CRITERIA FOR TOEPLITZ OPERATORS ON THE SPHERE

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Abstract. Let $H^2(S)$ be the Hardy space on the unit sphere S in $\mathbf{C}^n$. We show that a set of inner functions Λ is sufficient for the purpose of determining which $A \in \mathcal{B}(H^2(S))$ is a Toeplitz operator if and only if the multiplication operators $\{M_u : u \in \Lambda\}$ on $L^2(S, d\sigma)$ generate the von Neumann algebra $\{M_f : f \in L^\infty(S, d\sigma)\}$.

1. Introduction

Throughout the paper, we denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$ by S. Let σ be the positive, regular Borel measure on S which is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. Furthermore we normalize σ such that $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the closure in $L^2(S, d\sigma)$ of polynomials in the coordinate variables $z_1, \dots, z_n$. Let

$$P : L^2(S, d\sigma) \to H^2(S)$$

be the orthogonal projection. For each $f \in L^\infty(S, d\sigma)$, we have the Toeplitz operator T_f on $H^2(S)$ defined by the formula

$$T_f = PM_f|H^2(S).$$

That is, T_f is the compression of the multiplication operator M_f to the subspace $H^2(S)$.

Toeplitz operators, on various reproducing-kernel Hilbert spaces, have been extensively studied in the literature. This paper concerns one of the most elementary questions in the theory, namely, how does one characterize a Toeplitz operator on $H^2(S)$?

In the case where the complex dimension n equals 1, i.e., in the unit circle case, there is a very simple answer due to Brown and Halmos. In [2], Brown and Halmos showed that if A is a bounded operator on the Hardy space H^2 of the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, then A is a Toeplitz operator if and only if it satisfies the equation

$$T_{\bar{z}} A T_z = A. \tag{1.1}$$

This criterion for Toeplitz operator was later generalized to arbitrary complex dimension n. In [3], Davie and Jewell showed that, for whatever n, a bounded operator A on $H^2(S)$ is a Toeplitz operator if and only if it satisfies the equation

$$\sum_{j=1}^{n} T_{\bar{z}_j} A T_{z_j} = A. \tag{1.2}$$

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By any reasonable standard, (1.2) is a satisfactory generalization of (1.1) to the high-dimensional case. But the fact that (1.2) *is* a satisfactory generalization of (1.1) also causes one to neglect the other side of the story, namely, when $n \geq 2$, (1.2) is really a *different* kind of test for Toeplitz operators. The substantive difference between (1.1) and (1.2) is the simple fact that when $n = 1$, z is an inner function on the unit disc; in contrast, if $n \geq 2$, then the functions $z_1, \ldots, z_n$ are far from being inner.

Let B denote the open unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. Recall that an analytic function u on B is said to be *inner* if

$$\lim_{r \uparrow 1} |u(r\zeta)| = 1 \quad \text{for } \sigma\text{-a.e. } \zeta \in S.$$

As usual, we identify the function u on B with its boundary value on S. In this paper, we consider the problem of characterizing Toeplitz operators in terms of inner functions, which is not addressed by (1.2) in the case $n \geq 2$. When [3] was published in 1977, it was not yet known that there are non-constant inner functions on B in the case $n \geq 2$. Therefore (1.2) was the best that could be managed in terms of characterizing Toeplitz operators at the time. Later, Aleksandrov [1] and Løw [6] showed that there are non-constant inner functions on B for every $n \geq 2$. This makes it possible to consider the problem of characterizing Toeplitz operators in terms of inner functions.

Note that if u is an inner function, then for every $f \in L^\infty(S, d\sigma)$ we have

$$T_{\bar{u}} T_f T_u = T_{\bar{u} f u} = T_f.$$

Thus, in order for an operator A on $H^2(S)$ to be a Toeplitz operator, it is necessary that

$$T_{\bar{u}} A T_u = A \quad \text{for every inner function } u \text{ on } B.$$

Our question is the following. Suppose that Λ is a non-empty set of inner functions on B. If A is a bounded operator on $H^2(S)$ and if it satisfies the condition

$$T_{\bar{u}} A T_u = A \quad \text{for every } u \in \Lambda,$$

can we conclude that A is a Toeplitz operator on $H^2(S)$? Note that even in the case $n = 1$, this question goes beyond the Brown-Halmos criterion for Toeplitz operators.

We will characterize those sets Λ which yield the answer "yes" to the above question. Interestingly, this characterization involves von Neumann algebras and a rare use of the double-commutant relation of Murray and von Neumann. We will then deal with specific Λ's which yield the answer "yes" and specific Λ's which yield the answer "no" to the above question.

To conclude the introduction, let us summarize these specific results.

(1) Let $\mathrm{Aut}(B)$ denote the group of biholomorphic bijections on the ball B. Let u be any non-constant inner function on B. Then the set $\{u \circ \psi : \psi \in \mathrm{Aut}(B)\}$ is an example of

Λ that yields the answer "yes" to the above question. That is, if A is a bounded operator on $H^2(S)$ such that $T_{\overline{u\circ\psi}} A T_{u\circ\psi} = A$ for every $\psi \in \mathrm{Aut}(B)$, then A is a Toeplitz operator.

(2) Suppose that $n \geq 2$. Then for each singleton set $\Lambda = \{u\}$, the answer to the above question is always "no". In other words, for every inner function u on B, there is a bounded operator Y on $H^2(S)$ such that $T_{\bar{u}} Y T_u = Y$ and yet Y is not a Toeplitz operator.

(3) Suppose that $n = 1$. Let u be an inner function on the unit disc. If u does not have the form

$$e^{i\theta}\frac{a-z}{1-\bar{a}z},$$

where $\theta \in \mathbf{R}$ and $|a| < 1$, then there is a bounded operator Y on H^2 such that $T_{\bar{u}} Y T_u = Y$ and yet Y is not a Toeplitz operator. This puts the Brown-Halmos criterion in the proper perspective.

2. Transformations on the Unit Ball

For each $a \in B\backslash\{0\}$, we have the Möbius transform

$$\varphi_a(z) = \frac{1}{1-\langle z,a\rangle}\left\{a - \frac{\langle z,a\rangle}{|a|^2}a - (1-|a|^2)^{1/2}\left(z - \frac{\langle z,a\rangle}{|a|^2}a\right)\right\}, \qquad z \in B.$$

Each φ_a is an involution, i.e., $\varphi_a \circ \varphi_a = \mathrm{id}$ [7,Theorem 2.2.2]. We also define $\varphi_0(z) = -z$ on B. By Theorems 3.3.8 and 2.2.2 in [7], the formula

$$(2.1) \qquad (U_a f)(z) = \frac{(1-|a|^2)^{n/2}}{(1-\langle z,a\rangle)^n} f(\varphi_a(z)), \qquad f \in L^2(S, d\sigma),$$

defines a unitary operator with the property $[U_a, P] = 0$.

Let $\mathcal{U} = \mathcal{U}(n)$ denote the collection of unitary transformations on $\mathbf{C}^n$. For each $V \in \mathcal{U}$, define the operator $W_V : L^2(S, d\sigma) \to L^2(S, d\sigma)$ by the formula

$$(2.2) \qquad (W_V g)(z) = g(Vz),$$

$g \in L^2(S, d\sigma)$. By the invariance of σ, W_V is a unitary operator on $L^2(S, d\sigma)$.

Let $\mathrm{Aut}(B)$ denote the group of biholomorphic bijections on B. If $\psi \in \mathrm{Aut}(B)$ and if $a \in B$ is such that $\psi(a) = 0$, then

$$(2.3) \qquad \psi = V\varphi_a$$

for some $V \in \mathcal{U}$ [7,Theorem 2.2.5]. For such a ψ, set

$$(2.4) \qquad R_\psi = U_a W_V.$$

Then R_ψ is a unitary operator on $L^2(S, d\sigma)$ which has the properties that

$$R_\psi M_f R_\psi^* = M_{f\circ\psi}$$

for every $f \in L^\infty(S, d\sigma)$ and that $[R_\psi, P] = 0$. Consequently, $H^2(S)$ is a reducing subspace for R_ψ, and if we regard R_ψ as a unitary operator on $H^2(S)$, then

$$R_\psi T_f R_\psi^* = T_{f\circ\psi}$$

for every $f \in L^\infty(S, d\sigma)$.

Let $H^\infty(S)$ be the collection of bounded analytic functions on B. As usual, each $h \in H^\infty(S)$ is identified with its boundary value on S. Our first proposition is based on ideas in Section VI of the paper [4] by Feldman and Rochberg, where the main interest was Hankel operators with conjugate analytic symbols.

Proposition 2.1. *Let $h \in H^\infty(S)$. If h is not a constant, then, on the Hilbert space $L^2(S, d\sigma)$, the von Neumann algebra generated by*

$$\{M_{h\circ\psi} : \psi \in \mathrm{Aut}(B)\}$$

equals $\{M_f : f \in L^\infty(S, d\sigma)\}$.

Proof. Let $\partial_1, \ldots, \partial_n$ denote the differentiations with respect to the complex variables $z_1, \ldots, z_n$ respectively. We first show that if h is not a constant, then there exist $\psi_1, \ldots, \psi_n \in \mathrm{Aut}(B)$ such that

$$\partial_j(h \circ \psi_j)(0) \neq 0, \qquad j = 1, \ldots, n. \tag{2.5}$$

For any analytic function g on B, write $\mathrm{grad}(g) = (\partial_1 g, \ldots, \partial_n g)$. If h is not a constant, then there exists an $a \in B$ such that $\mathrm{grad}(h)(a) \neq 0$. That is, $\mathrm{grad}(h)(\varphi_a(0)) \neq 0$. Since the derivative $\varphi_a'(0)$ is an invertible $n \times n$ matrix [7,Theorem 2.2.2(ii)], by the chain rule, we have $\mathrm{grad}(h \circ \varphi_a)(0) \neq 0$. That is, there is at least one $\nu \in \{1, \ldots, n\}$ such that

$$\partial_\nu(h \circ \varphi_a)(0) \neq 0.$$

By simple transpositions of coordinates, we see that there are $V_1, \ldots, V_n \in \mathcal{U}$ such that $\partial_j(h \circ \varphi_a \circ V_j)(0) \neq 0$ for $j = 1, \ldots, n$. Thus (2.5) holds for $\psi_j = \varphi_a \circ V_j = V_j \varphi_{V_j^* a}$, $j = 1, \ldots, n$.

Let $\mathbf{T}^n$ denote the n-dimensional torus $\{(\tau_1, \ldots, \tau_n) : |\tau_1| = \cdots = |\tau_n| = 1\}$. Let dm_n be the Lebesgue measure on $\mathbf{T}^n$ with the normalization $m_n(\mathbf{T}^n) = 1$. Now, for each pair of $j \in \{1, \ldots, n\}$ and $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{T}^n$, define the function

$$\eta_\tau^{(j)}(z_1, \ldots, z_n) = (h \circ \psi_j)(\tau_1 z_1, \ldots, \tau_n z_n).$$

Then, of course, we still have $\eta_\tau^{(j)} \in \{h \circ \psi : \psi \in \mathrm{Aut}(B)\}$. Using power-series expansion, it is straightforward to verify that for each $j \in \{1, \ldots, n\}$,

$$\int \bar{\tau}_j M_{\eta_\tau^{(j)}} \, dm_n(\tau) = \partial_j(h \circ \psi_j)(0) M_{z_j}.$$

By (2.5), this means that the von Neumann algebra generated by $\{M_{h\circ\psi} : \psi \in \mathrm{Aut}(B)\}$ contains $M_{z_1}, \dots, M_{z_n}$. Since $M_{z_1}, \dots, M_{z_n}$ generate the von Neumann algebra $\{M_f : f \in L^\infty(S, d\sigma)\}$, this completes the proof. □

3. Main Results

To better state our results, let us introduce the following terminology:

Definition 3.1. Let Λ be a non-empty set of inner functions. We say that the set Λ is *Toeplitz-determining* if it has the property that for $A \in \mathcal{B}(H^2(S))$, the condition that $T_{\bar{u}} A T_u = A$ for every $u \in \Lambda$ implies that A is a Toeplitz operator on $H^2(S)$.

The following is our main result:

Theorem 3.2. *Let Λ be a non-empty set of inner functions. Let $\mathcal{N}(\Lambda)$ denote the von Neumann algebra generated by*

$$\{M_u : u \in \Lambda\}$$

on the Hilbert space $L^2(S, d\sigma)$. Then the set Λ is Toeplitz-determining if and only if $\mathcal{N}(\Lambda) = \{M_f : f \in L^\infty(S, d\sigma)\}$.

Proof. First, assuming that $\mathcal{N}(\Lambda) = \{M_f : f \in L^\infty(S, d\sigma)\}$, we will show that Λ is Toeplitz-determining. The set Λ is, of course, a subset of $H^2(S)$. Since $H^2(S)$ is separable and since separability is a hereditary property for metric spaces, there is a countable subset Λ_0 of Λ which is dense in Λ with respect to the norm topology of $H^2(S)$. We can always list Λ_0 as

$$\Lambda_0 = \{u_1, u_2, \dots, u_k, \dots\}$$

if we allow the possibility $u_j = u_k$ for distinct j and k.

Now suppose that A is a bounded operator on $H^2(S)$ such that

$$T_{\bar{u}} A T_u = A \quad \text{for every } u \in \Lambda. \tag{3.1}$$

To show that $A = T_\varphi$ for some $\varphi \in L^\infty(S, d\sigma)$, we follow the ideas in the proof of [3,Lemma 2.5]. Define the operator

$$\tilde{A} = A \oplus 0$$

on $L^2(S, d\sigma)$, where the direct sum corresponds to the space decomposition

$$L^2(S, d\sigma) = H^2(S) \oplus \{H^2(S)\}^\perp.$$

For each natural number k, define the operator

$$L_k = \frac{1}{k^k} \sum_{1 \le i_1, \dots, i_k \le k} M_{\bar{u}_k^{i_k}} \cdots M_{\bar{u}_1^{i_1}} \tilde{A} M_{u_1^{i_1}} \cdots M_{u_k^{i_k}}.$$

It is easy to see that if $1 \le j \le k$, then

$$\|M_{\bar{u}_j} L_k M_{u_j} - L_k\| \le 2\frac{k^{k-1}}{k^k}\|\tilde{A}\| = \frac{2}{k}\|A\|. \tag{3.2}$$

Since $\|L_k\| \leq \|\tilde{A}\| = \|A\|$ for every $k \geq 1$, there is a strictly increasing sequence of natural numbers $k(1) < k(2) < \cdots < k(\ell) < \cdots$ such that the limit

$$L = \lim_{\ell \to \infty} L_{k(\ell)}$$

exists in the weak operator topology. Clearly, it follows from (3.2) that $M_{\bar{u}_j} L M_{u_j} - L = 0$ for every $j \geq 1$. Since $|u_j| = 1$ a.e. on S, this means that

$$LM_{u_j} = M_{u_j} L \quad \text{for every } j \geq 1.$$

Since Λ_0 is dense in Λ with respect to the L^2-norm, the above implies that L commutes with $\{M_u : u \in \Lambda\}$. Since each M_u, $u \in \Lambda$, is a unitary operator, it follows that L also commutes with $\{M_{\bar{u}} : u \in \Lambda\}$. The assumption $\mathcal{N}(\Lambda) = \{M_f : f \in L^\infty(S, d\sigma)\}$ then leads to the conclusion that L commutes with $\{M_f : f \in L^\infty(S, d\sigma)\}$. Hence there is a $\varphi \in L^\infty(S, d\sigma)$ such that $L = M_\varphi$.

Thus to complete the proof that A is a Toeplitz operator, it suffices to show that the compression of L to the subspace $H^2(S)$ equals A. Note that for $h, g \in H^2(S)$ and natural numbers $1 \leq i_1, \ldots, i_k \leq k$, we have

$$\begin{aligned}
\langle M_{\bar{u}_k^{i_k}} \cdots M_{\bar{u}_1^{i_1}} \tilde{A} M_{u_1^{i_1}} \cdots M_{u_k^{i_k}} h, g \rangle &= \langle \tilde{A} M_{u_1^{i_1}} \cdots M_{u_k^{i_k}} h, M_{u_1^{i_1}} \cdots M_{u_k^{i_k}} g \rangle \\
&= \langle A T_{u_1}^{i_1} \cdots T_{u_k}^{i_k} h, T_{u_1}^{i_1} \cdots T_{u_k}^{i_k} g \rangle \\
&= \langle T_{\bar{u}_k}^{i_k} \cdots T_{\bar{u}_1}^{i_1} A T_{u_1}^{i_1} \cdots T_{u_k}^{i_k} h, g \rangle = \langle Ah, g \rangle,
\end{aligned}$$

where the last $=$ follows from repeated applications of (3.1). Thus the compression of each L_k to $H^2(S)$ equals A. Hence $A = PL|H^2(S)$ as promised. This proves the "if" part of the theorem.

To prove the "only if" part, let us now assume that $\mathcal{N}(\Lambda) \neq \{M_f : f \in L^\infty(S, d\sigma)\}$. We will find a bounded operator Y on $H^2(S)$ such that $T_{\bar{u}} Y T_u = Y$ for every $u \in \Lambda$ and such that $Y \notin \{T_f : f \in L^\infty(S, d\sigma)\}$.

By the double-commutant relation, the assumption $\mathcal{N}(\Lambda) \neq \{M_f : f \in L^\infty(S, d\sigma)\}$ implies that the commutant of $\mathcal{N}(\Lambda)$ is strictly larger than the commutant of $\{M_f : f \in L^\infty(S, d\sigma)\}$. That is, there is a bounded operator Z which commutes with $\mathcal{N}(\Lambda)$ but which does not commute with $\{M_f : f \in L^\infty(S, d\sigma)\}$.

Now take any non-constant inner function v constructed by Aleksandrov [1] or Løw [6]. By Proposition 2.1, the unitary operators $\{M_{v \circ \psi} : \psi \in \mathrm{Aut}(B)\}$ generate the von Neumann algebra $\{M_f : f \in L^\infty(S, d\sigma)\}$. Since Z does not commute with $\{M_f : f \in L^\infty(S, d\sigma)\}$, it follows that there is an inner function $w \in \{v \circ \psi : \psi \in \mathrm{Aut}(B)\}$ such that

$$M_{\bar{w}} Z M_w \neq Z.$$

We follow the usual multi-index notation [7,page 3]. For each $\alpha \in \mathbf{Z}_+^n$, define the function $\epsilon_\alpha(z) = z^\alpha$ on S. Then, of course, the linear span of $\{\epsilon_\alpha \bar{\epsilon}_\beta : \alpha, \beta \in \mathbf{Z}_+^n\}$ is dense in $L^2(S, d\sigma)$. Hence there are $a, b, c, d \in \mathbf{Z}_+^n$ such that

$$\langle M_{\bar{w}} Z M_w \epsilon_a \bar{\epsilon}_b, \epsilon_c \bar{\epsilon}_d \rangle \neq \langle Z \epsilon_a \bar{\epsilon}_b, \epsilon_c \bar{\epsilon}_d \rangle.$$

Now define the operator $X = M_{\epsilon_d} Z M_{\bar{\epsilon}_b}$. Then the above gives us

$$\langle M_{\bar{w}} X M_w \epsilon_a, \epsilon_c \rangle \neq \langle X \epsilon_a, \epsilon_c \rangle. \tag{3.3}$$

Let Y be the compression of X to the subspace $H^2(S)$. That is, $Y = PX|H^2(S)$. Since $\epsilon_a, \epsilon_c \in H^2(S)$ and since $(1-P)M_w P = 0$, (3.3) tells us that $T_{\bar{w}} Y T_w \neq Y$. Since w is an inner function, this means that Y is not a Toeplitz operator.

On the other hand, since Z is in the commutant of $\mathcal{N}(\Lambda)$, we have $M_{\bar{u}} Z M_u = Z$ for every $u \in \Lambda$. Since $X = M_{\epsilon_d} Z M_{\bar{\epsilon}_b}$, it follows that $M_{\bar{u}} X M_u = X$ for every $u \in \Lambda$. Compressing to the subspace $H^2(S)$, we see that $T_{\bar{u}} Y T_u = Y$ for every $u \in \Lambda$. Thus we have produced the promised Y. This completes the proof. □

As an immediate consequence of Theorem 3.2 and Proposition 2.1, we have

Corollary 3.3. *Let u be a non-constant inner function. Then a bounded operator A on $H^2(S)$ is a Toeplitz operator if and only if*

$$T_{\overline{u \circ \psi}} A T_{u \circ \psi} = A$$

for every $\psi \in \mathrm{Aut}(B)$.

Recall that the unitary operator R_ψ defined by (2.4) has the property that $R_\psi^* T_f R_\psi = T_{f \circ \psi^{-1}}$, $f \in L^\infty(S, d\sigma)$. In other words, an operator A is a Toeplitz operator if and only if $R_\psi^* A R_\psi$ is a Toeplitz operator. In light of this, let us restate Corollary 3.3 as

Corollary 3.4. *Let u be a non-constant inner function. Then a bounded operator A on $H^2(S)$ is a Toeplitz operator if and only if*

$$T_{\bar{u}} R_\psi^* A R_\psi T_u = R_\psi^* A R_\psi$$

for every $\psi \in \mathrm{Aut}(B)$.

4. Set of a Singleton

Let us now consider the case where Λ is a set of a single inner function. In this case, the story is actually simpler in complex dimensions $n \geq 2$.

Proposition 4.1. *Suppose that $n \geq 2$. Then for each inner function u, there is a bounded operator Y on $H^2(S)$ such that $T_{\bar{u}} Y T_u = Y$ and such that Y is not a Toeplitz operator.*

Proof. Let an inner function u be given and let $\mathcal{N}(u)$ denote the von Neumann algebra generated by the single unitary operator M_u. To prove the proposition, according to Theorem 3.2, it suffices to show that $\mathcal{N}(u) \neq \{M_f : f \in L^\infty(S, d\sigma)\}$.

Let $\mathcal{P}$ denote the linear span of all u^j and $\bar{u}^k$, $j, k = 0, 1, 2, \ldots$. Note that since u is uni-modulous, $\mathcal{N}(u)$ is just the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. Because of the assumption $n \geq 2$, we have the function

$$q(z_1, \ldots, z_n) = z_1 \bar{z}_2$$

on S. To complete the proof, it suffices to show that M_q is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. Assuming the contrary, there would be a sequence $\{\xi_i\} \subset \mathcal{P}$ such that $\langle M_{\xi_i}1, q\rangle \to \langle M_q 1, q\rangle$ as $i \to \infty$. In other words, we would have

$$\lim_{i\to\infty} \langle \xi_i, q\rangle = \langle q, q\rangle = \|q\|^2 > 0 \tag{4.1}$$

for some sequence $\{\xi_i\} \subset \mathcal{P}$. But for each integer $j \geq 0$, the analyticity of u^j gives us

$$\langle u^j, q\rangle = \langle z_2 u^j, z_1\rangle = 0.$$

Similarly, for each integer $k \geq 0$, we have

$$\langle \bar{u}^k, q\rangle = \langle z_2, z_1 u^k\rangle = 0.$$

Thus, in the Hilbert space $L^2(S, d\sigma)$, q is orthogonal to the set $\mathcal{P}$. This clearly contradicts (4.1). This contradiction shows that M_q is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. □

Proposition 4.2. *Suppose that $n = 1$. If u is an inner function on the unit disc and if u does not have the form*

$$e^{i\theta}\frac{a - z}{1 - \bar{a}z}, \quad \text{where } \theta \in \mathbf{R} \text{ and } |a| < 1, \tag{4.2}$$

then there is a bounded operator Y on H^2 such that $T_{\bar{u}} Y T_u = Y$ and such that Y is not a Toeplitz operator.

Proof. For the given u, again let $\mathcal{N}(u)$ denote the von Neumann algebra generated by the single unitary operator M_u. To prove the proposition, according to Theorem 3.2, it suffices to show that if u does not have the form (4.2), then $\mathcal{N}(u) \neq \{M_f : f \in L^\infty\}$.

We only need, of course, to consider the case where u is not a constant. Then

$$u = bs,$$

where b is a Blaschke product or a constant, and s is a so-called singular inner function or a constant. Suppose that u does not have the form (4.2). Then either b has at least two zeros (counting multiplicity), or s is a non-trivial singular inner function. For such a u it is well known (and easy to verify) that the dimension of the subspace $H^2 \ominus uH^2$ is at least 2. Hence there is a $q \in H^2 \ominus uH^2$ with $\|q\| \neq 0$ which is orthogonal to the one-dimensional subspace $\mathbf{C}$. That is,

$$q \in H^2, \quad q \perp uH^2, \quad \text{and } q \perp \mathbf{C}. \tag{4.3}$$

Since $\|q\| \neq 0$, there exists a $p \in L^\infty$ such that $\langle p, q\rangle \neq 0$.

As in the previous proof, let $\mathcal{P}$ denote the linear span of all u^j and $\bar{u}^k$, $j, k = 0, 1, 2, \ldots$. Again, $\mathcal{N}(u)$ is just the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. To complete the proof, it suffices to show that M_p is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. Assuming the contrary, there would be a sequence $\{\xi_i\} \subset \mathcal{P}$ such that $\langle M_{\xi_i} 1, q\rangle \to \langle M_p 1, q\rangle$ as $i \to \infty$. In other words, we would have

$$\lim_{i\to\infty} \langle \xi_i, q\rangle = \langle p, q\rangle \neq 0 \tag{4.4}$$

for some sequence $\{\xi_i\} \subset \mathcal{P}$. But clearly, (4.3) implies $q \perp \mathcal{P}$, which contradicts (4.4). This contradiction shows that M_p is not in the weak closure of $\{M_\xi : \xi \in \mathcal{P}\}$. $\square$

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