t)) \rangle|}{-r(g(t))} \le cM'.$$

Similarly, we have

$$\frac{\langle A(g(t))g'(t), g'(t) \rangle}{-r(g(t))} = \frac{\langle A(g(t))(u+v), u+v \rangle}{-r(g(t))} \le c^2 M''.$$

Combining these two inequalities with (2.5) and (2.4), we see that the smallness of c ensures $d(z, z+u+v) < a$. This proves the first inclusion in the proposition.

It is easy to see that the second inclusion, $D(z,a) \subset \cdots$, is simply a consequence of Lemma 2.2. This completes the proof. □

Proposition 2.5. *There is a $0 < C_{2.5} < \infty$ such that if $0 < a < 1/2$, then*

$$D(z,a) \subset z + \mathcal{P}((\bar{\partial} r)(z); C_{2.5} a\sqrt{-r(z)}, -C_{2.5} ar(z))$$

for every $z \in \Omega$ satisfying the condition $-r(z) < \theta$.

Proof. Suppose that $0 < a < 1/2$ and that $z, w \in \Omega$ satisfy the condition $d(z,w) < a$. Then there is a C^1 map $g : [0,1] \to \Omega$ with $g(0) = z$ and $g(1) = w$ such that

$$\int_0^1 \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle}dt < 2a.$$

Thus $d(z, g(t)) < 2a < 1$ for every $t \in [0,1]$. Suppose that $w = z + u + v$ with $u \perp (\bar{\partial} r)(z)$ and $v \in \{\xi(\bar{\partial} r)(z) : \xi \in \mathbf{C}\}$. To estimate $|u|$, we again apply Lemma 2.1, which gives us $-r(z) \geq -c(1)r(g(t))$ for every $t \in [0,1]$. Since $|u| \leq |z - w|$, we have

$$|u| \leq \int_0^1 |g'(t)|dt \leq C_3\sqrt{\frac{-r(z)}{c(1)}}\int_0^1 \sqrt{\langle \mathcal{B}(g(t))g'(t), g'(t)\rangle}dt < C_3\sqrt{\frac{-r(z)}{c(1)}} \cdot 2a. \tag{2.8}$$

To estimate $|v|$, we apply Lemma 2.2. Since $0 < a < 1/2$, Lemma 2.2 gives us

$$|\langle v, (\bar{\partial} r)(z)\rangle| \leq C_{2.2}d(z,w)(3/2)2^6(-r(z)). \tag{2.9}$$

Recall that θ was chosen so that $(\bar{\partial} r)(\zeta) \neq 0$ whenever $0 < -r(\zeta) < 3\theta$. Hence the proposition follows from (2.8) and (2.9). □

On the domain Ω we define the measure

$$d\mu(z) = \frac{dv(z)}{(-r(z))^{n+1}}. \tag{2.10}$$

Proposition 2.6. *For each $a \in (0, \infty)$, there are $0 < c(a) \leq C(a) < \infty$ such that*

$$c(a) \leq \mu(D(z,a)) \leq C(a)$$

for every $z \in \Omega$.

Proof. Since $v(\mathcal{P}(\eta; x, y)) = C_n x^{2n-2}y^2$, this follows immediately from Proposition 2.4 and Lemma 2.1. □

For each $0 \leq \rho < \theta$, define the surface

$$S_\rho = \{z \in \mathbf{C}^n : -r(z) = \rho\}.$$

In particular, we have $S_0 = \partial\Omega$, the boundary of the domain Ω.

Proposition 2.7. *There exist a finite open cover $U_1, \ldots, U_m$* of

$$H = \{z \in \mathbf{C}^n : 0 \leq -r(z) \leq \theta/2\}$$

in $\mathbf{C}^n$ and a $1 \leq C < \infty$ such that the following holds true: Suppose that $0 < \rho \leq \theta/2$ and that $z, w \in S_\rho \cap U_i$ for some $i \in \{1, \ldots, m\}$. Furthermore, suppose that there is an $R \geq 1$ such that $|z - w| \leq R\sqrt{\rho}$ and $|\langle z - w, (\bar{\partial} r)(z)\rangle| \leq R^2\rho$. Then $d(z,w) \leq CR^2$.

Proof. For $\zeta \in \mathbf{C}^n$ and $a > 0$, denote $B(\zeta, a) = \{\xi \in \mathbf{C}^n : |\zeta - \xi| < a\}$ as usual. Note that by assumption, H is a compact set on which $|\nabla r|$ does not vanish. By the usual open covering argument, there is a $\tau > 0$ such that if $z_0 \in H$, then the conclusion of the standard implicit function theorem holds on $B(z_0, \tau)$ for the equation $r = r(z_0)$. See, e.g., [17,page 74]. Since H is compact, there are $z_1, \ldots, z_m \in H$ such that $\cup_{i=1}^m B(z_i, \tau/2) \supset H$. We define $U_i = B(z_i, \tau/2)$, $i = 1, \ldots, m$.

Now let $0 < \rho \le \theta/2$, and let $z, w \in S_\rho \cap U_i$ satisfy the conditions $|z - w| \le R\sqrt{\rho}$ and $|\langle z - w, (\bar{\partial} r)(z)\rangle| \le R^2\rho$ for some $R \ge 1$. Then, of course, $|z - w| < \tau$. By the discussion in the first paragraph, every point in $S_\rho \cap B(z, \tau)$ can be expressed in the form

$$z + x + f_z(x),$$

where $x \in \mathbf{C}^n$ satisfies the conditions $\mathrm{Re}\langle x, (\bar{\partial} r)(z)\rangle = 0$ and $|x| < \tau'$, and where f_z satisfies the condition $|f_z(x)| \le C_1|x|^2$. Since the implicit function theorem provides bounds that are independent of the points in H, reducing the value of τ if necessary, we may assume that $|f_z(x)| \le (1/2)|x|$ when $|x + f_z(x)| < \tau$. Let $x_0 \in B(0, \tau')$ be such that $w = z + x_0 + f_z(x_0)$. Then $(1/2)|x_0| \le |x_0 + f_z(x_0)| = |z - w|$. Hence $|x_0| \le 2R\sqrt{\rho}$. We have

$$\begin{aligned} |\langle x_0, (\bar{\partial} r)(z)\rangle| &\le |\langle w - z, (\bar{\partial} r)(z)\rangle| + |\langle f_z(x_0), (\bar{\partial} r)(z)\rangle| \le R^2\rho + C_2|f_z(x_0)| \\ &\le R^2\rho + C_3|x_0|^2 \le R^2\rho + 4C_3R^2\rho = C_4R^2\rho. \end{aligned}$$

Now define the map $g : [0, 1] \to S_\rho$ by the formula

$$g(t) = z + tx_0 + f_z(tx_0),$$

$t \in [0, 1]$. We have $g'(t) = x_0 + (Df_z)(tx_0)x_0$, where Df_z is the derivative of f_z. Recalling (2.5), we have

$$\frac{1}{\rho}\langle A(g(t))g'(t), g'(t)\rangle \le \frac{C_5}{\rho}|x_0|^2 \le \frac{C_5}{\rho}4R^2\rho = C_6R^2.$$

Therefore

$$\int_0^1 \left(\frac{\langle A(g(t))g'(t), g'(t)\rangle}{-r(g(t))}\right)^{1/2} dt \le \sqrt{C_6}R = C_7R. \tag{2.11}$$

Note that the condition $|f_z(x)| \le C_1|x|^2$ implies that $(Df_z)(0) = 0$. Hence

$$g'(t) = x_0 + \{(Df_z)(tx_0) - (Df_z)(0)\}x_0 = x_0 + h(t)$$

with $|h(t)| \le C_8|x_0|^2 \le 4C_8R^2\rho = C_9R^2\rho$. Consequently

$$\begin{aligned} |\langle g'(t), (\bar{\partial} r)(g(t))\rangle| &\le |\langle g'(t), (\bar{\partial} r)(z)\rangle| + |\langle g'(t), (\bar{\partial} r)(g(t)) - (\bar{\partial} r)(z)\rangle| \\ &\le |\langle x_0, (\bar{\partial} r)(z)\rangle| + |\langle h(t), (\bar{\partial} r)(z)\rangle| + C_{11}|g'(t)||g(t) - z| \\ &\le C_4R^2\rho + C_{10}C_9R^2\rho + C_{12}|x_0|^2 \le C_{13}R^2\rho. \end{aligned}$$

Thus we have

$$\int_0^1 \frac{|\langle g'(t), (\bar\partial r)(g(t))\rangle|}{-r(g(t))} dt \le C_{13}R^2.$$

Recalling (2.5), (2.4) and combining the above with (2.11), we find that $d(z,w) \le C_7 R + C_{13}R^2$. Since we assume $R \ge 1$, it follows that $d(z,w) \le (C_7 + C_{11})R^2$. $\square$

For each $0 \le \rho \le \theta/2$, we write $d\sigma_\rho$ for the natural surface measure on S_ρ. For every triple of $0 \le \rho \le \theta/2$, $\zeta \in S_\rho$ and $t > 0$, we define

$$Q_\rho(\zeta, t) = \{\xi \in S_\rho : |\zeta - \xi|^2 + |\langle \zeta - \xi, (\bar\partial r)(\zeta)\rangle| < t\}.$$

Proposition 2.8. *There are constants* $0 < \tau \le \theta/2$ *and* $0 < c_{2.8} \le C_{2.8} < \infty$ *such that*

$$c_{2.8}t^n \le \sigma_\rho(Q_\rho(\zeta, t)) \le C_{2.8}t^n \tag{2.12}$$

for all $0 \le \rho \le \tau$, $\zeta \in S_\rho$ *and* $0 < t \le T_0$, *where* $T_0 = \sup\{|u - v|^2 + |\langle u - v, (\bar\partial r)(u)\rangle| : u, v \in \Omega\}$.

Proof. First, we remark that the restriction $t \le T_0$ is only necessary to guarantee the *lower bound* in (2.12). Second, adjusting the constants $c_{2.8}$ and $C_{2.8}$ if necessary, it suffices to find an $a \in (0, T_0]$ such that (2.12) holds for all $0 < t < a$.

For each $\zeta \in S_\rho$, $0 \le \rho \le \theta/2$, denote $T_\zeta = \{x \in \mathbf{C}^n : \mathrm{Re}\langle x, (\bar\partial r)(\zeta)\rangle = 0\}$, which is the real tangent space to S_ρ at ζ. For each pair of $\zeta \in S_\rho$ and $s > 0$, define $E_\zeta(s) = \{x \in T_\zeta : |x|^2 + |\langle x, (\bar\partial r)(\zeta)\rangle| < s\}$. Each $x \in T_\zeta$ has the decomposition $x = y + z$, where $\langle y, (\bar\partial r)(\zeta)\rangle = 0$ and $z \in \{w(\bar\partial r)(\zeta) : w \in \mathbf{C}\}$. Since $\mathrm{Re}\langle x, (\bar\partial r)(\zeta)\rangle = 0$, we have $z = ih(\bar\partial r)(\zeta)$ for some $h \in \mathbf{R}$. Let v_{2n-1} denote the real $(2n-1)$-dimensional volume measure on T_ζ. Using this $x = y + z$ decomposition, it is elementary that there are $0 < c_1 \le C_1 < \infty$ such that

$$c_1 s^n \le v_{2n-1}(E_\zeta(s)) \le C_1 s^n \tag{2.13}$$

for all $\zeta \in S_\rho$, $0 \le \rho \le \theta/2$, and $s > 0$.

As in the proof of Proposition 2.7, we apply the standard implicit function theorem. There is a $0 < \tau \le \theta/2$ such that the conclusion of the implicit function theorem holds on $\{z \in \mathbf{C}^n : 0 \le -r(z) \le \tau\}$ for r with uniform bounds. Namely, there are constants $b > 0$, $0 < c \le 1$ and $0 < C_2 < \infty$ such that if $0 \le \rho \le \tau$ and $\zeta \in S_\rho$, then every element in $B(\zeta, b) \cap S_\rho$ can be expressed in the form

$$\Phi_\zeta(x) = \zeta + x + f_\zeta(x)$$

for some $x \in T_\zeta \cap B(0, c)$, where f_ζ satisfies the conditions $|f_\zeta(x)| \le C_2|x|^2$ and $|f_\zeta(x)| \le (1/2)|x|$ when $|x| < c$. Furthermore, there are constants $0 < c_3 \le C_3 < \infty$ such that the matrix inequality $c_3 \le (D\Phi_\zeta(x))^* D\Phi_\zeta(x) \le C_3$ holds whenever $0 \le -r(\zeta) \le \tau$ and $x \in T_\zeta \cap B(0, c)$, where $D\Phi_\zeta$ is the derivative of Φ_ζ, which is a $2n \times (2n-1)$ real matrix.

Now take $a_1 = b^2$. Let $0 \le \rho \le \tau$ and $0 < t < a_1$. Then for any pair of $\zeta \in S_\rho$ and $\xi \in Q_\rho(\zeta, t)$, we have $\xi \in B(\zeta, b)$. Therefore we can write

$$\xi = \zeta + x + f_\zeta(x)$$

for some $x \in T_\zeta \cap B(0, c)$. We have $|\xi - \zeta| = |x + f_\zeta(x)| \ge (1/2)|x|$ and $\langle x, (\bar\partial r)(\zeta)\rangle = \langle \xi - \zeta, (\bar\partial r)(\zeta)\rangle - \langle f_\zeta(x), (\bar\partial r)(\zeta)\rangle$ with $|f_\zeta(x)| \le C_2|x|^2$. Hence

$$|x|^2 + |\langle x, (\bar\partial r)(\zeta)\rangle| \le C_4\{|\xi - \zeta|^2 + |\langle \xi - \zeta, (\bar\partial r)(\zeta)\rangle|\} \tag{2.14}$$

for some constant $1 \le C_4 < \infty$. Similarly, $|\xi - \zeta| = |x + f_\zeta(x)| \le (1 + C_3)|x|$ and $\langle \xi - \zeta, (\bar\partial r)(\zeta)\rangle = \langle x, (\bar\partial r)(\zeta)\rangle + \langle f_\zeta(x), (\bar\partial r)(\zeta)\rangle$ with $|f_\zeta(x)| \le C_2|x|^2$. Consequently

$$|\xi - \zeta|^2 + |\langle \xi - \zeta, (\bar\partial r)(\zeta)\rangle| \le C_5\{|x|^2 + |\langle x, (\bar\partial r)(\zeta)\rangle|\} \tag{2.15}$$

for some constant $1 \le C_5 < \infty$. Set $a_2 = c^2/C_4$ and $a = \min\{a_1, a_2, T_0\}$. If $0 < t < a$, then $E_\zeta(C_4 t) \subset T_\zeta \cap B(0, c)$. Thus (2.14) implies that for $0 < t < a$, we have

$$\Phi_\zeta(E_\zeta(C_4 t)) \supset Q_\rho(\zeta, t).$$

Combining the smoothness of Φ_ζ on $T_\zeta \cap B(0, c)$ with the upper bound in (2.13), we obtain

$$\sigma_\rho(Q_\rho(\zeta, t)) \le \sigma_\rho(\Phi_\zeta(E_\zeta(C_4 t))) \le C_6 v_{2n-2}(E_\zeta(C_4 t)) \le C_6 C_1 (C_4 t)^n = C_7 t^n,$$

which gives us the upper bound in (2.12). Similarly, since $a < c^2$ and $C_5 \ge 1$, for $0 < t < a$ we have $E_\zeta(t/C_5) \subset T_\zeta \cap B(0, c)$. Therefore for $0 < t < a$, (2.15) implies

$$\Phi_\zeta(E_\zeta(t/C_5)) \subset Q_\rho(\zeta, t).$$

From the non-singularity of Φ_ζ on $T_\zeta \cap B(0, c)$ and the lower bound in (2.13) we obtain

$$c_1(t/C_5)^n \le v_{2n-1}(E_\zeta(t/C_5)) \le C_8 \sigma_\rho(\Phi_\zeta(E_\zeta(t/C_5))) \le C_8 \sigma_\rho(Q_\rho(\zeta, t)),$$

proving the lower bound in (2.12). This completes the proof. □

Proposition 2.9. *There is a constant $0 < C_{2.9} < \infty$ such that the following holds true: Let $z \in \Omega$, $k \in \mathbf{Z}$ and $j \in \mathbf{Z}_+$. Then the volume of the set*

$$\begin{aligned} W_{z;k,j} = \{w \in \Omega : \ & 2^{k-1}(-r(z)) < -r(w) \le 2^k(-r(z)) \\ & \text{and } \ |z - w|^2 + |\langle z - w, (\bar\partial r)(z)\rangle| \le 2^{k+j}(-r(z))\} \end{aligned}$$

does not exceed $C_{2.9} 2^{nj}(-2^k r(z))^{n+1}$.

Proof. First of all, there is a C_1 such that $|(\bar\partial r)(\zeta) - (\bar\partial r)(\zeta')| \le C_1|\zeta - \zeta'|$ for all $\zeta, \zeta' \in \Omega$. Suppose that $2^k(-r(z)) \le \tau$, where τ is the same as in Proposition 2.8. For any value $2^{k-1}(-r(z)) < \rho \le 2^k(-r(z))$, denote

$$\Sigma(\rho; k, j) = \{w \in S_\rho : |z - w|^2 + |\langle z - w, (\bar\partial r)(z)\rangle| \le 2^{k+j}(-r(z))\}.$$

Suppose that $\Sigma(\rho; k, j) \neq \emptyset$, and pick a $w_\rho \in \Sigma(\rho; k, j)$. Then elementary estimates yield

$$\Sigma(\rho; k, j) \subset Q_\rho(w_\rho, C_2 2^{k+j}(-r(z))).$$

Applying Proposition 2.8, we obtain

$$\sigma_\rho(\Sigma(\rho; k, j)) \leq \sigma_\rho(Q_\rho(w_\rho, C_2 2^{k+j}(-r(z)))) \leq C_3(-2^{k+j} r(z))^n \tag{2.16}$$

for every $2^{k-1}(-r(z)) < \rho \leq 2^k(-r(z))$ under the condition $2^k(-r(z)) \leq \tau$.

Now consider Ω as a domain in the real space $\mathbf{R}^{2n}$ under the usual identification. We know that $(\nabla r)(x) \neq 0$ for every $x \in \partial\Omega$. For each $a \in \partial\Omega$, there is a $j = j(a) \in \{1, 2, \ldots, 2n-1, 2n\}$ such that the map

$$F_a(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = (x_1, \ldots, x_{j-1}, -r(x_1, x_2, \ldots, x_{2n-1}, x_{2n}), x_{j+1}, \ldots, x_{2n})$$

from Ω to $\mathbf{R}^{2n}$ has the property that the derivative $(DF_a)(a)$ is invertible. Thus there is an open neighborhood U_a of a in $\mathbf{R}^{2n}$ such that the inverse mapping theorem holds on U_a for F_a. Shrinking U_a slightly if necessary, we may assume that DF_a^{-1} is bounded on $F_a U_a$. By the compactness of $\partial\Omega$, there is a finite subset $A \subset \partial\Omega$ and a $0 < \tau_1 \leq \tau$ such that

$$\bigcup_{a \in A} U_a \supset \{w \in \Omega : 0 < -r(w) \leq \tau_1\}. \tag{2.17}$$

To complete the proof of the proposition, consider the following two cases.

(1) Suppose that $-2^k r(z) \geq \tau_1$. Then the conclusion of the proposition is trivial. (2) Suppose that $-2^k r(z) < \tau_1$. In this case (2.17) gives us

$$W_{z;k,j} = \bigcup_{a \in A} \{U_a \cap W_{z;k,j}\}.$$

Since A is an *a priori* determined finite set, it suffices to estimate the volume of $U_a \cap W_{z;k,j}$ for each $a \in A$. Given any $a \in A$, there is a $j = j(a)$ such that for $0 < \rho < \tau_1$,

$$S_\rho \cap U_a = \{F_a^{-1}(x_1, \ldots, x_{j-1}, \rho, x_{j+1}, \ldots, x_{2n}) : (x_1, \ldots, x_{j-1}, \rho, x_{j+1}, \ldots, x_{2n}) \in F_a U_a\}.$$

By (2.16), for each $0 < \rho < \tau_1$, the real $(2n-1)$-dimensional volume of the set

$$\{(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n}) : (x_1, \ldots, x_{j-1}, \rho, x_{j+1}, \ldots, x_{2n}) \in F_a(U_a \cap \Sigma(\rho; k, j))\}$$

does not exceed $C_9(-2^{k+j} r(z))^n$. Hence we have the (real) $2n$-dimensional volume estimate

$$v(F_a(U_a \cap W_{z;k,j})) \leq C_9(-2^{k+j} r(z))^n \cdot 2^k(-r(z)).$$

Applying F_a^{-1}, we find that

$$v(U_a \cap W_{z;k,j}) \leq C_{10} 2^{nj}(-2^k r(z))^{n+1}.$$

Since $\text{card}(A) < \infty$, this completes the proof. □

Definition 2.10. (i) Let a be a positive number. A subset Γ of Ω is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) A subset Γ of Ω is simply said to be separated if it is a-separated for some $a > 0$.

Lemma 2.11. (1) *For any pair of* $0 < a < \infty$ *and* $0 < R < \infty$, *there is a natural number* $N = N(a, R)$ *such that for every* a*-separated set* Γ *in* Ω *and every* $z \in \Omega$, *we have*

$$\text{card}\{u \in \Gamma : d(u, z) \leq R\} \leq N.$$

(2) *For any pair of* $0 < a \leq R < \infty$, *there is a natural number* $m = m(a, R)$ *such that every* a*-separated set* Γ in Ω *admits a partition* $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ *with the property that for every* $j \in \{1, \ldots, m\}$, *the set* Γ_j *is* R*-separated.*

Proof. By Proposition 2.6, any integer $N \geq C(R + a)/c(a)$ will do for (1). Then, by(1), for any $0 < a \leq R < \infty$, there is an $m \in \mathbf{N}$ such that if Γ is any a-separated set in Ω, then $\text{card}\{u \in \Gamma : d(u, v) \leq 2R\} \leq m$ for every $v \in \Gamma$. By a standard maximality argument, Γ admits a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $j \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_j$ and $u \neq v$ imply $d(u, v) > 2R$. Thus each Γ_j is R-separated, proving (2). □

3. Forelli-Rudin estimates on Ω

We will need the familiar functions

$$X(z, w) = -r(w) - \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(z_j - w_j) - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r(w)}{\partial w_j \partial w_k}(z_j - w_j)(z_k - w_k), \tag{3.1}$$

$$\rho(z, w) = |z - w|^2 + |\langle z - w, (\bar{\partial} r)(z)\rangle|$$

and

$$F(z, w) = |r(z)| + |r(w)| + \rho(z, w)$$

associated with Ω and r, which are standard fixtures on strongly pseudo-convex domains.

Lemma 3.1. [18,20] *There is a* $\delta > 0$ *such that*

$$|X(z, w)| \approx |r(z)| + |r(w)| + |\text{Im}X(z, w)| + |z - w|^2 \approx F(z, w)$$

in the region $\mathcal{R}_\delta = \{(z, w) \in \Omega \times \Omega : |r(z)| + |r(w)| + |z - w| < \delta\}$.

Below is what one usually refers to as the Forelli-Rudin estimates:

Lemma 3.2. [18,20] *Let* $a \in \mathbf{R}$ *and* $\kappa > -1$. *Then for* $z \in \Omega$,

$$\int_\Omega \frac{|r(w)|^\kappa}{F(z, w)^{n+1+\kappa+a}} dv(w) \approx \begin{cases} 1 & \text{if } a < 0 \\ \log\{|r(z)|^{-1}\} & \text{if } a = 0 \\ |r(z)|^{-a} & \text{if } a > 0 \end{cases}.$$

Recall that for any $z \in \overline{\Omega}$ with $0 \le -r(z) \le \theta$, we have $(\bar{\partial} r)(z) \ne 0$ as a vector in $\mathbf{C}^n$.

Definition 3.3. For $z \in \Omega$ satisfying the condition $0 < -r(z) < \theta$, let u_z denote the unit vector $(\bar{\partial} r)(z)/|(\bar{\partial} r)(z)|$ in $\mathbf{C}^n$.

Lemma 3.4. *There exist constants $\delta_0 > 0$ and $0 < C_{3.4} < \infty$ such that if $z \in \Omega$ and $i \in \mathbf{Z}_+$ satisfy the condition $-2^{i+1}r(z) < \delta_0$, then for every $x \in [1, 2]$ we have*

$$d(z + 2^i r(z)u_z, z + x2^i r(z)u_z) \le C_{3.4}. \tag{3.2}$$

Moreover, if $-r(z) < \delta_0$, then $d(z, z + su_z) \le C_{3.4}$ for every $s \in [r(z), 0]$.

Proof. Let $z \in \Omega$ be such that $0 < -r(z) < \theta$. Then

$$\frac{d}{dt} r(z + tu_z) = 2\mathrm{Re}\langle u_z, (\bar{\partial} r)(z + tu_z)\rangle = 2|(\bar{\partial} r)(z)| + O(|t|).$$

Thus there is a $0 < \delta_0 < \theta$ such that if $-r(z) < \delta_0$, then

$$\text{the function } \ t \mapsto r(z + tu_z) \ \text{ is increasing on } \ [-\delta_0, \delta_0].$$

Now let $z \in \Omega$ and $i \in \mathbf{Z}_+$ be such that $-2^{i+1}r(z) < \delta_0$. Let $x \in [1, 2]$. Then for any $s \le s'$ in the interval $[x2^i r(z), 2^i r(z)]$, the above monotonicity guarantees $r(z + su_z) \le r(z + s'u_z)$, i.e., $-r(z + s'u_z) \le -r(z + su_z)$. For such a pair of s and s', it follows from (2.3), (2.4) and the above monotonicity that

$$d(z + s'u_z, z + su_z) \le C\frac{|(z + s'u_z) - (z + su_z)|}{-r(z + s'u_z)} \le C\frac{| - 2^{i+1}r(z) - 2^i(-r(z))|}{-2^i r(z)} = C,$$

which proves (3.2). Similarly, if $-r(z) < \delta_0$, then for every $s \in [r(z), 0]$ we have $r(z+su_z) \le r(z)$, i.e., $-r(z + su_z) \ge -r(z)$. Hence the same argument shows that $d(z, z + su_z) \le C|z - (z + su_z)|/(-r(z)) \le C$. This proves the lemma. □

Lemma 3.5. *There exist constants $0 < c_{3.5} \le 1$ and $0 < \delta_1 \le \delta_0$, where δ_0 was given in Lemma* 3.4, *such that if $z \in \Omega$ satisfies the condition $-r(z) < \delta_1$ and if $-\delta_1 \le t \le 0$, then*

$$-r(z + tu_z) + r(z) \ge c_{3.5}|t|.$$

Proof. Taylor expansion gives us

$$r(z + tu_z) = r(z) + 2t\mathrm{Re}\langle u_z, (\bar{\partial} r)(z)\rangle + O(t^2) = r(z) + 2t|(\bar{\partial} r)(z)| + O(t^2).$$

In other words, $r(z + tu_z) - r(z) = \{2|(\bar{\partial} r)(z)| + O(t)\}t$. From this the desired conclusion becomes obvious. □

Proposition 3.6. *There is a constant $0 < C_{3.6} < \infty$ such that if $z, w \in \Omega$ satisfy the conditions $r(z) = r(w)$ and $|z - w|^2 + |\langle z - w, (\bar{\partial} r)(z)\rangle| \le -2^j r(z)$, $j \in \mathbf{Z}_+$, then $d(z, w) \le C_{3.6}(1 + j)$.*

Proof. Recall the sets $U_1, \dots, U_m$ from Proposition 2.7, which are an open cover of $H = \{\zeta \in \mathbf{C}^n : 0 \le -r(\zeta) \le \theta/2\}$ in $\mathbf{C}^n$. By general topology, there is an $a_1 > 0$ such that for any pair of $z, w \in H$, if $|z - w| < a_1$, then there is an $i(z, w) \in \{1, \dots, m\}$ such that $z, w \in U_{i(z,w)}$. Another elementary exercise gives us a pair of constants $0 < \theta_0 < \theta/2$ and $0 < a < \min\{1, a_1/4\}$ which have the following property: Suppose that $z, w \in \{\zeta \in \Omega : -r(\zeta) < \theta_0\}$ and that $z', w' \in \Omega$. If the inequalities $|z - w| < a$, $|z - z'| < a$ and $|w - w'| < a$ hold, then there is an $i^* \in \{1, \dots, m\}$ such that $z', w' \in U_{i^*}$.

Define $\delta = \min\{\theta_0, a^2, \delta_1/2\}$, where δ_1 is given in Lemma 3.5. We divide the rest of the proof into two cases.

(1) Suppose that $-2^j r(z) < \delta c_{3.5}$, where $c_{3.5}$ is also from Lemma 3.5. Then $|z-w|^2 < \delta$, which implies $|z - w| < a$. We set $s = 2^j r(z)/c_{3.5}$. Since $-\delta_1 < s < 0$, by Lemma 3.5,

$$-r(z + s u_z) \ge c_{3.5}|s| = -2^j r(z).$$

Since $j \ge 0$, there is an $s(z) \in [s, 0]$ such that $-r(z + s(z)u_z) = -2^j r(z)$. We set $z' = z + s(z)u_z$. Then $r(z') = 2^j r(z)$ and $|z - z'| = |s(z)| \le |s| < \delta < a$. Since $|s| < \delta \le \delta_0/2$, it follows from Lemma 3.4 that $d(z, z') \le C_1(1 + j)$.

Similarly, since $r(w) = r(z)$, there is an $s(w) \in [s, 0]$ such that if we set $w' = w + s(w)u_w$, then $r(w') = 2^j r(z)$, $|w - w'| \le |s| < a$ and $d(w, w') \le C_1(1 + j)$.

Since $\bar\partial r$ satisfies a Lipschitz condition on Ω, from the conditions

$$|z - w|^2 + |\langle z - w, (\bar\partial r)(z)\rangle| \le -2^j r(z),$$

$|z - z'| \le -2^j r(z)/c_{3.5}$, $|w - w'| \le -2^j r(z)/c_{3.5}$ and $-2^j r(z) < \delta c_{3.5}$ it is easy to deduce

$$|z' - w'|^2 + |\langle z' - w', (\bar\partial r)(z')\rangle| \le -C_2 2^j r(z) = C_2|r(z')|.$$

Since $|z - z'| < a$, $|w - w'| < a$ and $|z - w| < a$, by the first paragraph, there is an $i^* \in \{1, \dots, m\}$ such that $z', w' \in U_{i^*}$. Hence it follows from the above bound and Proposition 2.7 that $d(z', w') \le C_3$. Combining this with the last two paragraphs, we obtain

$$d(z, w) \le d(z, z') + d(z', w') + d(w, w') \le 2C_1(1 + j) + C_3 \le C_4(1 + j).$$

This proves the proposition under the condition $-2^j r(z) < \delta c_{3.5}$.

(2) Suppose that $-2^j r(z) \ge \delta c_{3.5}$. (2.a) Further, suppose that $-r(z) \ge \delta c_{3.5}/2$. In this case the conclusion is trivial, for $\{\zeta \in \mathbf{C}^n : -r(\zeta) \ge \delta c_{3.5}/2\}$ is a compact subset of Ω. (2.b) Suppose that $-r(z) < \delta c_{3.5}/2$, i.e., $-2r(z) < \delta c_{3.5}$. Let j_0 be the largest natural number such that $-2^{j_0} r(z) < \delta c_{3.5}$. Then obviously $j_0 < j$. By the work in case (1) we know that there are $\tilde z, \tilde w \in \Omega$ such that $r(\tilde z) = 2^{j_0} r(z) = r(\tilde w)$, $d(z, \tilde z) \le C_1(1 + j_0)$ and $d(w, \tilde w) \le C_1(1 + j_0)$. The choice of j_0 ensures that $-2^{j_0+1} r(z) \ge \delta c_{3.5}$, which means $-r(\tilde z) \ge \delta c_{3.5}/2$ and $-r(\tilde w) \ge \delta c_{3.5}/2$. Hence $d(\tilde z, \tilde w) \le C_5$, and consequently

$$d(z, w) \le d(z, \tilde z) + d(\tilde z, \tilde w) + d(w, \tilde w) \le 2C_1(1 + j_0) + C_5 \le C_6(1 + j)$$

in this subcase. This completes the proof. □

Lemma 3.7. *There is a* $1 \leq C_{3.7} < \infty$ *such that if* $z, w \in \Omega$ *satisfy the conditions* $2^{k-1}(-r(z)) \leq -r(w) \leq 2^k(-r(z))$ *and* $|z-w|^2 + |\langle z-w, (\bar{\partial}r)(z)\rangle| < 2^{k+j}(-r(z))$, *where* $k \in \mathbf{Z}$ *and* $j \in \mathbf{Z}_+$, *then* $d(z,w) < C_{3.7}(1+|k|+j)$.

Proof. (1) First, let us consider the case where $k \geq 1$. (1.a) Further, suppose that $2^k(-r(z)) \geq c_{3.5}\delta_1/4$, where $c_{3.5}$ and δ_1 are the constants in Lemma 3.5. Then the condition $2^{k-1}(-r(z)) \leq -r(w) \leq 2^k(-r(z))$ implies $-r(w) \geq c_{3.5}\delta_1/8$. If we also have $-r(z) \geq c_{3.5}\delta_1/4$, then of course, $d(z,w) \leq C_1$, regardless of other conditions. Suppose that $-r(z) < c_{3.5}\delta_1/4$. Let k' be the largest integer such that $-2^{k'+1}r(z) < c_{3.5}\delta_1$. Set

$$z' = z + 2^{k'+1}r(z)u_z.$$

Since $-2^k r(z) \geq c_{3.5}\delta_1/4$, we have $k'+1 < k+2$, i.e., $k' \leq k$. It follows from Lemma 3.4 that $d(z,z') \leq C_{3.4}(k'+2) \leq C_{3.4}(k+2)$. By Lemma 3.5, we have $c_{3.5}2^{k'+1}|r(z)| \leq -r(z')$. The choice of k' ensures that $-2^{k'+2}r(z) \geq c_{3.5}\delta_1$. Hence the above implies

$$c_{3.5}^2\delta_1/2 \leq -r(z').$$

Thus $d(z',w) \leq C_2$, and consequently $d(z,w) \leq C_{3.4}(k+2) + C_2 \leq C_3 k$ in this subcase.

(1.b) Suppose that $2^k(-r(z)) < c_{3.5}\delta_1/4$. Then $-r(w) \leq 2^k(-r(z)) < c_{3.5}\delta_1/4$. By Lemma 3.5, we have

$$c_{3.5}\delta_1 \leq -r(z - \delta_1 u_z).$$

Hence $-r(z-\delta_1 u_z) > -r(w)$. Since $-r(z) \leq 2^{k-1}(-r(z)) \leq -r(w)$, there is an $s \in [-\delta_1, 0]$ such that $r(z+su_z) = r(w)$. Also, Lemma 3.5 tells us that

$$c_{3.5}|s| \leq -r(z+su_z) = -r(w) \leq -2^k r(z). \tag{3.3}$$

Thus $|s| \leq -c_{3.5}^{-1}2^k r(z)$. Now the condition $2^k(-r(z)) < c_{3.5}\delta_1/4$ implies $-c_{3.5}^{-1}2^k r(z) \leq \delta_1/4$. Therefore it follows from Lemma 3.4 and the inequality $|s| \leq -c_{3.5}^{-1}2^k r(z)$ that

$$d(z, z+su_z) \leq C_{3.4}\{1 + C_3 \log(2^k/c_{3.5})\} \leq C_4 k.$$

Thus what remains for this subcase is to show that

$$d(z+su_z, w) \leq C_5(1+j).$$

For convenience, let us denote $\zeta = z+su_z$. Since $r(\zeta) = r(w)$, to prove the above inequality, by Proposition 3.6, it suffices to show that

$$|\zeta - w|^2 + |\langle \zeta - w, (\bar{\partial}r)(\zeta)\rangle| \leq C_6 2^j(-r(\zeta)). \tag{3.4}$$

By (3.3), $|\zeta - z| = |s| \leq c_{3.5}^{-1}2^k(-r(z)) \leq (2/c_{3.5})(-r(\zeta))$. Since Ω is bounded, we have $|\zeta - z|^2 \leq C_7|\zeta - z|$. Therefore

$$|\zeta - w|^2 \leq 2|\zeta - z|^2 + 2|z-w|^2 \leq C_8(-r(\zeta)) + 2^{k+j+1}(-r(z)) \leq C_9 2^j(-r(\zeta)).$$

We have $|\langle z-w, (\bar{\partial} r)(z)\rangle| < 2^{k+j}(-r(z))$ by assumption. Also,

$$|\langle \zeta - w, (\bar{\partial} r)(\zeta)\rangle - \langle z-w, (\bar{\partial} r)(z)\rangle| \le C_{10}|\zeta - z| \le C_{11}(-r(\zeta)).$$

Since $2^{k-1}(-r(z)) \le -r(\zeta)$, these inequalities prove (3.4). Thus the case $k \ge 1$ is proved.

(2) Now suppose that $k \le 0$. Note that the condition $2^{k-1}(-r(z)) \le -r(w) \le 2^k(-r(z))$ implies that $2^{-k}(-r(w)) \le -r(z) \le 2^{-k+1}(-r(w))$. Also, the condition $|z-w|^2 + |\langle z-w, (\bar{\partial} r)(z)\rangle| < 2^{k+j}(-r(z))$ can be rewritten as

$$|z-w|^2 + |\langle z-w, (\bar{\partial} r)(z)\rangle| < 2^{1+j}(-r(w)).$$

Since $|\langle z-w, (\bar{\partial} r)(z)\rangle - \langle z-w, (\bar{\partial} r)(w)\rangle| \le C_{12}|z-w|^2$, we now have

$$|z-w|^2 + |\langle z-w, (\bar{\partial} r)(w)\rangle| < C_{13}2^j(-r(w)).$$

Thus case (2) follows from case (1) by reversing the roles of z and w. □

We need the following "vanishing" version of Lemma 3.2.

Lemma 3.8. *Given any $a > 0$ and $\kappa > -1$, there are $0 < C < \infty$ and $s > 0$ such that*

$$\int_{\Omega \setminus D(z,R)} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \le C 2^{-sR}$$

for all $z \in \Omega$ and $R \ge 3C_{3.7}$, where $C_{3.7}$ is the constant in Lemma 3.7.

Proof. For $z \in \Omega$ and $k \in \mathbf{Z}$ we define the sets

$$\begin{aligned} Z_{z;k,0} = \{w \in \Omega : &\ 2^{k-1}(-r(z)) \le -r(w) < 2^k(-r(z)) \\ &\text{and} \ \ |z-w|^2 + |\langle z-w, (\bar{\partial} r)(z)\rangle| < 2^k(-r(z))\} \quad \text{and} \\ Z_{z;k,j} = \{w \in \Omega : &\ 2^{k-1}(-r(z)) \le -r(w) < 2^k(-r(z)) \\ &\text{and} \ \ 2^{k+j-1}(-r(z)) \le |z-w|^2 + |\langle z-w, (\bar{\partial} r)(z)\rangle| < 2^{k+j}(-r(z))\}, \quad j \ge 1. \end{aligned}$$

By the definition of $F(z,w)$, for all $k \ge 0$ and $j \ge 0$, if $w \in Z_{z;k,j}$, then

$$\frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} \le C_1 \frac{(2^k|r(z)|)^\kappa |r(z)|^a}{(2^{k+j}|r(z)|)^{n+1+\kappa+a}} = \frac{C_1}{2^{(n+1+a)k}2^{(n+1+\kappa+a)j}|r(z)|^{n+1}}.$$

In the case $k < 0$, $j \ge 0$ and $w \in Z_{z;k,j}$, we have

$$\frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} \le \frac{C_2(2^k|r(z)|)^\kappa |r(z)|^a}{(|r(z)| + 2^{k+j}|r(z)|)^{n+1+\kappa+a}} = \frac{C_2 2^{\kappa k}}{(1+2^{k+j})^{n+1+\kappa+a}|r(z)|^{n+1}}.$$

By Proposition 2.9, $v(Z_{z;k,j}) \le C_3 2^{nj}(-2^k r(z))^{n+1}$. Thus if $k \ge 0$ and $j \ge 0$, then

$$\int_{Z_{z;k,j}} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \le \frac{C_1 C_3 2^{nj}(-2^k r(z))^{n+1}}{2^{(n+1+a)k}2^{(n+1+\kappa+a)j}|r(z)|^{n+1}} = \frac{C_4}{2^{ak}2^{(1+\kappa+a)j}}. \tag{3.5}$$

Similarly, in the case $k < 0$ and $j \geq 0$, we have

$$(3.6)\quad \int_{Z_{z;k,j}} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \leq \frac{C_2 2^{\kappa k} C_3 2^{nj}(-2^k r(z))^{n+1}}{(1+2^{k+j})^{n+1+\kappa+a}|r(z)|^{n+1}} = \frac{C_5 2^{n(k+j)} 2^{(1+\kappa)k}}{(1+2^{k+j})^{n+1+\kappa+a}}$$

and $2^{(1+\kappa)k} = 2^{(1+\kappa)k/2} \cdot 2^{(1+\kappa)(k+j)/2} \cdot 2^{-(1+\kappa)j/2}$. Let $R \geq 3C_{3.7}$. By Lemma 3.7, the condition $Z_{z;k,j} \backslash D(z,R) \neq \emptyset$ implies either $|k| \geq (2C_{3.7})^{-1}R$ or $j \geq (2C_{3.7})^{-1}R$. Therefore

$$\int_{\Omega \backslash D(z,R)} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \leq \sum_{(k,j)\in E(R)} \int_{Z_{z;k,j}} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w),$$

where $E(R) = \{(k,j) \in \mathbf{Z} \times \mathbf{Z}_+ : \text{ either } |k| \geq (2C_{3.7})^{-1}R \text{ or } j \geq (2C_{3.7})^{-1}R\}$. Using (3.5) and (3.6), it is now elementary to verify that the lemma holds for every $0 < s < (2C_{3.7})^{-1}\min\{a, (1+\kappa)/2\}$. □

Lemma 3.9. *Given any $a > 0$ and $\kappa > -1$, there is a $0 < C < \infty$ such that*

$$\int_\Omega d(z,w) \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \leq C$$

for every $z \in \Omega$.

Proof. Given any $z \in \Omega$, define $E_0 = D(z, 3C_{3.7})$ and

$$E_i = D(z,(3+i)C_{3.7}) \backslash D(z,(3+i-1)C_{3.7})$$

for $i \geq 1$. For each $i \in \mathbf{Z}_+$, if $w \in E_i$, then $d(z,w) < (3+i)C_{3.7}$. Hence

$$\begin{aligned}\int_\Omega d(z,w) \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) &= \sum_{i=0}^\infty \int_{E_i} d(z,w) \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \\ &\leq \sum_{i=0}^\infty (3+i)C_{3.7} \int_{E_i} \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w).\end{aligned}$$

We now apply Lemma 3.2 to the term where $i = 0$ and Lemma 3.8 to the terms where $i \geq 1$. The result of this is

$$\int_\Omega d(z,w) \frac{|r(w)|^\kappa |r(z)|^a}{F(z,w)^{n+1+\kappa+a}} dv(w) \leq 3C_{3.7}C_0 + \sum_{i=1}^\infty (3+i)C_{3.7} C 2^{-s(3+i-1)C_{3.7}}.$$

Since Lemma 3.8 guarantees that $s > 0$, the right-hand side is finite. □

Lemma 3.10. *There exist constants $0 < a_0 < 1/2$ and $0 < C_{3.10} < \infty$ such that for any $z, z', w, w' \in \Omega$ satisfying the conditions $d(z,z') < a_0$ and $d(w,w') < a_0$, we have*

$$F(z,w) \leq C_{3.10} F(z',w').$$

Proof. By Lemma 2.1, it suffices to consider the case where $-r(\zeta) < \theta$ for every $\zeta \in \{z, z', w, w'\}$. Since $|(\bar{\partial} r)(z) - (\bar{\partial} r)(w)| \leq C_1|z-w|$ for all $z, w \in \Omega$, there is a C_2 such that

$$F(w,z) \leq C_2 F(z,w)$$

for all $z, w \in \Omega$. Therefore it suffices to find $0 < a_0 < 1/2$ and C such that

$$F(z,w) \leq CF(z',w) \tag{3.7}$$

for all $z, z', w \in \Omega$ satisfying the condition $d(z,z') < a_0$. Let $z, z' \in \Omega$ be such that $d(z,z') < a$ for some $0 < a < 1/2$. By Lemma 2.1, $-r(z) \leq (4/c_{2.1})(-r(z'))$. Hence, to prove (3.7), it suffices to consider the case where

$$\rho(z,w) = |z-w|^2 + |\langle z-w, (\bar{\partial} r)(z)\rangle| \geq -r(z). \tag{3.8}$$

Proposition 2.5 tells us that $z' = z + u + v$ with $u \perp (\bar{\partial} r)(z)$ and $v \in \{\xi(\bar{\partial} r)(z) : \xi \in \mathbf{C}\}$ satisfying the conditions $|u| \leq C_{2.5}a|r(z)|^{1/2}$ and $|v| \leq C_{2.5}a|r(z)|$. Therefore

$$|z-z'|^2 = |u|^2 + |v|^2 \leq C_3 a^2 |r(z)|.$$

By a simple completion of square, we find that

$$|z'-w|^2 \geq |z'-z|^2 - 2|z'-z||z-w| + |z-w|^2 \geq (1/2)|z-w|^2 - |z'-z|^2.$$

Also, $|\langle z-z', (\bar{\partial} r)(z)\rangle| = |\langle v, (\bar{\partial} r)(z)\rangle| \leq C_4 a|r(z)|$. Consequently

$$\begin{aligned} |\langle z'-w, (\bar{\partial} r)(z')\rangle| &\geq |\langle z'-w, (\bar{\partial} r)(z)\rangle| - C_1|z'-w||z-z'| \\ &\geq |\langle z-w, (\bar{\partial} r)(z)\rangle| - C_4 a|r(z)| - C_1|z-w||z-z'| - C_1|z-z'|^2. \end{aligned}$$

Combining the above inequalities, we see that there is a $1 \leq C_5 < \infty$ such that

$$\begin{aligned} \rho(z',w) &\geq \frac{1}{2}\rho(z,w) - C_5 a\{\rho^{1/2}(z,w)|r(z)|^{1/2} + |r(z)|\} \\ &= \frac{1}{2}(1-C_5 a)\rho(z,w) + \frac{1}{2}C_5 a\{\rho(z,w) - 2\rho^{1/2}(z,w)|r(z)|^{1/2} + |r(z)|\} - \frac{3}{2}C_5 a|r(z)|. \end{aligned}$$

Note that the $\{\cdots\}$ above is a complete square. Thus if $a \leq (2C_5)^{-1}$, then

$$\rho(z',w) \geq \frac{1}{4}\rho(z,w) - \frac{3}{2}C_5 a|r(z)|.$$

Recalling (3.8), if we further require that $a \leq (12C_5)^{-1}$, then

$$\rho(z',w) \geq (1/8)\rho(z,w).$$

Thus if we set $a_0 = (12C_5)^{-1}$, then there is a C such that (3.7) holds for all $z, z', w \in \Omega$ satisfying the condition $d(z,z') < a_0$. □

4. Estimates related to the Bergman kernel

Let $K(z,w)$ be the Bergman kernel for Ω. By definition, it has the symmetry $K(w,z) = \overline{K(z,w)}$. The following well-known result of Fefferman gives us a good handle on K:

Theorem 4.1. [10,Theorem 2] *The Bergman kernel has the form*

$$K(z,w) = C|(\nabla r)(w)|^2 \det\mathcal{L}(w) X^{-(n+1)}(z,w) + \tilde{K}(z,w)$$

on $\mathcal{R}_\delta = \{(z,w) \in \Omega \times \Omega : |r(z)| + |r(w)| + |z-w| < \delta\}$ *for some* $\delta > 0$, *where* $\mathcal{L}$ *is the Levi form for the domain* Ω, X *is given by* (3.1), *and* $\tilde{K}$ *is an admissible kernel of weight* $\geq -n-(1/2)$. *That is, there is a constant* C' *such that* $|\tilde{K}(z,w)| \leq C'F(z,w)^{-n-(1/2)}$.

For any $\delta > 0$, the Bergman kernel K is known to be bounded on $(\Omega\times\Omega)\backslash\mathcal{R}_\delta$ [14]. One obvious implication of Theorem 4.1 is that

$$c|r(z)|^{-n-1} \leq |K(z,z)| \leq C|r(z)|^{-n-1}, \quad z \in \Omega. \tag{4.1}$$

For each $z \in \Omega$, let us denote $K_z(w) = K(w,z)$. Then it has the reproducing property

$$h(z) = \langle h, K_z\rangle$$

for $h \in L^2_a(\Omega)$. We write k_z for the normalized reproducing kernel, i.e., $k_z = K_z/\|K_z\|$.

Lemma 4.2. *Given any* $0 < \eta < 1/2$ and $a > 0$, *there are constants* $s > 0$ *and* $0 < C_{4.2} < \infty$ *such that*

$$\sup_{z\in\Omega} |r(z)|^{-(n/2)-\eta} \sum_{w\in\Gamma\backslash D(z,R)} |r(w)|^{(n/2)+\eta}\left(\frac{|r(z)|^{1/2}|r(w)|^{1/2}}{F(z,w)}\right)^{n+1} \leq C_{4.2}2^{-sR}$$

for every a*-separated set* Γ *in* Ω *and every* $R \geq 3C_{3.7}+1$.

Proof. Let $0 < \eta < 1/2$ and $a > 0$ be given. Define $\alpha = (1/3)\min\{a_0, a\}$, where a_0 is the constants in Lemma 3.10. Suppose that Γ is an a-separated set in Ω. Then

$$D(w,\alpha) \cap D(w',\alpha) = \emptyset \quad \text{for all} \quad w \neq w' \quad \text{in} \quad \Gamma.$$

Applying Lemmas 2.1 and 3.10, for $\zeta \in D(w,\alpha)$ we have

$$\frac{|r(w)|^{n+(1/2)+\eta}|r(z)|^{(n+1)/2}}{F(z,w)^{n+1}} \leq C\frac{|r(\zeta)|^{n+(1/2)+\eta}|r(z)|^{(n+1)/2}}{F(z,\zeta)^{n+1}}.$$

Thus for $z \in \Omega$ we have

$$\begin{aligned}\sum_{w\in\Gamma\backslash D(z,R)} &|r(w)|^{(n/2)+\eta}\left(\frac{|r(z)|^{1/2}|r(w)|^{1/2}}{F(z,w)}\right)^{n+1}\\ &\leq \sum_{w\in\Gamma\backslash D(z,R)} \frac{C}{\mu(D(w,\alpha))}\int_{D(w,\alpha)} \frac{|r(\zeta)|^{n+(1/2)+\eta}|r(z)|^{(n+1)/2}}{F(z,\zeta)^{n+1}}d\mu(\zeta)\\ &\leq \frac{C}{c(\alpha)}\int_{\Omega\backslash D(z,R-\alpha)} \frac{|r(\zeta)|^{-(1/2)+\eta}|r(z)|^{(n+1)/2}}{F(z,\zeta)^{n+1}}dv(\zeta),\end{aligned}$$

where the second $\le$ is justified by Proposition 2.6. Applying Lemma 3.8 to the last integral, the desired conclusion follows. □

Lemma 4.3. *There is a constant* $0 < C_{4.3} < \infty$ *such that*

$$|f(z)| \le C_{4.3}|r(z)|^{-(n+1)/2}\|f\chi_{D(z,1)}\|$$

for all $f \in L^2_a(\Omega)$ *and* $z \in \Omega$, *where* $\|f\chi_{D(z,1)}\|$ *is the norm of* $f\chi_{D(z,1)}$ *in* $L^2(\Omega)$.

Proof. It is easy to see that the conclusion is trivial if $-r(z) \ge \theta$. Suppose that $-r(z) < \theta$. Then Proposition 2.4 provides a $c > 0$ such that

$$D(z,1) \supset z + \mathcal{P}((\bar{\partial}r)(z); c|r(z)|^{1/2}, c|r(z)|)$$

for every such z. Averaging on the polyball, for $f \in L^2_a(\Omega)$ we have

$$|f(z)| \le \frac{1}{v(\mathcal{P}((\bar{\partial}r)(z); c|r(z)|^{1/2}, c|r(z)|))}\int_{z+\mathcal{P}((\bar{\partial}r)(z);c|r(z)|^{1/2},c|r(z)|)}|f|dv.$$

Applying the Cauchy-Schwarz inequality on the right, the desired conclusion follows. □

Lemma 4.4. *Given any complex dimension* $m \in \mathbf{N}$, *there is a constant* $0 < C_{4.4}(m) < \infty$ *such that the following bound holds: Let* $0 < \rho < \infty$ *and define* $B(\rho) = \{z \in \mathbf{C}^m : |z| < \rho\}$. *Then for every* $u \in \mathbf{C}^m$ *with* $|u| < \rho/2$ *and every analytic function* f *on* $B(\rho)$, *we have*

$$|f(u) - f(0)| \le \frac{|u|}{\rho}\cdot\frac{C_{4.4}(m)}{v_m(B(\rho))}\int_{B(\rho)}|f|dv_m.$$

Proof. By standard integration formulas on the ball, there is a $C = C(m)$ such that

$$|(\partial_j g)(0)| \le \frac{C}{v_m(B(1))}\int_{B(1)}|g|dv_m \tag{4.2}$$

for every analytic function g on $B(1) = \{z \in \mathbf{C}^m : |z| < 1\}$ and every $j \in \{1, \dots, m\}$. Suppose that $u = (u_1, \dots, u_m)$. If f is analytic on $B(\rho)$, then

$$f(u) - f(0) = \int_0^1 \frac{d}{dt}f(tu)dt = \int_0^1 \sum_{j=1}^m (\partial_j f)(tu)u_j dt.$$

Since $|u| < \rho/2$, for every $t \in [0,1]$ we have $tu + B(\rho/2) \subset B(\rho)$. From (4.2) and the scaling properties of ∂_j and dv_m we deduce

$$|(\partial_j f)(tu)| \le \frac{2}{\rho}\cdot\frac{C}{v_m(B(\rho/2))}\int_{tu+B(\rho/2)}|f|dv_m.$$

Since $v_m(B(\rho/2)) = 2^{-2m}v_m(B(\rho))$, we see that the constant $C_{4.4}(m) = m2^{2m+1}C$ will do for the lemma. □

Lemma 4.5. *There exist constants $0 < C_{4.5} < \infty$ and $0 < c_{4.5} < 1$ such that*

$$|f(w) - f(z)| \le C_{4.5} d(z,w)|r(z)|^{-(n+1)/2}\|f\chi_{D(z,1)}\|$$

for every pair of $z, w \in \Omega$ satisfying the condition $d(z,w) < c_{4.5}$ and every $f \in L^2_a(\Omega)$.

Proof. By Lemma 2.1, there is a $0 < \theta_1 < \theta$ such that if $-r(z) \ge \theta$ and $d(z,w) \le 1$, then $-r(w) \ge \theta_1$. Since $\{\zeta \in \Omega : -r(\zeta) \ge \theta_1\}$ is a compact subset of Ω, we see that the case $-r(z) \ge \theta$ is trivial.

Suppose that $-r(z) < \theta$. Then Proposition 2.4 provides a $c > 0$ such that

$$D(z,1) \supset z + \mathcal{P}((\bar\partial r)(z); c|r(z)|^{1/2}, c|r(z)|) \tag{4.3}$$

for every such z. By Proposition 2.5, there is an $0 < \alpha < 1/2$ such that

$$D(z,\alpha) \subset z + \mathcal{P}((\bar\partial r)(z); (c/2)|r(z)|^{1/2}, (c/2)|r(z)|) \tag{4.4}$$

for every such z. Set $c_{4.5} = \alpha$. Let $w \in \Omega$ be such that $d(z,w) < \alpha$. Then we can write $w = z + x + y$, where $x \perp (\bar\partial r)(z)$ and $y \in \{\eta(\bar\partial r)(z) : \eta \in \mathbf{C}\}$. By (4.4), we have $|x| < (c/2)|r(z)|^{1/2}$ and $|y| < (c/2)|r(z)|$.

Let $f \in L^2_a(\Omega)$ be given. Define $F(\xi) = f(z+\xi+y)$ for $\xi \perp (\bar\partial r)(z)$ with $|\xi| < c|r(z)|^{1/2}$. Applying Lemma 4.4 to the case where $\rho = c|r(z)|^{1/2}$, we have

$$|f(w) - f(z+y)| = |F(x) - F(0)| \le \frac{|x|}{c|r(z)|^{1/2}} \cdot \frac{C_{4.4}(n-1)}{v_{n-1}(B)} \int_B |F| dv_{n-1}, \tag{4.5}$$

where $B = \{\xi \in \mathbf{C}^n : \langle \xi, (\bar\partial r)(z)\rangle = 0 \text{ and } |\xi| < c|r(z)|^{1/2}\}$. On the other hand, for every $\xi \in B$ we have

$$|F(\xi)| = |f(z+\xi+y)| \le \frac{1}{A(D((c/2)|r(z)|))} \int_{D((c/2)|r(z)|)} |f(z+\xi+y+\eta u_z)| dA(\eta), \tag{4.6}$$

where $u_z = (\bar\partial r)(z)/|(\bar\partial r)(z)|$ and $D((c/2)|r(z)|) = \{\eta \in \mathbf{C} : |\eta| < (c/2)|r(z)|\}$. Note that $v_{n-1}(B) = c_{n-1}(c|r(z)|^{1/2})^{2n-2} = a_1|r(z)|^{n-1}$. Combining (4.5), (4.6) and (4.3), we obtain

$$|f(w) - f(z+y)| \le \frac{|x|}{|r(z)|^{1/2}} \cdot \frac{C_1}{|r(z)|^{n+1}} \int_{D(z,1)} |f| dv.$$

Applying the Cauchy-Schwarz inequality and Proposition 2.6, we have

$$|f(w) - f(z+y)| \le \frac{|x|}{|r(z)|^{1/2}} \cdot \frac{C_1\sqrt{v(D(z,1))}}{|r(z)|^{n+1}} \|f\chi_{D(z,1)}\| \le \frac{|x|}{|r(z)|^{1/2}} \cdot \frac{C_2\|f\chi_{D(z,1)}\|}{|r(z)|^{(n+1)/2}}.$$

Since $d(z,w) < \alpha$ and $\alpha < 1/2$, (2.6) gives us that $|x|/|r(z)|^{1/2} \le C_3 d(z,w)$. Hence

$$|f(w) - f(z+y)| \le C_4 d(z,w)|r(z)|^{-(n+1)/2}\|f\chi_{D(z,1)}\|. \tag{4.7}$$

Next we show that

$$|f(z+y)-f(z)| \le C_5 d(z,w)|r(z)|^{-(n+1)/2}\|f\chi_{D(z,1)}\|, \tag{4.8}$$

which together with (4.7) will complete the proof of the lemma.

To prove (4.8), we write $y = \beta u_z$, where $\beta \in \mathbf{C}$. Define $G(\eta) = f(z+\eta u_z)$ for $\eta \in \mathbf{C}$ with $|\eta| < c|r(z)|$. Now, applying Lemma 4.4 to the case where $\rho = c|r(z)|$, we have

$$|f(z+y)-f(z)| = |G(\beta)-G(0)| \le \frac{|\beta|}{c|r(z)|}\cdot\frac{C_{4.4}(1)}{A(D(c|r(z)|))}\int_{D(c|r(z)|)}|G|dA, \tag{4.9}$$

where $D(c|r(z)|) = \{\eta \in \mathbf{C} : |\eta| < c|r(z)|\}$. For each $\eta \in D(c|r(z)|)$ we have

$$|G(\eta)| = |f(z+\eta u_z)| \le \frac{1}{v_{n-1}(B)}\int_B |f(z+\xi+\eta u_z)|dv_{n-1}(\xi), \tag{4.10}$$

where $B = \{\xi \in \mathbf{C}^n : \langle \xi, (\bar\partial r)(z)\rangle = 0 \text{ and } |\xi| < c|r(z)|^{1/2}\}$. Note that $|\beta| = |y|$. Thus (4.9), (4.10) and (4.3) together give us

$$|f(z+y)-f(z)| \le \frac{|y|}{|r(z)|}\cdot\frac{C_6}{|r(z)|^{n+1}}\int_{D(z,1)}|f|dv.$$

Since $d(z,w) < \alpha$ and $\alpha < 1/2$, Lemma 2.2 implies that $|y|/|r(z)| \le C_7 d(z,w)$. Applying the Cauchy-Schwarz inequality and Proposition 2.6 on the right-hand side, we obtain (4.8). This completes the proof. □

Proposition 4.6. *There is a constant $C_{4.6}$ such that if $z, w \in \Omega$ satisfies the condition $d(z,w) < c_{4.5}$, where $c_{4.5}$ was given in Lemma 4.5, then*

$$|\langle f, k_z - k_w\rangle| \le C_{4.6} d(z,w)\|f\chi_{D(z,1)}\|$$

for every $f \in L^2_a(\Omega)$. Consequently, if $d(z,w) < c_{4.5}$, then $\|k_z - k_w\| \le C_{4.6} d(z,w)$.

Proof. Write $K_z(\zeta) = K(\zeta, z)$, the unnormalized reproducing kernel. Note that Lemma 4.5 implies that $\|K_z - K_w\| \le C_{4.5} d(z,w)|r(z)|^{-(n+1)/2}$ if $d(z,w) < c_{4.5}$. Therefore

$$|\|K_z\| - \|K_w\|| \le C_{4.5} d(z,w)|r(z)|^{-(n+1)/2} \quad \text{if } \ d(z,w) < c_{4.5}.$$

Combining this with (4.1), the condition $d(z,w) < c_{4.5}$, and Lemma 2.1, we obtain

$$|\|K_z\|^{-1} - \|K_w\|^{-1}| = \frac{|\|K_z\| - \|K_w\||}{\|K_z\|\|K_w\|} \le C_1 d(z,w)|r(z)|^{(n+1)/2} \tag{4.11}$$

when $d(z,w) < c_{4.5}$. Let $f \in L^2_a(\Omega)$. Then

$$\begin{aligned}\langle f, k_z - k_w\rangle &= f(z)\|K_z\|^{-1} - f(w)\|K_w\|^{-1} \\ &= (f(z)-f(w))\|K_w\|^{-1} + f(z)(\|K_z\|^{-1} - \|K_w\|^{-1}).\end{aligned} \tag{4.12}$$

Applying Lemma 4.5, we have

$$
\begin{aligned}
|f(z) - f(w)| \|K_w\|^{-1} &\le C_{4.5} d(z,w) |r(z)|^{-(n+1)/2} \|f\chi_{D(z,1)}\| \|K_w\|^{-1} \\
&\le C_2 d(z,w) \|f\chi_{D(z,1)}\|,
\end{aligned}
\tag{4.13}
$$

where the second $\le$ follows from (4.1), the condition $d(z,w) < c_{4.5}$, and Lemma 2.1. On the other hand, Lemma 4.3 tells us that

$$|f(z)| \le C_{4.3} |r(z)|^{-(n+1)/2} \|f\chi_{D(z,1)}\|.$$

Combining this with (4.11), we obtain

$$|f(z)| |\|K_z\|^{-1} - \|K_w\|^{-1}| \le C_1 C_{4.3} d(z,w) \|f\chi_{D(z,1)}\|. \tag{4.14}$$

Obviously, the lemma follows from (4.12), (4.13) and (4.14). □

Lemma 4.7. *There is a $c_{4.7} > 0$ such that for any pair of $z, w \in \Omega$, if $d(z,w) \le c_{4.7}$, then $|\langle k_z, k_w \rangle| \ge 1/2$.*

Proof. We have $1 - \mathrm{Re}\langle k_z, k_w \rangle = 2^{-1} \|k_z - k_w\|^2$. By Proposition 4.6, there is a $c_{4.7} > 0$ such that for any pair of $z, w \in \Omega$, if $d(z,w) \le c_{4.7}$, then $\|k_z - k_w\| \le 1$. Thus if $d(z,w) \le c_{4.7}$, then $1 - \mathrm{Re}\langle k_z, k_w \rangle \le 1/2$, which implies $|\langle k_z, k_w \rangle| \ge 1/2$. □

5. Discrete sums

We now consider operators constructed from the Bergman kernel.

Lemma 5.1. *There is a constant $0 < C_{5.1} < \infty$ such that the following estimate holds: Let Γ be any 1-separated set in Ω. Suppose that $\{e_z : z \in \Gamma\}$ is an orthonormal set and $\{c_z : z \in \Gamma\}$ is a bounded set of complex coefficients. Then*

$$\left\| \sum_{z \in \Gamma} c_z k_z \otimes e_z \right\| \le C_{5.1} \sup_{z \in \Gamma} |c_z|.$$

Proof. Given such Γ, $\{e_z : z \in \Gamma\}$ and $\{c_z : z \in \Gamma\}$, define the operator

$$A = \sum_{z \in \Gamma} c_z k_z \otimes e_z.$$

Then for every $f \in L^2_a(\Omega)$ we have

$$A^* f = \sum_{z \in \Gamma} \bar{c}_z \|K_z\|^{-1} f(z) e_z.$$

From Lemma 4.3 and (4.1) we obtain

$$\|A^* f\|^2 \le C_{4.3}^2 \sum_{z \in \Gamma} |c_z|^2 \|K_z\|^{-2} |r(z)|^{-n-1} \|f\chi_{D(z,1)}\|^2 \le (C_{4.3}^2 / c) \sup_{z \in \Gamma} |c_z|^2 \|f\|^2.$$

Since $f \in L^2_a(\Omega)$ is arbitrary, this means that $\|A\| = \|A^*\| \le c^{-1/2} C_{4.3} \sup_{z\in\Gamma} |c_z|$. □

Lemma 5.2. *Let Γ be a 1-separated set in Ω. Suppose that for every $z \in \Gamma$, we have a $\zeta(z) \in \Omega$ with $d(z, \zeta(z)) < c_{4.5}$. Then for every orthonormal set $\{e_z : z \in \Gamma\}$ and for every bounded set of complex coefficients $\{c_z : z \in \Gamma\}$, we have*

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes e_z - \sum_{z\in\Gamma} c_z k_{\zeta(z)} \otimes e_z \right\| \le C_{4.6} \sup_{z\in\Gamma} |c_z| d(z, \zeta(z)).$$

Proof. Write

$$D = \sum_{z\in\Gamma} c_z k_z \otimes e_z - \sum_{z\in\Gamma} c_z k_{\zeta(z)} \otimes e_z = \sum_{z\in\Gamma} c_z (k_z - k_{\zeta(z)}) \otimes e_z.$$

For any $f \in L^2_a(\Omega)$, we have

$$D^* f = \sum_{z\in\Gamma} \bar{c}_z \langle f, k_z - k_{\zeta(z)} \rangle e_z.$$

Applying Proposition 4.6, if $d(z, \zeta(z)) < c_{4.5}$ for every $z \in \Gamma$, then

$$\begin{aligned} \|D^* f\|^2 &= \sum_{z\in\Gamma} |\bar{c}_z \langle f, k_z - k_{\zeta(z)} \rangle|^2 \le C_{4.6}^2 \sum_{z\in\Gamma} |c_z|^2 d^2(z, \zeta(z)) \|f \chi_{D(z,1)}\|^2 \\ &\le C_{4.6}^2 \sup_{z\in\Gamma} |c_z|^2 d^2(z, \zeta(z)) \|f\|^2. \end{aligned}$$

Since $f \in L^2_a(\Omega)$ is arbitrary, this implies $\|D\| = \|D^*\| \le C_{4.6} \sup_{z\in\Gamma} |c_z| d(z, \zeta(z))$. □

Corollary 5.3. *Given any $a > 0$, $0 \le C < \infty$ and $\epsilon > 0$, there is a $\delta > 0$ such that the following estimate holds: Let Γ be an a-separated set in Ω. Suppose that φ, φ', ψ and ψ' are maps from Γ into Ω. If the inequalities*

$$d(z, \varphi(z)) \le C, \quad d(z, \psi(z)) \le C, \quad d(\varphi(z), \varphi'(z)) \le \delta, \quad d(\psi(z), \psi'(z)) \le \delta$$

hold for every $z \in \Gamma$, then for any bounded set of coefficients $\{c_z : z \in \Gamma\}$ we have

$$\left\| \sum_{z\in\Gamma} c_z k_{\varphi(z)} \otimes k_{\psi(z)} - \sum_{z\in\Gamma} c_z k_{\varphi'(z)} \otimes k_{\psi'(z)} \right\| \le \epsilon \sup_{z\in\Gamma} |c_z|.$$

Proof. For $z, w \in \Gamma$, if $d(z, w) > 2C + 2$, then $d(\varphi(z), \varphi(w)) > 2$ and $d(\psi(z), \psi(w)) > 2$. By Lemma 2.11, there is an $N \in \mathbf{N}$ determined by a and C such that Γ admits a partition

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$$

with the property that for each $j \in \{1, \ldots, N\}$, the sets $\{\varphi(z) : z \in \Gamma_j\}$ and $\{\psi(z) : z \in \Gamma_j\}$ are 1-separated. Pick an orthonormal set $\{e_z : z \in \Gamma\}$. Fixing a $j \in \{1, \ldots, N\}$ for the moment, we have

$$\sum_{z \in \Gamma_j} c_z k_{\varphi(z)} \otimes k_{\psi(z)} - \sum_{z \in \Gamma_j} c_z k_{\varphi'(z)} \otimes k_{\psi'(z)} = AB^* - A'B'^*,$$

where

$$A = \sum_{z \in \Gamma_j} c_z k_{\varphi(z)} \otimes e_z, \quad B = \sum_{z \in \Gamma_j} k_{\psi(z)} \otimes e_z,$$
$$A' = \sum_{z \in \Gamma_j} c_z k_{\varphi'(z)} \otimes e_z, \quad B' = \sum_{z \in \Gamma_j} k_{\psi'(z)} \otimes e_z.$$

We have

$$AB^* - A'B'^* = (A - A')B^* + A'(B^* - B'^*).$$

Since $\{\varphi(z) : z \in \Gamma_j\}$ and $\{\psi(z) : z \in \Gamma_j\}$ are 1-separated, if we apply Lemma 5.2 to $A - A'$ and $B - B'$ and Lemma 5.1 to B and A', we see that

$$\|AB^* - A'B'^*\| \le \frac{\epsilon}{N} \sup_{z \in \Gamma} |c_z|$$

when δ is sufficiently small. This completes the proof. □

6. Operators in the Toeplitz algebra $\mathcal{T}$

Define the measure

$$d\tilde{\mu}(w) = K(w, w)dv(w)$$

on Ω. By (4.1), this is just a slightly different version of the measure $d\mu$ defined by (2.10). Given an $f \in L^\infty(\Omega)$, we have the integral representation

$$T_f = \int f(w) k_w \otimes k_w d\tilde{\mu}(w) \tag{6.1}$$

for the Toeplitz operator T_f. This formula is obtained by direct verification. Starting from this representation, we will show that the Toeplitz algebra $\mathcal{T}$ contains certain classes of operators. The two main steps in the section are Propositions 6.4 and 6.6 below.

Proposition 6.1. *Suppose that Γ is a separated set in Ω and that $\{c_z : z \in \Gamma\}$ is a bounded set of complex coefficients. Then the operator*

$$\sum_{z \in \Gamma} c_z k_z \otimes k_z$$

belongs to the closure of $\{T_f : f \in L^\infty(\Omega)\}$ with respect to the operator norm.

Proof. By Lemma 2.11, we may assume that Γ is 1-separated. Let $\epsilon > 0$ be given. Since $\sup_{z\in\Gamma} |c_z| < \infty$, it follows from Corollary 5.3 that there is a $\delta > 0$ such that

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes k_z - \sum_{z\in\Gamma} c_z k_{\zeta(z)} \otimes k_{\zeta(z)} \right\| \le \epsilon \tag{6.2}$$

if $\zeta(z) \in D(z,\delta)$ for every $z \in \Gamma$. We may, of course, assume that $\delta < 1$, consequently $D(z,\delta) \cap D(w,\delta) = \emptyset$ for all $z \neq w$ in Γ.

Define the function

$$\varphi = \sum_{z\in\Gamma} \frac{c_z}{\tilde{\mu}(D(z,\delta))} \chi_{D(z,\delta)} \tag{6.3}$$

on Ω. By (4.1) and Proposition 2.6, there is an $\beta > 0$ such that $\tilde{\mu}(D(z,\delta)) \ge \beta$ for every $z \in \Omega$. Hence $\varphi \in L^\infty(\Omega)$. We will show that

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes k_z - T_\varphi \right\| \le \epsilon. \tag{6.4}$$

To prove this, we define the measure $d\nu_z = \{\tilde{\mu}(D(z,\delta))\}^{-1} \chi_{D(z,\delta)} d\tilde{\mu}$ for every $z \in \Gamma$. Then it follows from (6.1) and (6.3) that

$$T_\varphi = \sum_{z\in\Gamma} c_z \int k_w \otimes k_w d\nu_z(w).$$

Note that each $d\nu_z$ is a probability measure concentrated on $D(z,\delta)$. Hence $d\nu_z$ is in the weak-* closure of convex combinations of unit point masses on $D(z,\delta)$. Therefore T_φ is the limit in weak operator topology of operators of the form

$$T = \frac{1}{k} \sum_{j=1}^{k} \sum_{z\in\Gamma} c_z k_{\zeta(z;j)} \otimes k_{\zeta(z;j)},$$

where $k \in \mathbf{N}$ and $\zeta(z;j) \in D(z,\delta)$ for all $z \in \Gamma$ and $j \in \{1,\dots,k\}$. By (6.2),

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes k_z - T \right\| \le \frac{1}{k} \sum_{j=1}^{k} \left\| \sum_{z\in\Gamma} c_z k_z \otimes k_z - \sum_{z\in\Gamma} c_z k_{\zeta(z;j)} \otimes k_{\zeta(z;j)} \right\| \le \epsilon.$$

Since this holds for every such T and since T_φ is the weak limit of such T's, (6.4) follows. Since $\epsilon > 0$ is arbitrary, this completes the proof. □

Next we remind the reader of a well-known fact:

Proposition 6.2. [22,Theorem 4.1.25] *The Toeplitz algebra $\mathcal{T}$ contains $\mathcal{K}$, the collection of compact operators on the Bergman space $L^2_a(\Omega)$.*

Definition 6.3. (a) Let $\mathcal{D}_0$ denote the collection of operators of the form

$$\sum_{z\in\Gamma} c_z k_z \otimes k_{\gamma(z)},$$

where Γ is any separated set in Ω, $\{c_z : z \in \Gamma\}$ is any bounded set of complex coefficients, and $\gamma : \Gamma \to \Omega$ is any map for which there is a $0 \le C < \infty$ such that

$$d(z, \gamma(z)) \le C \tag{6.5}$$

for every $z \in \Gamma$.
(b) Let $\mathcal{D}$ denote the closure of the linear span of $\mathcal{D}_0$ with respect to the operator norm.
(c) For any $A \in \mathcal{B}(L^2_a(\Omega))$, $\mathcal{D}_0(A)$ denotes the collection of operators of the form

$$\sum_{z\in\Gamma} c_z \langle A k_{\psi(z)}, k_{\varphi(z)}\rangle k_{\varphi(z)} \otimes k_{\psi(z)},$$

where Γ is a separated set in Ω, $\{c_z : z \in \Gamma\}$ is a bounded set of coefficients, and $\varphi, \psi : \Gamma \to \Omega$ are maps for which there is a $0 \le C < \infty$ such that $d(z, \varphi(z)) \le C$ and $d(z, \psi(z)) \le C$ for every $z \in \Gamma$.
(d) For any $A \in \mathcal{B}(L^2_a(\Omega))$, $\mathcal{D}(A)$ denotes the closure of the linear span of $\mathcal{D}_0(A)$ with respect to the operator norm.

Proposition 6.4. *We have the inclusion $\mathcal{D}_0 \subset \mathcal{T}$. Consequently, $\mathcal{D} \subset \mathcal{T}$.*

Proof. Let Γ, $\{c_z : z \in \Gamma\}$, γ and C be as described in Definition 6.3(a), and consider

$$T = \sum_{z\in\Gamma} c_z k_z \otimes k_{\gamma(z)}.$$

To show that $T \in \mathcal{T}$, by Lemma 2.11, we may assume that

$$d(z, w) > 4C + 2 \quad \text{for all} \quad z \ne w \quad \text{in} \quad \Gamma. \tag{6.6}$$

For each $z \in \Gamma$, since $d(z, \gamma(z)) \le C$, by (2.4) there is a C^1 map $g_z : [0, 1] \to \Omega$ such that $g_z(0) = z$, $g_z(1) = \gamma(z)$, and such that the number

$$\ell_z = \int_0^1 \sqrt{\langle \mathcal{B}(g_z(t))g_z'(t), g_z'(t)\rangle}dt$$

satisfies the condition $\ell_z \le 2C$. Pick a $k \in \mathbf{N}$ such that $2C/k < \min\{a_0, c_{4.7}\}$, where a_0 and $c_{4.7}$ are the constants in Lemmas 3.10 and 4.7 respectively. For each $z \in \Gamma$, there are

$$0 = x(z, 0) \le x(z, 1) \le \cdots \le x(z, k - 1) \le x(z, k) = 1$$

such that

$$\int_0^{x(z,j)} \sqrt{\langle \mathcal{B}(g_z(t))g_z'(t), g_z'(t)\rangle}dt = \frac{j}{k}\ell_z$$

for $j = 0, 1, \ldots, k$. For each pair of $z \in \Gamma$ and $j \in \{0, 1, \ldots, k\}$, we now define

$$\gamma_j(z) = g_z(x(z, j)).$$

We have $\gamma_0(z) = z$ and $\gamma_k(z) = \gamma(z)$, $z \in \Gamma$. Since $\ell_z \le 2C$, for all $0 \le j < k$ and $z \in \Gamma$,

$$d(\gamma_j(z), \gamma_{j+1}(z)) \le \int_{x(z,j)}^{x(z,j+1)} \sqrt{\langle \mathcal{B}(g_z(t))g_z'(t), g_z'(t)\rangle}dt = \ell_z/k < \min\{a_0, c_{4.7}\}.$$

By Lemma 4.7, this ensures that

$$|\langle k_{\gamma_j(z)}, k_{\gamma_{j+1}(z)}\rangle| \ge 1/2 \tag{6.7}$$

for all $0 \le j < k$ and $z \in \Gamma$.

To prove that $T \in \mathcal{T}$, it suffices to show that for every $j \in \{0, 1, \ldots, k\}$ and every subset E of Γ, we have

$$\sum_{z \in E} c_z k_z \otimes k_{\gamma_j(z)} \in \mathcal{T}. \tag{6.8}$$

We will accomplish this by an induction on j. Since $\gamma_0(z) = z$ for every $z \in \Gamma$, the case $j = 0$ follows from Proposition 6.1. Suppose now that $0 \le j < k$ and that (6.8) holds for this j and for every $E \subset \Gamma$. To simplify notation, for every $S \subset \Gamma$, let us denote

$$X_S = \sum_{z \in S} c_z k_z \otimes k_{\gamma_j(z)} \quad \text{and} \quad Y_S = \sum_{z \in S} \frac{1}{\langle k_{\gamma_{j+1}(z)}, k_{\gamma_j(z)}\rangle} k_{\gamma_{j+1}(z)} \otimes k_{\gamma_{j+1}(z)}.$$

By the induction hypothesis, we have $X_S \in \mathcal{T}$. By (6.7) and Proposition 6.1, we also have $Y_S \in \mathcal{T}$. Therefore $X_S Y_S \in \mathcal{T}$ for every $S \subset \Gamma$. To complete the induction, it suffices to show that given $E \subset \Gamma$ and $\epsilon > 0$, there is a finite partition $E = S_1 \cup \cdots \cup S_N$ such that

$$\left\| X_{S_1} Y_{S_1} + \cdots + X_{S_N} Y_{S_N} - \sum_{z \in E} c_z k_z \otimes k_{\gamma_{j+1}(z)} \right\| \le \epsilon. \tag{6.9}$$

To see how this is done, first note that for any partition $E = S_1 \cup \cdots \cup S_N$,

$$\begin{aligned} &X_{S_1} Y_{S_1} + \cdots + X_{S_N} Y_{S_N} - \sum_{z \in E} c_z k_z \otimes k_{\gamma_{j+1}(z)} \\ &\quad = \sum_{\nu=1}^{N} \sum_{\substack{z,w \in S_\nu \\ z \ne w}} c_z \frac{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(z)}\rangle}{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(w)}\rangle} k_z \otimes k_{\gamma_{j+1}(w)} = UWV^*, \end{aligned}$$

where

$$U = \sum_{z \in E} c_z k_z \otimes e_z, \quad V = \sum_{z \in E} k_{\gamma_{j+1}(z)} \otimes e_z \quad \text{and}$$

$$W = \sum_{\nu=1}^{N} \sum_{\substack{z,w \in S_\nu \\ z \ne w}} \frac{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(z)}\rangle}{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(w)}\rangle} e_z \otimes e_w,$$

where $\{e_z : z \in E\}$ is an orthonormal set.

By (6.6), $\{\gamma_{j+1}(z) : z \in E\}$ is a 1-separated set. Thus Lemma 5.1 provides the bound $\|V\| \leq C_{5.1}$. Similarly, $\|U\| \leq C_{5.1}c$, where $c = \sup_{z\in\Gamma} |c_z|$. Consequently

$$\left\| X_{S_1}Y_{S_1} + \cdots + X_{S_N}Y_{S_N} - \sum_{z\in E} c_z k_z \otimes k_{\gamma_{j+1}(z)} \right\| \leq C_{5.1}^2 c\|W\|. \tag{6.10}$$

Thus we need to find a partition $E = S_1 \cup \cdots \cup S_N$ such that $\|W\|$ is small. To do this, consider an $R > 3C_{3.7}+1$, whose value will be determined below. By Lemma 2.11, there is a partition $E = S_1 \cup \cdots \cup S_N$ such that for every $\nu \in \{1, \ldots, N\}$, the conditions $z, w \in S_\nu$ and $z \neq w$ imply $d(z, w) > R$. With $S_1, \ldots, S_N$ so chosen, we define

$$\mathcal{F} = \bigcup_{\nu=1}^{N} \{(z, w) \in S_\nu \times S_\nu : z \neq w\}.$$

We can rewrite W in the form

$$W = \sum_{(z,w)\in E\times E} a(z, w) e_z \otimes e_w,$$

where

$$a(z, w) = \begin{cases} \frac{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(z)}\rangle}{\langle k_{\gamma_{j+1}(w)}, k_{\gamma_j(w)}\rangle} & \text{if} \quad (z, w) \in \mathcal{F} \\ 0 & \text{if} \quad (z, w) \notin \mathcal{F} \end{cases}.$$

Recall that $d(\gamma_p(z), \gamma_{p+1}(z)) < a_0$ for all $z \in \Gamma$ and $0 \leq p < k$. Recalling (6.7) and applying Theorem 4.1 and Lemmas 3.1, 2.1, and Lemma 3.10 multiple times, we obtain

$$\begin{aligned} |a(z, w)| \leq 2\frac{|K(\gamma_j(z), \gamma_{j+1}(w))|}{\|K_{\gamma_{j+1}(w)}\|\|K_{\gamma_j(z)}\|} &\leq C_1 \left(\frac{|r(\gamma_j(z))|^{1/2}|r(\gamma_{j+1}(w))|^{1/2}}{F(\gamma_j(z), \gamma_{j+1}(w))} \right)^{n+1} \\ &\leq C_2 \left(\frac{|r(z)|^{1/2}|r(w)|^{1/2}}{F(z, w)} \right)^{n+1} \end{aligned}$$

for $(z, w) \in \mathcal{F}$. Pick an $\eta \in (0, 1/2)$ and define $h(w) = |r(w)|^{(n/2)+\eta}$, $w \in \Gamma$. If $(z, w) \in \mathcal{F}$, then $d(z, w) > R$ by design. Since E is 1-separated, it follows from Lemma 4.2 that

$$\sum_{w\in E} |a(z, w)|h(w) \leq C_2 \sum_{w\in E\setminus D(z,R)} |r(w)|^{(n/2)+\eta} \left(\frac{|r(z)|^{1/2}|r(w)|^{1/2}}{F(z, w)} \right)^{n+1} \leq \frac{C_2C_{4.2}}{2^{sR}} h(z)$$

for every $z \in E$. A similar inequality holds for $\sum_{z\in E} |a(z, w)|h(z)$, $w \in E$. By the standard Schur test, we conclude that $\|W\| \leq C_2C_{4.2}2^{-sR}$. Recalling (6.10), we see that (6.9) holds if we pick $R > 3C_{3.7} + 1$ such that $C_{5.1}^2 cC_2C_{4.2}2^{-sR} \leq \epsilon$. This completes the proof. □

Following the ideas in [28], we will now generalize the notion of *localized operators* to strongly pseudo-convex domains.

Definition 6.5. Let A be a bounded operator on the Bergman space $L^2_a(\Omega)$. Then $\mathrm{LOC}(A)$ denotes the collection of operators of the form

$$T = \sum_{z\in\Gamma} T_{f_z} A T_{f_z}, \tag{6.11}$$

where Γ is any separated set in Ω and $\{f_z : z \in \Gamma\}$ is any family of continuous functions on Ω satisfying the following three conditions:
(1) There is a $0 < \rho < \infty$ such that $f_z = 0$ on $\Omega\backslash D(z, \rho)$ for every $z \in \Gamma$.
(2) The inequality $0 \le f_z \le 1$ holds on Ω for every $z \in \Gamma$.
(3) The family $\{f_z : z \in \Gamma\}$ satisfies a uniform Lipschitz condition on Ω with respect to the metric d. That is, there is a $0 < C < \infty$ such that $|f_z(\zeta) - f_z(\xi)| \le Cd(\zeta, \xi)$ for all $z \in \Gamma$ and $\zeta, \xi \in \Omega$.

Proposition 6.6. *For every bounded operator A on $L^2_a(\Omega)$, we have* $\mathrm{LOC}(A) \subset \mathcal{D}(A)$.

Proof. Let A be a bounded operator on $L^2_a(\Omega)$, and consider a T given by (6.11). To prove that $T \in \mathcal{D}(A)$, by Lemma 2.11, we may assume that Γ is 1-separated. For convenience, let us define the product measure $\nu = \tilde{\mu} \times \tilde{\mu}$ on $\Omega \times \Omega$. By (6.1), for each $z \in \Gamma$ we have

$$T_{f_z} A T_{f_z} = \iint h_z(u, v) k_u \otimes k_v d\nu(u, v), \tag{6.12}$$

where

$$h_z(u, v) = f_z(u) f_z(v) \langle A k_v, k_u\rangle. \tag{6.13}$$

By condition (1) above, h_z vanishes on the complement of $D(z, \rho) \times D(z, \rho)$. It follows from Proposition 4.6 and condition (3) above that for any $a > 0$, there is a $b > 0$ such that

$$\sup_{z\in\Gamma} |h_z(u, v) - h_z(u', v')| \le a \quad \text{if} \;\; d(u, u') \le b \;\; \text{and} \;\; d(v, v') \le b. \tag{6.14}$$

The rest of the proof is divided into two steps.

Step I. We first show that for any $\epsilon > 0$, there is a $0 < \delta \le \rho$ such that the following holds true: Suppose that Λ is a subset of Γ. For each $z \in \Lambda$, let $\varphi(z), \psi(z) \in D(z, \rho)$. For each $z \in \Lambda$, suppose that we have a Borel set $E_z = F_z \times G_z$ with $F_z \subset D(\varphi(z), \delta)$, $G_z \subset D(\psi(z), \delta)$ and $\nu(E_z) > 0$. Finally, for each $z \in \Lambda$, let $a_z \in [0, 2]$. Then

$$\left\| \sum_{z\in\Lambda} \frac{a_z}{\nu(E_z)} \iint_{E_z} h_z(u, v) k_u \otimes k_v d\nu(u, v) - \sum_{z\in\Lambda} a_z h_z(\varphi(z), \psi(z)) k_{\varphi(z)} \otimes k_{\psi(z)} \right\| \le \epsilon. \tag{6.15}$$

To prove this, denote

$$W = \sum_{z\in\Lambda} \frac{a_z}{\nu(E_z)} \iint_{E_z} h_z(u, v) k_u \otimes k_v d\nu(u, v) \quad \text{and}$$
$$Z = \sum_{z\in\Lambda} a_z h_z(\varphi(z), \psi(z)) k_{\varphi(z)} \otimes k_{\psi(z)}.$$

Note that for each $z \in \Lambda$, $\{\chi_{E_z}/\nu(E_z)\}d\nu$ is a probability measure concentrated on E_z. Thus it is in the weak-* closure of convex combinations of unit point masses on E_z. Consequently W is in the closure in weak operator topology of operators of the form

$$W' = \frac{1}{k}\sum_{j=1}^{k}\sum_{z\in\Lambda} a_z h_z(u(z;j), v(z;j))k_{u(z;j)} \otimes k_{v(z;j)},$$

where $k \in \mathbf{N}$ and, for each $1 \le j \le k$, we have $(u(z;j), v(z;j)) \in E_z$, i.e., $u(z;j) \in F_z$ and $v(z;j) \in G_z$, $z \in \Lambda$. It is easy to see that

$$W' - Z = \frac{1}{k}\sum_{j=1}^{k}(X_j + Y_j),$$

where

$$X_j = \sum_{z\in\Lambda} a_z\{h_z(u(z;j), v(z;j)) - h_z(\varphi(z), \psi(z))\}k_{u(z;j)} \otimes k_{v(z;j)} \quad \text{and}$$

$$Y_j = \sum_{z\in\Lambda} a_z h_z(\varphi(z), \psi(z))k_{u(z;j)} \otimes k_{v(z;j)} - \sum_{z\in\Lambda} a_z h_z(\varphi(z), \psi(z))k_{\varphi(z)} \otimes k_{\psi(z)}.$$

From (6.14) and Lemmas 2.11 and 5.1 we see that there is a $\delta_1 > 0$ such that $\|X_j\| \le \epsilon/2$ for every $1 \le j \le k$ if $\delta \le \delta_1$. By Corollary 5.3, there is a $\delta_2 > 0$ such that $\|Y_j\| \le \epsilon/2$ for every $1 \le j \le k$ if $\delta \le \delta_2$. Hence for any $0 < \delta \le \min\{\delta_1, \delta_2, \rho\}$, we have $\|W' - Z\| \le \epsilon$. Since $W - Z$ is the weak limit of operators of the form $W' - Z$, we have $\|W - Z\| \le \epsilon$ for any choice of $0 < \delta \le \min\{\delta_1, \delta_2, \rho\}$. This proves (6.15) and completes Step I.

Step II. Recall that $\nu = \tilde{\mu} \times \tilde{\mu}$. By (4.1) and Proposition 2.6, there is an $N \in \mathbf{N}$ such that $N \ge \nu(D(w, 2\rho) \times D(w, 2\rho))$ for every $w \in \Omega$. Let $\epsilon > 0$ be given. We will now find a $B \in \mathrm{span}(\mathcal{D}_0(A))$ such that

$$\|T - B\| \le N\epsilon. \tag{6.16}$$

Since $\epsilon > 0$ is arbitrary, this will imply the membership $T \in \mathcal{D}(A)$. To find such a $B \in \mathrm{span}(\mathcal{D}_0(A))$, let δ be the number provided for this ϵ in Step I. For each $z \in \Gamma$, there is a subset S_z in $D(z, \rho)$ that is maximal with respect to the property

$$D(x, \delta/2) \cap D(y, \delta/2) = \emptyset \quad \text{for all} \quad x \ne y \quad \text{in} \quad S_z.$$

By Proposition 2.6 and the fact that $\mu(D(z, 2\rho)) < \infty$, we see that S_z is a finite set, and consequently we can represent it in the form $S_z = \{\varphi(z;1), \ldots, \varphi(z;m(z))\}$ with some $m(z) \in \mathbf{N}$. The maximality of S_z implies that $\cup_{j=1}^{m(z)} D(\varphi(z;j), \delta) \supset D(z, \rho)$. Thus for each $z \in \Gamma$, a standard set-theoretical argument gives us Borel sets

$$F(z;1), \ldots, F(z;m(z))$$

with the following properties:

(i) $D(\varphi(z;j),\delta/2) \subset F(z;j) \subset D(\varphi(z;j),\delta)$ for each $j \in \{1,\dots,m(z)\}$.

(ii) $F(z;i) \cap F(z;j) = \emptyset$ for all $i \neq j$ in $\{1,\dots,m(z)\}$.

(iii) $D(z,\rho) \subset \cup_{j=1}^{m(z)} F(z;j) \subset D(z,2\rho)$.

We now define $E_{z;i,j} = F(z;i) \times F(z;j)$ for $z \in \Gamma$ and $i,j \in \{1,\dots,m(z)\}$.

Let $z \in \Gamma$. Since h_z vanishes on $(\Omega \times \Omega)\backslash(D(z,\rho) \times D(z,\rho))$, (iii) and (ii) imply

$$T_{f_z}AT_{f_z} = \sum_{i=1}^{m(z)}\sum_{j=1}^{m(z)} \iint_{E_{z;i,j}} h_z(u,v)k_u \otimes k_v d\nu(u,v). \tag{6.17}$$

By (i) and Proposition 2.6, there is a $k \in \mathbf{N}$ such that $1/k < \nu(E_{z;i,j})$ for all $z \in \Gamma$ and $i,j \in \{1,\dots,m(z)\}$. For such a triple of z,i,j, we let $p(z;i,j)$ be the largest natural number satisfying the condition $p(z;i,j)/k \le \nu(E_{z;i,j})$. Define

$$a(z;i,j) = \frac{k}{p(z;i,j)}\nu(E_{z;i,j})$$

for $z \in \Gamma$ and $i,j \in \{1,\dots,m(z)\}$. Then $0 < a(z;i,j) \le 2$, because the definition of $p(z;i,j)$ ensures that $\{p(z;i,j)+1\}/k > \nu(E_{z;i,j})$. We can now rewrite (6.17) in the form

$$T_{f_z}AT_{f_z} = \frac{1}{k}\sum_{i=1}^{m(z)}\sum_{j=1}^{m(z)} p(z;i,j)\frac{a(z;i,j)}{\nu(E_{z;i,j})}\iint_{E_{z;i,j}} h_z(u,v)k_u \otimes k_v d\nu(u,v). \tag{6.18}$$

On the other hand, for every $z \in \Gamma$, we have

$$\begin{aligned}\sum_{i=1}^{m(z)}\sum_{j=1}^{m(z)} p(z;i,j) = k\sum_{i=1}^{m(z)}\sum_{j=1}^{m(z)}\frac{p(z;i,j)}{k} &\le k\sum_{i=1}^{m(z)}\sum_{j=1}^{m(z)}\nu(E_{z;i,j}) \\ &\le k\nu(D(z,2\rho)\times D(z,2\rho)) \le kN.\end{aligned}$$

We can regard $p(z;i,j)$ as the "multiplicity" with which the triple (z,i,j) appears in (6.18). The above estimate shows that for a fixed $z \in \Gamma$, all the multiplicities add up to something less than or equal to kN. Thus there are subsets $\Gamma_1,\Gamma_2,\dots,\Gamma_{kN}$ of Γ such that

$$\sum_{z\in\Gamma} T_{f_z}AT_{f_z} = \frac{1}{k}\sum_{\ell=1}^{kN}\sum_{z\in\Gamma_\ell}\frac{a(z;i(z,\ell),j(z,\ell))}{\nu(E_{z;i(z,\ell),j(z,\ell)})}\iint_{E_{z;i(z,\ell),j(z,\ell)}} h_z(u,v)k_u\otimes k_v d\nu(u,v), \tag{6.19}$$

where for each pair of $\ell \in \{1,\dots,kN\}$ and $z \in \Gamma_\ell$ we have $i(z,\ell), j(z,\ell) \in \{1,\dots,m(z)\}$. For each $\ell \in \{1,\dots,kN\}$, define

$$B_\ell = \sum_{z\in\Gamma_\ell} a(z;i(z,\ell),j(z,\ell))h_z(\varphi(z;i(z,\ell)),\varphi(z;j(z,\ell)))k_{\varphi(z;i(z,\ell))} \otimes k_{\varphi(z;j(z,\ell))}.$$

Since $\varphi(z;i(z,\ell)), \varphi(z;j(z,\ell)) \in D(z,\rho)$ for every $z \in \Gamma_\ell$, recalling Definition 6.3(c) and (6.13), we have $B_\ell \in \mathcal{D}_0(A)$. Therefore the operator

$$B = \frac{1}{k}\sum_{\ell=1}^{kN} B_\ell \tag{6.20}$$

belongs to the linear span of $\mathcal{D}_0(A)$. By the choice of δ and Step I, we have

$$\left\| \sum_{z\in\Gamma_\ell} \frac{a(z;i(z,\ell),j(z,\ell))}{\nu(E_{z;i(z,\ell),j(z,\ell)})} \iint_{E_{z;i(z,\ell),j(z,\ell)}} h_z(u,v)k_u \otimes k_v d\nu(u,v) - B_\ell \right\| \le \epsilon,$$

$1 \le \ell \le kN$. Combining this with (6.19) and (6.20), we see that $\|T - B\|$ does not exceed

$$\frac{1}{k}\sum_{\ell=1}^{kN} \left\| \sum_{z\in\Gamma_\ell} \frac{a(z;i(z,\ell),j(z,\ell))}{\nu(E_{z;i(z,\ell),j(z,\ell)})} \iint_{E_{z;i(z,\ell),j(z,\ell)}} h_z(u,v)k_u \otimes k_v d\nu(u,v) - B_\ell \right\| \le N\epsilon.$$

This proves (6.16) and completes the proof of the proposition. □

It follows from Lemma 2.11 that $\mathcal{D}(A) \subset \mathcal{D}$ for every $A \in \mathcal{B}(L^2_a(\Omega))$ (cf. Definition 6.3). Thus from Propositions 6.6 and 6.4 we immediately obtain

Corollary 6.7. *For every bounded operator A on $L^2_a(\Omega)$, we have* $\mathrm{LOC}(A) \subset \mathcal{T}$.

To conclude this section, we recall

Lemma 6.8. *Let $\{f_1, \dots, f_\ell\}$ be a finite set of functions in $L^\infty(\Omega)$ with the property that $f_j f_k = 0$ for all $j \neq k$ in $\{1, \dots, \ell\}$. Let A be any bounded operator on the Bergman space $L^2_a(\Omega)$. Then there exist complex numbers $\{\gamma_1, \dots, \gamma_\ell\}$ with $|\gamma_k| = 1$ for every $k \in \{1, \dots, \ell\}$ and a subset E of $\{1, \dots, \ell\}$ such that if we define*

$$F = \sum_{k\in E} f_k, \quad G = \sum_{k\in\{1,\dots,\ell\}\setminus E} f_k, \quad F' = \sum_{k\in E} \gamma_k f_k \quad \text{and} \quad G' = \sum_{k\in\{1,\dots,\ell\}\setminus E} \gamma_k f_k,$$

then

$$\left\| \sum_{j\neq k} T_{f_j} A T_{f_k} \right\| \le 4(\|T_{F'}AT_G\| + \|T_{G'}AT_F\|).$$

This lemma was proved in the case of the unit ball as Lemma 5.1 in [28]. But the proof in the case of a general Ω is exactly the same. The only property of Toeplitz operators that was used in the proof of [28,Lemma 5.1] was that a Toeplitz operator is the compression to a subspace of a multiplication operator on an L^2. Thus not only does Lemma 6.8 hold, its analogue also holds, for example, in the setting of Hardy spaces. For that reason we will not repeat the proof of Lemma 6.8 here.

7. Oscillation and compactness

For a continuous function f on Ω, we define

$$\operatorname{diff}(f) = \sup\{|f(z) - f(w)| : d(z,w) \le 1\}.$$

Lemma 7.1. *For any continuous function f on Ω and any $k \in \mathbf{N}$, we have*

$$|f(z) - f(w)| \le (k+1)\operatorname{diff}(f) \tag{7.1}$$

for any pair of $z, w \in \Omega$ satisfying the condition $d(z,w) \le k$.

Proof. Let $z, w \in \Omega$ be such that $d(z,w) \le k$. By (2.4), there is a C^1 map $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = z$, $\gamma(1) = w$ and

$$\int_0^1 \sqrt{\langle \mathcal{B}(\gamma(t))\gamma'(t), \gamma'(t)\rangle}dt \le k+1. \tag{7.2}$$

There are $0 = x_0 \le x_1 \le \cdots \le x_k \le x_{k+1} = 1$ such that

$$\int_{x_j}^{x_{j+1}} \sqrt{\langle \mathcal{B}(\gamma(t))\gamma'(t), \gamma'(t)\rangle}dt = \frac{1}{k+1}\int_0^1 \sqrt{\langle \mathcal{B}(\gamma(t))\gamma'(t), \gamma'(t)\rangle}dt \tag{7.3}$$

for $0 \le j \le k$. Define $z_j = \gamma(x_j)$, $j = 0, 1, \ldots, k+1$. Then $z_0 = z$ and $z_{k+1} = w$. It follows from (7.2), (7.3) and (2.4) that $d(z_j, z_{j+1}) \le 1$, consequently

$$|f(z_j) - f(z_{j+1})| \le \operatorname{diff}(f)$$

for $j = 0, 1, \ldots, k$. With this inequality, (7.1) follows from a standard telescoping sum. □

Recall that P denotes the orthogonal projection from $L^2(\Omega)$ onto $L^2_a(\Omega)$.

Lemma 7.2. *There is a constant $0 < C_{7.2} < \infty$ such that $\|[M_f, P]\| \le C_{7.2}\operatorname{diff}(f)$ for every bounded continuous function f on Ω.*

Proof. Let T be the integral operator on $L^2(\Omega)$ with the function

$$\{d(z,w) + 2\}|K(z,w)|$$

as its integral kernel. We know that $|K(z,w)| = |K(w,z)|$. Recall that the Bergman kernel K is bounded on $(\Omega \times \Omega)\backslash\mathcal{R}_\delta$ for any $\delta > 0$ [14]. Thus it follows from Theorem 4.1 and Lemma 3.1 that $|K(z,w)| \le C_1 F(z,w)^{-n-1}$ for all $z, w \in \Omega$. Combining this fact with Lemmas 3.9 and 3.2, and with the Schur test, we see that the operator T is bounded on $L^2(\Omega)$. Let f be a bounded continuous function on Ω. It follows from Lemma 7.1 that

$$|(f(z) - f(w))K(z,w)| \le \operatorname{diff}(f)\{d(z,w) + 2\}|K(z,w)|$$

for all $z, w \in \Omega$. Hence $\|[M_f, P]\| \le \operatorname{diff}(f)\|T\|$. □

Recall that a continuous function f on Ω is said to have *vanishing oscillation* if

$$\lim_{z\to\partial\Omega}\sup\{|f(z)-f(w)| : d(z,w)\le 1\}=0.$$

We denote by $\mathrm{VO}_{\mathrm{bdd}}$ the collection of continuous functions of vanishing oscillation on Ω that are also *bounded*.

Proposition 7.3. *For each $f\in\mathrm{VO}_{\mathrm{bdd}}$, the commutator $[M_f,P]$ is compact.*

Proof. It suffices to consider $f\in\mathrm{VO}_{\mathrm{bdd}}$ with $\|f\|_\infty\le 1$. For each $R>0$, we will decompose f in the form $f=g_R+h_R$, where g_R has a compact support and h_R satisfies the conditions $\mathrm{diff}(h_R)\le R^{-1}$ and $\|h_R\|_\infty\le 1$. Since g_R has a compact support, $[M_{g_R},P]$ is compact. On the other hand, Lemma 7.2 tells us that $\|[M_{h_R},P]\|\to 0$ as $R\to\infty$. Thus such a general decomposition implies the compactness of $[M_f,P]$.

To decompose f, let $R>0$ be given. Since $f\in\mathrm{VO}_{\mathrm{bdd}}$, there is a $t>0$ such that

$$|f(z)-f(w)|\le(2R)^{-1}\quad\text{if}\quad z\in H_t\quad\text{and}\quad d(z,w)\le 1, \tag{7.4}$$

where $H_t=\{\zeta\in\Omega:-r(\zeta)<t\}$, and we may assume $H_t\ne\Omega$. Define

$$\varphi_R(x)=\begin{cases}(2R)^{-1}x & \text{if}\quad 0\le x\le 2R\\ 1 & \text{if}\quad x>2R\end{cases}.$$

Then φ_R satisfies the Lipschitz condition $|\varphi_R(x)-\varphi_R(y)|\le(2R)^{-1}|x-y|$ for $x,y\in[0,\infty)$. For a non-empty set $E\subset\Omega$ and $z\in\Omega$, we denote $d(z,E)=\inf\{d(z,\zeta):\zeta\in E\}$ as usual. By the triangle inequality for d, $|d(z,E)-d(w,E)|\le d(z,w)$ for all $z,w\in\Omega$. Hence

$$|\varphi_R(d(z,E))-\varphi_R(d(w,E))|\le(2R)^{-1}|d(z,E)-d(w,E)|\le(2R)^{-1}d(z,w) \tag{7.5}$$

for all $z,w\in\Omega$. We now define

$$g_R(z)=f(z)(1-\varphi_R(d(z,\Omega_t)))\quad\text{and}\quad h_R(z)=f(z)\varphi_R(d(z,\Omega_t)),$$

where $\Omega_t=\{\zeta\in\Omega:-r(\zeta)\ge t\}$. Since $\|f\|_\infty\le 1$ and $\|\varphi_R\|_\infty=1$, we have

$$|h_R(z)-h_R(w)|\le|f(z)-f(w)|+|\varphi_R(d(z,\Omega_t))-\varphi_R(d(w,\Omega_t))|.$$

If $h_R(z)-h_R(w)\ne 0$, then either $z\in H_t$ or $w\in H_t$. Thus if $d(z,w)\le 1$ and $h_R(z)-h_R(w)\ne 0$, then it follows from (7.4) and (7.5) that $|h_R(z)-h_R(w)|\le 1/R$. That is, $\mathrm{diff}(h_R)\le 1/R$ as promised. On the other hand, if $g_R(z)\ne 0$, then $d(z,\Omega_t)<2R$. By Lemma 2.1, this means that $-r(z)\ge c(R)t$, where $c(R)>0$ is a constant determined by R. Hence the support of g_R is a compact set contained in Ω. This completes the proof. □

Lemma 7.4. *Let $f_1,\dots,f_k\dots$ be a sequence of continuous functions on Ω satisfying the following four conditions*:

(1) *There is a $0<C<\infty$ such that $\|f_k\|_\infty\le C$ for every $k\in\mathbf{N}$.*

(2) *For every* $k \in \mathbf{N}$, *there exist* $a_k > b_k > 0$ *such that* $f_k = 0$ *on* $\Omega_{a_k} \cup H_{b_k}$.
(3) $\lim_{k\to\infty} a_k = 0$.
(4) $\lim_{k\to\infty} \operatorname{diff}(f_k) = 0$.
Then there is an infinite subset I *of* $\mathbf{N}$ *such that* $f_J \in \mathrm{VO}_{\mathrm{bdd}}$ *for every* $J \subset I$, *where*

$$f_J = \sum_{k\in J} f_k.$$

Proof. By condition (3) and Lemma 2.1, we can inductively pick a sequence of natural numbers $k(1) < k(2) < \cdots < k(j) < \cdots$ such that $a_{k(j+1)} < b_{k(j)}$ and

$$d(z,w) \geq 2 \quad \text{if} \quad -r(z) \leq a_{k(j+1)} \quad \text{and} \quad -r(w) \geq b_{k(j)} \tag{7.6}$$

for every $j \in \mathbf{N}$. Let $I = \{k(1), k(2), \dots, k(j), \dots\}$.

For each $k \in \mathbf{N}$, define $\mathcal{R}_k = \{z \in \Omega : b_k \leq -r(z) \leq a_k\}$. Then (2) says that $f_k = 0$ on $\Omega \backslash \mathcal{R}_k$. It follows from (7.6) that

$$\text{if } z \in \mathcal{R}_{k(j)} \text{ and } w \in \mathcal{R}_{k(j')} \text{ for } j \neq j' \text{ in } \mathbf{N}, \text{ then } d(z,w) \geq 2. \tag{7.7}$$

This immediately implies that if $J \subset I$, then f_J is continuous on Ω. Moreover, since $\mathcal{R}_{k(j)} \cap \mathcal{R}_{k(j')} = \emptyset$ whenever $j \neq j'$, it follows from (1) and (2) that $\|f_J\|_\infty \leq C$ for every $J \subset I$. That is, such an f_J is bounded on Ω.

Let $j_0 \in \mathbf{N}$, and let $z, w \in \Omega$ satisfy the conditions $-r(z) \leq a_{k(j_0)}$ and $d(z,w) \leq 1$. Then it follows from (7.7) that there is at most one $j \in \mathbf{N}$ such that $f_{k(j)}(z) - f_{k(j)}(w) \neq 0$. Furthermore, by (7.6), if such a j exist, then it must satisfy the condition $j \geq j_0$. Thus for $z, w \in \Omega$ satisfying the conditions $-r(z) \leq a_{k(j_0)}$ and $d(z,w) \leq 1$, we have

$$|f_J(z) - f_J(w)| \leq \sup\{\operatorname{diff}(f_{k(j)}) : j \geq j_0\}$$

for every $J \subset I$. Applying conditions (3) and (4), this means that for every $J \subset I$, f_J has vanishing oscillation. □

Definition 7.5. (a) For each $t > 0$, the symbol $\Lambda(t)$ denotes the collection of *continuous* functions g on Ω satisfying the following three conditions:
(1) $0 \leq g(z) \leq 1$ for every $z \in \Omega$.
(2) $g(z) = 1$ when $z \in \Omega_t = \{\zeta \in \Omega : -r(\zeta) \geq t\}$.
(3) There is a $t' = t'(g) \in (0,t)$ such that $g(z) = 0$ whenever $-r(z) \leq t'$.
(b) Let $t > 0$ and $\delta > 0$. Then $\Lambda(t;\delta)$ denotes the collection of functions $g \in \Lambda(t)$ satisfying the additional condition $\operatorname{diff}(g) \leq \delta$.

Lemma 7.6. *For all* $t > 0$ *and* $\delta > 0$, *we have* $\Lambda(t;\delta) \neq \emptyset$.

Proof. This is similar to the proof of Proposition 7.3. Let $\psi : [0,\infty) \to [0,1]$ be a Lipschitz function with Lipschitz constant δ. Furthermore, suppose that $\psi(0) = 1$ and that $\psi = 0$ on $[R,\infty)$ for a sufficiently large R. Let $t > 0$ be such that $\Omega_t \neq \emptyset$ (otherwise, (2) is trivial). By Lemma 2.1, the function $f(z) = \psi(d(z,\Omega_t))$ is in $\Lambda(t;\delta)$. □

Lemma 7.7. *Given any pair of $f \in L^\infty(\Omega)$ and $h \in L^2_a(\Omega)$, we have*

$$\lim_{t\downarrow 0} \sup\{\|T_{fg}h - T_f h\| : g \in \Lambda(t)\} = 0. \tag{7.8}$$

Proof. Denote $H_t = \{z \in \Omega : -r(z) < t\}$ as before. By Definition 7.5(a), we have

$$\|T_{fg}h - T_f h\|^2 \le \|fgh - fh\|^2 \le \|f\|_\infty^2 \int_{H_t} |h(z)|^2 dv(z)$$

for all $g \in \Lambda(t)$, $f \in L^\infty(\Omega)$ and $h \in L^2_a(\Omega)$. This obviously implies (7.8). $\square$

For a bounded operator A on a Hilbert space $\mathcal{H}$, denote

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \text{ is any compact operator on } \mathcal{H}\},$$

which is the essential norm of A.

Lemma 7.8. [16,Lemma 2.1] *Let $\{B_i\}$ be a sequence of compact operators on a Hilbert space $\mathcal{H}$ satisfying the following conditions*:
(a) *Both sequences $\{B_i\}$ and $\{B_i^*\}$ converge to 0 in the strong operator topology.*
(b) *The limit $\lim_{i\to\infty} \|B_i\|$ exists.*
Then there exist natural numbers $i(1) < i(2) < \cdots < i(m) < \cdots$ such that the sum

$$\sum_{m=1}^{\infty} B_{i(m)} = \lim_{N\to\infty} \sum_{m=1}^{N} B_{i(m)}$$

exists in the strong operator topology and we have

$$\left\| \sum_{m=1}^{\infty} B_{i(m)} \right\|_{\mathcal{Q}} = \lim_{i\to\infty} \|B_i\|.$$

Definition 7.9. For $t > 0$ and $\delta > 0$, the symbol $\Phi(t;\delta)$ denotes the collection of *continuous* functions f on Ω satisfying the following three conditions:
(1) $0 \le f(z) \le 1$ for every $z \in \Omega$.
(2) $f(z) = 0$ whenever $-r(z) \ge t$.
(3) $\mathrm{diff}(f) \le \delta$.

In analogy with [28,Proposition 3.7], every operator in $\mathrm{EssCom}(\{T_g : g \in \mathrm{VO_{bdd}}\})$ satisfies the following "ϵ-δ" condition:

Proposition 7.10. *Let X be an operator in the essential commutant of $\{T_g : g \in \mathrm{VO_{bdd}}\}$. Then for every $\epsilon > 0$, there is a $\delta = \delta(X, \epsilon) > 0$ such that*

$$\lim_{t\downarrow 0} \sup\{\|[X, T_f]\| : f \in \Phi(t;\delta)\} \le \epsilon.$$

Using 7.4-7.9 above, the proof of Proposition 7.10 is a repeat of the proof of Proposition 3.7 in [28], modified in the obvious way. For that reason we will omit the proof of Proposition 7.10 here.

Lemma 7.11. *Let $h_1, \dots, h_k \dots$ be a sequence of continuous functions on Ω, and denote $U_k = \{z \in \Omega : h_k(z) \neq 0\}$, $k \in \mathbf{N}$. Suppose that this sequence has the property that there is an $a > 1$ such that $\inf\{d(z,w) : z \in U_j, w \in U_k\} \geq a$ for every pair of $j \neq k$ in $\mathbf{N}$. Then the function $h = \sum_{k=1}^{\infty} h_k$ has the property that $\operatorname{diff}(h) \leq \sup_{k \in \mathbf{N}} \operatorname{diff}(h_k)$.*

Proof. Observe that, under the assumption, for any pair of $z, w \in \Omega$ satisfying the condition $d(z,w) \leq 1$, the cardinality of the set $\{k \in \mathbf{N} : h_k(z) - h_k(w) \neq 0\}$ is at most 1. □

8. Approximate partition of unity

In this section the boundary $\partial\Omega$ of the domain plays a prominent role. It will be beneficial to make a simplification of notation: for $\zeta \in \partial\Omega$ and $t > 0$, let us write

$$Q(\zeta, t) = \{\xi \in \partial\Omega : |\zeta - \xi|^2 + |\langle \zeta - \xi, (\bar{\partial} r)(\zeta) \rangle| < t\}.$$

In other words, in terms of the notation in Section 2, we have $Q(\zeta, t) = Q_0(\zeta, t)$. Similarly, we will write $d\sigma$ for $d\sigma_0$. That is, $d\sigma$ is the surface measure on $\partial\Omega$.

Lemma 8.1. *There is a constant $1 \leq C_{8.1} < \infty$ such that for any triple of $\zeta, \xi \in \partial\Omega$ and $t > 0$, if $Q(\zeta, t) \cap Q(\xi, t) \neq \emptyset$, then $Q(\xi, t) \subset Q(\zeta, C_{8.1} t)$.*

Proof. By the assumption on the defining function r, there is an L such that $|(\bar{\partial} r)(x) - (\bar{\partial} r)(y)| \leq L|x - y|$ for all $x, y \in \partial\Omega$. Suppose that there is a $w \in Q(\zeta, t) \cap Q(\xi, t)$. Then

$$|\zeta - w|^2 + |\langle \zeta - w, (\bar{\partial} r)(\zeta) \rangle| < t \quad \text{and} \quad |\xi - w|^2 + |\langle \xi - w, (\bar{\partial} r)(\xi) \rangle| < t.$$

From this it is elementary to obtain $|\zeta - \xi|^2 \leq 2(|\zeta - w|^2 + |\xi - w|^2) < 2t$. Further,

$$\begin{aligned} |\langle \zeta - \xi, (\bar{\partial} r)(\zeta) \rangle| &\leq |\langle \zeta - w, (\bar{\partial} r)(\zeta) \rangle| + |\langle w - \xi, (\bar{\partial} r)(\zeta) \rangle| \\ &\leq |\langle \zeta - w, (\bar{\partial} r)(\zeta) \rangle| + |\langle w - \xi, (\bar{\partial} r)(\xi) \rangle| + |w - \xi| \cdot L|\xi - \zeta| \\ &< 2t + \sqrt{t} \cdot L\sqrt{2t} = (2 + \sqrt{2}L)t. \end{aligned}$$

Therefore if we set $C_1 = 4 + \sqrt{2}L$, then the condition $Q(\zeta, t) \cap Q(\xi, t) \neq \emptyset$ implies $\xi \in Q(\zeta, C_1 t)$. Suppose that $z \in Q(\xi, t)$, i.e., $|\xi - z|^2 + |\langle \xi - z, (\bar{\partial} r)(\xi) \rangle| < t$. Then

$$|\langle \xi - z, (\bar{\partial} r)(z) \rangle| \leq |\langle \xi - z, (\bar{\partial} r)(\xi) \rangle| + |\xi - z| \cdot L|\xi - z| < (1 + L)t.$$

That is, if $z \in Q(\xi, t)$, then $\xi \in Q(z, (2 + L)t) \subset Q(z, C_1 t)$. Thus the condition $Q(\zeta, t) \cap Q(\xi, t) \neq \emptyset$ implies $Q(\zeta, C_1 t) \cap Q(z, C_1 t) \neq \emptyset$ for every $z \in Q(\xi, t)$. By the first conclusion, we have $z \in Q(\zeta, C_1^2 t)$. Thus the lemma holds for $C_{8.1} = C_1^2 = (4 + \sqrt{2}L)^2$. □

Corollary 8.2. *Consider any $\zeta \in \partial\Omega$ and $t > 0$. If $x, y \in \partial\Omega$ are such that $x \in Q(\zeta, t)$ and $y \notin Q(\zeta, C_{8.1} t)$, then $y \notin Q(x, t)$.*

Proof. If $x \in Q(\zeta, t)$, then $Q(x,t) \cap Q(\zeta,t) \neq \emptyset$. By Lemma 8.1, we have $Q(x,t) \subset Q(\zeta, C_{8.1}t)$. Therefore if $y \notin Q(\zeta, C_{8.1}t)$, then $y \notin Q(x,t)$. □

Lemma 8.3. *There is a constant* $0 < C_{8.3} < \infty$ *such that the following bound holds*: Let $t > 0$, *and let* E *be a subset of* $\partial\Omega$ *that has the property* $Q(x,t) \cap Q(y,t) = \emptyset$ *for all* $x \neq y$ *in* E. *Then for any* $R \geq 1$ *and any* $\zeta \in \partial\Omega$,

$$\mathrm{card}\{x \in E : Q(x, Rt) \cap Q(\zeta, Rt) \neq \emptyset\} \leq C_{8.3}R^n.$$

Proof. It $t > T_0$ (see Proposition 2.8), then the property of E implies $\mathrm{card}(E) \leq 1$. Suppose that $0 < t \leq T_0$. If $Q(x,Rt) \cap Q(\zeta,Rt) \neq \emptyset$, then $Q(x,Rt) \subset Q(\zeta, C_{8.1}Rt)$ by Lemma 8.1. Let $E_0 = \{x \in E : Q(x,Rt) \cap Q(\zeta,Rt) \neq \emptyset\}$. Since $Q(x,t) \cap Q(y,t) = \emptyset$ for all $x \neq y$ in E, we have

$$\mathrm{card}(E_0) \inf_{x \in E_0} \sigma(Q(x,t)) \leq \sum_{x \in E_0} \sigma(Q(x,t)) = \sigma\left(\bigcup_{x \in E_0} Q(x,t) \right) \leq \sigma(Q(\zeta, C_{8.1}Rt)).$$

Applying Proposition 2.8 to the case $\rho = 0$, we have

$$\mathrm{card}(E_0) c_{2.8} t^n \leq C_{2.8}(C_{8.1}Rt)^n.$$

Cancelling out t^n and simplifying, we see that the lemma holds for the constant $C_{8.3} = (C_{2.8}/c_{2.8})C_{8.1}^n$. □

The first order Taylor expansion for r reads

$$r(z+u) = r(z) + 2\mathrm{Re}\langle u, (\bar{\partial}r)(z)\rangle + \int_0^1 2\mathrm{Re}\langle u, (\bar{\partial}r)(z+xu) - (\bar{\partial}r)(z)\rangle dx.$$

Thus $r(z + t(\bar{\partial}r)(z)) = r(z) + 2t|(\bar{\partial}r)(z)|^2 + O(t^2)$ when $|t|$ is small. Recall that $\bar{\partial}r$ does not vanish on $\partial\Omega$. Hence when z is close to $\partial\Omega$ and t is positive and small, we have

$$r(z + t(\bar{\partial}r)(z)) \geq r(z) + t|(\bar{\partial}r)(z)|^2.$$

Thus for each $z \in \Omega$ near $\partial\Omega$, there is a $t_z > 0$, $t_z \approx |r(z)|$, such that $r(z + t_z(\bar{\partial}r)(z)) = 0$. Let us restate this fact more precisely: There exist a $J \in \mathbf{N}$ and a $0 < C_p < \infty$ such that for every $z \in H_{2^{-J}} = \{\zeta \in \Omega : -r(\zeta) < 2^{-J}\}$, there is a $p(z) \in \partial\Omega$ such that

$$|z - p(z)| \leq C_p|r(z)|. \tag{8.1}$$

In other words, there is a map $p : H_{2^{-J}} \to \partial\Omega$ such that the above bound holds for every $z \in H_{2^{-J}}$. Note that our choice above does not promise any kind of continuity for the map p, but that does not matter for our purpose.

This p and the defining function r together allow us to decompose $H_{2^{-J}}$ in a manner that is analogous to the radial-spherical decomposition for the unit ball in [28]. More specifically, p plays the role of "spherical coordinates", while $-r$ is the analogue of "radial coordinate". Because we only need a large-scale, or "coarse", decomposition, (8.1) is all that we need to know about p.

Lemma 8.4. *There is a constant* $0 < C_{8.4} < \infty$ *such that*

$$|z' - w'|^2 + |\langle z' - w', (\bar{\partial} r)(z')\rangle| \\ \leq 3\{|z - w|^2 + |\langle z - w, (\bar{\partial} r)(z)\rangle|\} + C_{8.4}\{|z - z'| + |w - w'|\}$$

for all $z, w, z', w' \in \overline{\Omega} = \Omega \cup \partial\Omega$.

Proof. It is elementary that $|z' - w'|^2 \leq 3|z - w|^2 + 3|z - z'|^2 + 3|w - w'|^2$. Since Ω is bounded, there is a C_1 such that $|\zeta - \xi| \leq C_1$ for all $\zeta, \xi \in \overline{\Omega}$. Hence

$$|z' - w'|^2 \leq 3|z - w|^2 + 3C_1\{|z - z'| + |w - w'|\}. \tag{8.2}$$

Similarly, since $\bar{\partial} r$ is bounded and satisfies a Lipschitz condition on $\overline{\Omega}$, we have

$$\begin{aligned} |\langle z' - w', (\bar{\partial} r)(z')\rangle| &\leq (|z - z'| + |w - w'|)|(\bar{\partial} r)(z')| + |\langle z - w, (\bar{\partial} r)(z')\rangle| \\ &\leq C_2(|z - z'| + |w - w'|) + |\langle z - w, (\bar{\partial} r)(z)\rangle| + |z - w||(\bar{\partial} r)(z') - (\bar{\partial} r)(z)| \\ &\leq C_2(|z - z'| + |w - w'|) + |\langle z - w, (\bar{\partial} r)(z)\rangle| + C_3|z - z'|. \end{aligned} \tag{8.3}$$

Obviously, the lemma follows from (8.2) and (8.3). □

We begin the decomposition with natural numbers $m > J$ and $j \geq 1$. Define

$$d_{m,j} = m2^{-jm}, \quad a_{m,j} = C_{8.1}m2^{-jm} \quad \text{and} \quad b_{m,j} = C_{8.1}^2 m2^{-jm}, \tag{8.4}$$

where $C_{8.1}$ is the constant in Lemma 8.1. That is, $a_{m,j} = C_{8.1}d_{m,j}$ and $b_{m,j} = C_{8.1}^2 d_{m,j}$. Let $E_{m,j}$ be a subset of $\partial\Omega$ that is *maximal* with respect to the property

$$Q(u, d_{m,j}) \cap Q(v, d_{m,j}) = \emptyset \quad \text{for all} \quad u \neq v \in E_{m,j}. \tag{8.5}$$

By the maximality of $E_{m,j}$ and Lemma 8.1, we have

$$\bigcup_{u \in E_{m,j}} Q(u, a_{m,j}) = \partial\Omega. \tag{8.6}$$

Fix a natural number N_0 such that $N_0 \geq C_{8.3}(C_{8.1}^2)^n$, where $C_{8.3}$ is the constant in Lemma 8.3. Since $b_{m,j} = C_{8.1}^2 d_{m,j}$, it follows from (8.5) and Lemma 8.3 that

$$\text{card}\{v \in E_{m,j} : Q(v, b_{m,j}) \cap Q(u, b_{m,j}) \neq \emptyset\} \leq N_0 \tag{8.7}$$

for every $u \in E_{m,j}$. Now, given any $m > J$, $j \geq 1$ and $u \in E_{m,j}$, we define the sets

$$A_{m,j,u} = \{z \in \Omega : p(z) \in Q(u, a_{m,j}) \text{ and } 2^{-(j+1)m} > -r(z) \geq 2^{-(j+2)m}\} \quad \text{and}$$
$$B_{m,j,u} = \{z \in \Omega : p(z) \in Q(u, b_{m,j}) \text{ and } 2^{-jm} > -r(z) > 2^{-(j+3)m}\}.$$

It follows from (8.6) that

$$\bigcup_{j=1}^{\infty}\bigcup_{u\in E_{m,j}} A_{m,j,u} = H_{2^{-2m}} = \{z\in\Omega : -r(z) < 2^{-2m}\}. \tag{8.8}$$

Note that even though we have (8.8), we do not know that every $A_{m,j,u}$ is non-empty from its definition. Nevertheless, we have

Lemma 8.5. *There is a constant $J < M_{8.5} < \infty$ such that if $m \geq M_{8.5}$, then $A_{m,j,u} \neq \emptyset$ for all $j \geq 1$ and $u \in E_{m,j}$.*

Proof By the Taylor expansion for r, there are constants $J < M_1 < \infty$ and $0 < C_1 < \infty$ such that if $m \geq M_1$, then for every pair of $j \geq 1$ and $u \in E_{m,j}$ there is a u' such that $-r(u') = 2^{-(j+(3/2))m}$ and $|u - u'| \leq C_1(-r(u'))$. By Lemma 8.4, we have

$$|u-u'|^2 + |\langle u-u', (\bar\partial r)(u)\rangle| \leq C_{8.4}|u-u'| \leq C_{8.4}C_1(-r(u')) = C_{8.4}C_1 2^{-(j+(3/2))m}.$$

Applying Lemma 8.4 again and recalling (8.1), we have

$$\begin{aligned}|u-p(u')|^2+|\langle u-p(u'), (\bar\partial r)(u)\rangle| &\leq 3\{|u-u'|^2 + |\langle u-u', (\bar\partial r)(u)\rangle|\} + C_{8.4}|u'-p(u')| \\ &\leq 3C_{8.4}C_1 2^{-(j+(3/2))m} + C_{8.4}C_p 2^{-(j+(3/2))m}.\end{aligned}$$

Let $M_{8.5} \geq M_1$ be such that $M_{8.5} \geq 3C_{8.4}C_1 + C_{8.4}C_p$. If $m \geq M_{8.5}$, then $u' \in A_{m,j,u}$. □

Lemma 8.6. *There is a constant $M_{8.5} + 100 \leq M_{8.6} < \infty$ such that for $m \geq M_{8.6}$, $j \geq 1$, and $u \in E_{m,j}$, if $z \in A_{m,j,u}$ and $w \in \Omega\backslash B_{m,j,u}$, then $d(z,w) \geq (1/13)m$.*

Proof. Set $M_1 = \max\{M_{8.5} + 100, 10C_{8.4}\}$, where $C_{8.4}$ and $M_{8.5}$ are the constants in Lemmas 8.4 and 8.5 respectively. Consider any $m \geq M_1$, $j \geq 1$ and $u \in E_{m,j}$. For a pair of $z \in A_{m,j,u}$ and $w \in \Omega\backslash B_{m,j,u}$, there are three possibilities, depending on the value of $r(w)$.

(1) Suppose that $-r(w) \geq 2^{-jm}$. Then $r(z)/r(w) \leq 2^{-(j+1)m}/2^{-jm} = 2^{-m}$. Combining this with Lemma 2.1, we have $c_{2.1}2^{-4d(w,z)} \leq r(z)/r(w) \leq 2^{-m}$. Hence

$$d(z,w) \geq (1/4)m + (1/4)\{\log c_{2.1}/\log 2\}.$$

Let $M_2 \geq M_1$ be such that $(1/2)M_2 \geq |\log c_{2.1}/\log 2|$. Thus if $m \geq M_2$, then for all $j \geq 1$, $u \in E_{m,j}$, $z \in A_{m,j,u}$ and $w \in \Omega\backslash B_{m,j,u}$, we have

$$d(z,w) \geq (1/8)m \tag{8.9}$$

under the condition $-r(w) \geq 2^{-jm}$.

(2) Suppose that $-r(w) \leq 2^{-(j+3)m}$. Then $r(w)/r(z) \leq 2^{-(j+3)m}/2^{-(j+2)m} = 2^{-m}$. From Lemma 2.1 we now deduce $c_{2.1}2^{-4d(z,w)} \leq r(w)/r(z) \leq 2^{-m}$. Thus (8.9) again holds under the condition $-r(w) \leq 2^{-(j+3)m}$ when $m \geq M_2$.

(3) Suppose that $2^{-(j+3)m} < -r(w) < 2^{-jm}$. Then by the definition of $B_{m,j,u}$ we have $p(w) \notin Q(u, b_{m,j})$. In contrast, since $z \in A_{m,j,u}$, we have $p(z) \in Q(u, a_{m,j})$. Since $b_{m,j} = C_{8.1}a_{m,j}$, by Corollary 8.2 we have $p(w) \notin Q(p(z), a_{m,j})$. Recall that $a_{m,j} = C_{8.1}m2^{-jm}$ and that $C_{8.1} \geq 1$. Thus it follows from Lemma 8.4 and (8.1) that

$$\begin{aligned} m2^{-mj} \leq a_{m,j} &\leq |p(z) - p(w)|^2 + |\langle p(z) - p(w), (\bar{\partial}r)(p(z))\rangle| \\ &\leq 3\{|z - w|^2 + |\langle z - w, (\bar{\partial}r)(z)\rangle|\} + C_{8.4}C_p\{|r(z)| + |r(w)|\} \\ &\leq 3\{|z - w|^2 + |\langle z - w, (\bar{\partial}r)(z)\rangle|\} + C_{8.4}C_p\{2^{-(j+1)m} + 2^{-jm}\}. \end{aligned}$$

Now we pick an $M_3 \geq M_2$ such that $M_3 \geq 4C_{8.4}C_p$, i.e., $(1/2)M_3 \geq 2C_{8.4}C_p$. When $m \geq M_3$, elementary manipulations turn the above into the inequality

$$(1/6)m2^{-mj} \leq |z - w|^2 + |\langle z - w, (\bar{\partial}r)(z)\rangle|.$$

Combining this with Lemma 2.2, we obtain

$$(1/6)m2^{-mj} \leq C_{2.2}\{d(z, w) + d^2(z, w)\}2^{12d(z,w)}(-r(z)).$$

Since $-r(z) \leq 2^{-(j+1)m}$, this implies

$$(1/6)m2^{m} \leq C_{2.2}\{d(z, w) + d^2(z, w)\}2^{12d(z,w)}.$$

From this inequality it is elementary to deduce that there is an $M_{8.6} \geq M_3$ such that if $m \geq M_{8.6}$, then $d(z, w) \geq (1/13)m$. Combining this with (8.9), the proof is complete. □

By Lemma 8.5, for every triple of $m \geq M_{8.5}$, $j \geq 1$ and $u \in E_{m,j}$, we can pick a

$$z_{m,j,u} \in A_{m,j,u}. \tag{8.10}$$

This pick will be fixed for the rest of the paper.

Lemma 8.7. *There is a constant $M_{8.6} < M_{8.7} < \infty$ such that if $m \geq M_{8.7}$, then there is an $0 < R_m < \infty$ which has the property that*

$$B_{m,j,u} \subset D(z_{m,j,u}, R_m) \tag{8.11}$$

for all $j \geq 1$ and $u \in E_{m,j}$.

Proof. Suppose that $m \geq M_{8.5}$. Given any $j \geq 1$ and $u \in E_{m,j}$, we have $2^{-(j+2)m} \leq -r(z_{m,j,u}) < 2^{-(j+1)m}$ by (8.10). Now let $w \in B_{m,j,u}$. Then $2^{-(j+3)m} < -r(w) < 2^{-jm}$, which means $-2^{-2m}r(z_{m,j,u}) \leq -r(w) \leq -2^{2m}r(z_{m,j,u})$. In other words, we have

$$2^{k-1}(-r(z_{m,j,u})) \leq -r(w) \leq 2^{k}(-r(z_{m,j,u})) \quad \text{for some } k \in \mathbf{Z} \text{ with } |k| \leq 2m. \tag{8.12}$$

We have $p(w) \in Q(u, b_{m,j})$. Since $p(z_{m,j,u}) \in Q(u, a_{m,j}) \subset Q(u, b_{m,j})$, Lemma 8.1 gives us $Q(u, b_{m,j}) \subset Q(p(z_{m,j,u}), C_{8.1}b_{m,j})$. Hence $p(w) \in Q(p(z_{m,j,u}), C_{8.1}b_{m,j})$. That is,

$$|p(z_{m,j,u}) - p(w)|^2 + |\langle p(z_{m,j,u}) - p(w), (\bar{\partial}r)(p(z_{m,j,u}))\rangle| < C_{8.1}b_{m,j}.$$

Applying Lemma 8.4 and (8.1), we obtain

$$\begin{aligned}|z_{m,j,u}-w|^2+&|\langle z_{m,j,u}-w,(\bar\partial r)(z_{m,j,u})\rangle| < 3C_{8.1}b_{m,j}+C_{8.4}C_p(|r(z_{m,j,u})|+|r(w)|)\\ &\le 3C_{8.1}^3m2^{-jm}+2C_{8.4}C_p2^{-jm} < (3C_{8.1}^3m+2C_{8.4}C_p)2^{2m}(-r(z_{m,j,u})).\end{aligned}$$

Let $M_{8.7} > M_{8.6}$ be such that $(3C_{8.1}^3M_{8.7}+2C_{8.4}C_p)2^{-M_{8.7}} \le 1$. When $m \ge M_{8.7}$, the above inequality gives us

$$\begin{aligned}|z_{m,j,u}-w|^2+|\langle z_{m,j,u}-w,(\bar\partial r)(z_{m,j,u})\rangle| &< 2^{3m}(-r(z_{m,j,u}))\\ &\le 2^{k+3m+|k|}(-r(z_{m,j,u})).\end{aligned} \tag{8.13}$$

Combining (8.12) and (8.13) with Lemma 3.7, we obtain $d(z_{m,j,u},w) < C_{3.7}(1+|k|+3m+|k|) \le C_{3.7}(1+7m)$. Thus when $m \ge M_{8.7}$, (8.11) holds for $R_m = C_{3.7}(1+7m)$. □

In the above we picked constants such that $M_{8.7} > M_{8.6} \ge M_{8.5}+100$ and $M_{8.5} > J$. Thus if $m \ge M_{8.7}$, then $m/13 > 7$. Now, for every $m \ge M_{8.7}$, we define the function

$$\tilde f_m(x) = \begin{cases} 1-\{(m/13)-4\}^{-1}x & \text{for} \quad 0 \le x \le (m/13)-4 \\ 0 & \text{for} \quad (m/13)-4 < x < \infty \end{cases}. \tag{8.14}$$

Obviously, $\tilde f_m$ satisfies the Lipschitz condition $|\tilde f_m(x)-\tilde f_m(y)| \le \{(m/13)-4\}^{-1}|x-y|$, $x,y \in [0,\infty)$. For every triple of $m \ge M_{8.7}$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, we define

$$f_{m,j,u}(z) = \tilde f_m(d(z,A_{m,j,u})) \quad \text{for} \quad z \in \Omega.$$

Lemma 8.8. *For every triple of $m \ge M_{8.7}$, $j \in \mathbf{N}$ and $u \in E_{m,j}$, the function $f_{m,j,u}$ defined above has the following five properties*:
(a) *The inequality $0 \le f_{m,j,u} \le 1$ holds on* $\mathbf{B}$.
(b) *$f_{m,j,u} = 1$ on the set $A_{m,j,u}$.*
(c) *$|f_{m,j,u}(z)-f_{m,j,u}(w)| \le \{(m/13)-4\}^{-1}d(z,w)$ for all $z,w \in \Omega$.*
(d) *If $f_{m,j,u}(z) \ne 0$ and $w \in \Omega\backslash B_{m,j,u}$, then $d(z,w) \ge 4$.*
(e) *We have $\operatorname{diff}(f_{m,j,u}) \le \{(m/13)-4\}^{-1}$.*

Proof. (a) and (b) follow directly from the definitions of $\tilde f_m$ and $f_{m,j,u}$. Then note that

$$\begin{aligned}|f_{m,j,u}(z)-f_{m,j,u}(w)| &= |\tilde f_m(d(z,A_{m,j,u}))-\tilde f_m(d(w,A_{m,j,u}))|\\ &\le \frac{1}{(m/13)-4}|d(z,A_{m,j,u})-d(w,A_{m,j,u})| \le \frac{d(z,w)}{(m/13)-4},\end{aligned}$$

which proves (c). For (d), observe that if $f_{m,j,u}(z) \ne 0$, then $d(z,A_{m,j,u}) < (m/13)-4$. This means that there is a $z' \in A_{m,j,u}$ such that $d(z,z') \le (m/13)-4$. If $w \in \Omega\backslash B_{m,j,u}$, then Lemma 8.6 tells us that $d(z',w) \ge m/13$. By the triangle inequality,

$$d(z,w) \ge d(z',w)-d(z,z') \ge (m/13)-\{(m/13)-4\} = 4.$$

Hence (d) holds. Finally, note that (e) is an immediate consequence of (c). □

By (8.7) and a standard maximality argument, each $E_{m,j}$ admits a partition

$$E_{m,j} = E_{m,j}^{(1)} \cup \cdots \cup E_{m,j}^{(N_0)} \tag{8.15}$$

such that for every $\nu \in \{1, \ldots, N_0\}$, we have $Q(u, b_{m,j}) \cap Q(v, b_{m,j}) = \emptyset$ for all $u \neq v$ in $E_{m,j}^{(\nu)}$. Therefore for each $\nu \in \{1, \ldots, N_0\}$, the conditions $u, v \in E_{m,j}^{(\nu)}$ and $u \neq v$ imply $B_{m,j,u} \cap B_{m,j,v} = \emptyset$.

Definition 8.9. Let $m \geq M_{8.7}$ be given. (a) For each pair of $\kappa \in \{1, 2, 3\}$ and $\nu \in \{1, \ldots, N_0\}$, where N_0 is the integer that appears in (8.7) and (8.15), let $I_m^{(\nu,\kappa)}$ denote the collection of all triples $m, 3j + \kappa, u$ satisfying the conditions $j \in \mathbf{Z}_+$ and $u \in E_{m,3j+\kappa}^{(\nu)}$.
(b) For $\kappa \in \{1, 2, 3\}$, $\nu \in \{1, \ldots, N_0\}$ and $q \in \mathbf{N}$, let $I_{m,q}^{(\nu,\kappa)}$ denote the collection of all triples $m, 3j + \kappa, u$ satisfying the conditions $0 \leq j \leq q$ and $u \in E_{m,3j+\kappa}^{(\nu)}$.
(c) Denote $I_m = \cup_{\kappa=1}^{3} \cup_{\nu=1}^{N_0} I_m^{(\nu,\kappa)}$.

The elements in I_m, equivalently the subscripts in $A_{m,j,u}$, $B_{m,j,u}$ and $f_{m,j,u}$, are obviously quite cumbersome to write as triples. For this we have the following remedy:

Notation 8.10. (1) We will use the symbol ω to represent the triple m, j, u.
(2) For any subset I of I_m, denote $f_I = \sum_{\omega \in I} f_\omega$ and $F_I = \sum_{\omega \in I} f_\omega^2$.

Lemma 8.11. *Let $m \geq M_{8.7}$, $\kappa \in \{1, 2, 3\}$ and $\nu \in \{1, \ldots, N_0\}$. Then for any $\omega \neq \omega'$ in $I_m^{(\nu,\kappa)}$, we have $B_\omega \cap B_{\omega'} = \emptyset$.*

Proof. If $\omega = (m, 3j + \kappa, u)$ and $\omega' = (m, 3j + \kappa, v)$ for a pair of $u \neq v$ in $E_{m,3j+\kappa}^{(\nu)}$, then by the property of the partition (8.15) we already know that $B_\omega \cap B_{\omega'} = \emptyset$. The other possibility is that $\omega = (m, 3j + \kappa, u)$ and $\omega' = (m, 3j' + \kappa, v)$ with $u \in E_{m,3j+\kappa}^{(\nu)}$ and $v \in E_{m,3j'+\kappa}^{(\nu)}$, where $j \neq j'$. If $j \neq j'$, then $|(3j + \kappa) - (3j' + \kappa)| \geq 3$, which ensures $B_\omega \cap B_{\omega'} = \emptyset$ by the values of $-r$ on B_ω and $B_{\omega'}$. □

Lemma 8.12. *Let $m \geq M_{8.7}$, $\kappa \in \{1, 2, 3\}$ and $\nu \in \{1, \ldots, N_0\}$. Then for every subset I of $I_m^{(\nu,\kappa)}$, we have $f_I \in \Phi(2^{-m}; ((m/13) - 4)^{-1})$.*

Proof. Let $I \subset I_m^{(\nu,\kappa)}$. For each $\omega \in I$, f_ω is continuous on Ω and satisfies the condition $0 \leq f_\omega \leq 1$. Lemma 8.11 tells us that for $\omega \neq \omega'$ in I, we have $B_\omega \cap B_{\omega'} = \emptyset$. By Lemma 8.8(d), if $z, w \in \Omega$ are such that $f_\omega(z) \neq 0$ and $f_{\omega'}(w) \neq 0$, then $d(z, w) \geq 4$. It follows that f_I is continuous on $\mathbf{B}$ and that $0 \leq f_I \leq 1$. Furthermore, we can invoke Lemma 7.11 to obtain $\mathrm{diff}(f_I) \leq \sup_{\omega \in I} \mathrm{diff}(f_\omega) \leq ((m/13) - 4)^{-1}$, where the second $\leq$ follows from Lemma 8.8(e).

Since $I \subset I_m^{(\nu,\kappa)}$, if $\omega \in I$, then $B_\omega \subset H_{2^{-\kappa m}} = \{\zeta \in \Omega : -r(\zeta) < 2^{-\kappa m}\}$. Since Lemma 8.8(d) says that $f_\omega = 0$ on $\Omega \backslash B_\omega$, we conclude that $f_I = 0$ on $\{\zeta \in \Omega : -r(\zeta) \geq 2^{-\kappa m}\}$. Recalling Definition 7.9, this completes the verification of the membership $f_I \in \Phi(2^{-m}; ((m/13) - 4)^{-1})$. □

Lemma 8.13. *Let $m \geq M_{8.7}$, $\kappa \in \{1,2,3\}$ and $\nu \in \{1,\dots,N_0\}$, and let I be any subset of $I_m^{(\nu,\kappa)}$. Then for every bounded operator A on $L^2_a(\Omega)$, we have*

$$\sum_{\omega \in I} T_{f_\omega} A T_{f_\omega} \in \mathrm{LOC}(A).$$

Proof. Given any $I \subset I_m^{(\nu,\kappa)}$, consider the set $\Gamma = \{z_\omega : \omega \in I\}$, where z_ω was picked in (8.10). By Lemmas 8.11 and 8.6, Γ is an $(m/26)$-separated set in Ω. Define $f_{z_\omega} = f_\omega$ for each $\omega \in I$. We need to verify that the functions $\{f_{z_\omega} : z_\omega \in \Gamma\}$ satisfy conditions (1)-(3) in Definition 6.5. First of all, (2) follows from Lemma 8.8(a). Lemma 8.7 tells us that for each $\omega \in I$, we have $B_\omega \subset D(z_\omega, R_m)$. By Lemma 8.8(d), we have $f_{z_\omega} = 0$ on $\Omega \backslash D(z_\omega, R_m)$, verifying (1). Finally, condition (3) follows from Lemma 8.8(c). □

9. The essential commutant of $\{T_f : f \in \mathrm{VO_{bdd}}\}$

Recall that we write $\mathcal{K}$ for the collection of compact operators on the Bergman space $L^2_a(\Omega)$. Furthermore, Proposition 6.2 tells us that $\mathcal{K} \subset \mathcal{T}$. Also recall that for each $f \in L^\infty(\Omega)$, we have the Hankel operator H_f defined by the formula

$$H_f h = (1-P)(fh), \quad h \in L^2_a(\Omega).$$

Proof of Theorem 1.1(i). Obviously, Proposition 7.3 implies that $\mathrm{EssCom}\{T_f : f \in \mathrm{VO_{bdd}}\} \supset \mathcal{T}$. Thus we only need to prove that $\mathrm{EssCom}\{T_f : f \in \mathrm{VO_{bdd}}\} \subset \mathcal{T}$.

Let $X \in \mathrm{EssCom}\{T_f : f \in \mathrm{VO_{bdd}}\}$ be given. To show that $X \in \mathcal{T}$, pick any $\epsilon > 0$. It suffices to produce a decomposition $X = Y + Z$ such that $Y \in \mathcal{T}$ and

$$\|Z\| \leq 3N_0\{16(2 + \|X\|) + \|X\| + 2\}\epsilon, \tag{9.1}$$

where N_0 is the constant that appears in (8.7) and (8.15).

First, we apply Proposition 7.10, which provides a $\delta > 0$ and a $t^* > 0$ such that

$$\|[X, T_f]\| \leq 2\epsilon \quad \text{for every} \quad f \in \Phi(t^*; \delta). \tag{9.2}$$

Then we apply Lemma 7.2, which tells us that there is a $\delta' > 0$ such that

$$\|H_g\| \leq \epsilon \tag{9.3}$$

for every bounded continuous function g on Ω with $\mathrm{diff}(g) \leq \delta'$. With δ, t^* and δ' so fixed, we pick an integer $m \geq M_{8.7}$ satisfying the conditions

$$((m/13) - 4)^{-1} \leq \min\{\epsilon, \delta, \delta'\} \quad \text{and} \quad 2^{-m} \leq t^*. \tag{9.4}$$

With m so fixed, let us consider the function F_{I_m} given in Notation 8.10(2). Since

$$F_{I_m} = \sum_{\kappa=1}^{3} \sum_{\nu=1}^{N_0} F_{I_m^{(\nu,\kappa)}} \tag{9.5}$$

and since by Lemma 8.12 each $F_{I_m^{(\nu,\kappa)}}$ satisfies the inequality $0 \le F_{I_m^{(\nu,\kappa)}} \le 1$ on Ω, we have $0 \le F_{I_m} \le 3N_0$ on Ω. By Lemma 8.8(b) and (8.8), we have $F_{I_m}(z) \ge 1$ whenever $-r(z) < 2^{-2m}$. Thus we have shown that the function

$$h = \chi_{\Omega_{2^{-2m}}} + F_{I_m} \tag{9.6}$$

satisfies the inequality $1 \le h \le 3N_0 + 1$ on Ω, where $\Omega_{2^{-2m}} = \{\zeta \in \Omega : -r(\zeta) \ge 2^{-2m}\}$. This guarantees that the positive Toeplitz operator T_h is both bounded and invertible on $L_a^2(\Omega)$. Moreover, $\|T_h^{-1}\| \le 1$. Since $T_h \in \mathcal{T}$ and $\mathcal{T}$ is a C^*-algebra, we have $T_h^{-1} \in \mathcal{T}$.

By (9.6) and (9.5), we have the decomposition

$$X = XT_hT_h^{-1} = X_0 + \sum_{\kappa=1}^{3}\sum_{\nu=1}^{N_0} X_{\nu,\kappa}, \tag{9.7}$$

where

$$X_0 = XT_{\chi_{\Omega_{2^{-2m}}}}T_h^{-1} \quad \text{and} \quad X_{\nu,\kappa} = XT_{F_{I_m^{(\nu,\kappa)}}}T_h^{-1}$$

for $1 \le \kappa \le 3$ and $1 \le \nu \le N_0$. Obviously, the Toeplitz operator $T_{\chi_{\Omega_{2^{-2m}}}}$ is compact. Hence, by Proposition 6.2, $X_0 \in \mathcal{K} \subset \mathcal{T}$.

We further decompose each $X_{\nu,\kappa}$. To do that, define the operators

$$Y_{\nu,\kappa} = \sum_{\omega \in I_m^{(\nu,\kappa)}} T_{f_\omega}XT_{f_\omega}T_h^{-1} \quad \text{and} \quad A_{\nu,\kappa} = \sum_{\substack{\omega,\omega' \in I_m^{(\nu,\kappa)} \\ \omega \ne \omega'}} T_{f_\omega}XT_{f_{\omega'}}T_h^{-1}. \tag{9.8}$$

Obviously, $Y_{\nu,\kappa} + A_{\nu,\kappa} = T_{f_{I_m^{(\nu,\kappa)}}}XT_{f_{I_m^{(\nu,\kappa)}}}T_h^{-1}$ (cf. Notation 8.10). We further define

$$B_{\nu,\kappa} = [X, T_{f_{I_m^{(\nu,\kappa)}}}]T_{f_{I_m^{(\nu,\kappa)}}}T_h^{-1} + XH^*_{f_{I_m^{(\nu,\kappa)}}}H_{f_{I_m^{(\nu,\kappa)}}}T_h^{-1}. \tag{9.9}$$

It follows from Lemmas 8.8(d) and 8.11 that $F_{I_m^{(\nu,\kappa)}} = f^2_{I_m^{(\nu,\kappa)}}$. For any real-valued $f \in L^\infty(\Omega)$, we have $T_{f^2} = T_f^2 + H_f^*H_f$. Therefore

$$X_{\nu,\kappa} = Y_{\nu,\kappa} + A_{\nu,\kappa} + B_{\nu,\kappa}. \tag{9.10}$$

Since $T_h^{-1} \in \mathcal{T}$, it follows from Lemma 8.13 and Corollary 6.7 that $Y_{\nu,\kappa} \in \mathcal{T}$.

To estimate $\|A_{\nu,\kappa}\|$, first observe that on $L^2(\Omega)$, we have the strong convergence

$$\sum_{\substack{\omega,\omega' \in I_{m,q}^{(\nu,\kappa)} \\ \omega \ne \omega'}} M_{f_\omega}XPM_{f_{\omega'}} \to \sum_{\substack{\omega,\omega' \in I_m^{(\nu,\kappa)} \\ \omega \ne \omega'}} M_{f_\omega}XPM_{f_{\omega'}} \quad \text{as} \quad q \to \infty,$$

where $I_{m,q}^{(\nu,\kappa)}$ was given by Definition 8.9(b). Compressing this strong convergence to $L_a^2(\Omega)$ and using the bound $\|T_h^{-1}\| \le 1$, we see that there is a $q \in \mathbf{N}$ such that

$$\|A_{\nu,\kappa}\| \le 2\|Z_{\nu,\kappa}\|, \quad \text{where} \quad Z_{\nu,\kappa} = \sum_{\substack{\omega,\omega' \in I_{m,q}^{(\nu,\kappa)} \\ \omega \neq \omega'}} T_{f_\omega} X T_{f_{\omega'}}. \tag{9.11}$$

Since $f_\omega f_{\omega'} = 0$ for $\omega \neq \omega'$ in $I_{m,q}^{(\nu,\kappa)}$, by Lemma 6.8, there are complex numbers $\{\gamma_\omega : \omega \in I_{m,q}^{(\nu,\kappa)}\}$ of modulus 1 and a subset I of $I_{m,q}^{(\nu,\kappa)}$ such that if we define

$$F = \sum_{\omega \in I} f_\omega, \quad G = \sum_{\omega \in I_{m,q}^{(\nu,\kappa)} \setminus I} f_\omega, \quad F' = \sum_{\omega \in I} \gamma_\omega f_\omega \quad \text{and} \quad G' = \sum_{\omega \in I_{m,q}^{(\nu,\kappa)} \setminus I} \gamma_\omega f_\omega,$$

then

$$\|Z_{\nu,\kappa}\| \le 4(\|T_{F'} X T_G\| + \|T_{G'} X T_F\|). \tag{9.12}$$

Note that $T_{G'} X T_F = T_{G'}[X, T_F] + T_{G'} T_F X$. We have $F \in \Phi(2^{-m}; ((m/13) - 4)^{-1})$ by Lemma 8.12. Hence it follows from (9.4) and (9.2) that

$$\|T_{G'}[X, T_F]\| \le \|[X, T_F]\| \le 2\epsilon. \tag{9.13}$$

Since $B_\omega \cap B_{\omega'} = \emptyset$ for all $\omega \neq \omega'$ in $I_m^{(\nu,\kappa)}$, we have $G'F = 0$ on Ω, and consequently $T_{G'} T_F = -H^*_{\overline{G'}} H_F$. Since $\mathrm{diff}(F) \le ((m/13) - 4)^{-1}$, by (9.4) and (9.3), we have

$$\|T_{G'} T_F X\| \le \|H_F\| \|X\| \le \|X\|\epsilon.$$

Combining this with (9.13), we see that $\|T_{G'} X T_F\| \le (2 + \|X\|)\epsilon$. The same argument also shows that $\|T_{F'} X T_G\| \le (2 + \|X\|)\epsilon$. Substituting these in (9.12) and recalling (9.11), we obtain

$$\|A_{\nu,\kappa}\| \le 16(2 + \|X\|)\epsilon. \tag{9.14}$$

Next we estimate $\|B_{\nu,\kappa}\|$.

Lemma 8.12 tells us that $\mathrm{diff}(f_{I_m^{(\nu,\kappa)}}) \le ((m/13) - 4)^{-1}$. Combining this with (9.4) and (9.3), and with the fact $\|T_h^{-1}\| \le 1$, we obtain

$$\|X H^*_{f_{I_m^{(\nu,\kappa)}}} H_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1}\| \le \|X\| \|H_{f_{I_m^{(\nu,\kappa)}}}\| \le \|X\|\epsilon.$$

Again, Lemma 8.12 says that $f_{I_m^{(\nu,\kappa)}} \in \Phi(2^{-m}; ((m/13) - 4)^{-1})$. Hence it follows from (9.4) and (9.2) that

$$\|[X, T_{f_{I_m^{(\nu,\kappa)}}}] T_{f_{I_m^{(\nu,\kappa)}}} T_h^{-1}\| \le \|[X, T_{f_{I_m^{(\nu,\kappa)}}}]\| \le 2\epsilon.$$

Recalling (9.9), from the above two inequalities we obtain

$$\|B_{\nu,\kappa}\| \le (\|X\| + 2)\epsilon. \tag{9.15}$$

To summarize, we have shown that for each pair of $1 \le \kappa \le 3$ and $1 \le \nu \le N_0$, we have the decomposition (9.10) where $Y_{\nu,\kappa} \in \mathcal{T}$ and where $A_{\nu,\kappa}$, $B_{\nu,\kappa}$ satisfy estimates (9.14) and (9.15) respectively. Combining (9.10) with (9.7), we have $X = Y + Z$, where

$$Y = X_0 + \sum_{\kappa=1}^{3}\sum_{\nu=1}^{N_0} Y_{\nu,\kappa} \quad \text{and} \quad Z = \sum_{\kappa=1}^{3}\sum_{\nu=1}^{N_0}(A_{\nu,\kappa} + B_{\nu,\kappa}). \tag{9.16}$$

Now, (9.1) follows from (9.14) and (9.15), and we have shown that $Y \in \mathcal{T}$. This completes the proof of part (i) in Theorem 1.1. □

Proposition 9.1. *For $X \in \mathcal{T}$, if $\mathrm{LOC}(X) \subset \mathcal{K}$, then X is compact.*

Proof. Let $X \in \mathcal{T}$ and suppose that $\mathrm{LOC}(X) \subset \mathcal{K}$. As we showed above, for every $\epsilon > 0$, X admits a decomposition $X = Y + Z$, where Y and Z are given by (9.16), with X_0 known to be compact. By (9.8) and Lemma 8.13, the condition $\mathrm{LOC}(X) \subset \mathcal{K}$ implies $Y_{\nu,\kappa} \in \mathcal{K}$. Thus Y is compact. Since Z satisfies (9.1), this shows that X is compact. □

Proposition 9.2. *Let $X \in \mathcal{T}$. Suppose that X has the property that for every $0 < R < \infty$,*

$$\lim_{z\to\partial\Omega} \sup\{|\langle Xk_w, k_z\rangle| : d(z,w) < R\} = 0. \tag{9.17}$$

Then X is a compact operator.

Proof. Recall from Proposition 6.6 that $\mathrm{LOC}(X) \subset \mathcal{D}(X)$. Combining this with Proposition 9.1, it suffices to prove the inclusion $\mathcal{D}_0(X) \subset \mathcal{K}$ under the assumption that (9.17) holds for every $0 < R < \infty$. By Definition 6.3(c), we need to show that the operator

$$T = \sum_{z\in\Gamma} c_z\langle Xk_{\psi(z)}, k_{\varphi(z)}\rangle k_{\varphi(z)} \otimes k_{\psi(z)}$$

is compact, where Γ is a separated set in Ω, $\{c_z : z \in \Gamma\}$ is a bounded set of coefficients, and $\varphi, \psi : \Gamma \to \Omega$ are maps for which there is a $0 \le C < \infty$ such that $d(z, \varphi(z)) \le C$ and $d(z, \psi(z)) \le C$ for every $z \in \Gamma$.

By the assumption on φ, ψ and Lemma 2.11, there is a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k$ such that for each $1 \le j \le k$, the conditions $z, w \in \Gamma_j$ and $z \ne w$ imply $d(\varphi(z), \varphi(w)) > 2$ and $d(\psi(z), \psi(w)) > 2$. Hence for each $1 \le j \le k$, the sets $\{\varphi(z) : z \in \Gamma_j\}$ and $\{\psi(z) : z \in \Gamma_j\}$ are 1-separated. This leads to the decomposition $T = T_1 + \cdots + T_k$, where

$$T_j = \sum_{z\in\Gamma_j} c_z\langle Xk_{\psi(z)}, k_{\varphi(z)}\rangle k_{\varphi(z)} \otimes k_{\psi(z)}$$

for every $1 \le j \le k$. Thus it suffices to show that $T_j \in \mathcal{K}$ for every $1 \le j \le k$. Fix such a j for the moment. For each $\delta > 0$, denote $\Gamma_{j,\delta} = \{z \in \Gamma_j : -r(z) \le \delta\}$. Using an obvious finite-rank approximation and applying Lemma 5.1, for each $\delta > 0$, we have

$$\|T_j\|_{\mathcal{Q}} \le \left\| \sum_{z \in \Gamma_{j,\delta}} c_z \langle X k_{\psi(z)}, k_{\varphi(z)} \rangle k_{\varphi(z)} \otimes k_{\psi(z)} \right\| \le C_{5.1}^2 c \sup_{z \in \Gamma_{j,\delta}} |\langle X k_{\psi(z)}, k_{\varphi(z)} \rangle|,$$

where $c = \sup_{z\in\Gamma} |c_z|$. Since $d(z, \varphi(z)) \le C$ and $d(z, \psi(z)) \le C$ for every $z \in \Gamma$, it follows from (9.17) that the right-hand side tends to 0 as $\delta \downarrow 0$. Thus $\|T_j\|_{\mathcal{Q}} = 0$, i.e., T_j is a compact operator. This completes the proof. □

As an immediate consequence of Proposition 9.2, we have

Corollary 9.3. *Let $X \in \mathcal{T}$. Then X is compact if and only if*

$$\lim_{z\to\partial\Omega} \|X k_z\| = 0.$$

10. The essential commutant of the Toeplitz algebra

We now turn to the proof of part (ii) in Theorem 1.1.

Proposition 10.1. *If $f \in \mathrm{VO}_{\mathrm{bdd}}$, then*

$$\lim_{z\to\partial\Omega} \|(f - f(z))k_z\| = 0.$$

Proof. Let $f \in \mathrm{VO}_{\mathrm{bdd}}$ and consider a large $R > 0$. We have

$$\begin{aligned} \|(f - f(z))k_z\|^2 &= \int_{D(z,R)} |f(w) - f(z)|^2 |k_z(w)|^2 dv(w) \\ &\quad + \int_{\Omega\setminus D(z,R)} |f(w) - f(z)|^2 |k_z(w)|^2 dv(w) \\ &\le \sup_{d(z,w)<R} |f(w) - f(z)|^2 + C_1 \|f\|_\infty^2 \int_{\Omega\setminus D(z,R)} \frac{|r(z)|^{n+1}}{F(z,w)^{2n+2}} dv(w). \end{aligned}$$

Applying Lemma 3.8, there are constants $0 < C_2 < \infty$ and $s > 0$ such that

$$\|(f - f(z))k_z\|^2 \le \sup_{d(z,w)<R} |f(w) - f(z)|^2 + C_2 \|f\|_\infty^2 2^{-sR}. \tag{10.1}$$

Now we use the fact that f has vanishing oscillation: Using the cutoff functions provided by Lemma 7.6, for any $\delta > 0$, we can write $f = f_1 + f_2$, where f_1 has a compact support in Ω and $\mathrm{diff}(f_2) \le \delta$. Combining this fact with Lemma 7.1, we see that

$$\lim_{z\to\partial\Omega} \sup_{d(z,w)<R} |f(w) - f(z)| = 0$$

once an $R > 0$ is given. This and (10.1) together imply that $\|(f - f(z))k_z\| \to 0$ as $z \to \partial\Omega$. This completes the proof. □

Proposition 10.2. *Suppose that* $\{z_j\}$ *and* $\{w_j\}$ *are sequences in* Ω *satisfying the following two conditions*:

(1) $\lim_{j\to\infty} r(z_j) = 0$.

(2) *There is a constant* $0 < C < \infty$ *such that* $d(z_j, w_j) \le C$ *for every* $j \in \mathbf{N}$.

Then for every $A \in \mathrm{EssCom}(\mathcal{T})$ *we have*

$$\lim_{j\to\infty} \|[A, k_{z_j} \otimes k_{w_j}]\| = 0. \tag{10.2}$$

Proof. For the given $\{z_j\}$, $\{w_j\}$ and A, suppose that (10.2) did not hold. Then, replacing $\{z_j\}$, $\{w_j\}$ by subsequences if necessary, we may assume that there is a $c > 0$ such that

$$\lim_{j\to\infty} \|[A, k_{z_j} \otimes k_{w_j}]\| = c. \tag{10.3}$$

We will show that this leads to a contradiction.

By condition (1) and Lemma 2.1, there is a sequence $j_1 < j_2 < \cdots < j_\nu < \cdots$ of natural numbers such that $-r(z_{j_{\nu+1}}) < -r(z_{j_\nu})$ for every $\nu \in \mathbf{N}$ and such that the set $\{z_{j_\nu} : \nu \in \mathbf{N}\}$ is 1-separated. For each $\nu \in \mathbf{N}$, we now define the operator

$$B_\nu = [A, k_{z_{j_\nu}} \otimes k_{w_{j_\nu}}],$$

whose rank is at most 2. By conditions (1), (2) and Lemma 2.1, we also have that $r(w_j) \to 0$ as $j \to \infty$. Thus both sequences of vectors $\{k_{z_j}\}$ and $\{k_{w_j}\}$ converge to 0 weakly in $L^2_a(\Omega)$. Consequently we have the convergence

$$\lim_{\nu\to\infty} B_\nu = 0 \quad \text{and} \quad \lim_{\nu\to\infty} B^*_\nu = 0$$

in the strong operator topology. Thus by (10.3) and Lemma 7.8, there is a subsequence $\nu(1) < \nu(2) < \cdots < \nu(m) < \cdots$ of natural numbers such that the sum

$$B = \sum_{m=1}^{\infty} B_{\nu(m)}$$

converges strongly with $\|B\|_{\mathcal{Q}} = c > 0$. Thus B is not compact. Now define the operator

$$Y = \sum_{m=1}^{\infty} k_{z_{j_{\nu(m)}}} \otimes k_{w_{j_{\nu(m)}}}.$$

Since the set $\{z_{j_\nu} : \nu \in \mathbf{N}\}$ is 1-separated and since condition (2) holds, by Proposition 6.4 we have $Y \in \mathcal{T}$. Since $A \in \mathrm{EssCom}(\mathcal{T})$, the commutator $[A, Y]$ is compact. On the other hand, we clearly have $[A, Y] = B$, which is not compact because $\|B\|_{\mathcal{Q}} > 0$. This gives us the contradiction promised earlier. $\square$

Lemma 10.3. [27,Lemma 5.1] *Let T be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for each unit vector $x \in \mathcal{H}$ we have $\|[T, x \otimes x]\| = \|(T - \langle Tx, x\rangle)x\|$.*

Lemma 10.4. [27,Lemma 5.2] *Let T be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then for every pair of unit vectors $x, y \in \mathcal{H}$ we have*

$$|\langle Tx, x\rangle - \langle Ty, y\rangle| \le \|[T, x \otimes y]\| + \|[T, x \otimes x]\| + \|[T, y \otimes y]\|.$$

For a bounded operator A on $L^2_a(\Omega)$, we define the function

$$\tilde{A}(z) = \langle Ak_z, k_z\rangle, \quad z \in \Omega.$$

Recall that $\tilde{A}$ is commonly called the *Berezin transform* of the operator A.

Proposition 10.5. *If $A \in \mathrm{EssCom}(\mathcal{T})$, then its Berezin transform $\tilde{A}$ is in* $\mathrm{VO}_{\mathrm{bdd}}$.

Proof. It suffices to consider a self-adjoint $A \in \mathrm{EssCom}(\mathcal{T})$. Obviously, $\tilde{A}$ is bounded, and Proposition 4.6 tells us that it is continuous on Ω. If it were true that $\tilde{A} \notin \mathrm{VO}$, then there would be a $c > 0$ and sequences $\{z_j\}$, $\{w_j\}$ in Ω with

$$\lim_{j\to\infty} r(z_j) = 0 \tag{10.4}$$

such that for every $j \in \mathbf{N}$, we have $d(z_j, w_j) \le 1$ and

$$|\langle Ak_{z_j}, k_{z_j}\rangle - \langle Ak_{w_j}, k_{w_j}\rangle| = |\tilde{A}(z_j) - \tilde{A}(w_j)| \ge c. \tag{10.5}$$

But on the other hand, it follows from Lemma 10.4 that

$$|\langle Ak_{z_j}, k_{z_j}\rangle - \langle Ak_{w_j}, k_{w_j}\rangle| \le \|[A, k_{z_j} \otimes k_{w_j}]\| + \|[A, k_{z_j} \otimes k_{z_j}]\| + \|[A, k_{w_j} \otimes k_{w_j}]\|. \tag{10.6}$$

By (10.4) and the condition $d(z_j, w_j) \le 1$, $j \in \mathbf{N}$, we can apply Proposition 10.2 to obtain

$$\lim_{j\to\infty} \|[A, k_{z_j} \otimes k_{w_j}]\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \|[A, k_{z_j} \otimes k_{z_j}]\| = 0. \tag{10.7}$$

By Lemma 2.1, conditions (10.4) and $d(z_j, w_j) \le 1$, $j \in \mathbf{N}$, also imply $\lim_{j\to\infty} r(w_j) = 0$. Thus Proposition 10.2 also provides that

$$\lim_{j\to\infty} \|[A, k_{w_j} \otimes k_{w_j}]\| = 0. \tag{10.8}$$

Obviously, (10.6), (10.7) and (10.8) together contradict (10.5). $\square$

Lemma 10.6. *If $A \in \mathrm{EssCom}(\mathcal{T})$, then*

$$\lim_{z\to\partial\Omega} \|(A - T_{\tilde{A}})k_z\| = 0.$$

Proof. Again, it suffices to consider a self-adjoint $A \in \mathrm{EssCom}(\mathcal{T})$. Then it follows from Lemma 10.3 and Proposition 10.2 that

$$\lim_{z\to\partial\Omega} \|(A - \tilde{A}(z))k_z\| = \lim_{z\to\partial\Omega} \|[A, k_z \otimes k_z]\| = 0.$$

Therefore it suffices to show that

$$\lim_{z\to\partial\Omega} \|(T_{\tilde{A}} - \tilde{A}(z))k_z\| = 0.$$

Since $\|(T_{\tilde{A}} - \tilde{A}(z))k_z\| \le \|(\tilde{A} - \tilde{A}(z))k_z\|$, this follows from Propositions 10.5 and 10.1. □

Finally, we are ready to determine the essential commutant of $\mathcal{T}$.

Proof of Theorem 1.1(ii). Again, it follows from Proposition 7.3 that $\mathrm{EssCom}(\mathcal{T}) \supset \{T_f : f \in \mathrm{VO_{bdd}}\} + \mathcal{K}$.

For the reverse inclusion, consider any $A \in \mathrm{EssCom}(\mathcal{T})$. We need to show that $A \in \{T_f : f \in \mathrm{VO_{bdd}}\} + \mathcal{K}$. We know that $\tilde{A} \in \mathrm{VO_{bdd}}$ from Proposition 10.5. Hence it suffices to show that $A - T_{\tilde{A}}$ is compact. For this we apply Lemma 10.6, which gives us

$$\lim_{z\to\partial\Omega} \|(A - T_{\tilde{A}})k_z\| = 0. \tag{10.9}$$

The membership $A \in \mathrm{EssCom}(\mathcal{T})$ implies, of course, that $A \in \mathrm{EssCom}\{T_f : f \in \mathrm{VO_{bdd}}\}$. Hence Theorem 1.1(i) tells us that $A \in \mathcal{T}$. Consequently, $A - T_{\tilde{A}} \in \mathcal{T}$. By Corollary 9.3, the membership $A - T_{\tilde{A}} \in \mathcal{T}$ and (10.9) together imply that $A - T_{\tilde{A}}$ is compact. □

11. Berezin transform near the boundary

The purpose of this section is to show that condition (9.17) is implied by the vanishing of Berezin transform near $\partial\Omega$. This along with Proposition 9.2 will give us the proof of Theorem 1.2. To begin, we need to fix some necessary constants:

Lemma 11.1. (1) *There is a* $0 < c_0 < 1$ *such that* $z + \mathcal{P}((\bar{\partial}r)(z); 2c_0\sqrt{-r(z)}, -2c_0 r(z)) \subset D(z,1)$ *for every* $z \in \Omega$ *satisfying the condition* $-r(z) < \theta$.
(2) *There is a* $b_0 > 0$ *such that* $D(z, 3b_0) \subset z + \mathcal{P}((\bar{\partial}r)(z); c_0\sqrt{-r(z)}, -c_0 r(z))$ *for every* $z \in \Omega$ *satisfying the condition* $-r(z) < \theta$.
(3) *There is an* $a_0 > 0$ *such that* $z + \mathcal{P}((\bar{\partial}r)(z); a_0\sqrt{-r(z)}, -a_0 r(z)) \subset D(z, b_0)$ *for every* $z \in \Omega$ *satisfying the condition* $-r(z) < \theta$.

Proof. By Proposition 2.4, there is a $0 < c < 1$ such that $z + \mathcal{P}((\bar{\partial}r)(z); c\sqrt{-r(z)}, -cr(z)) \subset D(z,1)$ for every $z \in \Omega$ satisfying the condition $-r(z) < \theta$. Then $c_0 = c/2$ will do for (1).

To prove (2), take any $0 < b < 1/2$ such that $C_{2.5}b < c_0$. By Proposition 2.5, we have

$$D(z,b) \subset z + \mathcal{P}((\bar{\partial}r)(z); c_0\sqrt{-r(z)}, -c_0 r(z))$$

whenever $-r(z) < \theta$. Thus (2) holds for the constant $b_0 = b/3$.

Finally, note that (3) is a direct consequence of Proposition 2.4. □

Once the above constants are fixed, we can introduce the following "polyballs":

Definition 11.2. (1) Let

$$\begin{aligned}\mathcal{P} &= \{(u_1, u_2, \dots, u_u) \in \mathbf{C}^n : |u_1| < a_0 \text{ and } (|u_2|^2 + \cdots + |u_n|^2)^{1/2} < a_0\},\\ \mathcal{Q} &= \{(u_1, u_2 \dots, u_u) \in \mathbf{C}^n : |u_1| \le c_0 \text{ and } (|u_2|^2 + \cdots + |u_n|^2)^{1/2} \le c_0\} \quad \text{and}\\ \mathcal{R} &= \{(u_1, u_2 \dots, u_u) \in \mathbf{C}^n : |u_1| < 2c_0 \text{ and } (|u_2|^2 + \cdots + |u_n|^2)^{1/2} < 2c_0\}.\end{aligned}$$

(2) For each $z \in \Omega$ satisfying the condition $-r(z) < \theta$, let $\mathcal{S}_z$ be the linear transformation on $\mathbf{C}^n$ given by the formula

$$\mathcal{S}_z(u_1, u_2, \dots u_n) = (-r(z)u_1, \sqrt{-r(z)}u_2, \dots, \sqrt{-r(z)}u_n), \quad (u_1, u_2, \dots u_n) \in \mathbf{C}^n.$$

(3) For each $z \in \Omega$ satisfying the condition $-r(z) < \theta$, let $\mathcal{U}_z$ be a unitary transformation on $\mathbf{C}^n$ such that $\mathcal{U}_z\{(0, u_2, \dots, u_n) : u_2, \dots, u_n \in \mathbf{C}\} = \{u \in \mathbf{C}^n : \langle u, (\bar\partial r)(z)\rangle = 0\}$.
(4) For each $z \in \Omega$ satisfying the condition $-r(z) < \theta$, denote $\mathcal{V}_z = \mathcal{U}_z\mathcal{S}_z$.

Proposition 11.3. *Suppose that U is a connected open set in $\mathbf{C}^n$ that is symmetric with respect to conjugation. That is, $(w_1, \dots, w_n) \in U$ if and only if $(\bar w_1, \dots, \bar w_n) \in U$. Let F be an analytic function on the domain $U \times U$ in $\mathbf{C}^n \times \mathbf{C}^n$. If $F(\bar z, z) = 0$ for every $z \in U$, then F is identically zero on $U \times U$.*

Proof. For each $j \in \{1, \dots, n\}$, let e_j denote the vector in $\mathbf{C}^n$ whose j-th component is 1 and whose other components are 0. We then define

$$\begin{aligned}(d_jF)(w, z) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F(w + \overline{(x+iy)e_j}, z + (x+iy)e_j)\Big|_{x=0=y} \quad \text{and}\\ (\partial_jF)(w, z) &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)F(w + (x+iy)e_j, z)\Big|_{x=0=y}\end{aligned}$$

for $j \in \{1, \dots, n\}$ and $w, z \in U$. It is straightforward to verify that for every multi-index $\alpha \in \mathbf{Z}_+^n$, we have $d^\alpha F = \partial^\alpha F$. Since $F(\bar z, z) = 0$ for every $z \in U$, an easy induction on $|\alpha|$ yields $(d^\alpha F)(\bar z, z) = 0$ for every $z \in U$ and every $\alpha \in \mathbf{Z}_+^n$. Thus if we fix any $z \in U$, then $(\partial^\alpha F)(\bar z, z) = 0$ for every $\alpha \in \mathbf{Z}_+^n$. By the standard power-series expansion, this means that the analytic function $f_z(\zeta) = F(\zeta, z)$, $\zeta \in U$, vanishes on a small open ball containing $\bar z$. Since U is connected, we conclude that $f_z = 0$ on U. Since this is true for every $z \in U$, it follows that F is identically zero on $U \times U$. □

Proposition 11.4. *Let A be a bounded operator on the Bergman space $L^2_a(\Omega)$. If*

$$\lim_{z\to\partial\Omega}\langle Ak_z, k_z\rangle = 0, \tag{11.1}$$

then for every given $0 < R < \infty$ we have

$$\lim_{z\to\partial\Omega}\sup\{|\langle Ak_w, k_z\rangle| : w \in D(z, R)\} = 0. \tag{11.2}$$

Proof. Given (11.1), suppose that (11.2) failed for some $0 < R < \infty$. We will show that this results in a contradiction. First of all, the failure of (11.2) for this particular R means that there is an $\epsilon > 0$ and sequences $\{z_j\}$, $\{w_j\}$ in Ω such that

$$\lim_{j\to\infty} r(z_j) = 0 \tag{11.3}$$

and at the same time, $d(z_j, w_j) < R$ and

$$|\langle Ak_{w_j}, k_{z_j}\rangle| \geq \epsilon \tag{11.4}$$

for every $j \geq 1$. Since $d(z_j, w_j) < R$, for every $j \geq 1$ we have a C^1 map $g_j : [0,1] \to \Omega$ such that $g_j(0) = z_j$, $g_j(1) = w_j$ and

$$\int_0^1 \sqrt{\langle \mathcal{B}(g_j(t))g_j'(t), g_j'(t)\rangle}dt \leq R + 1. \tag{11.5}$$

By (11.3), (11.5) and Lemma 2.1, discarding a finite number of j's if necessary, we may assume that $-r(g_j(t)) < \theta$ for all j and $t \in [0,1]$. Thus Lemma 11.1 can be applied on all these paths. With the b_0 provided by Lemma 11.1, we pick an $m \in \mathbf{N}$ such that $(R+1)/m < b_0$. Thus for every $j \geq 1$, there is a partition

$$0 = x_j(0) < x_j(1) < \cdots < x_j(m) = 1$$

of the interval $[0,1]$ such that

$$\int_{x_j(i-1)}^{x_j(i)} \sqrt{\langle \mathcal{B}(g_j(t))g_j'(t), g_j'(t)\rangle}dt \leq \frac{R+1}{m} < b_0 \tag{11.6}$$

for every $1 \leq i \leq m$. Now, for every pair of $j \geq 1$ and $0 \leq i \leq m$, we define

$$z_j^{(i)} = g_j(x_j(i)).$$

In particular, we have $z_j^{(0)} = z_j$ and $z_j^{(m)} = w_j$ for all j.

Recall that we write $K_z(\zeta) = K(\zeta, z)$, which is the (unnormalized) reproducing kernel for $L_a^2(\Omega)$. Let us denote

$$\Phi(w, z) = \langle AK_w, K_z\rangle,$$

$w, z \in \Omega$. For every pair of $j \geq 1$ and $0 \leq i \leq m$, we define the function

$$F_j^{(i)}(\zeta, \xi) = |r(z_j^{(0)})|^{n+1}\Phi\left(z_j^{(i)} + \mathcal{V}_{z_j^{(i)}}\bar{\zeta}, z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\xi\right), \tag{11.7}$$

$\zeta, \xi \in \mathcal{R}$. A review of Definitions 11.2 and 2.3 gives us the identity

$$z_j^{(i)} + \mathcal{V}_{z_j^{(i)}}\mathcal{R} = z_j^{(i)} + \mathcal{P}((\bar{\partial}r)(z_j^{(i)}); 2c_0\sqrt{-r(z_j^{(i)})}, -2c_0 r(z_j^{(i)})). \tag{11.8}$$

Therefore Lemma 11.1 ensures that each $F_j^{(i)}$ is well defined, and it is obviously an analytic function of $\mathcal{R} \times \mathcal{R}$. By (11.8), Lemma 11.1(1) and (11.5), if $w = z_j^{(i)} + \mathcal{V}_{z_j^{(i)}}\bar{\zeta}$ for some $\zeta \in \mathcal{R}$, then $d(w, z_j^{(0)}) \le R+2$. Thus by (4.1) and Lemma 2.1, there is a $C_1 = C_1(R)$ such that

$$|F_j^{(i)}(\zeta, \xi)| \le C_1 \|A\|$$

for all $\zeta, \xi \in \mathcal{R}$, $j \ge 1$ and $0 \le i \le m$. Hence for each $0 \le i \le m$, $\{F_j^{(i)} : j \ge 1\}$ is a normal family of analytic functions on $\mathcal{R} \times \mathcal{R}$. Consequently there is a sequence

$$j_1 < j_2 < \cdots < j_\nu < \cdots$$

in $\mathbf{N}$ such that for every $0 \le i \le m$, the sequence $\{F_{j_\nu}^{(i)}\}_{\nu \in \mathbf{N}}$ is uniformly convergent on each compact subset of $\mathcal{R} \times \mathcal{R}$. For every $0 \le i \le m$, define the function

$$F^{(i)} = \lim_{\nu \to \infty} F_{j_\nu}^{(i)} \tag{11.9}$$

on $\mathcal{R} \times \mathcal{R}$. Next we show that every $F^{(i)}$ is identically zero on $\mathcal{R} \times \mathcal{R}$.

We will accomplish this by an induction on i. First, let us show that $F^{(0)}$ is the zero function. For $j \ge 1$ and $\zeta \in \mathcal{R}$, we have

$$F_j^{(0)}(\bar{\zeta}, \zeta) = |r(z_j^{(0)})|^{n+1} \Phi\left(z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\zeta, z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\zeta\right).$$

As we explained above, (4.1) and Lemma 2.1 together guarantee that

$$|F_j^{(0)}(\bar{\zeta}, \zeta)| \le C_1 \left|\left\langle A k_{z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\zeta}, k_{z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\zeta} \right\rangle\right|.$$

By (11.3) and Lemmas 11.1(1) and 2.1, for each $\zeta \in \mathcal{R}$ we have $r\big(z_j^{(0)} + \mathcal{V}_{z_j^{(0)}}\zeta\big) \to 0$ as $j \to \infty$. Thus, combining the above inequality with (11.1) and (11.9), we find that $F^{(0)}(\bar{\zeta}, \zeta) = 0$ for every $\zeta \in \mathcal{R}$. By Proposition 11.3, $F^{(0)}$ is identically zero on $\mathcal{R} \times \mathcal{R}$.

Now suppose that $0 \le i < m$ and that we have shown that $F^{(i)}$ is identically zero on $\mathcal{R} \times \mathcal{R}$. We need to show that $F^{(i+1)}$ is also identically zero on $\mathcal{R} \times \mathcal{R}$. By (11.6), we have $d(z_j^{(i)}, z_j^{(i+1)}) < b_0$. A review of Definition 11.2 and Lemma 11.1 gives us

$$z_j^{(i+1)} + \mathcal{V}_{z_j^{(i+1)}}\mathcal{P} \subset D(z_j^{(i+1)}, b_0) \subset D(z_j^{(i)}, 3b_0) \subset z_j^{(i)} + \mathcal{V}_{z_j^{(i)}}\mathcal{Q}. \tag{11.10}$$

Let $\xi \in \mathcal{R}$ be given. By (11.7) and (11.10), for any $\zeta \in \mathcal{P}$, there is an $\eta_j(\zeta) \in \mathcal{Q}$ such that

$$F_j^{(i+1)}(\zeta, \xi) = F_j^{(i)}(\eta_j(\zeta), \xi). \tag{11.11}$$

Since $\mathcal{Q}$ is a compact set in $\mathcal{R}$ and since $F^{(i)} = 0$, by (11.9) we have

$$\lim_{\nu\to\infty} \sup\{|F^{(i)}_{j_\nu}(\eta,\xi)| : \eta \in \mathcal{Q}\} = 0.$$

Combining this with (11.11) and (11.9), we find that $F^{(i+1)}(\zeta,\xi) = 0$ for every $\zeta \in \mathcal{P}$. Since $\mathcal{P}$ is a non-empty open subset of $\mathcal{R}$, this implies that $F^{(i+1)}(\zeta,\xi) = 0$ for every $\zeta \in \mathcal{R}$. Since this is true for every $\xi \in \mathcal{R}$, we conclude that $F^{(i+1)}$ is identically zero on $\mathcal{R} \times \mathcal{R}$. This completes the induction on i.

In particular, the above tells us that $F^{(m)} = 0$ on $\mathcal{R} \times \mathcal{R}$, and consequently

$$\lim_{\nu\to\infty} F^{(m)}_{j_\nu}(0,0) = F^{(m)}(0,0) = 0. \tag{11.12}$$

Recalling (11.7), we have

$$F^{(m)}_{j_\nu}(0,0) = |r(z^{(0)}_{j_\nu})|^{n+1}\Phi\big(z^{(m)}_j, z^{(0)}_j\big) = |r(z_{j_\nu})|^{n+1}\langle AK_{w_{j_\nu}}, K_{z_{j_\nu}}\rangle.$$

Since $d(w_{j_\nu}, z_{j_\nu}) < R$, from (4.1) and Lemma 2.1 we obtain

$$|\langle Ak_{w_{j_\nu}}, k_{z_{j_\nu}}\rangle| \le C_2|F^{(m)}_{j_\nu}(0,0)|.$$

This and (11.12) together contradict (11.4). This completes the proof. □

Proof of Theorem 1.2. This follows immediately from Propositions 11.4 and 9.2. □

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