TOEPLITZ OPERATORS ASSOCIATED WITH MEASURES AND THE DIXMIER TRACE ON THE HARDY SPACE

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Abstract. Let μ be a regular Borel measure on the open unit ball $\mathbf{B}$ in $\mathbf{C}^n$. By a natural formula, it gives rise to a Toeplitz operator T_μ on the Hardy space $H^2(S)$. We characterize the membership of T_μ^s, $0 < s \le 1$, in any norm ideal $\mathcal{C}_\Phi$ that satisfies condition (DQK). The same techniques allow us to compute the Dixmier trace of T_μ when $T_\mu \in \mathcal{C}_1^+$.

1. Introduction

Toeplitz operators are usually associated with symbols that are functions. But in this paper we only consider Toeplitz operators whose symbols are measures. Moreover, the underlying space will be the Hardy space on the sphere.

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Write $d\sigma$ for the standard spherical measure on S with the normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is simply the norm closure of the analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Suppose that μ is a regular Borel measure on the open unit ball $\mathbf{B} = \{z \in \mathbf{C}^n : |z| < 1\}$. On the Hardy space $H^2(S)$, we define the Toeplitz operator T_μ by the formula

$$(1.1) \qquad (T_\mu f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^n} d\mu(w), \quad f \in H^2(S).$$

It is well known that the Toeplitz operator T_μ is bounded on $H^2(S)$ if and only if μ is a Carleson measure for the Hardy space. In the case where $n = 1$, Luecking characterized the membership of T_μ in the Schatten class $\mathcal{C}_p$ for all $0 < p < \infty$ [12]. Recently in [13], Pau and Perälä generalized this Schatten-class characterization to cover all $n \ge 1$.

There are, however, many more important operator ideals other than the Schatten classes. For example, if one is interested in the Dixmier trace [1,5,6,16], one considers the ideal $\mathcal{C}_1^+$, which is strictly larger than the trace class $\mathcal{C}_1$ but contained in every $\mathcal{C}_{1+\epsilon}$, $\epsilon > 0$. In this paper we will take up the task of determining the membership of T_μ in some of these other operator ideals. But, as the reader will see, the techniques required to handle these other ideals are completely different from those employed in [12,13].

First, let us discuss general operator ideals, and the standard reference for these ideals is [11]. Let $\hat{c}$ denote the linear space of sequences $\{a_j\}_{j\in\mathbf{N}}$, where $a_j \in \mathbf{R}$ and for every sequence the set $\{j \in \mathbf{N} : a_j \ne 0\}$ is finite. A *symmetric gauge function* is a map

$$\Phi : \hat{c} \to [0, \infty)$$

Keywords: Hardy space, Toeplitz operator associated with measure, norm ideal, Dixmier trace.
[1] Supported by the National Natural Science Foundation of China (No.11571256 and No.11601400).

that has the following properties:

(a) Φ is a norm on $\hat{c}$.

(b) $\Phi(\{1, 0, \ldots, 0, \ldots\}) = 1$.

(c) $\Phi(\{a_j\}_{j\in\mathbf{N}}) = \Phi(\{|a_{\pi(j)}|\}_{j\in\mathbf{N}})$ for every bijection $\pi : \mathbf{N} \to \mathbf{N}$.

See [11,page 71]. Each symmetric gauge function Φ gives rise to the *symmetric norm*

$$\|A\|_\Phi = \sup_{j\geq 1} \Phi(\{s_1(A), \ldots, s_j(A), 0, \ldots, 0, \ldots\})$$

for bounded operators, where $s_1(A), \ldots, s_j(A), \ldots$ are the singular numbers of A. On any separable Hilbert space $\mathcal{H}$, the set of operators

$$\mathcal{C}_\Phi = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_\Phi < \infty\} \tag{1.2}$$

is a norm ideal [11,page 68]. That is, $\mathcal{C}_\Phi$ has the following properties:

- For any B, $C \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{C}_\Phi$, $BAC \in \mathcal{C}_\Phi$ and $\|BAC\|_\Phi \leq \|B\|\|A\|_\Phi\|C\|$.
- If $A \in \mathcal{C}_\Phi$, then $A^* \in \mathcal{C}_\Phi$ and $\|A^*\|_\Phi = \|A\|_\Phi$.
- For any $A \in \mathcal{C}_\Phi$, $\|A\| \leq \|A\|_\Phi$, and the equality holds when $\text{rank}(A) = 1$.
- $\mathcal{C}_\Phi$ is complete with respect to $\|.\|_\Phi$.

Now an obvious question is, how do we characterize the membership

$$T_\mu \in \mathcal{C}_\Phi \tag{1.3}$$

for the Toeplitz operator defined by (1.1)? Before we discuss this membership problem, let us first look at some classes of examples of $\mathcal{C}_\Phi$.

There are many familiar examples of symmetric gauge functions. For each $1 \leq p < \infty$, the formula $\Phi_p(\{a_j\}_{j\in\mathbf{N}}) = (\sum_{j=1}^\infty |a_j|^p)^{1/p}$ defines a symmetric gauge function on $\hat{c}$, and the corresponding ideal $\mathcal{C}_{\Phi_p}$ defined by (1.2) is just the Schatten class $\mathcal{C}_p$.

But there are plenty of important ideals $\mathcal{C}_\Phi$ beyond the Schatten classes. For each $1 \leq p < \infty$, we have the symmetric gauge function Φ_p^- defined by the formula

$$\Phi_p^-(\{a_j\}_{j\in\mathbf{N}}) = \sum_{j=1}^\infty \frac{|a_{\pi(j)}|}{j^{(p-1)/p}}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq \cdots \geq |a_{\pi(j)}| \geq \cdots$, which exists because each $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$ only has a finite number of nonzero terms. In this case, the ideal $\mathcal{C}_{\Phi_p^-}$ defined by (1.2) is called a Lorentz ideal and often simply denoted by the symbol $\mathcal{C}_p^-$. When $p = 1$, $\mathcal{C}_1^-$ is just the trace class $\mathcal{C}_1$. But when $1 < p < \infty$, $\mathcal{C}_p^-$ is strictly smaller than the Schatten class $\mathcal{C}_p$.

Similarly, for each $1 \leq p < \infty$ we have the symmetric gauge function

$$\Phi_p^+(\{a_j\}_{j\in\mathbf{N}}) = \sup_{j\geq 1} \frac{|a_{\pi(1)}| + |a_{\pi(2)}| + \cdots + |a_{\pi(j)}|}{1^{-1/p} + 2^{-1/p} + \cdots + j^{-1/p}}, \quad \{a_j\}_{j\in\mathbf{N}} \in \hat{c},$$

where, again, $\pi : \mathbf{N} \to \mathbf{N}$ is any bijection such that $|a_{\pi(j)}| \ge |a_{\pi(j+1)}|$ for every $j \in \mathbf{N}$. Then $\mathcal{C}_{\Phi_p^+}$ is usually denoted by the symbol $\mathcal{C}_p^+$, and we will write $\|\cdot\|_p^+$ for $\|\cdot\|_{\Phi_p^+}$. Moreover, for $1 \le p < \infty$ we have

$$\mathcal{C}_p^- \subset \mathcal{C}_p \subset \mathcal{C}_p^+,$$

and, with the only exception $\mathcal{C}_1^- = \mathcal{C}_1$, these inclusions are all strict. Furthermore, for each $1 < p < \infty$, $\mathcal{C}_{p/(p-1)}^+$ is the dual of $\mathcal{C}_p^-$ [11]. Also, it is well known that for each $1 \le p < \infty$, the ideal $\mathcal{C}_p^+$ is not separable with respect to the norm $\|\cdot\|_p^+$.

Because of the structure of the Hardy space $H^2(S)$, it does not appear easy to answer the membership question (1.3) for *all* symmetric gauge functions Φ. We need to impose a condition on Φ. But this condition is satisfied by Φ_p, Φ_p^- and Φ_p^+. Thus we will characterize the memberships $T_\mu \in \mathcal{C}_p^-$ and $T_\mu \in \mathcal{C}_p^+$, and we will do even more. Note that T_μ is a positive operator, so we can consider its powers. Thus, in addition to the membership problem (1.3), we can more generally consider the problem $T_\mu^s \in \mathcal{C}_\Phi$ for $0 < s \le 1$.

The reader will see that our techniques are so general that if we consider the analogue of the membership problem $T_\mu^s \in \mathcal{C}_\Phi$ on the Bergman space $L_a^2(\mathbf{B}, dv)$, then *no* condition needs to be imposed on Φ. In other words, in the Bergman space case our techniques can handle *all* symmetric gauge functions Φ. This is due to the structural difference between $L_a^2(\mathbf{B}, dv)$ and $H^2(S)$, which will be further explained later. But first let us discuss the condition that we do need to impose in the Hardy-space case.

For any $a = \{a_j\}_{j\in\mathbf{N}}$ and $N \in \mathbf{N}$, define the sequence $a^{[N]} = \{a_j^N\}_{j\in\mathbf{N}}$ by the formula

$$a_j^N = a_i \quad \text{if} \ \ (i-1)N + 1 \le j \le iN, \ i \in \mathbf{N}. \tag{1.4}$$

In other words, $a^{[N]}$ is obtained from a by repeating each term N times. Alternately, we can think of $a^{[N]}$ as $a \oplus \cdots \oplus a$, the "direct sum" of N copies of a.

Definition 1.1. [17,Definition 2.2] A symmetric gauge function Φ is said to satisfy condition (DQK) if there exist constants $0 < \theta < 1$ and $0 < \alpha < \infty$ such that

$$\Phi(a^{[N]}) \ge \alpha N^\theta \Phi(a)$$

for every $a \in \hat{c}$ and every $N \in \mathbf{N}$.

Obviously, the symmetric gauge functions Φ_p, $1 < p < \infty$, and $\Phi_1 = \Phi_1^-$ satisfy condition (DQK). In fact, one can think of (DQK) as an inherent property of the Schatten classes. But this is one property that is shared by many other classes:

Proposition 1.2. [17,Proposition 5.1] (Also see [9,Section 6].) *For each* $1 < p < \infty$, *both symmetric gauge functions* Φ_p^- *and* Φ_p^+ *satisfy condition* (DQK).

The case of $\mathcal{C}_1^+$ and Dixmier trace will be considered separately in Sections 7 and 8.

We will determine the membership $T_\mu^s \in \mathcal{C}_\Phi$ for Φ satisfying condition (DQK). Next we discuss the membership criterion, which involves the Bergman-metric structure of $\mathbf{B}$.

Throughout the paper, β denotes the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}, \quad z,w \in \mathbf{B},$$

where φ_z is the Möbius transform of $\mathbf{B}$ [14,Section 2.2]. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z,a) = \{w \in \mathbf{B} : \beta(z,w) < a\}$.

Definition 1.3. (i) Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.
(ii) Let $0 < a < b < \infty$. A subset Γ of $\mathbf{B}$ is said to be an a,b-lattice if it is a-separated and has the property $\cup_{z\in\Gamma} D(z,b) = \mathbf{B}$.
(iii) A subset Γ of $\mathbf{B}$ is simply said to be separated if it is a-separated for some $a > 0$.

Our investigation fits nicely in the following broader context. Given an operator A, particularly an operator on a reproducing-kernel Hilbert space, one is always interested in formulas for its set of singular numbers. But as a practical matter, a formula that is both explicit and exact, is usually not available. Thus one is frequently forced to search for alternatives: are there quantities given by simple formulas that are *equivalent* to $\{s_1(A), s_2(A), \dots, s_j(A), \dots\}$ in some clearly-defined sense?

Intuitively, for the Toeplitz operator T_μ defined by (1.1), if Γ is an a,b-lattice in $\mathbf{B}$, then the set of scalar quantities

$$\left\{\frac{\mu(D(z,b))}{(1-|z|^2)^n} : z \in \Gamma\right\} \tag{1.5}$$

should be equivalent to the set of singular numbers $\{s_1(T_\mu), s_2(T_\mu), \dots, s_j(T_\mu), \dots\}$. The main results of the paper confirm our intuition in two different ways. First, we have

Theorem 1.4. *Suppose that Φ is a symmetric gauge function satisfying condition* (DQK). *Let $0 < s \le 1$, and let $0 < a < b < \infty$ be given such that $b \ge 2a$. Then there exist constants $0 < c \le C < \infty$ which depend only on Φ, s, a, b and the complex dimension n such that*

$$c\Phi\left(\left\{\left(\frac{\mu(D(z,b))}{(1-|z|^2)^n}\right)^s\right\}_{z\in\Gamma}\right) \le \|T_\mu^s\|_\Phi \le C\Phi\left(\left\{\left(\frac{\mu(D(z,b))}{(1-|z|^2)^n}\right)^s\right\}_{z\in\Gamma}\right)$$

for every regular Borel measure μ on $\mathbf{B}$ and every a,b-lattice $\Gamma \subset \mathbf{B}$.

Second, the connection between (1.5) and $\{s_1(T_\mu), s_2(T_\mu), \dots, s_j(T_\mu), \dots\}$ can be seen through Dixmier trace. As it turns out, the techniques that allow us to prove Theorem 1.4, also allow us to compute the Dixmier trace of T_μ when $T_\mu \in \mathcal{C}_1^+$. In fact, to compute the Dixmier trace of T_μ, we just need a more refined version of (1.5), which is understandable because computation is more precise than general estimates. Suppose that Γ is an a,b-lattice in $\mathbf{B}$ with $b \ge 2a$. Then $\mathbf{B}$ admits a partition $\mathbf{B} = \cup_{z\in\Gamma} E_z$ such that $E_z \subset D(z,b)$ for every $z \in \Gamma$. We will show that T_μ has the same Dixmier trace as the diagonal operator

$$\sum_{z\in\Gamma} c_z e_z \otimes e_z,$$

where $\{e_z : z \in \Gamma\}$ is any orthonormal set and

$$c_z = \int_{E_z} \frac{d\mu(w)}{(1-|w|^2)^n}, \tag{1.6}$$

$z \in \Gamma$. In other words, Dixmier trace cannot distinguish between the singular numbers $\{s_j(T_\mu) : j \in \mathbf{N}\}$ and the scalar quantities $\{c_z : z \in \Gamma\}$ explicitly given by (1.6). This fits nicely in our broader context mentioned earlier.

Let us explain a little more of the underlying intuition for both Theorem 1.4 and the computation of Dixmier trace mentioned above. The determining factor here is the behavior of the normalized reproducing kernel k_z for the Hardy space $H^2(S)$. We have

$$\langle k_z, k_w \rangle = \left(\frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{1-\langle w, z\rangle} \right)^n, \tag{1.7}$$

$z, w \in \mathbf{B}$. The most important thing in the above is the power n, which is what distinguishes the Hardy space from other reproducing-kernel Hilbert spaces on $\mathbf{B}$. To prove a result such as Theorem 1.4, one needs control in both radial and spherical directions of a certain decomposition. Of the two, the radial direction is more problematic. If we had a power $n+\epsilon$ in (1.7) for some $\epsilon > 0$, then it would give us enough control in the radial direction to handle all norm ideals $\mathcal{C}_\Phi$. But n itself just misses being enough of a power, if we consider Φ unconditionally. Then came the realization that in the case where Φ satisfies condition (DQK), we can "manufacture" an additional power ϵ for control in the necessary estimates. That is why we are able to prove what we prove in this paper.

In the Bergman-space analogue of (1.7), the corresponding power is $n+1$. That, as we explained above, makes the Bergman-space case a much easier case. More to the point, condition (DQK) is not needed for the analogue of Theorem 1.4 on $L^2_a(\mathbf{B}, dv)$.

To conclude the Introduction, let us briefly describe the rest of the paper. Section 2 contains a number of preliminaries concerning the Bergman metric and related estimates. In Section 3, we state an operator form of the atomic decomposition on $H^2(S)$. Since we need a more precise statement than what can be found in standard references, we work out the details in Section 3.

In Section 4 we present a number of properties of symmetric gauge functions and symmetric norms. We would like to call particular attention to Proposition 4.6, which is how condition (DQK) enters our estimates.

With the above preparations, the upper bound in Theorem 1.4 is proved in Section 5, and the lower bound is proved in Section 6. The proofs of these two bounds are based on various decompositions in terms of radial and spherical coordinates, and judicious regrouping of the terms, which ultimately produce "small factors". The best way to explain this is to take a look at (5.24), where we see two small factors on the right-hand side,

$$2^{-2(s(n+t)-n)p} \quad \text{and} \quad 2^{-2\epsilon n \ell}.$$

The factor $2^{-2(s(n+t)-n)p}$, which represents decay in the spherical direction, is obtained through the use of the modified kernel $\psi_{z,t}$, whereas the factor $2^{-2\epsilon n\ell}$, which represents decay in the radial direction, is obtained through condition (DQK). But it takes the long, tedious work up to (5.24) to actually produce these small factors.

Sections 7 and 8 contain calculations of the Dixmier trace of T_μ when $T_\mu \in \mathcal{C}_1^+$. More specifically, in Section 7 we deal with the case where T_μ is a discrete sum. As it turns out, this discrete case embodies most of the difficulties and is more tedious than the estimates in Section 5. For example, it requires not one, but two applications of Proposition 4.6, which take quite a bit of work to set up. The reason for the added difficulty is that computation of Dixmier trace does not allow the use of the modified kernel $\psi_{z,t}$. Then in Section 8, we deduce the Dixmier trace of a general $T_\mu \in \mathcal{C}_1^+$ from the discrete case in Section 7, which also takes some work.

Finally, in Section 9 we show that the membership criterion in Theorem 1.4 is equivalent to a condition stated in terms of modified Berezin transform.

2. Preliminaries

The work in this paper relies heavily on the Bergman-metric structure of the ball. Let $d\lambda$ denote the standard Möbius invariant measure on $\mathbf{B}$. That is,

$$d\lambda(\zeta) = \frac{dv(\zeta)}{(1-|\zeta|^2)^{n+1}}.$$

Lemma 2.1. (1) *For any pair of* $0 < a < \infty$ *and* $0 < R < \infty$, *there is a natural number* $N = N(a, R)$ *such that for every* a*-separated set* Γ *in* $\mathbf{B}$ *and every* $z \in \mathbf{B}$, *we have*

$$\text{card}\{u \in \Gamma : \beta(u, z) \le R\} \le N.$$

(2) *For any pair of* $0 < a \le R < \infty$, *there is a natural number* $m = m(a, R)$ *such that every* a*-separated set* Γ *in* $\mathbf{B}$ *admits a partition* $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ *with the property that each* Γ_j *is* R*-separated*, $j = 1, \dots, m$.

Proof. (1) is a simple consequence of the fact that, for any $0 < r < \infty$, the value of $\lambda(D(w, r))$ is independent of $w \in \mathbf{B}$. Then, by (1), for any $0 < a \le R < \infty$, there is an $m \in \mathbf{N}$ such that if Γ is any a-separated set in $\mathbf{B}$, then $\text{card}\{u \in \Gamma : \beta(u, v) \le 2R\} \le m$ for every $v \in \Gamma$. By a standard maximality argument, Γ admits a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ such that for every $j \in \{1, \dots, m\}$, the conditions $u, v \in \Gamma_j$ and $u \ne v$ imply $\beta(u, v) > 2R$. Thus each Γ_j is R-separated, proving (2). □

Lemma 2.2. *Given any pair of* $0 < R_1 < \infty$ *and* $0 < R_2 < \infty$, *there is an* $m \in \mathbf{N}$ *which has the following property*: *Suppose that* Γ *is a* 1*-separated set in* $\mathbf{B}$. *Then for each* $z \in D(0, R_1)$, *there is a partition* $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$ *such that for every* $j \in \{1, \dots, m\}$, if $u, v \in \Gamma_j$ *and if* $u \ne v$, *then* $\beta(\varphi_u(z), \varphi_v(z)) > R_2$.

Proof. It suffices to note that for all $z, u, v \in \mathbf{B}$ we have

$$\beta(u, v) \le \beta(u, \varphi_u(z)) + \beta(\varphi_u(z), \varphi_v(z)) + \beta(\varphi_v(z), v) = 2\beta(0, z) + \beta(\varphi_u(z), \varphi_v(z)).$$

Then the desired conclusion follows from Lemma 2.1(2). □

Lemma 2.3. [20,Lemma 2.3] *For all $u, v, x, y \in \mathbf{B}$ we have*

$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_v(y)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_v(y)\rangle|} \le 2e^{\beta(x,0)+\beta(y,0)}\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}.$$

Lemma 2.4. [10,Lemma 3.9] *The inequality $1-|z|^2 \le 4e^{2\beta(z,w)}(1-|w|^2)$ holds for all $z, w \in \mathbf{B}$.*

Lemma 2.5. *For each $t > 0$, there is a constant $C_{2.5} = C_{2.5}(t)$ such that the inequality*

$$\sum_{\substack{v\in\Gamma\\ \beta(v,\xi)\ge R}} \left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+t}(1-|v|^2)^{n/2} \le C_{2.5}e^{-tR/2}(1-|\xi|^2)^{n/2}$$

holds for every 1-separated set Γ in $\mathbf{B}$, every $\xi \in \mathbf{B}$ and every $R \ge 0$.

Proof. This is similar to [20,Lemma 2.4], but we include the details here for the convenience of the reader. If $w \in D(v,1)$, then $v \in D(w,1) = \varphi_w(D(0,1))$. Thus if $w \in D(v,1)$, then $v = \varphi_w(y)$ for some $y \in D(0,1)$. Let $\xi \in \mathbf{B}$. Since $\xi = \varphi_\xi(0)$, from Lemma 2.3 we obtain

$$\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|} \le 2e\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}$$

for every $w \in D(v,1)$. Similarly, for $w \in D(v,1)$, Lemma 2.4 gives us

$$1-|v|^2 \le 4e^2(1-|w|^2).$$

Set $C_1 = (2e)^{n+t}(4e^2)^{n/2}$. Then the above two inequalities lead to

$$\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+t}(1-|v|^2)^{n/2}$$
$$\le C_1\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+t}(1-|w|^2)^{n/2} \tag{2.1}$$

for every $w \in D(v,1)$. Suppose that Γ is a 1-separated set in $\mathbf{B}$. Then by definition $D(v,1)\cap D(v',1) = \emptyset$ for $v \ne v'$ in Γ. Hence for all $\xi \in \mathbf{B}$ and $R \ge 0$ we have

$$\sum_{\substack{v\in\Gamma\\ \beta(v,\xi)\ge R}} \left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+t}(1-|v|^2)^{n/2}$$
$$\le \sum_{\substack{v\in\Gamma\\ \beta(v,\xi)\ge R}} \frac{C_1}{\lambda(D(v,1))}\int_{D(v,1)}\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+t}(1-|w|^2)^{n/2}d\lambda(w)$$
$$\le \frac{C_1}{\lambda(D(0,1))}\int_{\beta(w,\xi)\ge R-1}\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+t}(1-|w|^2)^{n/2}d\lambda(w). \tag{2.2}$$

To estimate the last integral, note that

$$\frac{(1-|\xi|^2)^{1/2}(1-|\varphi_\xi(\zeta)|^2)^{1/2}}{|1-\langle \xi, \varphi_\xi(\zeta)\rangle|} = (1-|\zeta|^2)^{1/2}.$$

Thus, making the substitution $w = \varphi_\xi(\zeta)$ and using the Möbius invariance of $d\lambda$, we obtain

$$\begin{aligned}\int_{\beta(w,\xi)\geq R-1} &\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle \xi, w\rangle|}\right)^{n+t}(1-|w|^2)^{n/2}d\lambda(w)\\ &= \int_{\beta(0,\zeta)\geq R-1}(1-|\zeta|^2)^{(n+t)/2}(1-|\varphi_\xi(\zeta)|^2)^{n/2}d\lambda(\zeta)\\ &= (1-|\xi|^2)^{n/2}\int_{\beta(0,\zeta)\geq R-1}\frac{dv(\zeta)}{|1-\langle \xi,\zeta\rangle|^n(1-|\zeta|^2)^{1-(t/2)}} = (**).\end{aligned}$$

It follows from [14,Proposition 1.4.10] that there is a $C_2 = C_2(t)$ such that

$$\int \frac{d\sigma(x)}{|1-\langle z,x\rangle|^n} \leq \frac{C_2}{(1-|z|^2)^{t/4}} \tag{2.3}$$

for every $z \in \mathbf{B}$. The condition $\beta(0,\zeta) \geq R-1$ implies $1-|\zeta| \leq 2e^{-2R+2}$. Combining (2.3) with the decomposition $dv = 2nr^{2n-1}drd\sigma$ of the volume measure, we have

$$\begin{aligned}\int_{\beta(0,\zeta)\geq R-1}\frac{dv(\zeta)}{|1-\langle \xi,\zeta\rangle|^n(1-|\zeta|^2)^{1-(t/2)}} &\leq \int_{\max\{1-2e^{-2R+2},0\}}^1 \frac{C_2 2nr^{2n-1}dr}{(1-r^2)^{1-(t/4)}}\\ &\leq nC_2\int_{\max\{1-2e^{-2R+2},0\}}^1\frac{dy}{(1-y)^{1-(t/4)}} \leq \frac{4}{t}nC_2(2e^{-2R+2})^{t/4}.\end{aligned}$$

Therefore

$$(**) \leq \frac{4}{t}(2e^2)^{t/4}nC_2e^{-tR/2}(1-|\xi|^2)^{n/2}.$$

Substituting this in (2.2), we conclude that the lemma holds for the constant

$$C_{2.5} = \frac{4n(2e^2)^{t/4}C_1C_2}{t\lambda(D(0,1))}.$$

This completes the proof. □

The proofs in Sections 5-8 rely on a standard radial-spherical decomposition of the ball introduced in [19], which we now review. First of all, the formula

$$d(u,\xi) = |1-\langle u,\xi\rangle|^{1/2}, \qquad u,\xi \in S, \tag{2.4}$$

defines a metric on the unit sphere S [14]. Denote

$$B(u,r) = \{\xi \in S : |1-\langle u,\xi\rangle|^{1/2} < r\}$$

for $u \in S$ and $r > 0$. There is a constant $A_0 \in (2^{-n}, \infty)$ such that

$$\min\{2^{-n}, \pi^{-1}\}r^{2n} \leq \sigma(B(u,r)) \leq A_0 r^{2n} \tag{2.5}$$

for all $u \in S$ and $0 < r \leq \sqrt{2}$ [14,Proposition 5.1.4].

For each integer $k \geq 0$, let $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ be a subset of S which is *maximal* with respect to the property

$$B(u_{k,j}, 2^{-k-1}) \cap B(u_{k,j'}, 2^{-k-1}) = \emptyset \quad \text{for all} \;\; 1 \leq j < j' \leq m(k). \tag{2.6}$$

The maximality of $\{u_{k,1}, \ldots, u_{k,m(k)}\}$ implies that

$$\cup_{j=1}^{m(k)} B(u_{k,j}, 2^{-k}) = S. \tag{2.7}$$

For each pair of $k \geq 0$ and $1 \leq j \leq m(k)$, define the subset

$$T_{k,j} = \{ru : 1 - 2^{-2k} \leq r < 1 - 2^{-2(k+1)}, u \in B(u_{k,j}, 2^{-k})\} \tag{2.8}$$

of $\mathbf{B}$. Let us also introduce the index set

$$I = \{(k,j) : k \geq 0, 1 \leq j \leq m(k)\}. \tag{2.9}$$

However cumbersome the above system is, it is essential for the proofs in Sections 5-8.

Lemma 2.6. [19,Lemma 2.4] *Given any $0 < a < \infty$, there exists a natural number K such that every a-separated set Γ in $\mathbf{B}$ admits a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_K$ which has the property that* $\text{card}(\Gamma_i \cap T_{k,j}) \leq 1$ *for all $i \in \{1, \ldots, K\}$ and $(k,j) \in I$.*

Last but not least, we remind the reader of the following counting lemma:

Lemma 2.7. [18,Lemma 4.1] *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\text{card}\{y \in X : (x,y) \in E\} \leq m \quad \textit{and} \quad \text{card}\{y \in X : (y,x) \in E\} \leq m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup \ldots \cup E_{2m}$$

and such that for each $1 \leq j \leq 2m$, the conditions $(x,y), (x',y') \in E_j$ and $(x,y) \neq (x',y')$ imply both $x \neq x'$ and $y \neq y'$.

3. Discrete sums on the Hardy space

The proof of Theorem 1.4 requires a class of operators on the Hardy space $H^2(S)$ that are constructed from separated sequences and *modified kernel functions*. One can view this section as an operator form of atomic decomposition [21].

First, recall that the formula

$$k_w(\zeta) = \frac{(1-|w|^2)^{n/2}}{(1-\langle \zeta, w\rangle)^n}$$

gives us the normalized reproducing kernel for the Hardy space $H^2(S)$. With that in mind, for each pair of $0 \le t < \infty$ and $w \in \mathbf{B}$, we define

$$\psi_{w,t}(\zeta) = \frac{(1-|w|^2)^{(n/2)+t}}{(1-\langle \zeta, w\rangle)^{n+t}}, \tag{3.1}$$

$\zeta \in \mathbf{B}$. In terms of the multiplier

$$m_w(\zeta) = \frac{1-|w|^2}{1-\langle \zeta, w\rangle}, \tag{3.2}$$

and the normalized reproducing kernel k_w, we have the relation

$$\psi_{w,t} = m_w^t k_w.$$

In particular, $\psi_{w,0} = k_w$. For $t > 0$, we think of $\psi_{w,t}$ as a modified version of k_w. This modification improves the "decaying rate" of the kernel, as can be seen below:

Proposition 3.1. [8,Proposition 3.1] *Given any $t > 0$, there is a constant $0 < C_{3.1} < \infty$ that depends only on t and the complex dimension n such that*

$$|\langle \psi_{z,t}, \psi_{w,t}\rangle| \le C_{3.1}\left(\frac{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w, z\rangle|}\right)^{n+t}$$

for all $z, w \in \mathbf{B}$.

The main purpose of the section is to establish Propositions 3.2 and 3.8 below.

Proposition 3.2. *Given any $t > 0$, there is a constant $0 < C_{3.2} < \infty$ that depends only on t and the complex dimension n such that*

$$\left\| \sum_{w\in\Gamma} \psi_{w,t} \otimes e_w \right\| \le C_{3.2}$$

for every 1-separated set Γ in $\mathbf{B}$, where $\{e_w : w \in \Gamma\}$ is any orthonormal set.

Proof. Given a 1-separated set Γ and an orthonormal set $\{e_w : w \in \Gamma\}$, let us write

$$B = \sum_{w\in\Gamma} \psi_{w,t} \otimes e_w.$$

Then

$$B^*B = \sum_{u,w\in\Gamma} \langle \psi_{w,t}, \psi_{u,t}\rangle e_u \otimes e_w.$$

Consider any vector $h = \sum_{w\in\Gamma} c_w e_w$. We have

$$B^*Bh = \sum_{u\in\Gamma} y_u e_u, \tag{3.3}$$

where

$$y_u = \sum_{w\in\Gamma} \langle \psi_{w,t}, \psi_{u,t}\rangle c_w,$$

$u \in \Gamma$. Applying Proposition 3.1, the Cauchy-Schwarz inequality and the case $R = 0$ in Lemma 2.5, we have

$$\begin{aligned}
|y_u|^2 &\le C_{3.1}^2 \left(\sum_{w\in\Gamma} \left(\frac{(1-|u|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,u\rangle|} \right)^{n+t} |c_w| \right)^2 \\
&\le C_{3.1}^2 \sum_{w\in\Gamma} \left(\frac{(1-|u|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,u\rangle|} \right)^{n+t} (1-|w|^2)^{n/2} \\
&\times \sum_{w\in\Gamma} \left(\frac{(1-|u|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,u\rangle|} \right)^{n+t} \frac{|c_w|^2}{(1-|w|^2)^{n/2}} \\
&\le C_{3.1}^2 C_{2.5} \sum_{w\in\Gamma} \left(\frac{(1-|u|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,u\rangle|} \right)^{n+t} \left(\frac{1-|u|^2}{1-|w|^2} \right)^{n/2} |c_w|^2
\end{aligned}$$

for every $u \in \Gamma$. Applying Lemma 2.5 again with $R = 0$, we have

$$\begin{aligned}
\sum_{u\in\Gamma} |y_u|^2 &\le C_{3.1}^2 C_{2.5} \sum_{w\in\Gamma} \sum_{u\in\Gamma} \left(\frac{(1-|u|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle w,u\rangle|} \right)^{n+t} \left(\frac{1-|u|^2}{1-|w|^2} \right)^{n/2} |c_w|^2 \\
&\le C_{3.1}^2 C_{2.5}^2 \sum_{w\in\Gamma} |c_w|^2.
\end{aligned}$$

By (3.3), this means $\|B^*Bh\|^2 \le C_{3.1}^2 C_{2.5}^2 \|h\|^2$. Since the vector $h = \sum_{w\in\Gamma} c_w e_w$ is arbitrary, it follows that $\|B\| \le (C_{3.1}C_{2.5})^{1/2}$. This completes the proof. □

Proposition 3.3. *Given any $t > 0$, consider the positive operator*

$$R_t = \int \psi_{z,t} \otimes \psi_{z,t} d\lambda(z)$$

on the Hardy space $H^2(S)$. There are constants $0 < a \le b < \infty$ such that $a\|h\|^2 \le \langle R_t h, h\rangle \le b\|h\|^2$ for every $h \in H^2(S)$.

Proof. The upper bound was explicitly stated in [7,Proposition 3.1]. The lower bound was not explicitly stated there, because it was not need in [7]. But the proof of [7,Proposition 3.1] clearly contains the lower bound. Indeed identity (3.6) in [7] gives us

$$\int \psi_{z,t}(w)\overline{\psi_{z,t}(w')} d\lambda(z) = \sum_{k=0}^{\infty} b_{k,t} C_k^{n-1+k} \langle w, w'\rangle^k = \sum_{k=0}^{\infty} b_{k,t} \sum_{|\alpha|=k} e_\alpha(w)\overline{e_\alpha(w')},$$

where $e_\alpha(w) = \left\{ \frac{(n-1+k)!}{\alpha!(n-1)!} \right\}^{1/2} w^\alpha$, $\alpha \in \mathbf{Z}_+^n$, and

$$b_{k,t} = n \left(\frac{\prod_{j=0}^{k-1}(n+t+j)}{k! C_k^{n-1+k}} \right)^2 \frac{(n-1+k)!}{\prod_{j=0}^{n-1+k}(2t+j)}$$

when $k \geq 1$. By standard asymptotic expansion (see, e.g., (3.3) in [7]), there is an $a > 0$ such that $b_{k,t} \geq a$ for every $k \geq 0$. Recall that $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ is the standard orthonormal basis in $H^2(S)$. Therefore the lower bound $R_t \geq a$ holds. □

Let $\mathcal{L}$ be a subset of $\mathbf{B}$ that is maximal with respect to the property of being 1-separated. This $\mathcal{L}$ will be fixed for the rest of the section. Define the function

$$F = \sum_{u \in \mathcal{L}} \chi_{D(u,2)}$$

on $\mathbf{B}$. By Lemma 2.1, there is a natural number $\mathcal{N} \in \mathbf{N}$ such that

$$\operatorname{card}\{v \in \mathcal{L} : D(u,2) \cap D(v,2) \neq \emptyset\} \leq \mathcal{N}$$

for every $u \in \mathcal{L}$. The maximality of $\mathcal{L}$ implies $\cup_{u \in \mathcal{L}} D(u,2) = \mathbf{B}$. Hence the inequality

$$1 \leq F \leq \mathcal{N} \tag{3.4}$$

holds on the unit ball $\mathbf{B}$. For each $t > 0$, define the operator

$$R_t' = \int F(w) \psi_{w,t} \otimes \psi_{w,t} d\lambda(w).$$

By Proposition 3.3 and (3.4), the operator inequality

$$a \leq R_t' \leq b\mathcal{N} \tag{3.5}$$

holds on $H^2(S)$. By the definition of F and the Möbius invariance of $d\lambda$,

$$R_t' = \sum_{u \in \mathcal{L}} \int_{D(u,2)} \psi_{w,t} \otimes \psi_{w,t} d\lambda(w) = \sum_{u \in \mathcal{L}} \int_{D(0,2)} \psi_{\varphi_u(z),t} \otimes \psi_{\varphi_u(z),t} d\lambda(z).$$

Now, for each $z \in \mathbf{B}$, define

$$Y_{z,t} = \sum_{u \in \mathcal{L}} \psi_{\varphi_u(z),t} \otimes \psi_{\varphi_u(z),t}.$$

Thus we have

$$R_t' = \int_{D(0,2)} Y_{z,t} d\lambda(z). \tag{3.6}$$

Definition 3.4. For any $t > 0$ and any separated set Γ in $\mathbf{B}$, we denote

$$E_{\Gamma,t} = \sum_{w\in\Gamma} \psi_{w,t} \otimes \psi_{w,t}.$$

Lemma 3.5. (1) *Given any* $0 < R < \infty$, *there is an* $N = N(R) \in \mathbf{N}$ *which has the following property: For every pair of* $t > 0$ *and* $\xi \in D(0, R)$, *there are* 1*-separated sets* $\Gamma_1, \dots, \Gamma_N$ *in* $\mathbf{B}$ *such that*

$$Y_{\xi,t} = E_{\Gamma_1,t} + \cdots + E_{\Gamma_N,t}.$$

(2) *For every* $0 < r < 1$, *we have* $\sup_{|z|\leq r} \|Y_{z,t}\| < \infty$.

Proof. For (1), it suffices to take the m provided by Lemma 2.2 for the case where $R_1 = R$ and $R_2 = 2$ to be the $N(R)$. Then (2) follows from (1) and Proposition 3.2. □

Lemma 3.6. *Let* $t \geq 0$ *be given. Then there is a constant* $C_{3.6} = C_{3.6}(t)$ *such that*

$$\|\psi_{z,t} - \psi_{w,t}\| \leq C_{3.6}\beta(z, w) \tag{3.7}$$

for all $z, w \in \mathbf{B}$. *Similarly, there is a constant* $C'_{3.6} = C'_{3.6}(t)$ *such that*

$$|\langle\psi_{\gamma,t}, k_z - k_w\rangle| \leq C'_{3.6}\beta(z, w)(1 - |z|^2)^{n/2}|\psi_{\gamma,t}(z)| \tag{3.8}$$

for every $\gamma \in \mathbf{B}$ *and all* $z, w \in \mathbf{B}$ *satisfying the condition* $\beta(z, w) < 1$.

Proof. First of all, by elementary analysis, there is a $C = C(n, t)$ such that

$$\left|1 - \left(\frac{1 - |u|^2}{|1 - \langle u, z\rangle|^2}\right)^{(n/2)+t} \left(\frac{1 - \langle z, u\rangle}{1 - \langle y, u\rangle}\right)^{n+t}\right| \leq C|u| \tag{3.9}$$

for all $u \in D(0, 1)$, $z \in \mathbf{B}$ and $y \in \overline{\mathbf{B}}$.

We have $\|m_z\|_\infty = 1 + |z| \leq 2$, consequently $\|\psi_{z,t}\| \leq 2^t$, $z \in \mathbf{B}$. Thus, to prove (3.7), it suffices to consider $z, w \in \mathbf{B}$ satisfying the condition $\beta(z, w) < 1$. For such a pair of z, w, we can write $w = \varphi_z(\xi)$ with $\beta(0, \xi) = \beta(z, w) < 1$. Then

$$\psi_{w,t}(\zeta) = \psi_{\varphi_z(\xi),t}(\zeta) = \psi_{z,t}(\zeta)\left(\frac{1 - |\varphi_z(\xi)|^2}{1 - |z|^2}\right)^{(n/2)+t} \left(\frac{1 - \langle\zeta, z\rangle}{1 - \langle\zeta, \varphi_z(\xi)\rangle}\right)^{n+t}.$$

By [14,Theorem 2.2.2], if we write $x = \varphi_z(\zeta)$, then $\zeta = \varphi_z(x)$ and

$$\frac{1 - \langle\zeta, z\rangle}{1 - \langle\zeta, \varphi_z(\xi)\rangle} = \frac{1 - \langle\varphi_z(x), \varphi_z(0)\rangle}{1 - \langle\varphi_z(x), \varphi_z(\xi)\rangle} = \frac{1 - \langle z, \xi\rangle}{1 - \langle x, \xi\rangle} = \frac{1 - \langle z, \xi\rangle}{1 - \langle\varphi_z(\zeta), \xi\rangle}.$$

Similarly,

$$\frac{1 - |\varphi_z(\xi)|^2}{1 - |z|^2} = \frac{1 - |\xi|^2}{|1 - \langle\xi, z\rangle|^2}.$$

Thus we can represent $\psi_{w,t}$ as the following "multiplicative perturbation" of $\psi_{z,t}$:

$$\psi_{w,t}(\zeta) = \psi_{z,t}(\zeta)\left(\frac{1-|\xi|^2}{|1-\langle\xi,z\rangle|^2}\right)^{(n/2)+t}\left(\frac{1-\langle z,\xi\rangle}{1-\langle\varphi_z(\zeta),\xi\rangle}\right)^{n+t}. \tag{3.10}$$

Since $\|\psi_{z,t}\| \le 2^t$, combining this identity with (3.9), we find that

$$\|\psi_{z,t}-\psi_{w,t}\| \le 2^t C|\xi|.$$

We have

$$\beta(0,\xi) = \frac{1}{2}\log\frac{1+|\xi|}{1-|\xi|} \ge \frac{1}{2}\log\frac{1}{1-|\xi|}.$$

From this it is elementary to derive that $|\xi| \le 1-e^{-2\beta(0,\xi)} \le 2\beta(0,\xi)$. Hence

$$\|\psi_{z,t}-\psi_{w,t}\| \le 2^t C\cdot 2\beta(0,\xi) = 2^{t+1}C\beta(z,w),$$

which proves (3.7).

To prove (3.8), note that

$$\langle\psi_{\gamma,t}, k_z - k_w\rangle = (1-|z|^2)^{n/2}\psi_{\gamma,t}(z) - (1-|w|^2)^{n/2}\psi_{\gamma,t}(w).$$

Writing $w = \varphi_z(\xi)$ as in the proof of (3.10), we have

$$\begin{aligned}(1-|w|^2)^{n/2}\psi_{\gamma,t}(w) &= (1-|z|^2)^{n/2}\psi_{\gamma,t}(z)\left(\frac{1-|\varphi_z(\xi)|^2}{1-|z|^2}\right)^{n/2}\left(\frac{1-\langle z,\gamma\rangle}{1-\langle\varphi_z(\xi),\gamma\rangle}\right)^{n+t}\\ &= (1-|z|^2)^{n/2}\psi_{\gamma,t}(z)\left(\frac{1-|\xi|^2}{|1-\langle\xi,z\rangle|^2}\right)^{n/2}\left(\frac{1-\langle\xi,z\rangle}{1-\langle\xi,\varphi_z(\gamma)\rangle}\right)^{n+t}.\end{aligned}$$

Combining these identities with an obvious variant of (3.9), (3.8) follows. □

Proposition 3.7. *For any given value $t > 0$, the map $z \mapsto Y_{z,t}$ from $\mathbf{B}$ into $\mathcal{B}(H^2(S))$ is continuous with respect to the operator norm.*

Proof. Let $z \in \mathbf{B}$ and consider $w \in U = D(z,1)$. By Lemma 3.5(2), we have $\sup_{\zeta\in U}\|Y_{\zeta,t}\| < \infty$. To estimate $\|Y_{z,t} - Y_{w,t}\|$, we pick an orthonormal set $\{f_u : u \in \mathcal{L}\}$ and define

$$X_{\zeta,t} = \sum_{u\in\mathcal{L}} \psi_{\varphi_u(\zeta),t} \otimes f_u$$

for each $\zeta \in U$. Since $Y_{\zeta,t} = X_{\zeta,t}X^*_{\zeta,t}$, we have $\sup_{\zeta\in U}\|X_{\zeta,t}\| < \infty$. Thus it suffices to estimate $\|X_{z,t} - X_{w,t}\|^2 = \|(X_{z,t} - X_{w,t})^*(X_{z,t} - X_{w,t})\|$.

To do this, we write $\rho = \beta(z,0)$. Since $w \in D(z,1)$, we have $w \in D(0,\rho+1)$. Then by Lemma 2.2, there is an $m \in \mathbf{N}$ determined by $\rho + 1$ such that $\|X_{z,t} - X_{w,t}\|^2$ is less than or equal to the sum of at most $2m$ terms of the form $\|A(X_{z,t} - X_{w,t})\|$, where

$$A = \sum_{v\in\Gamma} e_v \otimes \psi_{v,t},$$

Γ is a 1-separated set in $\mathbf{B}$, and $\{e_v : v \in \Gamma\}$ is an orthonormal set. Note that

$$A(X_{z,t} - X_{w,t}) = \sum_{(v,u)\in\Gamma\times\mathcal{L}} \langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle e_v \otimes f_u.$$

Thus for each $R > 0$, we can write

$$A(X_{z,t} - X_{w,t}) = S_{z,w;R} + T_{z,w;R}, \tag{3.11}$$

where

$$S_{z,w;R} = \sum_{\substack{(v,u)\in\Gamma\times\mathcal{L}\\ \beta(v,u)\le R}} \langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle e_v \otimes f_u \quad \text{and}$$

$$T_{z,w;R} = \sum_{\substack{(v,u)\in\Gamma\times\mathcal{L}\\ \beta(v,u)> R}} \langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle e_v \otimes f_u.$$

Let $\epsilon > 0$ be given. We first show that there is an $R > 0$ such that

$$\|T_{z,w;R}\| \le \epsilon/2 \quad \text{for every} \quad w \in U = D(z,1). \tag{3.12}$$

To prove this, note that since $\beta(w,0) < \rho + 1$, Lemma 2.3 gives us

$$\frac{(1-|\varphi_u(w)|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle \varphi_u(w), v\rangle|} \le 2e^{\rho+1}\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}$$

for $v \in \Gamma$ and $u \in \mathcal{L}$. A similar inequality holds with $\varphi_u(z)$ in place of $\varphi_u(w)$. Combining these facts with Proposition 3.1, we obtain

$$\begin{aligned}|\langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle| &\le |\langle \psi_{\varphi_u(z),t}, \psi_{v,t}\rangle| + |\langle \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle| \\ &\le C_1\left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+t},\end{aligned}$$

where $C_1 = 2(2e^{\rho+1})^{n+t}C_{3.1}$. Consider an arbitrary vector $h = \sum_{u\in\mathcal{L}} c_u f_u$. Then

$$T_{z,w;R}h = \sum_{v\in\Gamma} y_v e_v, \tag{3.13}$$

where each y_v satisfies the estimate

$$|y_v| \le C_1 \sum_{\substack{u\in\mathcal{L}\\ \beta(v,u)>R}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+t} |c_u|.$$

Applying Lemma 2.5 and the Schur-test as in the proof of Proposition 3.2, we obtain

$$\sum_{v\in\Gamma}|y_v|^2 \le C_1^2C_{2.5}^2e^{-tR}\sum_{u\in\mathcal{L}}|c_u|^2.$$

By (3.13), this means $\|T_{z,w;R}h\|^2 \le C_1^2C_{2.5}^2e^{-tR}\|h\|^2$. Since the vector h is arbitrary, we conclude that $\|T_{z,w;R}\| \le C_1C_{2.5}e^{-tR/2}$. Hence there is an $R > 0$ such that (3.12) holds.

Fix such an R. Next we show that for this fixed R, there is a $0 < \delta < 1$ such that if $\beta(z,w) \le \delta$, then $\|S_{z,w;R}\| \le \epsilon/2$. By (3.11) and (3.12), this will complete our proof. Since Γ and $\mathcal{L}$ are 1-separated, by Lemma 2.1, there is an $N \in \mathbf{N}$ such that

$$\mathrm{card}\{v \in \Gamma : \beta(v,x) \le R\} \le N \quad \text{and} \quad \mathrm{card}\{u \in \mathcal{L} : \beta(u,x) \le R\} \le N$$

for every $x \in \mathbf{B}$. By a standard maximality argument similar to Lemma 2.7, the set

$$E = \{(v,u) \in \Gamma \times \mathcal{L} : \beta(v,u) \le R\}$$

admits a partition $E = E_1 \cup \cdots \cup E_{2N}$ with the property that for every $j \in \{1,\dots,2N\}$, the conditions $(v,u),(v',u') \in E_j$ and $(v,u) \ne (v',u')$ imply both $v \ne v'$ and $u \ne u'$. Accordingly, we have the decomposition

$$S_{z,w;R} = S_1 + \cdots + S_{2N}, \tag{3.14}$$

where

$$S_j = \sum_{(v,u)\in E_j}\langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle e_v \otimes f_u$$

for each $j \in \{1,\dots,2N\}$. The property of E_j ensures that

$$\|S_j\| = \sup_{(v,u)\in E_j}|\langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle|. \tag{3.15}$$

On the other hand, it follows from Lemma 3.6 that

$$\begin{aligned}|\langle \psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}, \psi_{v,t}\rangle| &\le \|\psi_{\varphi_u(z),t} - \psi_{\varphi_u(w),t}\|\|\psi_{v,t}\| \\ &\le 2^tC_{3.6}\beta(\varphi_u(z),\varphi_u(w)) = 2^tC_{3.6}\beta(z,w).\end{aligned}$$

Combining this with (3.14) and (3.15), we find that $\|S_{z,w;R}\| \le 2N2^tC_{3.6}\beta(z,w)$. Thus if we choose $0 < \delta < 1$ such that $2N2^tC_{3.6}\delta \le \epsilon/2$, then for every w satisfying the condition $\beta(z,w) \le \delta$, we have $\|S_{z,w;R}\| \le \epsilon/2$. This completes the proof. □

Proposition 3.8. *Given any $t > 0$, there exists a constant $\delta > 0$ and a finite number of 1-separated sets $\Gamma_1,\dots,\Gamma_m$ in $\mathbf{B}$ such that*

$$\langle E_{\Gamma_1,t}f,f\rangle + \cdots + \langle E_{\Gamma_m,t}f,f\rangle \ge \delta\|f\|^2$$

for every $f \in H^2(S)$.

Proof. The closure of $D(0,2)$ is, of course, a compact subset of $\mathbf{B}$. Recall that we have the integral formula (3.6) for R_t'. It follows from the norm-continuity provided by Proposition 3.7 that the integral on the right-hand side of (3.6) is the limit in operator norm of Riemann sums. In particular, for the $a > 0$ that appears in (3.5), there is a Riemann sum $\mathcal{S}$ such that $\|R_t' - \mathcal{S}\| \le a/2$. Then, by (3.5), the operator inequality

$$\mathcal{S} \ge a/2 \tag{3.16}$$

holds on $H^2(S)$. Since $\mathcal{S}$ is a Riemann sum for the integral in (3.6), there are pairwise disjoint Borel subsets $G_1, \dots, G_\nu$ in $D(0,2)$ and $z_j \in G_j$, $j = 1, \dots, \nu$, such that

$$\mathcal{S} = \lambda(G_1)Y_{z_1,t} + \cdots + \lambda(G_\nu)Y_{z_\nu,t}. \tag{3.17}$$

If we set $\delta = a/\{2\lambda(D(0,2))\}$, then from (3.16) and (3.17) we obtain

$$Y_{z_1,t} + \cdots + Y_{z_\nu,t} \ge \delta.$$

Now an application of Lemma 3.5(1) completes the proof. □

4. Norm ideals and condition (DQK)

We need a number of basic facts about $\|\cdot\|_\Phi$.

Lemma 4.1. [19,Lemma 3.1] *Suppose that $A_1, \dots, A_m$ are finite-rank operators on a Hilbert space $\mathcal{H}$ and let $A = A_1 + \cdots + A_m$. Then for each symmetric gauge function Φ and each $0 < s \le 1$,*

$$\||A|^s\|_\Phi \le 2^{1-s}(\||A_1|^s\|_\Phi + \cdots + \||A_m|^s\|_\Phi).$$

Lemma 4.2. [10,Lemma 3.3] *Let A and B be two bounded operators. Then the inequalities*

$$\||AB|^s\|_\Phi \le \|B\|^s\||A|^s\|_\Phi \quad \text{and} \quad \||BA|^s\|_\Phi \le \|B\|^s\||A|^s\|_\Phi$$

hold for every symmetric gauge function Φ and every $0 < s \le 1$.

Lemma 4.3. [19,Lemma 5.1] *Let $\{A_k\}$ be a sequence of bounded operators on a separable Hilbert space $\mathcal{H}$. If $\{A_k\}$ weakly converges to an operator A, then the inequality*

$$\|A\|_\Phi \le \sup_k \|A_k\|_\Phi$$

holds for each symmetric gauge function Φ.

Recall from [11,page 125] that given a symmetric gauge function Φ, the formula

$$\Phi^*(\{b_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^\infty a_j b_j\right| : \{a_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi(\{a_j\}_{j\in\mathbf{N}}) \le 1\right\}, \quad \{b_j\}_{j\in\mathbf{N}} \in \hat{c},$$

defines the symmetric gauge function that is dual to Φ. Moreover, we have the relation $\Phi^{**} = \Phi$ [11,page 125]. This relation implies that for every $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, we have

$$\Phi(\{a_j\}_{j\in\mathbf{N}}) = \sup\left\{\left|\sum_{j=1}^{\infty} a_j b_j\right| : \{b_j\}_{j\in\mathbf{N}} \in \hat{c}, \Phi^*(\{b_j\}_{j\in\mathbf{N}}) \le 1\right\}. \tag{4.1}$$

Lemma 4.4. *Let Φ be a symmetric gauge function. Suppose that A and B are operators such that $A^*A \in \mathcal{C}_\Phi$ and $B^*B \in \mathcal{C}_\Phi$. Then $AB \in \mathcal{C}_\Phi$. Moreover,*

$$\|AB\|_\Phi \le \{\|A^*A\|_\Phi \|B^*B\|_\Phi\}^{1/2}.$$

Proof. Let Φ^* be the symmetric gauge function that is dual to Φ. Consider any finite-rank operator F. We have the polar decomposition $F = U|F|$, where U is a partial isometry and $|F| = (F^*F)^{1/2}$. We can factor F in the form $F = F_1F_2$, where $F_1 = U|F|^{1/2}$ and $F_2 = |F|^{1/2}$. Note that $\|F_1F_1^*\|_{\Phi^*} = \|F\|_{\Phi^*} = \|F_2^*F_2\|_{\Phi^*}$. Write $\|\cdot\|_2$ for the Hilbert-Schmidt norm. By (7.9) on page 63 in [11] and the duality between Φ and Φ^*, we have

$$\begin{aligned}|\mathrm{tr}(ABF)| &= |\mathrm{tr}(ABF_1F_2)| = |\mathrm{tr}(F_2ABF_1)| \le \|F_2A\|_2\|BF_1\|_2\\ &= \{\mathrm{tr}(A^*F_2^*F_2A)\mathrm{tr}(F_1^*B^*BF_1)\}^{1/2} = \{\mathrm{tr}(F_2^*F_2AA^*)\mathrm{tr}(B^*BF_1F_1^*)\}^{1/2}\\ &\le \{\|F_2^*F_2\|_{\Phi^*}\|AA^*\|_\Phi\|B^*B\|_\Phi\|F_1F_1^*\|_{\Phi^*}\}^{1/2} = \{\|AA^*\|_\Phi\|B^*B\|_\Phi\}^{1/2}\|F\|_{\Phi^*}.\end{aligned}$$

Since this holds for every finite-rank operator F, the lemma now follows from (4.1). □

Suppose that Φ is a symmetric gauge function. For each $1 < p < \infty$, we define

$$\Phi^{(p)}(\{a_j\}_{j\in\mathbf{N}}) = \{\Phi(\{|a_j|^p\}_{j\in\mathbf{N}})\}^{1/p}$$

for $\{a_j\}_{j\in\mathbf{N}} \in \hat{c}$. Using the duality mentioned above, it is easy to verify that $\Phi^{(p)}$ satisfies the triangle inequality and is, therefore, a symmetric gauge function.

Lemma 4.5. *Let Φ be a symmetric gauge function that satisfies condition* (DQK). *Then for every $1 < p < \infty$, the $\Phi^{(p)}$ defined above also satisfies condition* (DQK).

Proof. By Definition 1.1, there are α and θ such that $\Phi(h^{[N]}) \ge \alpha N^\theta \Phi(h)$ for all $h \in \hat{c}$ and $N \in \mathbf{N}$. Let $1 < p < \infty$. Given an $a = \{a_j\}_{j\in\mathbf{N}} \in \hat{c}$, denote $b = \{|a_j|^p\}_{j\in\mathbf{N}}$. Then

$$\Phi^{(p)}(a^{[N]}) = \{\Phi(b^{[N]})\}^{1/p} \ge \{\alpha N^\theta \Phi(b)\}^{1/p} = \alpha^{1/p}N^{\theta/p}\Phi^{(p)}(a)$$

for every $N \in \mathbf{N}$. Thus $\Phi^{(p)}$ satisfies condition (DQK) with constants $\alpha^{1/p}$ and θ/p. □

An obvious question is, how do we actually use condition (DQK) in the proof of Theorem 1.4 and in calculation of Dixmier trace? It will be used in the following way:

Proposition 4.6. *Suppose that Φ is a symmetric gauge function satisfying condition* (DQK), *and let $0 < s \le 1$. Then there exist constants $0 < \epsilon < 1$ and $1 \le C < \infty$ which depend only on Φ and s such that the following estimate holds*: *Let $N \in \mathbf{N}$. Suppose that*

$$A_1, A_2, \ldots, A_j, \ldots$$

are pairwise disjoint subsets of $\mathbf{N}$ *satisfying the condition* $\mathrm{card}(A_j) \le N$ *for every* $j \ge 1$. *Given a sequence* $a = \{a_i\}_{i\in\mathbf{N}}$ *of complex numbers, define*

$$b_j = \left(\frac{1}{N}\sum_{i\in A_j}|a_i|^2\right)^{1/2}$$

for every $j \in \mathbf{N}$. *Then we have*

$$\Phi(\{b_j^s\}_{j\in\mathbf{N}}) \le N^{-\epsilon}C\Phi(\{|a_i|^s\}_{i\in\mathbf{N}}).$$

Proof. By definition, there are $0 < \theta < 1$ and $0 < C < \infty$ such that

$$\Phi(a) \le Cm^{-\theta}\Phi(a^{[m]}) \quad \text{for all} \quad a \in \hat{c} \quad \text{and} \quad m \in \mathbf{N}. \tag{4.2}$$

Given any $N \in \mathbf{N}$, let $M \in \mathbf{N}$ be such that $N^{1/2} \le M < N^{1/2} + 1$. Given any sequence $a = \{a_i\}_{i\in\mathbf{N}}$ of complex numbers, define b_j as above, $j \in \mathbf{N}$. Let $E = \{j \in \mathbf{N} : b_j \ne 0\}$. Obviously, $\Phi(\{b_j^s\}_{j\in\mathbf{N}}) = \Phi(\{b_j^s\}_{j\in E})$. For each $j \in E$, define

$$B_j = \{i \in A_j : |a_i|^2 \ge b_j^2/2\}.$$

Finally, define

$$J_1 = \{j \in E : \mathrm{card}(B_j) > M\} \quad \text{and} \quad J_2 = \{j \in E : \mathrm{card}(B_j) \le M\}.$$

Write $\beta = \{b_j^s\}_{j\in J_1}$. Since $b_j^s \le 2^{s/2}|a_i|^s$ for every $i \in B_j$ and since $B_j \cap B_{j'} = \emptyset$ when $j \ne j'$, we have $\Phi(\beta^{[M]}) \le 2^{s/2}\Phi(\{|a_i|^s\}_{i\in\mathbf{N}})$. Combining this with (4.2), we find that

$$\Phi(\beta) \le CM^{-\theta}\Phi(\beta^{[M]}) \le 2^{s/2}CM^{-\theta}\Phi(\{|a_i|^s\}_{i\in\mathbf{N}}) \le 2^{s/2}CN^{-\theta/2}\Phi(\{|a_i|^s\}_{i\in\mathbf{N}}). \tag{4.3}$$

On the other hand, if $i \in A_j\backslash B_j$, then $|a_i|^2 < b_j^2/2$. Since $\mathrm{card}(A_j) \le N$, we have

$$\frac{1}{N}\sum_{i\in A_j\backslash B_j}|a_i|^2 < \frac{b_j^2}{2}.$$

Consequently, for each $j \in E$,

$$\frac{1}{M}\sum_{i\in B_j}\frac{M}{N}|a_i|^2 = \frac{1}{N}\sum_{i\in B_j}|a_i|^2 \ge \frac{b_j^2}{2}.$$

For each $j \in J_2$, since $\mathrm{card}(B_j) \le M$, the above implies that there is an $i(j) \in B_j$ such that $(M/N)|a_{i(j)}|^2 \ge b_j^2/2$. Obviously, for $j \ne j'$ in J_2 we have $i(j) \ne i(j')$. Hence

$$\Phi(\{b_j^s\}_{j\in J_2}) \le 2^{s/2}(M/N)^{s/2}\Phi(\{|a_{i(j)}|^s\}_{j\in J_2}) \le 2^sN^{-s/4}\Phi(\{|a_i|^s\}_{i\in\mathbf{N}}), \tag{4.4}$$

where for the second $\leq$ we use the fact that $M < N^{1/2} + 1$. Since $E = J_1 \cup J_2$, the proposition follows from (4.3) and (4.4). $\square$

We conclude the section with two basic lemmas.

Lemma 4.7. [19,Lemma 6.2] *If $A_1, \ldots, A_m, \ldots$ are trace-class operators, then the inequality*

$$\|A_1 \oplus \cdots \oplus A_m \oplus \cdots\|_\Phi \leq \Phi(\{\|A_1\|_1, \ldots, \|A_m\|_1, \ldots\})$$

holds for every symmetric gauge function Φ, where $\|\cdot\|_1$ is the norm of the trace class.

Lemma 4.8. [19,Lemma 2.2] *Suppose that X and Y are countable sets and that N is a natural number. Suppose that $T : X \to Y$ is a map that is at most N-to-1. That is, for every $y \in Y$, $\mathrm{card}\{x \in X : T(x) = y\} \leq N$. Then for every set of real numbers $\{b_y\}_{y \in Y}$ and every symmetric gauge function Φ, we have $\Phi(\{b_{T(x)}\}_{x \in X}) \leq N\Phi(\{b_y\}_{y \in Y})$.*

5. Proof of Theorem 1.4 — the upper bound

To prove the upper bound in Theorem 1.4, consider a regular Borel measure μ on $\mathbf{B}$. Given such a μ, we define the measure $\tilde{\mu}$ on $\mathbf{B}$ by the formula

$$d\tilde{\mu}(w) = \frac{d\mu(w)}{(1 - |w|^2)^n}. \tag{5.1}$$

It is straightforward to verify that we have the integral representation

$$T_\mu = \int k_w \otimes k_w d\tilde{\mu}(w)$$

for the Toeplitz operator T_μ defined by (1.1). Let $0 < a \leq b < \infty$. Suppose that Γ is an a, b-lattice in $\mathbf{B}$. We define

$$T_\Gamma = \sum_{z \in \Gamma} \int_{D(z,b)} k_w \otimes k_w d\tilde{\mu}(w).$$

Since $\cup_{z \in \Gamma} D(z, b) = \mathbf{B}$, the operator inequality $T_\mu \leq T_\Gamma$ holds on $H^2(S)$. It follows from this operator inequality that for every $0 < s \leq 1$ and every symmetric gauge function Φ,

$$\|T_\mu^s\|_\Phi \leq \|T_\Gamma^s\|_\Phi.$$

Thus it suffices to estimate $\|T_\Gamma^s\|_\Phi$. But this estimate can be further reduced.

Consider any *finite* subset F of Γ that has the property $\tilde{\mu}(D(z, b)) \neq 0$ for every $z \in F$. For such an F, we define

$$T_F = \sum_{z \in F} \int_{D(z,b)} k_w \otimes k_w d\tilde{\mu}(w).$$

Lemma 4.3 implies that $\|T^s_\Gamma\|_\Phi$ is the supremum of $\|T^s_F\|_\Phi$ over all such possible F's. Thus it suffices to consider an individual T_F.

To estimate $\|T^s_F\|_\Phi$, by Lemmas 2.6 and 4.1, partitioning F by a fixed number of subsets if necessary, we may assume that F has the additional property that

$$\text{card}(F \cap T_{k,j}) \leq 1 \quad \text{for every} \quad (k,j) \in I, \tag{5.2}$$

where $T_{k,j}$ and I are given by (2.8) and (2.9) respectively. For convenience, let us write $c_z = \tilde{\mu}(D(z,b))$ for each $z \in F$. Define the measure

$$d\nu_z(w) = c_z^{-1}\chi_{D(z,b)}(w)d\tilde{\mu}(w) = \frac{\chi_{D(z,b)}(w)}{c_z(1-|w|^2)^n}d\mu(w)$$

for each $z \in F$. Then

$$T_F = \sum_{z\in F} c_z \int k_w \otimes k_w d\nu_z(w). \tag{5.3}$$

Obviously, $d\nu_z$ is a probability measure concentrated on $D(z,b)$. Therefore each $d\nu_z$ is in the weak-* closure of the convex hull of unit point masses on $D(z,b)$. Consequently, T_F is in the closure in strong operator topology of operators of the form

$$T = \frac{1}{d}\sum_{z\in F} c_z \sum_{i=1}^{d} k_{w(z;i)} \otimes k_{w(z;i)}, \tag{5.4}$$

where $d \in \mathbf{N}$ and for each $z \in F$, we have $w(z;i) \in D(z,b)$ for every $i \in \{1, \ldots, d\}$. Thus, for any given $0 < s \leq 1$, it suffices to estimate $\|T^s\|_\Phi$.

Now we factor T. Pick an orthonormal set $\{\epsilon(z;i) : z \in F, 1 \leq i \leq d\}$ and define

$$W = \frac{1}{\sqrt{d}}\sum_{z\in F} c_z^{1/2} \sum_{i=1}^{d} k_{w(z;i)} \otimes \epsilon(z;i). \tag{5.5}$$

Obviously, we have $T = WW^*$. Denote $\Psi = \Phi^{(2)}$. Then

$$\|T^s\|_\Phi = \|(WW^*)^s\|_\Phi = \|(W^*W)^s\|_\Phi = \||W|^{2s}\|_\Phi = \||W|^s\|^2_{\Phi^{(2)}} = \||W|^s\|^2_\Psi. \tag{5.6}$$

This reduces the problem to the estimate of $\||W|^s\|_\Psi$.

To estimate $\||W|^s\|_\Psi$, pick a t such that $st > n$. By Proposition 3.8, there are 1-separated sets $\Gamma_1, \ldots, \Gamma_m$ in $\mathbf{B}$ such that the operator

$$A = E_{\Gamma_1,t} + \cdots + E_{\Gamma_m,t} \tag{5.7}$$

satisfies the inequality $A \geq \delta$ on $H^2(S)$ for some $\delta > 0$. By Lemma 4.2, we have

$$\||W|^s\|_\Psi = \||A^{-1}AW|^s\|_\Psi \leq \delta^{-s}\||AW|^s\|_\Psi.$$

For each $1 \leq r \leq m$, we pick an orthonormal set $\{e(r;w) : w \in \Gamma_r\}$ and factor $E_{\Gamma_r,t}$ in the form $E_{\Gamma_r,t} = B_r B_r^*$, where

$$B_r = \sum_{w\in\Gamma_r} \psi_{w,t} \otimes e(r;w).$$

Since A is given by (5.7), applying Lemmas 4.1, 4.2 and Proposition 3.2, we obtain

$$\||W|^s\|_\Psi \leq 2m\delta^{-s}C_{3.2}^s \max_{1\leq r\leq m} \||B_r^*W|^s\|_\Psi. \tag{5.8}$$

To summarize, we have now reduced the proof of the upper bound in Theorem 1.4 to the estimate of $\||B^*W|^s\|_\Psi$, where

$$B = \sum_{\gamma\in G} \psi_{\gamma,t} \otimes e_\gamma,$$

G is a 1-separated set in $\mathbf{B}$ and $\{e_\gamma : \gamma \in G\}$ is an orthonormal set. Invoking Lemma 2.6 again, we may further assume that G has the additional property

$$\mathrm{card}(G \cap T_{k,j}) \leq 1 \quad \text{for every} \quad (k,j) \in I, \tag{5.9}$$

which, along with (5.2), will be needed for our counting argument below.

Recalling (5.5) and using the reproducing property of k_w, we have

$$\begin{aligned} B^*W &= \sum_{\gamma\in G}\sum_{z\in F} c_z^{1/2}\frac{1}{\sqrt{d}}\sum_{i=1}^{d}(1-|w(z;i)|^2)^{n/2}\overline{\psi_{\gamma,t}(w(z;i))}e_\gamma \otimes \epsilon(z;i) \\ &= \sum_{\gamma\in G}\sum_{z\in F} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}, \end{aligned} \tag{5.10}$$

where

$$f_{z;\gamma} = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}(1-|w(z;i)|^2)^{n/2}\psi_{\gamma,t}(w(z;i))\epsilon(z;i) \tag{5.11}$$

for $\gamma \in G$ and $z \in F$. For each pair of $z \in F$ and $i \in \{1,\ldots,d\}$, we have $w(z;i) \in D(z,b)$. Thus there is an $x(z;i) \in D(0,b)$ such that $w(z;i) = \varphi_z(x(z;i))$. By Lemmas 2.3 and 2.4, there is a constant C_1 such that

$$(1-|w(z;i)|^2)^{n/2}|\psi_{\gamma,t}(w(z;i))| \leq C_1(1-|z|^2)^{n/2}|\psi_{\gamma,t}(z)|$$

for all $\gamma \in G$, $z \in F$ and $i \in \{1,\ldots,k\}$. Hence

$$\|f_{z;\gamma}\| \leq C_1(1-|z|^2)^{n/2}|\psi_{\gamma,t}(z)| \tag{5.12}$$

for all $\gamma \in G$ and $z \in F$.

At this point, we need to organize the pairs $(\gamma, z) \in G \times F$ using the decomposition scheme in Section 2. First of all, for each integer $k \geq 0$ we define

$$H_k = \{w \in \mathbf{B} : 1 - 2^{-2k} \leq |w| < 1 - 2^{-2(k+1)}\}.$$

The point is that $H_k = \cup_{j=1}^{m(k)} T_{k,j}$. Then, for each $k \geq 0$, define

$$G_k = G \cap H_k \quad \text{and} \quad F_k = F \cap H_k.$$

By (5.10), we have

$$B^*W = \sum_{\ell=0}^{\infty} Y_\ell + \sum_{\ell=1}^{\infty} Z_\ell, \tag{5.13}$$

where

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z)\in G_k \times F_{k+\ell}} c_z^{1/2} e_\gamma \otimes f_{z;\gamma} \quad \text{and} \quad Z_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z)\in G_{k+\ell} \times F_k} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}.$$

Next, from (2.7) we see that there exist Borel sets $\{S_{k,j} : (k,j) \in I\}$ in the sphere S that satisfy the following three conditions:

(1) For every $(k,j) \in I$, we have $S_{k,j} \subset B(u_{k,j}, 2^{-k})$.

(2) For every $k \geq 0$ and every pair of $j \neq j'$ in $\{1, \ldots, m(k)\}$, we have $S_{k,j} \cap S_{k,j'} = \emptyset$.

(3) For every $k \geq 0$, we have $\cup_{j=1}^{m(k)} S_{k,j} = S$.

We will use these sets to further decompose Y_ℓ.

We write each $z \in F$ in the form $z = |z|\xi_z$ with $\xi_z \in S$. For each pair of $k \geq 0$ and $\ell \geq 0$, we have a partition

$$F_{k+\ell} = F_{k,\ell,1} \cup \cdots \cup F_{k,\ell,m(k)}, \tag{5.14}$$

where

$$F_{k,\ell,j} = \{z \in F_{k+\ell} : \xi_z \in S_{k,j}\}, \tag{5.15}$$

$1 \leq j \leq m(k)$. By (5.9), for each $k \geq 0$ there is a $J_k \subset \{1, \ldots, m(k)\}$ such that $G_k = \{\gamma_{k,j} : j \in J_k\}$ and such that for each $j \in J_k$, $\gamma_{k,j} \in T_{k,j}$. For $k \geq 0$, $\ell \geq 0$, $j \in J_k$ and $j' \in \{1, \ldots, m(k)\}$, we now define

$$f_{k;j,j'}^{(\ell)} = \sum_{z\in F_{k,\ell,j'}} c_z^{1/2} f_{z;\gamma_{k,j}}. \tag{5.16}$$

Then

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{j\in J_k} \sum_{j'=1}^{m(k)} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)}.$$

We further decompose Y_ℓ according to spherical separation. For each $k \geq 0$, define

$$Q_{k,0} = \{(j,j') : j \in J_k, 1 \leq j' \leq m(k), d(u_{k,j}, u_{k,j'}) < 2^{-k+2}\} \quad \text{and}$$
$$Q_{k,p} = \{(j,j') : j \in J_k, 1 \leq j' \leq m(k), 2^{-k+p+1} \leq d(u_{k,j}, u_{k,j'}) < 2^{-k+p+2}\}, \quad p \geq 1.$$

Accordingly, we define

$$Y_\ell^{(p)} = \sum_{k=0}^{\infty} \sum_{(j,j') \in Q_{k,p}} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)}$$

for $p = 0, 1, 2, \ldots$. Then, of course,

$$Y_\ell = Y_\ell^{(0)} + Y_\ell^{(1)} + Y_\ell^{(2)} + \cdots + Y_\ell^{(p)} + \cdots . \tag{5.17}$$

By (2.6), the definition of $Q_{k,p}$ and (2.5), there is a constant $M \in \mathbf{N}$ such that for each pair of $k \geq 0$, $p \geq 0$ and each $j \in J_k$, we have

$$\mathrm{card}\{j' : (j,j') \in Q_{k,p}\} \leq M 2^{2np}. \tag{5.18}$$

Similarly, for $k \geq 0$, $p \geq 0$ and $j' \in \{1, \ldots, m(k)\}$, we have

$$\mathrm{card}\{j : (j,j') \in Q_{k,p}\} \leq M 2^{2np}. \tag{5.19}$$

By Lemma 2.7, each $Q_{k,p}$ admits a partition

$$Q_{k,p} = Q_{k,p}^{(1)} \cup \cdots \cup Q_{k,p}^{(2M2^{2np})}$$

such that for every $1 \leq i \leq 2M2^{2np}$, the conditions $(j,j'), (h,h') \in Q_{k,p}^{(i)}$ and $(j,j') \neq (h,h')$ imply both $j \neq h$ and $j' \neq h'$. Accordingly, for every $p \geq 0$ we have

$$Y_\ell^{(p)} = Y_\ell^{(p,1)} + \cdots + Y_\ell^{(p,2M2^{2np})}, \tag{5.20}$$

where

$$Y_\ell^{(p,i)} = \sum_{k=0}^{\infty} \sum_{(j,j') \in Q_{k,p}^{(i)}} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)},$$

$i = 1, \ldots, 2M2^{2np}$. If $k_1 \neq k_2$, then obviously $e_{\gamma_{k_1,j_1}} \perp e_{\gamma_{k_2,j_2}}$ for all $j_1 \in J_{k_1}$ and $j_2 \in J_{k_2}$. Similarly, when $k_1 \neq k_2$, a chase of definitions shows that $f_{k_1;j_1,j_1'}^{(\ell)} \perp f_{k_2;j_2,j_2'}^{(\ell)}$ for all $j_1 \in J_{k_1}$, $j_2 \in J_{k_2}$, $j_1' \in \{1, \ldots, m(k_1)\}$ and $j_2' \in \{1, \ldots, m(k_2)\}$. Now the property of each $Q_{k,p}^{(i)}$ guarantees that if $(j,j'), (h,h') \in Q_{k,p}^{(i)}$ and $(j,j') \neq (h,h')$, then we have both

$$e_{\gamma_{k,j}} \perp e_{\gamma_{k,h}} \quad \text{and} \quad f_{k;j,j'}^{(\ell)} \perp f_{k;h,h'}^{(\ell)}.$$

Because of all this orthogonality, for each pair of $p \geq 0$ and $1 \leq i \leq 2M2^{2np}$ we have

$$\| |Y_\ell^{(p,i)}|^s \|_\Psi = \Psi(\{\|f_{k;j,j'}^{(\ell)}\|^s\}_{(k,j,j') \in L_p^{(i)}}), \tag{5.21}$$

where

$$L_p^{(i)} = \bigcup_{k=0}^{\infty} \left\{ (k,j,j') : (j,j') \in Q_{k,p}^{(i)} \right\}.$$

Our next task is to estimate the vector norm $\|f_{k;j,j'}^{(\ell)}\|$, $(k,j,j') \in L_p^{(i)}$.

By (5.11), for $z \neq z'$ in F, we have $\langle f_{z;\gamma}, f_{z';\gamma'} \rangle = 0$ for all $\gamma, \gamma' \in G$. Therefore it follows from (5.16) and (5.12) that

$$\|f_{k;j,j'}^{(\ell)}\|^2 = \sum_{z \in F_{k,\ell,j'}} c_z \|f_{z;\gamma_{k,j}}\|^2 \leq C_1^2 \sum_{z \in F_{k,\ell,j'}} c_z (1-|z|^2)^n |\psi_{\gamma_{k,j},t}(z)|^2.$$

For $z \in F_{k,\ell,j'}$, we have $(1-|\gamma_{k,j}|^2)^n |\psi_{\gamma_{k,j},t}(z)|^2 = |m_{\gamma_{k,j}}(z)|^{2n+2t}$ (cf. (3.1), (3.2)) and

$$\left(\frac{1-|z|^2}{1-|\gamma_{k,j}|^2} \right)^n \leq 2^n \left(\frac{1-|z|}{1-|\gamma_{k,j}|} \right)^n \leq 2^n \left(\frac{2^{-2(k+\ell)}}{2^{-2(k+1)}} \right)^n = C_2 2^{-2n\ell}.$$

Writing $C_3 = C_1^2 C_2$, this gives us

$$\|f_{k;j,j'}^{(\ell)}\|^2 \leq C_3 2^{-2n\ell} \sum_{z \in F_{k,\ell,j'}} c_z |m_{\gamma_{k,j}}(z)|^{2n+2t}. \tag{5.22}$$

Since $\gamma_{k,j} \in T_{k,j}$, there is a $\zeta_{k,j} \in B(u_{k,j}, 2^{-k})$ such that $\gamma_{k,j} = |\gamma_{k,j}|\zeta_{k,j}$. For $z \in F_{k,\ell,j'}$, we have $\xi_z \in S_{k,j'}$, consequently $d(\xi_z, u_{k,j'}) \leq 2^{-k}$. Hence

$$\begin{aligned} \{2|1-\langle z,\gamma_{k,j}\rangle|\}^{1/2} &\geq |1-\langle \xi_z,\zeta_{k,j}\rangle|^{1/2} = d(\xi_z,\zeta_{k,j}) \\ &\geq d(u_{k,j'},u_{k,j}) - d(\xi_z,u_{k,j'}) - d(\zeta_{k,j},u_{k,j}) \\ &\geq d(u_{k,j'},u_{k,j}) - 2^{-k+1}. \end{aligned}$$

Thus if $(k,j,j') \in L_p^{(i)}$ for some $p \geq 1$ and $z \in F_{k,\ell,j'}$, then

$$\{2|1-\langle z,\gamma_{k,j}\rangle|\}^{1/2} \geq 2^{-k+p+1} - 2^{-k+1} \geq 2^{-k+p}.$$

Since $1-|\gamma_{k,j}|^2 \leq 2 \cdot 2^{-2k}$, we have $|m_{\gamma_{k,j}}(z)| \leq 4 \cdot 2^{-2p}$ for $z \in F_{k,\ell,j'}$ and $(k,j,j') \in L_p^{(i)}$, $p \geq 0$. Substituting this in (5.22), we find that

$$\|f_{k;j,j'}^{(\ell)}\|^2 \leq C_4 2^{-4(n+t)p} 2^{-2n\ell} \sum_{z \in F_{k,\ell,j'}} c_z \tag{5.23}$$

for $(k,j,j') \in L_p^{(i)}$, $p \geq 0$.

Recall that $F_{k,\ell,j'} \subset F_{k+\ell} \subset H_{k+\ell}$. Thus if $z \in F_{k,\ell,j'}$, then by (2.8) there is an $h \in \{1,\dots,m(k+\ell)\}$ such that $\xi_z \in B(u_{k+\ell,h}, 2^{-k-\ell})$. We have $S_{k,j'} \subset B(u_{k,j'}, 2^{-k})$ by choice. Combining these facts with (5.2) and (5.15), we find that

$$\mathrm{card}(F_{k,\ell,j'}) \leq \mathrm{card}\{h : B(u_{k+\ell,h}, 2^{-k-\ell}) \cap B(u_{k,j'}, 2^{-k}) \neq \emptyset\} \leq C_5 2^{2n\ell},$$

where the second $\leq$ is justified by (2.6) and (2.5). Also, the definition of $L_p^{(i)}$ ensures that $F_{k_1,\ell,j_1'} \cap F_{k_2,\ell,j_2'} = \emptyset$ for any pair of $(k_1, j_1, j_1') \neq (k_2, j_2, j_2')$ in $L_p^{(i)}$.

Suppose that our symmetric gauge function Φ satisfies condition (DQK). By Lemma 4.5, $\Psi = \Phi^{(2)}$ also satisfies condition (DQK). We now continue with (5.21) and (5.23). An application of Proposition 4.6 (for which the necessary verification of conditions was carried out in the preceding paragraph) to Ψ and s gives us

$$\begin{aligned}\||Y_\ell^{(p,i)}|^s\|_\Psi &\leq C_4^{s/2} 2^{-2s(n+t)p} \Psi\left(\left\{\left(2^{-2n\ell} \sum_{z \in F_{k,\ell,j'}} c_z\right)^{s/2}\right\}_{(k,j,j') \in L_p^{(i)}}\right) \\ &\leq C_4^{s/2} 2^{-2s(n+t)p} C(1+C_5)^{s/2} (C_5 2^{2n\ell})^{-\epsilon} \Psi(\{c_z^{s/2}\}_{z \in F}) \\ &= C_6 2^{-2s(n+t)p} 2^{-2\epsilon n\ell} \{\Phi(\{c_z^s\}_{z \in F})\}^{1/2}.\end{aligned}$$

Recalling (5.20) and applying Lemma 4.1, we obtain

$$\begin{aligned}\||Y_\ell^{(p)}|^s\|_\Psi &\leq 2 \sum_{i=1}^{2M2^{2np}} \||Y_\ell^{(p,i)}|^s\|_\Psi \leq 4MC_6 2^{-2(s(n+t)-n)p} 2^{-2\epsilon n\ell} \{\Phi(\{c_z^s\}_{z \in F})\}^{1/2} \\ &= C_7 2^{-2(s(n+t)-n)p} 2^{-2\epsilon n\ell} \{\Phi(\{c_z^s\}_{z \in F})\}^{1/2}. \qquad (5.24)\end{aligned}$$

Proposition 4.6 guarantees that $\epsilon > 0$. Also, we have $s(n+t) - n > 0$ by the choice of t. Recalling (5.17) and applying Lemma 4.1 again, we now have

$$\begin{aligned}\left\|\left|\sum_{\ell=0}^{\infty} Y_\ell\right|^s\right\|_\Psi &\leq 2 \sum_{\ell=0}^{\infty} \sum_{p=0}^{\infty} \||Y_\ell^{(p)}|^s\|_\Psi \leq 2C_7 \sum_{\ell=0}^{\infty} \sum_{p=0}^{\infty} 2^{-2(s(n+t)-n)p} 2^{-2\epsilon n\ell} \{\Phi(\{c_z^s\}_{z \in F})\}^{1/2} \\ &= C_8 \{\Phi(\{c_z^s\}_{z \in F})\}^{1/2}. \qquad (5.25)\end{aligned}$$

Next we turn to the operators Z_ℓ, which are much easier to handle because condition (DQK) will not be needed.

First of all, recall that $G_{k+\ell} = \{\gamma_{k+\ell,h} : h \in J_{k+\ell}\}$, where $\gamma_{k+\ell,h} \in T_{k+\ell,h}$ for every $h \in J_{k+\ell}$. By (5.2), for each $k \geq 0$ there is an $I_k \subset \{1, \ldots, m(k)\}$ such that $F_k = \{z_{k,j} : j \in I_k\}$ and such that for each $j \in I_k$, $z_{k,j} \in T_{k,j}$. For convenience, let us write

$$e_{k,h}^{(\ell)} = e_{\gamma_{k+\ell,h}} \quad \text{and} \quad \varphi_{k,h,j}^{(\ell)} = f_{z_{k,j};\gamma_{k+\ell,h}}$$

(cf. (5.11)). With this new notation we have

$$Z_\ell = \sum_{k=0}^{\infty} \sum_{(h,j) \in J_{k+\ell} \times I_k} c_{z_{k,j}}^{1/2} e_{k,h}^{(\ell)} \otimes \varphi_{k,h,j}^{(\ell)}.$$

Now define

$$\begin{aligned}&Q_{k,\ell;0} = \{(h,j) \in J_{k+\ell} \times I_k : d(u_{k,j}, u_{k+\ell,h}) < 2^{-k+2}\} \quad \text{and} \\ &Q_{k,\ell;p} = \{(h,j) \in J_{k+\ell} \times I_k : 2^{-k+p+1} \leq d(u_{k,j}, u_{k+\ell,h}) < 2^{-k+p+2}\}, \quad p \geq 1.\end{aligned}$$

Accordingly, we define

$$Z_\ell^{(p)} = \sum_{k=0}^{\infty} \sum_{(h,j)\in Q_{k,\ell;p}} c_{z_{k,j}}^{1/2} e_{k,h}^{(\ell)} \otimes \varphi_{k,h,j}^{(\ell)}.$$

for $p = 0, 1, 2, \ldots$. Then, of course,

$$Z_\ell = Z_\ell^{(0)} + Z_\ell^{(1)} + Z_\ell^{(2)} + \cdots + Z_\ell^{(p)} + \cdots . \tag{5.26}$$

As in (5.18) and (5.19), from (2.6) and (2.5) we deduce

$$\begin{aligned} \operatorname{card}\{h \in J_{k+\ell} : (h,j) \in Q_{k,\ell;p}\} &\le M2^{2n(\ell+p)} \quad \text{for every} \;\; j \in I_k \;\; \text{and} \\ \operatorname{card}\{j \in I_k : (h,j) \in Q_{k,\ell;p}\} &\le M2^{2np} \quad \text{for every} \;\; h \in J_{k+\ell}. \end{aligned}$$

Thus, as in Lemma 2.7, a standard maximality argument gives us a partition

$$Q_{k,\ell;p} = Q_{k,\ell;p}^{(1)} \cup \cdots \cup Q_{k,\ell;p}^{(2M2^{2n(\ell+p)})}$$

such that for every $i \in \{1, \ldots, 2M2^{2n(\ell+p)}\}$, the conditions $(h, j), (h', j') \in Q_{k,\ell;p}^{(i)}$ and $(h, j) \neq (h', j')$ imply both $h \neq h'$ and $j \neq j'$. Accordingly,

$$Z_\ell^{(p)} = Z_\ell^{(p,1)} + \cdots + Z_\ell^{(p,2M2^{2n(\ell+p)})}, \tag{5.27}$$

where

$$Z_\ell^{(p,i)} = \sum_{k=0}^{\infty} \sum_{(h,j)\in Q_{k,\ell;p}^{(i)}} c_{z_{k,j}}^{1/2} e_{k,h}^{(\ell)} \otimes \varphi_{k,h,j}^{(\ell)},$$

$i = 1, \ldots, 2M2^{2n(\ell+p)}$. Define

$$L_{\ell,p}^{(i)} = \bigcup_{k=0}^{\infty} \left\{ (k,h,j) : (h,j) \in Q_{k,\ell;p}^{(i)} \right\}.$$

The property of $Q_{k,\ell;p}^{(i)}$ ensures that for $(k, h, j) \neq (k', h', j')$ in $Q_{k,\ell;p}^{(i)}$, we have both $\varphi_{k,h,j}^{(\ell)} \perp \varphi_{k',h',j'}^{(\ell)}$ and $e_{k,h}^{(\ell)} \perp e_{k',h'}^{(\ell)}$. Moreover, the projection $(k, h, j) \mapsto (k, j)$ is injective on $L_{\ell,p}^{(i)}$. Therefore

$$\begin{aligned} \| |Z_\ell^{(p,i)}|^s \|_\Psi &= \Psi(\{c_{z_{k,j}}^{s/2} \|\varphi_{k,h,j}^{(\ell)}\|^s\}_{(k,h,j)\in L_{\ell,p}^{(i)}}) \\ &\le \sup_{(k,h,j)\in L_{\ell,p}^{(i)}} \|\varphi_{k,h,j}^{(\ell)}\|^s \Psi(\{c_z^{s/2}\}_{z\in F}) = \sup_{(k,h,j)\in L_{\ell,p}^{(i)}} \|\varphi_{k,h,j}^{(\ell)}\|^s \{\Phi(\{c_z^s\}_{z\in F})\}^{1/2}. \end{aligned} \tag{5.28}$$

Obviously, we need to estimate $\|\varphi_{k,h,j}^{(\ell)}\|$. By (5.12), for each $(k, h, j) \in L_{\ell,p}^{(i)}$ we have

$$\|\varphi_{k,h,j}^{(\ell)}\| = \|f_{z_{k,j};\gamma_{k+\ell,h}}\| \le C_1(1-|z_{k,j}|^2)^{n/2} |\psi_{\gamma_{k+\ell,h},t}(z_{k,j})| \le C_9 \left| \frac{1 - |\gamma_{k+\ell,h}|}{1 - \langle z_{k,j}, \gamma_{k+\ell,h}\rangle} \right|^{(n/2)+t}.$$

Since $\gamma_{k+\ell,h} \in T_{k+\ell,h}$, we write $\gamma_{k+\ell,h} = |\gamma_{k+\ell,h}|\zeta_{\gamma_{k+\ell,h}}$ with $\zeta_{\gamma_{k+\ell,h}} \in B(u_{k+\ell,h}, 2^{-k-\ell})$ as before. Similarly, $z_{k,j} = |z_{k,j}|\xi_{z_{k,j}}$, where $\xi_{z_{k,j}} \in B(u_{k,j}, 2^{-k})$. We have

$$2|1 - \langle z_{k,j}, \gamma_{k+\ell,h}\rangle| \geq |1 - \langle \xi_{z_{k,j}}, \zeta_{\gamma_{k+\ell,h}}\rangle| = d^2(\xi_{z_{k,j}}, \zeta_{\gamma_{k+\ell,h}})$$

and

$$d(\xi_{z_{k,j}}, \zeta_{\gamma_{k+\ell,h}}) \geq d(u_{k,j}, u_{k+\ell,h}) - 2^{-k} - 2^{-k-\ell}.$$

Thus in the case $p \geq 1$, we have

$$\frac{1}{|1 - \langle z_{k,j}, \gamma_{k+\ell,h}\rangle|} \leq \frac{2}{(2^{-k+p})^2} \leq 4 \cdot 2^{2(k-p)}.$$

Since $z_{k,j} \in T_{k,j}$, the conclusion also holds in the case $p = 0$. Therefore

$$\begin{aligned}\|\varphi^{(\ell)}_{k,h,j}\| &\leq C_{10}\{2^{2(k-p)}(1 - |\gamma_{k+\ell,h}|)\}^{(n/2)+t} \leq C_{10}\{2^{2(k-p)} \cdot 2^{-2(k+\ell)}\}^{(n/2)+t} \\ &= C_{10}2^{-(n+2t)(p+\ell)}\end{aligned}$$

for every $(k, h, j) \in L^{(i)}_{\ell,p}$. Substituting this in (5.28), we obtain

$$\||Z^{(p,i)}_\ell|^s\|_\Psi \leq C^s_{10}2^{-s(n+2t)(p+\ell)}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2}.$$

Applying Lemma 4.1 to (5.27), we have

$$\||Z^{(p)}_\ell|^s\|_\Psi \leq 2\sum_{i=1}^{2M2^{2n(p+\ell)}}\||Z^{(p,i)}_\ell|^s\|_\Psi \leq 4MC^s_{10}2^{-\kappa(p+\ell)}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2}, \tag{5.29}$$

where $\kappa = s(n + 2t) - 2n$. The choice $st > n$ ensures that $\kappa > 0$. Recalling (5.26), another application of Lemma 4.1 leads to

$$\begin{aligned}\left\|\left|\sum_{\ell=1}^\infty Z_\ell\right|^s\right\|_\Psi &\leq 2\sum_{\ell=1}^\infty\sum_{p=0}^\infty\||Z^{(p)}_\ell|^s\|_\Psi \leq 8MC^s_{10}\sum_{\ell=1}^\infty\sum_{p=0}^\infty 2^{-\kappa(p+\ell)}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2} \\ &= C_{11}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2}.\end{aligned} \tag{5.30}$$

Recalling (5.25) and applying Lemma 4.1 to (5.13), we find that

$$\||B^*W|^s\|_\Psi \leq C_{12}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2},$$

where $C_{12} = 2(C_8 + C_{11})$. This and (5.8) together give us

$$\||W|^s\|_\Psi \leq C_{13}\{\Phi(\{c^s_z\}_{z\in F})\}^{1/2}.$$

Substituting the above in (5.6), we obtain

$$\|T^s\|_\Phi = \||W|^s\|^2_\Psi \leq C^2_{13}\Phi(\{c^s_z\}_{z\in F}).$$

Since T approximates T_F (cf. (5.3) and (5.4)), Lemma 4.3 allows us to conclude that

$$\|T_F^s\|_\Phi \le C_{13}^2 \Phi(\{c_z^s\}_{z\in F}).$$

As we recall, F is an arbitrary finite subset of Γ satisfying (5.2) and the condition that $c_z = \tilde{\mu}(D(z,b)) \neq 0$ for every $z \in F$. Thus it follows from Lemmas 2.6, 4.1 and 4.3 that

$$\|T_\Gamma^s\|_\Phi \le 2KC_{13}^2 \Phi(\{\tilde{\mu}^s(D(z,b))\}_{z\in\Gamma}).$$

We know that $\tilde{\mu}(D(z,b)) \le C_{14}(1-|z|^2)^{-n}\mu(D(z,b))$ from Lemma 2.4. Since $\|T_\mu^s\|_\Phi \le \|T_\Gamma^s\|_\Phi$, this proves the upper bound for $\|T_\mu^s\|_\Phi$ in Theorem 1.4. □

Denote $K_w(\zeta) = (1-\langle \zeta, w\rangle)^{-n}$. Having proved the upper bound in Theorem 1.4, next we state a consequence of it, which will be convenient for application in Section 8.

Proposition 5.1. *Let $0 < a < \infty$ and $0 < b < \infty$ be positive numbers. Suppose that Φ is a symmetric gauge function satisfying condition* (DQK). *Then for any regular Borel measure μ on $\mathbf{B}$ and any a-separated set Γ in $\mathbf{B}$, we have*

$$\left\| \sum_{z\in\Gamma} \int_{D(z,b)} K_w \otimes K_w d\mu(w) \right\|_\Phi \le C_{5.1} \Phi\left(\left\{ \frac{\mu(D(z,b))}{(1-|z|^2)^n} \right\}_{z\in\Gamma} \right),$$

where $C_{5.1}$ is a constant that depends only on a, b, Φ and the complex dimension n.

Proof. Obviously,

$$\sum_{z\in\Gamma} \int_{D(z,b)} K_w \otimes K_w d\mu(w) = T_\nu,$$

where ν is the measure defined by the formula

$$d\nu = \sum_{z\in\Gamma} \chi_{D(z,b)} d\mu.$$

Since Γ is a-separated, there is a Γ' containing Γ that is maximal with respect to the property of being a-separated. Thus Γ' is an $a, 2a$-lattice in $\mathbf{B}$. By the upper bound in Theorem 1.4, the proposition will follow if we can find a constant C such that

$$\Phi\left(\left\{ \frac{\nu(D(w,2a))}{(1-|w|^2)^n} \right\}_{w\in\Gamma'} \right) \le C\Phi\left(\left\{ \frac{\mu(D(z,b))}{(1-|z|^2)^n} \right\}_{z\in\Gamma} \right). \tag{5.31}$$

Since Γ is a-separated, by Lemma 2.1, there is an $N \in \mathbf{N}$ determined by a, b such that for any $w \in \Gamma'$, $\mathrm{card}\{z \in \Gamma : D(w,2a) \cap D(z,b) \neq \emptyset\} \le N$. Let $\Gamma'' = \{w \in \Gamma' : \nu(D(w,2a)) \neq 0\}$. Then for each $w \in \Gamma''$, there is a $z(w) \in \Gamma$ such that

$$\nu(D(w,2a)) \le N\mu(D(z(w),b)) \quad \text{and} \quad \beta(w,z(w)) \le b + 2a.$$

Combining these two conditions with Lemma 2.4, we see that

$$\Phi\left(\left\{ \frac{\nu(D(w,2a))}{(1-|w|^2)^n} \right\}_{w\in\Gamma''} \right) \le C_1 N\Phi\left(\left\{ \frac{\mu(D(z(w),b))}{(1-|z(w)|^2)^n} \right\}_{w\in\Gamma''} \right). \tag{5.32}$$

If $w, \xi \in \Gamma''$ are such that $z(w) = z(\xi)$, then $\beta(w, \xi) \le 2b + 4a$. Thus, by Lemma 2.1, there is an $M \in \mathbf{N}$ such that the map $w \mapsto z(w)$ from Γ'' to Γ is at most M-to-1. Applying Lemma 4.8, we have

$$\Phi\left(\left\{\frac{\mu(D(z(w), b))}{(1 - |z(w)|^2)^n}\right\}_{w \in \Gamma''}\right) \le M\Phi\left(\left\{\frac{\mu(D(z, b))}{(1 - |z|^2)^n}\right\}_{z \in \Gamma}\right).$$

Combining this inequality with (5.32), (5.31) follows. □

6. Proof of Theorem 1.4 — the lower bound

The main part of the proof of the lower bound consists of estimates similar to those in Section 5. Therefore many of the notations below are the same as in Section 5. But some modifications and new ideas are necessary for the lower bound.

To prove the lower bound in Theorem 1.4, we again define $\tilde{\mu}$ by (5.1) when a measure μ is given. Let $0 < a \le b < \infty$. In contrast to Section 5, we now need the inequality

$$\frac{\mu(D(z, b))}{(1 - |z|^2)^n} \le \left(4e^{2b}\right)^n \tilde{\mu}(D(z, b)),$$

$z \in \mathbf{B}$, which also follows from Lemma 2.4. Suppose that Γ is an a, b-lattice in $\mathbf{B}$. As in Section 5, we again write $c_z = \tilde{\mu}(D(z, b))$ for $z \in \Gamma$.

Consider any *finite* subset F of Γ satisfying the following three conditions:
(a) $c_z \ne 0$ for every $z \in F$.
(b) F is R-separated for a sufficiently large $R > \max\{1, 2b\}$, to be determined later.
(c) F satisfies (5.2).
With such an F, we again define the operator T_F by (5.3). Let $0 < s \le 1$ be given. Pick a $t > 0$ such that $st > n$. But instead of the operator B in Section 5, here we need

$$E = \sum_{z \in F} \psi_{z,t} \otimes e_z,$$

where $\{e_z : z \in F\}$ is an orthonormal set. Then $\|E\| \le C_{3.2}$ by Proposition 3.2. For any symmetric gauge function Φ, it follows from Lemma 4.2 that

$$\|(E^* T_F E)^s\|_\Phi \le C_{3.2}^{2s} \|T_F^s\|_\Phi \le C_{3.2}^{2s} \|T_\mu^s\|_\Phi,$$

where the second $\le$ holds because $T_F \le T_\mu$, which is guaranteed by the condition $R > 2b$.

Recall from Section 5 that operators T given by (5.4) strongly approximate T_F. Consider $\mathcal{H} = \text{span}\{e_z : z \in F\}$, which is a finite-dimensional Hilbert space. We can regard $E^* T_F E$ as an operator on $\mathcal{H}$. Since $\dim(\mathcal{H}) < \infty$, all operator topologies on $\mathcal{H}$ are equivalent. Therefore there is a T given by (5.4) such that

$$\|(E^* T E)^s\|_\Phi \le 2\|(E^* T_F E)^s\|_\Phi \le 2C_{3.2}^{2s} \|T_\mu^s\|_\Phi.$$

Once we have this T, we again factor it in the form $T = WW^*$, where W is given by (5.5). Writing $\Psi = \Phi^{(2)}$ as in Section 5, we have

$$\|(E^*TE)^s\|_\Phi = \|\{E^*W(E^*W)^*\}^s\|_\Phi = \||E^*W|^{2s}\|_\Phi = \||E^*W|^s\|_\Psi^2.$$

Writing $C_1 = \{2C_{3.2}^{2s}\}^{1/2}$, the above gives us

$$\||E^*W|^s\|_\Psi \le C_1 \|T_\mu^s\|_\Phi^{1/2}. \tag{6.1}$$

Similar to (5.10), we have

$$E^*W = \sum_{\gamma, z \in F} c_z^{1/2} e_\gamma \otimes f_{z;\gamma},$$

where $f_{z;\gamma}$ is given by (5.11). Thus $E^*W = D + X$, where

$$D = \sum_{z \in F} c_z^{1/2} e_z \otimes f_{z;z} \quad \text{and} \quad X = \sum_{\substack{\gamma, z \in F \\ \gamma \ne z}} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}.$$

Since $D = E^*W - X$, it follows from Lemma 4.1 and (6.1) that

$$\||D|^s\|_\Psi \le 2C_1 \|T_\mu^s\|_\Phi^{1/2} + 2\||X|^s\|_\Psi. \tag{6.2}$$

First, let us look at the operator D.

Because $\{e_z : z \in F\}$ and $\{\epsilon(z;i) : z \in F, 1 \le i \le d\}$ are orthonormal sets, we have

$$\||D|^s\|_\Psi = \Psi(\{c_z^{s/2}\|f_{z;z}\|^s\}_{z \in F}).$$

We need a lower bound for $\|f_{z;z}\|$. By (5.11), we have

$$\|f_{z;z}\| \ge \min_{1 \le i \le d} (1 - |w(z;i)|^2)^{n/2} |\psi_{z,t}(w(z;i))|.$$

Recall that $w(z;i) \in D(z,b)$ for every $1 \le i \le d$. Thus it follows from Lemmas 2.3 and 2.4 that there is a $\delta > 0$ which is determined by b, n and t such that

$$(1 - |w(z;i)|^2)^{n/2} |\psi_{z,t}(w(z;i))| \ge \delta (1 - |z|^2)^{n/2} |\psi_{z,t}(z)| = \delta$$

for every $1 \le i \le d$ and every $z \in F$. Hence

$$\delta^s \Psi(\{c_z^{s/2}\}_{z \in F}) \le \||D|^s\|_\Psi. \tag{6.3}$$

Next we consider X, which will be handled in a way similar to the B^*W in Section 5.

Similar to (5.13), we have the decomposition

$$X = Y_0 + \sum_{\ell=1}^\infty Y_\ell + \sum_{\ell=1}^\infty Z_\ell,$$

where

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z)\in F_k\times F_{k+\ell}} c_z^{1/2} e_\gamma \otimes f_{z;\gamma} \quad \text{and} \quad Z_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z)\in F_{k+\ell}\times F_k} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}$$

for $\ell \geq 1$, and where

$$Y_0 = \sum_{k=0}^{\infty} \sum_{\substack{(\gamma,z)\in F_k\times F_k \\ \gamma\neq z}} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}.$$

As in Section 5, we first consider Y_ℓ.

By (5.2), for each $k \geq 0$ there is a $J_k \subset \{1, \ldots, m(k)\}$ such that $F_k = \{\gamma_{k,j} : j \in J_k\}$ and such that for each $j \in J_k$, $\gamma_{k,j} \in T_{k,j}$. Recall (5.15) for the definition of $F_{k,\ell,j}$. For $k \geq 0$, $\ell \geq 0$, $j \in J_k$ and $j' \in \{1, \ldots, m(k)\}$, we now define $f^{(\ell)}_{k;j,j'}$ by the formula

$$f^{(\ell)}_{k;j,j'} = \sum_{\substack{z\in F_{k,\ell,j'} \\ z\neq\gamma_{k,j}}} c_z^{1/2} f_{z;\gamma_{k,j}}, \tag{6.4}$$

which is a necessary modification of (5.16). (Here, we would like to remind the reader of the common convention that a summation over the empty index set means 0.) Then

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{j\in J_k} \sum_{j'=1}^{m(k)} e_{\gamma_{k,j}} \otimes f^{(\ell)}_{k;j,j'} = \sum_{p=0}^{\infty} Y_\ell^{(p)}$$

as in Section 5, where

$$Y_\ell^{(p)} = \sum_{k=0}^{\infty} \sum_{(j,j')\in Q_{k,p}} e_{\gamma_{k,j}} \otimes f^{(\ell)}_{k;j,j'} \tag{6.5}$$

for $p = 0, 1, 2, \ldots$, where $Q_{k,p}$ is the same as in Section 5.

Lemma 6.1. *Let $L \in \mathbf{N}$. If $R > 3L + 13$, then $Y_\ell^{(p)} = 0$ whenever we have both $\ell \leq L$ and $p \leq L$.*

Proof. Consider any pair of $\gamma_{k,j} \in F_k$ and $z \in F_{k,\ell,j'}$, $z \neq \gamma_{k,j}$. Furthermore, suppose that $(j, j') \in Q_{k,p}$, which, as we recall from Section 5, implies

$$d(u_{k,j}, u_{k,j'}) < 2^{-k+p+2}.$$

We have $z = |z|\xi_z$ and $\gamma_{k,j} = |\gamma_{k,j}|\xi_{\gamma_{k,j}}$. The membership $z \in F_{k,\ell,j'}$ means $2^{-2(k+\ell+1)} \leq 1 - |z| \leq 2^{-2(k+\ell)}$ and $\xi_z \in S_{k,j'}$, i.e., $d(\xi_z, u_{k,j'}) < 2^{-k}$. Similarly, since $\gamma_{k,j} \in T_{k,j}$, we have $2^{-2(k+1)} \leq 1 - |\gamma_{k,j}| \leq 2^{-2k}$ and $d(\xi_{\gamma_{k,j}}, u_{k,j}) < 2^{-k}$. Hence

$$|1 - \langle \xi_z, \xi_{\gamma_{k,j}} \rangle| = d^2(\xi_z, \xi_{\gamma_{k,j}}) \leq (2^{-k+p+2} + 2^{-k} + 2^{-k})^2 \leq 2^{-2k+2p+8}.$$

This leads to

$$|1 - \langle z, \gamma_{k,j}\rangle| \le |1 - \langle \xi_z, \xi_{\gamma_{k,j}}\rangle| + 1 - |z| + 1 - |\gamma_{k,j}| \le 2^{-2k+2p+10}.$$

Therefore

$$1 - |\varphi_z(\gamma_{k,j})|^2 = \frac{(1-|z|^2)(1-|\gamma_{k,j}|^2)}{|1-\langle z,\gamma_{k,j}\rangle|^2} \ge \frac{2^{-2(k+\ell+1)} \cdot 2^{-2(k+1)}}{(2^{-2k+2p+10})^2} = 2^{-(2\ell+4p+24)}.$$

Consequently

$$\beta(z, \gamma_{k,j}) \le \frac{1}{2}\log\frac{4}{1-|\varphi_z(\gamma_{k,j})|^2} \le \ell + 2p + 13.$$

Thus if we have both $\ell \le L$ and $p \le L$, then $\beta(z, \gamma_{k,j}) \le 3L + 13$. But if $R > 3L + 13$, then there is no such a pair of $z \ne \gamma_{k,j}$ in F, because F is supposed to be R-separated. By (6.4) and (6.5), this means that $Y_\ell^{(p)} = 0$ under the conditions $R > 3L + 13$, $\ell \le L$ and $p \le L$. This completes the proof. □

Now let $L \in \mathbf{N}$, whose value will be determined momentarily. We choose R such that $R > \max\{3L + 13, 2b\}$. By (5.24), for all $\ell \ge 0$ and $p \ge 0$,

$$\||Y_\ell^{(p)}|^s\|_\Psi \le C_7 2^{-2(s(n+t)-n)p} 2^{-2\epsilon n\ell}\Psi(\{c_z^{s/2}\}_{z\in F}),$$

where, as we recall, the $\epsilon > 0$ resulted from the (DQK) condition for Ψ. Taking Lemma 6.1 into account and applying Lemma 4.1, we obtain

$$\begin{aligned}\left\|\left|\sum_{\ell=0}^{\infty} Y_\ell\right|^s\right\|_\Psi &\le 2 \sum_{\substack{\ell,p\in\mathbf{Z}_+ \\ \max\{\ell,p\}\ge L}} \||Y_\ell^{(p)}|^s\|_\Psi \\ &\le 2C_7 \sum_{\substack{\ell,p\in\mathbf{Z}_+ \\ \max\{\ell,p\}\ge L}} 2^{-2(s(n+t)-n)p}2^{-2\epsilon n\ell}\Psi(\{c_z^{s/2}\}_{z\in F}) \le C_8 2^{-\omega L}\Psi(\{c_z^{s/2}\}_{z\in F}),\end{aligned}$$

where $\omega = 2\min\{s(n+t) - n, \epsilon n\}$.

For Z_ℓ, we similarly retrace the second half of Section 5. In particular, (5.29) still holds. Then, similar to Lemma 6.1, we find that $Z_\ell^{(p)} = 0$ if we have both $\ell \le L$ and $p \le L$, because $R > 3L + 13$ and F is R-separated. Thus

$$\left\|\left|\sum_{\ell=1}^{\infty} Z_\ell\right|^s\right\|_\Psi \le C_9 2^{-\kappa L}\Psi(\{c_z^{s/2}\}_{z\in F}),$$

where $\kappa = s(n + 2t) - 2n$. Then another application of Lemma 4.1 gives us

$$\||X|^s\|_\Psi \le 2\left\|\left|\sum_{\ell=0}^{\infty} Y_\ell\right|^s\right\|_\Psi + 2\left\|\left|\sum_{\ell=1}^{\infty} Z_\ell\right|^s\right\|_\Psi \le 2(C_8 2^{-\omega L} + C_9 2^{-\kappa L})\Psi(\{c_z^{s/2}\}_{z\in F}).$$

Combining this with (6.2) and (6.3), we obtain

$$\delta^s\Psi(\{c_z^{s/2}\}_{z\in F}) \le 2C_1\|T_\mu^s\|_\Phi^{1/2} + 4(C_8 2^{-\omega L} + C_9 2^{-\kappa L})\Psi(\{c_z^{s/2}\}_{z\in F}).$$

We pick L large enough so that $4(C_8 2^{-\omega L}+C_9 2^{-\kappa L}) \le \delta^s/2$, and set $R > \max\{3L+13, 2b\}$ accordingly. Then the obvious cancellation and simplification in the above leads to

$$\Psi(\{c_z^{s/2}\}_{z\in F}) \le 4\delta^{-s}C_1\|T_\mu^s\|_\Phi^{1/2}.$$

Since $\Psi = \Phi^{(2)}$, this implies that

$$\Phi(\{c_z^s\}_{z\in F}) \le \{4\delta^{-s}C_1\}^2\|T_\mu^s\|_\Phi.$$

Recall that F is any finite subset of Γ satisfying conditions (a), (b), (c). Combining this inequality with Lemmas 2.1 and 2.6, the desired lower bound in Theorem 1.4 follows. □

7. Dixmier trace — the case of discrete sums

In addition to Proposition 1.2, Φ_1^+ is another example of symmetric gauge function that satisfies condition (DQK). To see this, consider an $a = \{a_j\}_{j\in\mathbf{N}} \in \hat{c}$. It suffices to consider the case where $a_j \ge 0$ for every j and we have the descending arrangement

$$a_1 \ge a_2 \ge \cdots \ge a_j \ge \cdots .$$

Since $a_j = 0$ for all but a finite number of j's, there is a $k \in \mathbf{N}$ such that

$$\Phi_1^+(a) = \frac{a_1 + \cdots + a_k}{1^{-1} + \cdots + k^{-1}}.$$

On the other hand, by (1.4), for any $N \in \mathbf{N}$ we have

$$\Phi_1^+(a^{[N]}) \ge \frac{a_1^N + \cdots + a_{Nk}^N}{1^{-1} + \cdots + (Nk)^{-1}} = \frac{Na_1 + \cdots + Na_k}{1^{-1} + \cdots + (Nk)^{-1}}.$$

Obviously, for any $0 < \epsilon < 1$, $1^{-1} + \cdots + (Nk)^{-1} \le C_\epsilon N^\epsilon(1^{-1} + \cdots + k^{-1})$. Therefore

$$\Phi_1^+(a^{[N]}) \ge C_\epsilon^{-1}N^{1-\epsilon}\frac{a_1 + \cdots + a_k}{1^{-1} + \cdots + k^{-1}} = C_\epsilon^{-1}N^{1-\epsilon}\Phi_1^+(a).$$

This shows that Φ_1^+ satisfies condition (DQK), and we can take any value less than 1 to be its "θ". In particular, Theorem 1.4 determines the membership $T_\mu^s \in \mathcal{C}_1^+$, $0 < s \le 1$.

This enables us to consider the Dixmier trace of T_μ. But before we do that, let us briefly review the definition of Dixmier trace for the benefit of the reader. First of all, we cite [2,4,15] as general references. To define the Dixmier trace, one starts with a Banach limit ω on $\ell^\infty(\mathbf{N})$. But in addition to the properties that Banach limits [3,Section III.7] possess in general, ω is required to have the following "doubling" property:

(D) For each $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, $\omega(\{a_k\}_{k\in\mathbf{N}}) = \omega(\{a_1, a_1, a_2, a_2, \dots, a_k, a_k, \dots\})$.
Such an ω can be easily constructed. For example, one can start with the doubling operator $D : \ell^\infty(\mathbf{N}) \to \ell^\infty(\mathbf{N})$. That is,

$$D\{a_1, a_2, \dots, a_k, \dots\} = \{a_1, a_1, a_2, a_2, \dots, a_k, a_k, \dots\}$$

for $\{a_k\}_{k\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$. Take any Banach limits L_1 and L_2, distinct or identical. Then an elementary exercise shows that the formula

$$\omega(a) = L_2\biggl(\biggl\{\frac{1}{k}\sum_{j=1}^{k} L_1(D^j a)\biggr\}_{k\in\mathbf{N}}\biggr),$$

$a \in \ell^\infty(\mathbf{N})$, defines a Banach limit that has the doubling property (D).

With such an ω, for any *positive* operator $A \in \mathcal{C}_1^+$, its Dixmier trace is defined to be

$$\mathrm{Tr}_\omega(A) = \omega\biggl(\biggl\{\frac{1}{\log(k+1)}\sum_{j=1}^{k} s_j(A)\biggr\}_{k\in\mathbf{N}}\biggr).$$

The doubling property of ω ensures the additivity $\mathrm{Tr}_\omega(A+B) = \mathrm{Tr}_\omega(A) + \mathrm{Tr}_\omega(B)$ for positive operators $A, B \in \mathcal{C}_1^+$. Thus Tr_ω naturally extends to a linear functional on $\mathcal{C}_1^+$. This definition guarantees unitary invariance: $\mathrm{Tr}_\omega(U^*TU) = \mathrm{Tr}_\omega(T)$ for every $T \in \mathcal{C}_1^+$ and every unitary operator U. Since UT is unitarily equivalent to TU, we have $\mathrm{Tr}_\omega(UT) = \mathrm{Tr}_\omega(TU)$. From this it follows that $\mathrm{Tr}_\omega(XT) = \mathrm{Tr}_\omega(TX)$ for every $T \in \mathcal{C}_1^+$ and every bounded operator X, which is what one expects of a trace.

Previous calculations of Dixmier trace (see, e.g., [1,5,6,16]) relied heavily on the principle that if A is in the trace class, then $\mathrm{Tr}_\omega(A) = 0$. In this paper, our calculation of Dixmier trace will be based on two different vanishing principles.

Lemma 7.1. *Let $A \in \mathcal{C}_1^+$. If the kernel of A contains its range, then* $\mathrm{Tr}_\omega(A) = 0$.

Proof. Let P be the orthogonal projection onto the range of A. If the kernel of A contains the range of A, then $\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(PA) = \mathrm{Tr}_\omega(AP) = \mathrm{Tr}_\omega(0) = 0$. □

Even though our next lemma is trivial, we would like to state it for the record anyway. We remind the reader that we write $\|\cdot\|_1^+$ for $\|\cdot\|_{\Phi_1^+}$.

Lemma 7.2. *Let $Y_1, \dots, Y_j, \dots$ be operators in $\mathcal{C}_1^+$ such that $\sum_{j=1}^\infty \|Y_j\|_1^+ < \infty$. Define $Y = \sum_{j=1}^\infty Y_j$. If $\mathrm{Tr}_\omega(Y_j) = 0$ for every $j \in \mathbf{N}$, then $\mathrm{Tr}_\omega(Y) = 0$.*

Lemmas 7.1 and 7.2 will guide our calculation of Dixmier trace. Our task is to extract non-trivial results from these seemingly trivial principles.

Lemma 7.3. *Suppose that B is a set and that A is a subset of B. Let $h : A \to B$ be an injective map which has the property that $h(a) \neq a$ for every $a \in A$. Then there is a partition $A = E_1 \cup E_2 \cup E_3$ such that for every $i \in \{1, 2, 3\}$, we have $h(E_i) \cap E_i = \emptyset$.*

Proof. By Zorn's lemma, there is a subset E_1 of A that is maximal with respect to the property $h(E_1) \cap E_1 = \emptyset$. If $E_1 \neq A$, then there is a subset E_2 of $A \backslash E_1$ that is maximal with respect to the property $h(E_2) \cap E_2 = \emptyset$. Similarly, if $E_1 \cup E_2 \neq A$, then there is a subset E_3 of $A \backslash \{E_1 \cup E_2\}$ that is maximal with respect to the property $h(E_3) \cap E_3 = \emptyset$.

To complete the proof, it suffices to show that $E_1 \cup E_2 \cup E_3 = A$. Suppose that there were some $x \in A \backslash \{E_1 \cup E_2 \cup E_3\}$. It follows from the maximality of E_1, E_2 and E_3 that for each $i \in \{1, 2, 3\}$, if we define $F_i = E_i \cup \{x\}$, then $h(F_i) \cap F_i \neq \emptyset$. Since $h(x) \neq x$, this means that we have either $x \in h(E_i)$ or $h(x) \in E_i$ for each $i \in \{1, 2, 3\}$. Our construction ensures that $E_i \cap E_j = \emptyset$ when $i \neq j$. Therefore there is at most one $i \in \{1, 2, 3\}$ such that $h(x) \in E_i$. This leaves a pair of $j \neq k$ in $\{1, 2, 3\}$ such that $x \in h(E_j)$ and $x \in h(E_k)$. Since $E_j \cap E_k = \emptyset$, this contradicts the injectivity of h. Hence no such x exists. □

The computation of Dixmier trace is trivial when the operator in question is *explicitly* given as a diagonal operator with respect to an orthonormal set. Even though it is trivial, we state the case as a proposition below, which will serve as a convenient reference:

Proposition 7.4. *Let E be a countable index set and consider an operator of the form*

$$D = \sum_{z \in E} c_z e_z \otimes e_z,$$

where $\{c_z\}_{z \in E}$ are non-negative numbers such that $\Phi_1^+(\{c_z\}_{z \in E}) < \infty$, and, most important, $\{e_z : z \in E\}$ is an orthonormal set. Let $E' = \{z \in E : c_z \neq 0\}$. If $\text{card}(E') = \infty$, then

$$\text{Tr}_\omega(D) = \omega\left(\left\{ \frac{1}{\log(k+1)} \sum_{j=1}^{k} c_{z_j} \right\}_{k \in \mathbf{N}} \right),$$

where $z_1, z_2, \ldots, z_k, \ldots$ are an enumeration of the elements in E' such that $c_{z_j} \geq c_{z_{j+1}}$ for every $j \in \mathbf{N}$ (the condition $\Phi_1^+(\{c_z\}_{z \in E}) < \infty$ ensures that such an enumeration is possible). If $\text{card}(E') < \infty$, then, of course, $\text{Tr}_\omega(D) = 0$.

We first consider T_μ where μ is discrete. Our computation shows that for any separated set Γ in $\mathbf{B}$, Dixmier trace cannot distinguish $\{k_z : z \in \Gamma\}$ from an orthonormal set.

Theorem 7.5. *Suppose that Γ is an a-separated set in $\mathbf{B}$ for some $a > 0$. Let $\{c_z\}_{z \in \Gamma}$ be non-negative numbers such that $\Phi_1^+(\{c_z\}_{z \in \Gamma}) < \infty$. Then the operator*

$$T = \sum_{z \in \Gamma} c_z k_z \otimes k_z$$

is in the ideal $\mathcal{C}_1^+$. Moreover, its Dixmier trace is explicitly given by the formula

$$\text{Tr}_\omega(T) = \text{Tr}_\omega\left(\sum_{z \in \Gamma} c_z e_z \otimes e_z \right), \tag{7.1}$$

where $\{e_z : z \in \Gamma\}$ is any orthonormal set.

Proof. Obviously, the membership $T \in \mathcal{C}_1^+$ follows from Proposition 5.1 by applying it to the symmetric gauge function Φ_1^+ and the discrete measure $\nu = \sum_{z\in\Gamma} c_z(1-|z|^2)^n\delta_z$, where δ_z denotes the unit point mass at z. Next we compute the Dixmier trace $\mathrm{Tr}_\omega(T)$.

Since this calculation is quite long, let us first explain the main idea involved. Consider an arbitrary positive operator A in $\mathcal{C}_1^+$. Let $\{u_j : j \in \mathbf{N}\}$ be an orthonormal basis for the underlying Hilbert space, and define the operator

$$A' = \sum_{j=1}^{\infty} \langle Au_j, u_j\rangle u_j \otimes u_j.$$

It follows from [11,Lemma III.3.1] that $\|A'\|_1^+ \le \|A\|_1^+$. Hence $A' \in \mathcal{C}_1^+$. Note that $A - A'$ is an operator whose diagonal with respect to the orthonormal basis $\{u_j : j \in \mathbf{N}\}$ vanishes. Therefore one's first instinct is to say

$$\mathrm{Tr}_\omega(A - A') = 0, \tag{7.2}$$

and consequently $\mathrm{Tr}_\omega(A) = \mathrm{Tr}_\omega(A')$. But unfortunately, in such generality this is a wrong argument for the Dixmier trace [15,Section 7.5]. The main effort below amounts to proving (7.2) for our particular A and A', using the specifics of the operators.

Let $\{S_{k,j} : (k, j) \in I\}$ be the Borel sets introduced in Section 5, satisfying conditions (1), (2), (3) there. Again, we write each $z \in \Gamma$ in the form $z = |z|\xi_z$ with $\xi_z \in S$. Define

$$\Gamma_k = \{z \in \Gamma : 1 - 2^{-2k} \le |z| < 1 - 2^{-2(k+1)}\} \tag{7.3}$$

for each $k \ge 0$. Since the Dixmier trace is linear, decomposing Γ by a finite partition if necessary, Lemma 2.6 allows us to assume that

$$\mathrm{card}\{z \in \Gamma_k : \xi_z \in S_{k,j}\} \le 1 \tag{7.4}$$

for every $(k, j) \in I$. We pick an orthonormal set $\{e_z : z \in \Gamma\}$ and define

$$B = \sum_{z\in\Gamma} c_z^{1/2} k_z \otimes e_z.$$

Obviously, $T = BB^*$. Define $A = B^*B$. Since B^*B and BB^* have identical singular numbers, we have $\mathrm{Tr}_\omega(T) = \mathrm{Tr}_\omega(A)$. Thus our task becomes the computation of $\mathrm{Tr}_\omega(A)$. Then note that

$$A = A' + Y,$$

where

$$A' = \sum_{z\in\Gamma} c_z e_z \otimes e_z \quad \text{and} \quad Y = \sum_{\substack{w,z\in\Gamma \\ w\neq z}} c_z^{1/2} c_w^{1/2} \langle k_z, k_w\rangle e_w \otimes e_z.$$

Obviously, $A' \in \mathcal{C}_1^+$ and $\mathrm{Tr}_\omega(A')$ is the right-hand side of (7.1). Thus, as we explained earlier, our main task is to show that $\mathrm{Tr}_\omega(Y) = 0$.

The proof of $\mathrm{Tr}_\omega(Y) = 0$ requires two applications of Proposition 4.6 to the symmetric gauge function $\Psi = \Phi_1^{+(2)}$, which produce two "small factors", which in turn allow Lemma 7.2 to be applied. This involves a decomposition scheme similar to the one in Section 5, but only more complicated. To begin, we have

$$Y = Y_0 + \sum_{\ell=1}^{\infty}(Y_\ell + Y_\ell^*), \tag{7.5}$$

where

$$Y_0 = \sum_{k=0}^{\infty} \sum_{\substack{w,z\in\Gamma_k \\ w\neq z}} c_z^{1/2} c_w^{1/2} \langle k_z, k_w\rangle e_w \otimes e_z \quad \text{and}$$

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{(w,z)\in\Gamma_k\times\Gamma_{k+\ell}} c_z^{1/2} c_w^{1/2} \langle k_z, k_w\rangle e_w \otimes e_z, \quad \ell \geq 1. \tag{7.6}$$

For each pair of $k \geq 0$ and $\ell \geq 0$, we have a partition

$$\Gamma_{k+\ell} = \Gamma_{k,\ell,1} \cup \cdots \cup \Gamma_{k,\ell,m(k)},$$

where

$$\Gamma_{k,\ell,j} = \{z \in \Gamma_{k+\ell} : \xi_z \in S_{k,j}\}, \tag{7.7}$$

$1 \leq j \leq m(k)$. By (7.4), for each $k \geq 0$ there is a $J_k \subset \{1, \ldots, m(k)\}$ such that $\Gamma_k = \{\gamma_{k,j} : j \in J_k\}$ and such that $\xi_{\gamma_{k,j}} \in S_{k,j}$ for each $j \in J_k$.

For $k \geq 0$, $\ell \geq 0$, $j \in J_k$ and $j' \in \{1, \ldots, m(k)\}$, define

$$f_{k;j,j'}^{(\ell)} = \sum_{z\in\Gamma_{k,\ell,j'}} c_z^{1/2} \langle k_{\gamma_{k,j}}, k_z\rangle e_z. \tag{7.8}$$

Then

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{j\in J_k} \sum_{j'=1}^{m(k)} c_{\gamma_{k,j}}^{1/2} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)}$$

for $\ell \geq 1$. By (7.4), (7.7) and (7.8), we have

$$Y_0 = \sum_{k=0}^{\infty} \sum_{\substack{(j,j')\in J_k\times\{1,\ldots,m(k)\} \\ j\neq j'}} c_{\gamma_{k,j}}^{1/2} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(0)}.$$

Now we further decompose Y_ℓ according to spherical separation. For each $k \geq 0$, define

$$Q_{k,0} = \{(j,j') : j \in J_k, 1 \leq j' \leq m(k), d(u_{k,j}, u_{k,j'}) < 2^{-k+3}\} \quad \text{and}$$
$$Q_{k,p} = \{(j,j') : j \in J_k, 1 \leq j' \leq m(k), 2^{-k+p+2} \leq d(u_{k,j}, u_{k,j'}) < 2^{-k+p+3}\}, \quad p \geq 1.$$

Accordingly, we define

$$Y_\ell^{(p)} = \sum_{k=0}^{\infty} \sum_{(j,j')\in Q_{k,p}} c_{\gamma_{k,j}}^{1/2} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)} \tag{7.9}$$

if either $p \geq 1$ or $\ell \geq 1$. In the case $p = 0$ and $\ell = 0$, we define $Y_0^{(0)}$ by the above sum with the extra constraint that the inner sum be taken over all $(j, j') \in Q_{k,0}$ satisfying the condition $j \neq j'$. Then, of course,

$$Y_\ell = Y_\ell^{(0)} + Y_\ell^{(1)} + Y_\ell^{(2)} + \cdots + Y_\ell^{(p)} + \cdots , \tag{7.10}$$

$\ell \geq 0$. So far, this resembles a portion of Section 5. Next we will decompose each $Y_\ell^{(p)}$. Because we no longer have the benefit of the modified kernel $\psi_{z,t}$, the decomposition of $Y_\ell^{(p)}$ here is much more complicated than the corresponding part in Section 5.

For each pair of $k \geq 0$ and $p \geq 0$, let $F_{k;p}$ be a subset of S that is maximal with respect to the property

$$B(\xi, 2^{-k+p}) \cap B(\xi', 2^{-k+p}) = \emptyset \quad \text{for all} \quad \xi \neq \xi' \quad \text{in} \quad F_{k;p}. \tag{7.11}$$

From this we obtain Borel sets $\{E_{k;p}^\xi : \xi \in F_{k;p}\}$ with the following three properties:

(a) $\cup_{\xi \in F_{k;p}} E_{k;p}^\xi = S$

(b) $E_{k;p}^\xi \subset B(\xi, 2^{-k+p+1})$ for every $\xi \in F_{k;p}$.

(c) $E_{k;p}^\xi \cap E_{k;p}^{\xi'} = \emptyset$ for all $\xi \neq \xi'$ in $F_{k;p}$.

Now we define the operator

$$Z_{k,\ell;p}^{\xi,\xi'} = \sum_{\substack{u_{k,j}\in E_{k;p}^\xi, u_{k,j'}\in E_{k;p}^{\xi'} \\ (j,j')\in Q_{k,p}}} c_{\gamma_{k,j}}^{1/2} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(\ell)} \tag{7.12}$$

if either $p \geq 1$ or $\ell \geq 1$. Also, in the case where we have both $\ell = 0$ and $p = 0$, define

$$Z_{k,0;0}^{\xi,\xi'} = \sum_{\substack{u_{k,j}\in E_{k;0}^\xi, u_{k,j'}\in E_{k;0}^{\xi'} \\ (j,j')\in Q_{k,0},\ j\neq j'}} c_{\gamma_{k,j}}^{1/2} e_{\gamma_{k,j}} \otimes f_{k;j,j'}^{(0)}.$$

Furthermore, define the set

$$G_{k;p} = \{(\xi, \xi') \in F_{k;p} \times F_{k;p} : \text{there is at least one } (j, j') \in Q_{k,p} \text{ such that } u_{k,j} \in E_{k;p}^\xi \text{ and } u_{k,j'} \in E_{k;p}^{\xi'}\}.$$

This allows us to rewrite (7.9) as

$$Y_\ell^{(p)} = \sum_{k=0}^{\infty} \sum_{(\xi,\xi')\in G_{k;p}} Z_{k,\ell;p}^{\xi,\xi'}.$$

Now suppose that the conditions $(\xi,\xi') \in F_{k;p} \times F_{k;p}$, $u_{k,j} \in E^{\xi}_{k;p}$, $u_{k,j'} \in E^{\xi'}_{k;p}$ and $(j,j') \in Q_{k,p}$ are simultaneously satisfied. Then

$$\begin{aligned} d(\xi,\xi') &\le d(\xi,u_{k,j}) + d(u_{k,j},u_{k,j'}) + d(u_{k,j'},\xi') \\ &\le 2^{-k+p+1} + 2^{-k+p+3} + 2^{-k+p+1} < 2^{-k+p+4}. \end{aligned}$$

Combining this with (7.11) and (2.5), we see that there is a constant $N \in \mathbf{N}$ such that

$$\operatorname{card}\{\xi' : (\xi,\xi') \in G_{k;p}\} \le N \quad \text{and} \quad \operatorname{card}\{\xi' : (\xi',\xi) \in G_{k;p}\} \le N$$

for all $k \ge 0$, $p \ge 0$ and $\xi \in F_{k;p}$. Thus for each $G_{k;p}$, Lemma 2.7 provides a partition

$$G_{k;p} = G^{(1)}_{k;p} \cup \cdots \cup G^{(2N)}_{k;p}$$

such that for every $i \in \{1,\dots,2N\}$, the conditions $(\xi,\xi'),(\eta,\eta') \in G^{(i)}_{k;p}$ and $(\xi,\xi') \ne (\eta,\eta')$ imply both $\xi \ne \eta$ and $\xi' \ne \eta'$. Accordingly, we have

$$Y^{(p)}_\ell = Y^{(p,1)}_\ell + \cdots + Y^{(p,2N)}_\ell, \tag{7.13}$$

where

$$Y^{(p,i)}_\ell = \sum_{k=0}^{\infty} \sum_{(\xi,\xi') \in G^{(i)}_{k;p}} Z^{\xi,\xi'}_{k,\ell;p}. \tag{7.14}$$

for each $i \in \{1,\dots,2N\}$.

Now define

$$W^{\xi,\xi'}_{k,\ell;p} = \sum_{\substack{u_{k,j} \in E^{\xi}_{k;p},\, u_{k,j'} \in E^{\xi'}_{k;p} \\ (j,j') \in Q_{k,p}}} e_{\gamma_{k,j}} \otimes f^{(\ell)}_{k;j,j'} \tag{7.15}$$

if either $p \ge 1$ or $\ell \ge 1$, and impose the extra condition $j \ne j'$ in the sum when $\ell = 0 = p$ (the same will be assumed below). It is clear from (7.12) that $Y^{(p,i)}_\ell = VW^{(p,i)}_\ell$, where

$$V = \sum_{k=0}^{\infty} \sum_{j \in J_k} c^{1/2}_{\gamma_{k,j}} e_{\gamma_{k,j}} \otimes e_{\gamma_{k,j}} \quad \text{and} \quad W^{(p,i)}_\ell = \sum_{k=0}^{\infty} \sum_{(\xi,\xi') \in G^{(i)}_{k;p}} W^{\xi,\xi'}_{k,\ell;p}.$$

Applying Lemma 4.4, we have

$$\begin{aligned} \|Y^{(p,i)}_\ell\|^+_1 &\le \{\|V^*V\|^+_1 \|W^{(p,i)}_\ell W^{(p,i)*}_\ell\|^+_1\}^{1/2} \\ &= \{\Phi^+_1(\{c_z\}_{z\in\Gamma}) \|W^{(p,i)}_\ell W^{(p,i)*}_\ell\|^+_1\}^{1/2}. \end{aligned} \tag{7.16}$$

Thus we need to estimate $\|W_\ell^{(p,i)}W_\ell^{(p,i)*}\|_1^+$.

For any given $k \geq 0$ and (ξ,ξ'), the range of $W_{k,\ell;p}^{\xi,\xi'}$ is contained in the linear span of $\{e_{\gamma_{k,j}} : u_{k,j} \in E_{k;p}^{\xi}\}$, whereas the range of $W_{k,\ell;p}^{\xi,\xi'*}$ is contained in the linear span of $\{e_z : z \in \Gamma_{k,\ell,j'}$ and $u_{k,j'} \in E_{k;p}^{\xi'}\}$. Thus for each $i \in \{1,\dots,2N\}$, by the property of $G_{k;p}^{(i)}$, the conditions $(\xi,\xi'),(\eta,\eta') \in G_{k;p}^{(i)}$ and $(\xi,\xi') \neq (\eta,\eta')$ imply both

$$\mathrm{range}(W_{k,\ell;p}^{\xi,\xi'}) \perp \mathrm{range}(W_{k,\ell;p}^{\eta,\eta'}) \quad \text{and} \quad \mathrm{range}(W_{k,\ell;p}^{\xi,\xi'*}) \perp \mathrm{range}(W_{k,\ell;p}^{\eta,\eta'*}).$$

If $k \neq \kappa$, then, of course, we have

$$\mathrm{range}(W_{k,\ell;p}^{\xi,\xi'}) \perp \mathrm{range}(W_{\kappa,\ell;p}^{\eta,\eta'}) \quad \text{and} \quad \mathrm{range}(W_{k,\ell;p}^{\xi,\xi'*}) \perp \mathrm{range}(W_{\kappa,\ell;p}^{\eta,\eta'*})$$

for all $(\xi,\xi') \in G_{k;p}^{(i)}$ and $(\eta,\eta') \in G_{\kappa;p}^{(i)}$. From the above orthogonality it follows that

$$W_\ell^{(p,i)}W_\ell^{(p,i)*} = \sum_{k=0}^{\infty} \sum_{(\xi,\xi')\in G_{k;p}^{(i)}} W_{k,\ell;p}^{\xi,\xi'}W_{k,\ell;p}^{\xi,\xi'*},$$

and that the right-hand side is an orthogonal sum. Thus Lemma 4.7 gives us

$$\|W_\ell^{(p,i)}W_\ell^{(p,i)*}\|_1^+ \leq \Phi_1^+\left(\{\|W_{k,\ell;p}^{\xi,\xi'}W_{k,\ell;p}^{\xi,\xi'*}\|_1\}_{(\xi,\xi')\in G_{k;p}^{(i)},k\geq 0}\right). \tag{7.17}$$

On the other hand, it follows from (7.15), (7.7) and (7.8) that

$$W_{k,\ell;p}^{\xi,\xi'}W_{k,\ell;p}^{\xi,\xi'*} = \sum_{\substack{u_{k,j}\in E_{k;p}^{\xi},u_{k,j'}\in E_{k;p}^{\xi'}\\(j,j')\in Q_{k,p}}} \sum_{\substack{u_{k,h}\in E_{k;p}^{\xi},u_{k,j'}\in E_{k;p}^{\xi'}\\(h,j')\in Q_{k,p}}} \langle f_{k;h,j'}^{(\ell)}, f_{k;j,j'}^{(\ell)}\rangle e_{\gamma_{k,j}} \otimes e_{\gamma_{k,h}}.$$

Consequently

$$\|W_{k,\ell;p}^{\xi,\xi'}W_{k,\ell;p}^{\xi,\xi'*}\|_1 = \mathrm{tr}(W_{k,\ell;p}^{\xi,\xi'}W_{k,\ell;p}^{\xi,\xi'*}) = \sum_{\substack{u_{k,j}\in E_{k;p}^{\xi},u_{k,j'}\in E_{k;p}^{\xi'}\\(j,j')\in Q_{k,p}}} \|f_{k;j,j'}^{(\ell)}\|^2.$$

Similar to the proof of (5.22), in the current situation we have

$$\begin{aligned}\|f_{k;j,j'}^{(\ell)}\|^2 &= \sum_{z\in\Gamma_{k,\ell,j'}} c_z(1-|z|^2)^n|k_{\gamma_{k,j}}(z)|^2 = \sum_{z\in\Gamma_{k,\ell,j'}} c_z\left(\frac{1-|z|^2}{1-|\gamma_{k,j}|^2}\right)^n|m_{\gamma_{k,j}}(z)|^{2n} \\ &\leq C_0 2^{-2n\ell}\sum_{z\in\Gamma_{k,\ell,j'}} c_z|m_{\gamma_{k,j}}(z)|^{2n}.\end{aligned}$$

For any $(j,j') \in Q_{k,p}$ and $z \in \Gamma_{k,\ell,j'}$, we have $|m_{\gamma_{k,j}}(z)| \le C_1 2^{-2p}$ as the argument following (5.22) shows. (We emphasize that this includes the case where $p = 0$.) Define

$$d^{(\ell)}_{k,j'} = \left(2^{-2n\ell} \sum_{z\in\Gamma_{k,\ell,j'}} c_z\right)^{1/2} \tag{7.18}$$

for $(k,j') \in I$. Then the above estimates tell us that

$$\|W^{\xi,\xi'}_{k,\ell;p} W^{\xi,\xi'*}_{k,\ell;p}\|_1 \le C_2 2^{-4np} \sum_{\substack{u_{k,j}\in E^{\xi}_{k;p}, u_{k,j'}\in E^{\xi'}_{k;p}\\ (j,j')\in Q_{k,p}}} \left(d^{(\ell)}_{k,j'}\right)^2.$$

By (b), (2.6) and (2.5), we have $\mathrm{card}\{j : u_{k,j} \in E^{\xi}_{k;p}\} \le C_3 2^{2np}$. Thus

$$\|W^{\xi,\xi'}_{k,\ell;p} W^{\xi,\xi'*}_{k,\ell;p}\|_1 \le C_4 2^{-2np} \sum_{u_{k,j'}\in E^{\xi'}_{k;p}} \left(d^{(\ell)}_{k,j'}\right)^2 = C_4 2^{-2np} \sum_{(k,j')\in A^{\xi'}_{k;p}} \left(d^{(\ell)}_{k,j'}\right)^2,$$

where $A^{\xi'}_{k;p} = \{(k,j') : u_{k,j'} \in E^{\xi'}_{k;p}\}$. This suggests that we should define

$$\varphi^{\xi,\xi'}_{k,\ell;p} = \left(2^{-2np} \sum_{(k,j')\in A^{\xi'}_{k;p}} \left(d^{(\ell)}_{k,j'}\right)^2\right)^{1/2}$$

for $(\xi,\xi') \in G^{(i)}_{k;p}$. The above now becomes

$$\|W^{\xi,\xi'}_{k,\ell;p} W^{\xi,\xi'*}_{k,\ell;p}\|_1 \le C_4 \left(\varphi^{\xi,\xi'}_{k,\ell;p}\right)^2.$$

Denote $\Psi = \Phi_1^{+(2)}$. Since Φ_1^+ satisfies condition (DQK), Lemma 4.5 says that Ψ also satisfies condition (DQK), which enables us to apply Proposition 4.6 here.

For $(\xi,\xi') \ne (\eta,\eta')$ in $G^{(i)}_{k;p}$, since $\xi' \ne \eta'$, we have $A^{\xi'}_{k;p} \cap A^{\eta'}_{k;p} = \emptyset$. Also, $\mathrm{card}(A^{\xi'}_{k;p}) \le C_3 2^{2np}$ as we explained above. Applying Proposition 4.6 to Ψ, we have

$$\Phi_1^+\left(\{\|W^{\xi,\xi'}_{k,\ell;p} W^{\xi,\xi'*}_{k,\ell;p}\|_1\}_{(\xi,\xi')\in G^{(i)}_{k;p},k\ge 0}\right) \le C_4 \Phi_1^+\left(\left\{\left(\varphi^{\xi,\xi'}_{k,\ell;p}\right)^2\right\}_{(\xi,\xi')\in G^{(i)}_{k;p},k\ge 0}\right)$$

$$= C_4\left(\Psi\big(\{\varphi^{\xi,\xi'}_{k,\ell;p}\}_{(\xi,\xi')\in G^{(i)}_{k;p},k\ge 0}\big)\right)^2 \le C_4\left(C_5 2^{-2\epsilon np}\Psi\big(\{d^{(\ell)}_{k,j'}\}_{(k,j')\in I}\big)\right)^2. \tag{7.19}$$

From (7.7) and the properties of $\{S_{k,j} : (k,j) \in I\}$ stated in Section 5 we see that $\Gamma_{k,\ell,j} \cap \Gamma_{k,\ell,j'} = \emptyset$ if $j \ne j'$. For $k \ne \kappa$, we have $\Gamma_{k,\ell,j} \cap \Gamma_{\kappa,\ell,h} = \emptyset$ for all possible j and h.

Furthermore, from (7.7), (7.3), (7.4), (2.6) and (2.5) we obtain $\text{card}(\Gamma_{k,\ell,j'}) \le C_6 2^{2n\ell}$. Recalling (7.18) and applying Proposition 4.6 again, we have

$$\Psi\big(\{d^{(\ell)}_{k,j'}\}_{(k,j')\in I}\big) \le C_7 2^{-2\epsilon n\ell}\Psi(\{c_z^{1/2}\}_{z\in\Gamma}).$$

Substituting this in (7.19) and recalling the relation $\Psi = \Phi_1^{+(2)}$, we find that

$$\Phi_1^+\left(\{\|W^{\xi,\xi'}_{k,\ell;p}W^{\xi,\xi'*}_{k,\ell;p}\|_1\}_{(\xi,\xi')\in G^{(i)}_{k;p},k\ge 0}\right) \le C_8 2^{-4\epsilon np}2^{-4\epsilon n\ell}\Phi_1^+(\{c_z\}_{z\in\Gamma}).$$

Combining this with (7.17) and (7.16), we obtain

$$\|Y^{(p,i)}_\ell\|_1^+ \le C_8^{1/2}2^{-2\epsilon n(p+\ell)}\Phi_1^+(\{c_z\}_{z\in\Gamma}).$$

Recalling (7.13), we now have

$$\|Y^{(p)}_\ell\|_1^+ \le 2NC_8^{1/2}2^{-2\epsilon n(p+\ell)}\Phi_1^+(\{c_z\}_{z\in\Gamma})$$

for all $\ell \ge 0$ and $p \ge 0$. Thus

$$\sum_{\ell=0}^\infty\sum_{p=0}^\infty \|Y^{(p)}_\ell\|_1^+ + \sum_{\ell=1}^\infty\sum_{p=0}^\infty \|Y^{(p)*}_\ell\|_1^+ < \infty.$$

Combining this fact with (7.5), (7.10) and with Lemma 7.2, the conclusion $\text{Tr}_\omega(Y) = 0$ will follow if we can show that $\text{Tr}_\omega(Y^{(p)}_\ell) = 0$ for every pair of $\ell \ge 0$ and $p \ge 0$.

To prove that $\text{Tr}_\omega(Y^{(p)}_\ell) = 0$, let a pair of $\ell \ge 0$ and $p \ge 0$ be given. By (7.9), (7.8) and (7.7), we need to consider $\gamma_{k,j} = |\gamma_{k,j}|\xi_{\gamma_{k,j}} \in \Gamma_k$ and $z = |z|\xi_z \in \Gamma_{k+\ell}$, where $\xi_{\gamma_{k,j}} \in S_{k,j}$, $\xi_z \in S_{k,j'}$ and $(j,j') \in Q_{k,p}$. For such a pair of $\gamma_{k,j}$ and z, we have

$$d(\xi_{\gamma_{k,j}},\xi_z) \le d(\xi_{\gamma_{k,j}},u_{k,j}) + d(u_{k,j},u_{k,j'}) + d(u_{k,j'},\xi_z) < 2^{-k} + 2^{-k+p+3} + 2^{-k} \le 2^{-k+p+4}.$$

Therefore

$$|1-\langle z,\gamma_{k,j}\rangle| \le |1-\langle \xi_z,\xi_{\gamma_{k,j}}\rangle| + 1 - |z| + 1 - |\gamma_{k,j}| \le 3\cdot 2^{-2k+2p+8}.$$

Consequently

$$1-|\varphi_{\gamma_{k,j}}(z)|^2 = \frac{(1-|\gamma_{k,j}|^2)(1-|z|^2)}{|1-\langle z,\gamma_{k,j}\rangle|^2} \ge \frac{2^{-2(k+1)}\cdot 2^{-2(k+\ell+1)}}{(3\cdot 2^{-2k+2p+8})^2} = \frac{1}{3^2\cdot 2^{20}\cdot 2^{2\ell+4p}}.$$

This implies that there is a constant $0 < R_{\ell,p} < \infty$ such that for $\gamma_{k,j} = |\gamma_{k,j}|\xi_{\gamma_{k,j}} \in \Gamma_k$ and $z = |z|\xi_z \in \Gamma_{k+\ell}$ satisfying the conditions $\xi_{\gamma_{k,j}} \in S_{k,j}$, $\xi_z \in S_{k,j'}$ and $(j,j') \in Q_{k,p}$, we have $\beta(\gamma_{k,j},z) < R_{\ell,p}$. Thus another look at (7.9) and (7.8) gives us the new representation

$$Y^{(p)}_\ell = \sum_{(w,z)\in\Omega_{\ell,p}} c_z^{1/2}c_w^{1/2}\langle k_z,k_w\rangle e_w\otimes e_z,$$

where $\Omega_{\ell,p}$ is a subset of the set

$$\{(w,z) \in \Gamma \times \Gamma : \beta(w,z) < R_{\ell,p} \ \text{ and } \ w \neq z\}. \tag{7.20}$$

Since Γ is a-separated and $R_{\ell,p} < \infty$, Lemma 2.1 provides an $M_{\ell,p} \in \mathbf{N}$ such that $\text{card}\{w : (w,z) \in \Omega_{\ell,p}\} \le M_{\ell,p}$ for every z and $\text{card}\{z : (w,z) \in \Omega_{\ell,p}\} \le M_{\ell,p}$ for every w. By Lemma 2.7, we have a partition

$$\Omega_{\ell,p} = \Omega_{\ell,p}^{(1)} \cup \cdots \cup \Omega_{\ell,p}^{(2M_{\ell,p})}$$

such that for each $i \in \{1, \dots, 2M_{\ell,p}\}$, the conditions $(w,z),(w',z') \in \Omega_{\ell,p}^{(i)}$ and $(w,z) \neq (w',z')$ imply both $w \neq w'$ and $z \neq z'$. Accordingly, we have

$$Y_\ell^{(p)} = Y_{\ell;p}^{(1)} + \cdots + Y_{\ell;p}^{(2M_{\ell,p})}, \tag{7.21}$$

where

$$Y_{\ell;p}^{(i)} = \sum_{(w,z)\in\Omega_{\ell,p}^{(i)}} c_z^{1/2} c_w^{1/2} \langle k_z, k_w \rangle e_w \otimes e_z$$

for each $i \in \{1, \dots, 2M_{\ell,p}\}$. Obviously, we have $Y_{\ell;p}^{(i)} \in \mathcal{C}_1^+$.

Fix an $i \in \{1, \dots, 2M_{\ell,p}\}$ for the moment. The property of $\Omega_{\ell,p}^{(i)}$ ensures that the membership $(w,z) \in \Omega_{\ell,p}^{(i)}$ defines z as a function of w, and vice versa. Thus there is a subset E of Γ and an injective map $h : E \to \Gamma$ such that $\Omega_{\ell,p}^{(i)} = \{(w, h(w)) : w \in E\}$. Hence

$$Y_{\ell;p}^{(i)} = \sum_{w\in E} c_{h(w)}^{1/2} c_w^{1/2} \langle k_{h(w)}, k_w \rangle e_w \otimes e_{h(w)}.$$

By (7.20) we have $h(w) \neq w$ for every $w \in E$. Applying Lemma 7.3, we obtain a partition $E = E_1 \cup E_2 \cup E_3$ such that $h(E_\nu) \cap E_\nu = \emptyset$ for $\nu = 1,2,3$. For each $\nu \in \{1,2,3\}$, define the orthogonal projection

$$P_\nu = \sum_{w\in E_\nu} e_w \otimes e_w.$$

The property $h(E_\nu) \cap E_\nu = \emptyset$ obviously translates to $P_\nu Y_{\ell;p}^{(i)} P_\nu = 0$. Hence $\text{Tr}_\omega(P_\nu Y_{\ell;p}^{(i)}) = \text{Tr}_\omega(P_\nu Y_{\ell;p}^{(i)} P_\nu) = 0$. Since $Y_{\ell;p}^{(i)} = (P_1 + P_2 + P_3) Y_{\ell;p}^{(i)}$, we conclude that $\text{Tr}_\omega(Y_{\ell;p}^{(i)}) = 0$.

Combining the last conclusion with (7.21), we now have $\text{Tr}_\omega(Y_\ell^{(p)}) = 0$ for all $\ell \ge 0$ and $p \ge 0$. As we explained earlier, this completes the proof of Theorem 7.5. □

8. Dixmier trace — the general case

Having computed the Dixmier trace for the discrete sum T in Theorem 7.5, we will now use that result to compute the Dixmier trace for a general Toeplitz operator T_μ defined by (1.1). The gap between T and T_μ concerns "small perturbations of Γ", which is handled by the same techniques that proved the upper bound in Theorem 1.4.

Proposition 8.1. *Let Φ be a symmetric gauge function satisfying condition* (DQK). *Then there is a constant $0 < C_{8.1} < \infty$ such that the following holds*: *Let $0 < a < 1$. If Γ is any 1-separated set in $\mathbf{B}$ and if we have a set $\{w(z) : z \in \Gamma\} \subset \mathbf{B}$ satisfying the condition $\beta(z, w(z)) \leq a$ for every $z \in \Gamma$, then*

$$\left\| \sum_{z\in\Gamma} c_z k_z \otimes k_z - \sum_{z\in\Gamma} c_z k_{w(z)} \otimes k_{w(z)} \right\|_\Phi \leq C_{8.1} a \Phi(\{c_z\}_{z\in\Gamma})$$

for every set of non-negative coefficients $\{c_z\}_{z\in\Gamma}$.

Proof. By Lemma 2.6, we may assume that Γ satisfies the additional condition

$$\text{card}(\Gamma \cap T_{k,j}) \leq 1 \quad \text{for every} \quad (k,j) \in I. \tag{8.1}$$

Let us write

$$D = \sum_{z\in\Gamma} c_z k_z \otimes k_z - \sum_{z\in\Gamma} c_z k_{w(z)} \otimes k_{w(z)}.$$

Then $D = D_1 + D_2$, where

$$D_1 = \sum_{z\in\Gamma} c_z (k_z - k_{w(z)}) \otimes k_z \quad \text{and} \qquad D_2 = \sum_{z\in\Gamma} c_z k_{w(z)} \otimes (k_z - k_{w(z)}).$$

Since the estimates of $\|D_1\|_\Phi$ and $\|D_2\|_\Phi$ are similar, we will only consider the former.

To estimate $\|D_1\|_\Phi$, we pick an orthonormal set $\{\tilde{e}_z : z \in \Gamma\}$ and factor D_1 in the form $D_1 = WL$, where

$$W = \sum_{z\in\Gamma} c_z^{1/2} (k_z - k_{w(z)}) \otimes \tilde{e}_z \quad \text{and} \qquad L = \sum_{z\in\Gamma} c_z^{1/2} \tilde{e}_z \otimes k_z.$$

By Lemma 4.4, $\|D_1\|_\Phi \leq \|W^*W\|_\Phi^{1/2} \|L^*L\|_\Phi^{1/2}$. Note that

$$L^*L = \sum_{z\in\Gamma} c_z k_z \otimes k_z,$$

the Toeplitz operator associated with the discrete measure $\nu = \sum_{z\in\Gamma} c_z (1 - |z|^2)^n \delta_z$. Applying Proposition 5.1 to ν, we obtain

$$\|L^*L\|_\Phi \leq C\Phi(\{c_z\}_{z\in\Gamma}). \tag{8.2}$$

To complete the proof, we need to estimate $\|W^*W\|_\Phi^{1/2}$.

For the given Φ, we again have the symmetric gauge function $\Psi = \Phi^{(2)}$ defined in Section 4. Furthermore, $\|W^*W\|_\Phi^{1/2} = \|W\|_\Psi$ as before. Thus it suffices to estimate $\|W\|_\Psi$.

We again take advantage of the fact that the operator A given by (5.7) is invertible on $H^2(S)$. By Propositions 3.8 and 3.2, it suffices to estimate $\|B^*W\|_\Psi$, where

$$B = \sum_{\gamma \in G} \psi_{\gamma,t} \otimes e_\gamma,$$

$t > n$, G is a 1-separated set in $\mathbf{B}$ and $\{e_\gamma : \gamma \in G\}$ is an orthonormal set. By Lemma 2.6, we can further assume that the 1-separated set G has the property that

$$\operatorname{card}(G \cap T_{k,j}) \le 1 \quad \text{for every} \quad (k, j) \in I,$$

which, along with (8.1), allows us to repeat the counting argument in Section 5. But now

$$B^*W = \sum_{\gamma \in G} \sum_{z \in \Gamma} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}, \tag{8.3}$$

where

$$f_{z;\gamma} = \langle \psi_{\gamma,t}, k_z - k_{w(z)} \rangle \tilde{e}_z$$

for $\gamma \in G$ and $z \in \Gamma$. Since $\beta(z, w(z)) \le a$, Lemma 3.6 gives us

$$\|f_{z;\gamma}\| \le C'_{3.6} a (1 - |z|^2)^{n/2} |\psi_{\gamma,t}(z)|, \tag{8.4}$$

$\gamma \in G$ and $z \in \Gamma$. Obviously, the main difference between this and (5.12) is the factor a.

Following Section 5, for each integer $k \ge 0$ we define $H_k = \{w \in \mathbf{B} : 1 - 2^{-2k} \le |w| < 1 - 2^{-2(k+1)}\}$, $G_k = G \cap H_k$ and $F_k = \Gamma \cap H_k$. By (8.3), we have

$$B^*W = \sum_{\ell=0}^{\infty} Y_\ell + \sum_{\ell=1}^{\infty} Z_\ell, \tag{8.5}$$

where

$$Y_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z) \in G_k \times F_{k+\ell}} c_z^{1/2} e_\gamma \otimes f_{z;\gamma} \quad \text{and} \quad Z_\ell = \sum_{k=0}^{\infty} \sum_{(\gamma,z) \in G_{k+\ell} \times F_k} c_z^{1/2} e_\gamma \otimes f_{z;\gamma}.$$

We then decompose Y_ℓ and Z_ℓ as in Section 5, using the same sets $\{S_{k,j} : (k, j) \in I\}$, $Q_{k,p}$ and $Q_{k,\ell;p}$ introduced there. Taking $s = 1$, the argument that precedes (5.25) gives us

$$\left\| \sum_{\ell=0}^{\infty} Y_\ell \right\|_\Psi \le C_8 a \{\Phi(\{c_z\}_{z \in F})\}^{1/2}, \tag{8.6}$$

where the factor a comes from the fact that here we use (8.4) in place of (5.12). Similarly, the proof of (5.30) now gives us

$$\left\| \sum_{\ell=1}^{\infty} Z_\ell \right\|_\Psi \le C_{11} a \{\Phi(\{c_z\}_{z \in F})\}^{1/2}, \tag{8.7}$$

where a appears for the same reason. Combining (8.5), (8.6) and (8.7), we have $\|B^*W\|_\Psi \le C_{12}a\{\Phi(\{c_z\}_{z\in\Gamma})\}^{1/2}$. As we explained in the third paragraph of the proof, we can remove the B^* from $\|B^*W\|_\Psi$ by applying Propositions 3.8 and 3.2. Hence

$$\|W\|_\Psi \le C_{13}a\{\Phi(\{c_z\}_{z\in\Gamma})\}^{1/2}.$$

Recall that $\|W\|_\Psi = \|W^*W\|_\Phi^{1/2}$ and that $\|D_1\|_\Phi \le \|W^*W\|_\Phi^{1/2}\|L^*L\|_\Phi^{1/2}$. Thus the desired bound on $\|D_1\|_\Phi$ follows from the above inequality and (8.2). □

Finally, we will show that for a general Toeplitz operator T_μ defined by (1.1) on the Hardy space $H^2(S)$, we also have a formula for its Dixmier trace in the style of (7.1).

Theorem 8.2. *Let μ be a regular Borel measure on $\mathbf{B}$ such that $T_\mu \in \mathcal{C}_1^+$. Let Γ be an a,b-lattice in $\mathbf{B}$, where $0 < a < b < \infty$ and $b \ge 2a$. (Since $b \ge 2a$, such a Γ always exists.) By Theorem 1.4, we have*

$$\Phi_1^+\left(\left\{\frac{\mu(D(z,b))}{(1-|z|^2)^n}\right\}_{z\in\Gamma}\right) < \infty. \tag{8.8}$$

Since Γ is an a,b-lattice in $\mathbf{B}$, there is a partition $\mathbf{B} = \cup_{z\in\Gamma}E_z$ such that for every $z \in \Gamma$, we have $E_z \subset D(z,b)$. For each $z \in \Gamma$, define

$$c_z = \int_{E_z} \frac{d\mu(w)}{(1-|w|^2)^n}. \tag{8.9}$$

By (8.8) and Lemma 2.4, we have $\Phi_1^+(\{c_z\}_{z\in\Gamma}) < \infty$. The Dixmier trace of the Toeplitz operator T_μ is given by the formula

$$\mathrm{Tr}_\omega(T_\mu) = \mathrm{Tr}_\omega\left(\sum_{z\in\Gamma} c_z e_z \otimes e_z\right), \tag{8.10}$$

where $\{e_z : z \in \Gamma\}$ is any orthonormal set.

Proof. Let $\Gamma' = \{z \in \Gamma : c_z \ne 0\}$. Given a partition $\Gamma' = \Gamma'^{(1)} \cup \Gamma'^{(2)}$, for $i = 1,2$ we can define $E^{(i)} = \cup_{z\in\Gamma'^{(i)}}E_z$. Accordingly, $\mu = \mu^{(1)} + \mu^{(2)}$, where $\mu^{(i)}(\Delta) = \mu(\Delta \cap E^{(i)})$ for Borel sets $\Delta \subset \mathbf{B}$, $i = 1,2$. Obviously, both sides of (8.10) are additive with respect to such a decomposition. Therefore, by Lemma 2.1, it suffices to prove (8.10) under the additional assumption that Γ' is $2b+2$-separated. This implies that if we pick an arbitrary $\zeta(z) \in D(z,b)$ for each $z \in \Gamma'$, then the set $\{\zeta(z) : z \in \Gamma'\}$ is 1-separated.

We will prove (8.10) by using Theorem 7.5 and approximation in the ideal $\mathcal{C}_1^+$. This scheme proceeds as follows. Let an $\epsilon > 0$ be given. Then by the above-mentioned property of Γ' and Proposition 8.1, there is a $\delta > 0$ such that if $\zeta(z) \in D(z,b)$ for every $z \in \Gamma'$, and if a set $\{w(z) : z \in \Gamma'\}$ has the property that $\beta(\zeta(z), w(z)) \le \delta$ for every $z \in \Gamma'$, then

$$\left\|\sum_{z\in G} c_z k_{\zeta(z)} \otimes k_{\zeta(z)} - \sum_{z\in G} c_z k_{w(z)} \otimes k_{w(z)}\right\|_1^+ \le \epsilon\Phi_1^+(\{c_z\}_{z\in\Gamma}) \tag{8.11}$$

for every $G \subset \Gamma'$. For each $z \in \Gamma'$, we define the measure ν_z by the formula $\nu_z(\Delta) = c_z^{-1}\tilde{\mu}(\Delta \cap E_z)$, where Δ is any Borel set in $\mathbf{B}$ and the relation between $\tilde{\mu}$ and μ was given by (5.1). By (8.9), each ν_z is a probability measure on $\mathbf{B}$. Furthermore,

$$T_\mu = \sum_{z\in\Gamma'} c_z \int_{E_z} k_w \otimes k_w d\nu_z(w).$$

By Lemma 2.1(1), for the δ chosen above, there is an $N \in \mathbf{N}$ that has the following property: For each $z \in \Gamma'$, there are $\xi_{z,1}, \dots, \xi_{z,N} \in D(z,b)$ such that $\cup_{i=1}^N D(\xi_{z,i}, \delta/2) \supset D(z,b)$. Thus for each $z \in \Gamma'$, E_z admits a partition $E_z = E_{z,1} \cup \cdots \cup E_{z,N}$ such that

$$\sup_{u,v\in E_{z,i}} \beta(u,v) \le \delta, \tag{8.12}$$

$1 \le i \le N$. Accordingly, we rewrite the Toeplitz operator T_μ in the form

$$T_\mu = \sum_{z\in\Gamma'} \sum_{i=1}^{N} c_z \int_{E_{z,i}} k_w \otimes k_w d\nu_z(w). \tag{8.13}$$

With this N so fixed, we pick a $k \in \mathbf{N}$ such that $N/k \le \epsilon$.

For each $z \in \Gamma'$, denote $J_z = \{i \in \{1, \dots, N\} : \nu_z(E_{z,i}) \ne 0\}$. Then for every pair of $z \in \Gamma'$ and $i \in J_z$, define the probability measure $d\nu_{z,i} = \{\nu_z(E_{z,i})\}^{-1}\chi_{E_{z,i}} d\nu_z$. This allows us to rewrite (8.13) in the form

$$T_\mu = \sum_{z\in\Gamma'} \sum_{i\in J_z} c_z \nu(E_{z,i}) \int_{E_{z,i}} k_w \otimes k_w d\nu_{z,i}(w).$$

For every pair of $z \in \Gamma'$ and $i \in J_z$, there is an $m(z,i) \in \mathbf{Z}_+$ such that $m(z,i)/k \le \nu_z(E_{z,i}) < (m(z,i)+1)/k$. Thus for every such pair of z, i we have

$$\nu_z(E_{z,i}) = \frac{m(z,i)}{k} + a(z,i), \quad \text{where } \ 0 \le a(z,i) \le 1/k. \tag{8.14}$$

Accordingly, we have $T_\mu = T_1 + T_2$, where

$$T_1 = \frac{1}{k} \sum_{z\in\Gamma'} \sum_{i\in J_z} c_z m(z,i) \int_{E_{z,i}} k_w \otimes k_w d\nu_{z,i}(w) \quad \text{and} \tag{8.15}$$

$$T_2 = \sum_{z\in\Gamma'} \sum_{i\in J_z} c_z a(z,i) \int_{E_{z,i}} k_w \otimes k_w d\nu_{z,i}(w).$$

We will show that $\mathrm{Tr}_\omega(T_1)$ is close to the right-hand side of (8.10) and that $\|T_2\|_1^+$ is small.

To estimate $\mathrm{Tr}_\omega(T_1)$, observe that for every $z \in \Gamma'$, we have

$$\sum_{i\in J_z} m(z,i) = k \sum_{i\in J_z} \frac{m(z,i)}{k} \le k \sum_{i\in J_z} \nu_z(E_{z,i}) = k\nu_z(\cup_{i\in J_z} E_{z,i}) = k\nu_z(E_z) = k.$$

That is, there is a natural number $k' \leq k$ such that

$$\sum_{i \in J_z} m(z,i) \leq k' \quad \text{for every} \quad z \in \Gamma'.$$

We can think of $m(z,i)$ as the "multiplicity" with which $E_{z,i}$ appears in the sum (8.15). Once this is clear, we see that there are subsets $\Gamma_1 \supset \cdots \supset \Gamma_{k'}$ of Γ' such that

$$T_1 = (1/k)(S_1 + \cdots + S_{k'}), \tag{8.16}$$

where, for each $1 \leq j \leq k'$,

$$S_j = \sum_{z \in \Gamma_j} c_z \int_{E_{z,\iota(j,z)}} k_w \otimes k_w d\nu_{z,\iota(j,z)}(w)$$

with $\iota(j,z) \in J_z$ for every $z \in \Gamma_j$. Furthermore, to match multiplicities, for every pair of $z \in \Gamma'$ and $i \in J_z$ we have

$$\text{card}\{j \in \{1, \dots, k'\} : z \in \Gamma_j \text{ and } \iota(j,z) = i\} = m(z,i). \tag{8.17}$$

For each pair of $1 \leq j \leq k'$ and $z \in \Gamma_j$, we pick a $\zeta(z,j) \in E_{z,\iota(j,z)}$. Accordingly, we define the operators

$$D_j = \sum_{z \in \Gamma_j} c_z k_{\zeta(z,j)} \otimes k_{\zeta(z,j)},$$

$1 \leq j \leq k'$. We need to estimate $\|S_j - D_j\|_1^+$.

Fix a $j \in \{1, \dots, k'\}$ for the moment. For each $z \in \Gamma_j$, $\nu_{z,\iota(j,z)}$ is a probability measure concentrated on $E_{z,\iota(j,z)}$. It is, therefore, in the weak-* closure of convex combinations of unit point masses on $E_{z,\iota(j,z)}$. Consequently, S_j is the weak limit of operators of the form

$$H_j = \frac{1}{d} \sum_{r=1}^{d} \sum_{z \in \Gamma_j} c_z k_{w(z,r)} \otimes k_{w(z,r)},$$

where $d \in \mathbf{N}$ and $w(z,r) \in E_{z,\iota(j,z)}$ for every pair of $z \in \Gamma_j$ and $r \in \{1, \dots, d\}$. For a given $r \in \{1, \dots, d\}$, since $w(z,r) \in E_{z,\iota(j,z)}$ and $\zeta(z,j) \in E_{z,\iota(j,z)}$, by (8.12) we have $\beta(\zeta(z,j), w(z,r)) \leq \delta$ for every $z \in \Gamma_j$. Applying (8.11), we find that

$$\|H_j - D_j\|_1^+ \leq \frac{1}{d} \sum_{r=1}^{d} \left\| \sum_{z \in \Gamma_j} c_z k_{w(z,r)} \otimes k_{w(z,r)} - \sum_{z \in \Gamma_j} c_z k_{\zeta(z,j)} \otimes k_{\zeta(z,j)} \right\|_1^+ \leq \epsilon \Phi_1^+(\{c_z\}_{z \in \Gamma}).$$

Since $S_j - D_j$ is in the weak closure of operators of the form $H_j - D_j$, combining the above estimate with Lemma 4.3, we obtain $\|S_j - D_j\|_1^+ \leq \epsilon \Phi_1^+(\{c_z\}_{z \in \Gamma})$. Recalling (8.16) and the fact that $k' \leq k$, we now have $\|T_1 - (1/k)(D_1 + \cdots + D_{k'})\|_1^+ \leq \epsilon \Phi_1^+(\{c_z\}_{z \in \Gamma})$. Thus

$$|\text{Tr}_\omega(T_1) - \text{Tr}_\omega((1/k)(D_1 + \cdots + D_{k'}))| \leq \epsilon \Phi_1^+(\{c_z\}_{z \in \Gamma}). \tag{8.18}$$

Recall that for each pair of $1 \le j \le k'$ and $z \in \Gamma_j$, we have $\zeta(z,j) \in E_{z,\iota(j,z)}$. Thus, by the assumption on Γ', every $\{\zeta(z,j) : z \in \Gamma_j\}$ is a 1-separated set, $1 \le j \le k'$. Hence Theorem 7.5 can be applied to every D_j. Pick an orthonormal set $\{e_z : z \in \Gamma\}$. By Theorem 7.5, we have

$$\mathrm{Tr}_\omega\left(\frac{1}{k}(D_1 + \cdots + D_{k'})\right) = \mathrm{Tr}_\omega\left(\frac{1}{k}\sum_{j=1}^{k'}\sum_{z\in\Gamma_j} c_z e_z \otimes e_z\right).$$

Applying (8.17) on the right-hand side, we obtain

$$\mathrm{Tr}_\omega\left(\frac{1}{k}(D_1 + \cdots + D_{k'})\right) = \mathrm{Tr}_\omega\left(\sum_{z\in\Gamma'}\sum_{i\in J_z} c_z \frac{m(z,i)}{k} e_z \otimes e_z\right).$$

Combining the above with (8.14) and with the fact that $\sum_{i\in J_z} \nu_z(E_{z,i}) = \nu_z(E_z) = 1$ for every $z \in \Gamma'$, we have

$$\mathrm{Tr}_\omega\left(\frac{1}{k}(D_1 + \cdots + D_{k'})\right) = \mathrm{Tr}_\omega\left(\sum_{z\in\Gamma'} c_z e_z \otimes e_z\right) - \mathcal{E}, \tag{8.19}$$

where

$$\mathcal{E} = \mathrm{Tr}_\omega\left(\sum_{z\in\Gamma'}\sum_{i\in J_z} c_z a(z,i) e_z \otimes e_z\right).$$

We have $0 \le a(z,i) \le 1/k$ for every pair of $z \in \Gamma'$ and $i \in J_z$. Since $\mathrm{card}(J_z) \le N$ for every $z \in \Gamma'$, it is easy to see that $\mathcal{E} \le (N/k)\Phi_1^+(\{c_z\}_{z\in\Gamma})$. Recall that k was chosen so that $N/k \le \epsilon$. Combining these facts with (8.18) and (8.19), we conclude that

$$\left|\mathrm{Tr}_\omega(T_1) - \mathrm{Tr}_\omega\left(\sum_{z\in\Gamma'} c_z e_z \otimes e_z\right)\right| \le 2\epsilon\Phi_1^+(\{c_z\}_{z\in\Gamma}). \tag{8.20}$$

Next we estimate $\|T_2\|_1^+$.

A retrace of the definitions of the measures ν_z and $\nu_{z,i}$ gives us $T_2 = T_\alpha$, where

$$d\alpha = \sum_{z\in\Gamma'}\sum_{i\in J_z} \frac{a(z,i)}{\nu_z(E_{z,i})}\chi_{E_{z,i}} d\mu.$$

Recall that Γ' is $2b+2$-separated. This guarantees that $D(z,b) \cap D(z',b) = \emptyset$ for $z \ne z'$ in Γ'. Therefore it follows from Proposition 5.1 that

$$\|T_2\|_1^+ = \|T_\alpha\|_1^+ \le C_{5.1}\Phi_1^+\left(\left\{\frac{\alpha(D(z,b))}{(1-|z|^2)^n}\right\}_{z\in\Gamma'}\right). \tag{8.21}$$

Furthermore, for each $z \in \Gamma'$ we have

$$\alpha(D(z,b)) = \alpha(E_z) = \sum_{i\in J_z} a(z,i)\frac{\mu(E_{z,i})}{\nu_z(E_{z,i})} = c_z \sum_{i\in J_z} a(z,i)\frac{\mu(E_{z,i})}{\tilde{\mu}(E_{z,i})}.$$

Since $E_{z,i} \subset E_z \subset D(z,b)$, Lemma 2.4 tells us that $\mu(E_{z,i}) \le C_1(1-|z|^2)^n\tilde{\mu}(E_{z,i})$. Thus

$$\alpha(D(z,b)) \le C_1(1-|z|^2)^n c_z \sum_{i\in J_z} a(z,i) \le C_1(1-|z|^2)^n c_z(N/k) \le C_1(1-|z|^2)^n c_z\epsilon$$

for every $z \in \Gamma'$, where the second $\le$ follows from the facts that $0 \le a(z,i) \le 1/k$ and that $J_z \subset \{1,\dots,N\}$. Substituting this in (8.21), we obtain

$$\|T_2\|_1^+ \le C_1C_{5.1}\epsilon\Phi_1^+(\{c_z\}_{z\in\Gamma'}). \tag{8.22}$$

Since $T_\mu = T_1 + T_2$ and since $\epsilon > 0$ is arbitrary, (8.10) follows from (8.20) and (8.22). □

9. Modified Berezin transforms and an equivalent condition

Recall that for an operator A on the Hardy space $H^2(S)$, the function

$$\hat{A}(z) = \langle Ak_z, k_z\rangle, \quad z \in \mathbf{B},$$

is called the *Berezin transform* of A. But for our purpose we need the scalar quantity $\langle A\psi_{z,t}, \psi_{z,t}\rangle$ with $t > 0$, which is really a modified version of Berezin transform. If μ is a Borel measure on $\mathbf{B}$, then for the Toeplitz operator T_μ defined by (1.1) we have

$$\langle T_\mu\psi_{z,t}, \psi_{z,t}\rangle = \int \frac{(1-|z|^2)^{n+2t}}{|1-\langle w,z\rangle|^{2n+2t}}d\mu(w).$$

This quantity gives us a condition that is equivalent to the condition in Theorem 1.4:

Theorem 9.1. *Let $0 < s \le 1$ be given. Pick a $t > 0$ such that $s(n+2t) > n$. Suppose that Φ is a symmetric gauge function satisfying condition* (DQK). *Let $0 < a < b < \infty$ also be given such that $b \ge 2a$. Then there exist constants $0 < c \le C < \infty$ which depend only on s, t, a, b, Φ and the complex dimension n such that*

$$c\Phi(\{\langle T_\mu\psi_{z,t}, \psi_{z,t}\rangle^s\}_{z\in\Gamma}) \le \Phi\left(\left\{\left(\frac{\mu(D(z,b))}{(1-|z|^2)^n}\right)^s\right\}_{z\in\Gamma}\right) \le C\Phi(\{\langle T_\mu\psi_{z,t}, \psi_{z,t}\rangle^s\}_{z\in\Gamma})$$

for every regular Borel measure μ on $\mathbf{B}$ and every a,b-lattice $\Gamma \subset \mathbf{B}$.

The proof of Theorem 9.1 will take a few steps. First of all, we need to introduce more sets based on the radial-spherical decomposition in Section 2, which proved valuable in [10,19]. Let $(k,j) \in I$. In addition to the $T_{k,j}$ given by (2.8), we define

$$Q_{k,j} = \{ru : 1-2^{-2k} \le r < 1-2^{-2(k+2)}, u \in B(u_{k,j}, 9\cdot 2^{-k})\}. \tag{9.1}$$

For each $(k,j) \in I$, we define the subset

$$F_{k,j} = \{(\ell,i) : \ell > k, 1 \le i \le m(\ell), B(u_{\ell,i}, 2^{-\ell}) \cap B(u_{k,j}, 3\cdot 2^{-k}) \ne \emptyset\}$$

of I. We then define

$$W_{k,j} = Q_{k,j} \cup \{\cup_{(\ell,i)\in F_{k,j}} Q_{\ell,i}\}, \tag{9.2}$$

$(k, j) \in I$. By (2.7) and (9.1), we have $W_{k,j} \supset \{ru : 1-2^{-2k} \le r < 1, u \in B(u_{k,j}, 3\cdot 2^{-k})\}$.

As before, $\tilde{\mu}$ will always be defined in terms of μ by (5.1).

Lemma 9.2. *Let Φ be a symmetric gauge function satisfying condition* (DQK) *and let $0 < s \le 1$. Then there is constant $0 < C_{9.2} < \infty$ such that*

$$\Phi(\{2^{2snk}\mu^s(W_{k,j})\}_{(k,j)\in I}) \le C_{9.2}\Phi(\{\tilde{\mu}^s(Q_{k,j})\}_{(k,j)\in I})$$

for every regular Borel measure μ on $\mathbf{B}$.

Proof. From (9.1) and (9.2) it is obvious that

$$2^{2nk}\mu(W_{k,j}) \le C_1\tilde{\mu}(Q_{k,j}) + C_1 \sum_{(\ell,i)\in F_{k,j}} 2^{-2n(\ell-k)}\tilde{\mu}(Q_{\ell,i}). \tag{9.3}$$

We then repurpose the symbol ℓ. For $\ell \ge 0$ and $(k, j) \in I$, define the set

$$G^{(\ell)}_{k,j} = \{(k+\ell, h) : 1 \le h \le m(k+\ell), B(u_{k+\ell,h}, 2^{-k-\ell}) \cap B(u_{k,j}, 3\cdot 2^{-k}) \ne \emptyset\}$$

as in [19]. By (2.6) and (2.5), there is a natural number M such that

$$\mathrm{card}(G^{(\ell)}_{k,j}) \le M2^{2n\ell} \tag{9.4}$$

for all $(k, j) \in I$ and $\ell \ge 0$. Similarly, there is an $N \in \mathbf{N}$ such that

$$\mathrm{card}\{j' \in \{1, \dots, m(k)\} : G^{(\ell)}_{k,j'} \cap G^{(\ell)}_{k,j} \ne \emptyset\} \le N$$

for all $(k, j) \in I$ and $\ell \ge 0$. By a standard maximality argument, for each $\ell \ge 0$, there is a partition

$$I = I^{(1)}_\ell \cup \dots \cup I^{(N)}_\ell$$

such that for every $\nu \in \{1, \dots, N\}$, the conditions $(k, j), (k', j') \in I^{(\nu)}_\ell$ and $(k, j) \ne (k', j')$ imply $G^{(\ell)}_{k,j} \cap G^{(\ell)}_{k',j'} = \emptyset$.

For $(k, j) \in I$ and $\ell \ge 0$, define

$$E_{\ell;k,j} = 2^{-2n\ell} \sum_{(k+\ell,h)\in G^{(\ell)}_{k,j}} \tilde{\mu}(Q_{k+\ell,h}). \tag{9.5}$$

Continuing with (9.3), we have

$$2^{2nk}\mu(W_{k,j}) \le C_1 \sum_{\ell=0}^{\infty} E_{\ell;k,j}.$$

Since $0 < s/2 < 1$, this leads to

$$2^{snk}\mu^{s/2}(W_{k,j}) \le C_1^{s/2}\sum_{\ell=0}^{\infty} E_{\ell;k,j}^{s/2}. \tag{9.6}$$

Again we write $\Psi = \Phi^{(2)}$. Since Φ satisfies condition (DQK), Lemma 4.5 tells us that Ψ also satisfies condition (DQK). For each pair of $\ell \ge 0$ and $\nu \in \{1, \dots, N\}$, the sets in the family $\{G_{k,j}^{(\ell)} : (k,j) \in I_\ell^{(\nu)}\}$ are pairwise disjoint. By (9.4) and (9.5), we can apply Proposition 4.6 in the present situation to obtain

$$\Psi\left(\{E_{\ell;k,j}^{s/2}\}_{(k,j)\in I_\ell^{(\nu)}}\right) \le C_2 M^{s/2} 2^{-2\epsilon n\ell}\Psi\left(\{\tilde{\mu}^{s/2}(Q_{k,j})\}_{(k,j)\in I}\right).$$

Thus, writing $C_3 = NC_2M^{s/2}$, for every $\ell \ge 0$ we have

$$\Psi\left(\{E_{\ell;k,j}^{s/2}\}_{(k,j)\in I}\right) \le \sum_{\nu=1}^{N}\Psi\left(\{E_{\ell;k,j}^{s/2}\}_{(k,j)\in I_\ell^{(\nu)}}\right) \le C_3 2^{-2\epsilon n\ell}\Psi\left(\{\tilde{\mu}^{s/2}(Q_{k,j})\}_{(k,j)\in I}\right).$$

Combining this with (9.6), we find that

$$\begin{aligned}\Psi\left(\{2^{snk}\mu^{s/2}(W_{k,j})\}_{(k,j)\in I}\right) &\le C_1^{s/2}\sum_{\ell=0}^{\infty}\Psi\left(\{E_{\ell;k,j}^{s/2}\}_{(k,j)\in I}\right)\\ &\le C_1^{s/2}C_3\sum_{\ell=0}^{\infty}2^{-2\epsilon n\ell}\Psi\left(\{\tilde{\mu}^{s/2}(Q_{k,j})\}_{(k,j)\in I}\right).\end{aligned}$$

Since Proposition 4.6 guarantees $\epsilon > 0$, the above can be rewritten as

$$\Psi\left(\{2^{snk}\mu^{s/2}(W_{k,j})\}_{(k,j)\in I}\right) \le C_4\Psi\left(\{\tilde{\mu}^{s/2}(Q_{k,j})\}_{(k,j)\in I}\right).$$

Squaring both sides and using the relation $\Psi = \Phi^{(2)}$, we now have

$$\Phi\left(\{2^{2snk}\mu^{s}(W_{k,j})\}_{(k,j)\in I}\right) \le C_4^2\Phi\left(\{\tilde{\mu}^{s}(Q_{k,j})\}_{(k,j)\in I}\right),$$

proving the lemma. □

As on page 996 in [19], for each $(k,j) \in I$ we define

$$H_{k,j} = \{(\ell,h) \in I : 0 \le \ell \le k, 1 \le h \le m(\ell), B(u_{\ell,h}, 2^{-\ell}) \cap B(u_{k,j}, 2^{-k}) \ne \emptyset\}. \tag{9.7}$$

Lemma 9.3. *Given any $t > 0$, there is a constant $C_{9.3}$ which depends only on t and n such that the following estimate holds: Let $(k,j) \in I$ and $z \in T_{k,j}$. Then there exist $(\ell,\nu(\ell)) \in H_{k,j}$ for $\ell = 0, \dots, k$ such that for every Borel measure μ on $\mathbf{B}$, we have*

$$\langle T_\mu\psi_{z,t}, \psi_{z,t}\rangle \le C_{9.3}\sum_{\ell=0}^{k}2^{-2(n+2t)(k-\ell)}\cdot 2^{2n\ell}\mu(W_{\ell,\nu(\ell)}).$$

Proof. Let $(k,j) \in I$ and $z \in T_{k,j}$ be given. Then $z = |z|\xi$ for some $\xi \in B(u_{k,j}, 2^{-k})$. Set $\nu(k) = j$. If $0 \le \ell < k$, by (2.7), there is a $\nu(\ell) \in \{1, \dots, m(\ell)\}$ such that $\xi \in B(u_{\ell,\nu(\ell)}, 2^{-\ell})$. Then the lemma will follow if we can prove that the inequality

$$|\psi_{z,t}|^2 \le C_1 \sum_{\ell=0}^{k} 2^{-2(n+2t)(k-\ell)} 2^{2n\ell} \chi_{W_{\ell,\nu(\ell)}} \tag{9.8}$$

holds on $\mathbf{B}$, where C_1 depends only on n and t.

To prove (9.8), first observe that $W_{0,\nu(0)} = \mathbf{B}$. By the argument on the bottom of page 996 in [19], we have $|1 - \langle w, z\rangle|^{-1} \le 4 \cdot 2^{2(\ell-1)}$ for $w \in W_{\ell-1,\nu(\ell-1)} \backslash W_{\ell,\nu(\ell)}$. On the other hand, the definition of $T_{k,j}$ gives us $1 - |z| \le 2^{-2k}$. Combining these two inequalities, we see that (9.8) holds on $\mathbf{B} \backslash W_{k,\nu(k)} = \mathbf{B} \backslash W_{k,j}$. But on the set $W_{k,j}$, (9.8) follows from the simple fact that $|1 - \langle w, z\rangle| \ge 1 - |z| \ge 2^{-2(k+1)} = (1/4)2^{-2k}$ since $z \in T_{k,j}$. □

Lemma 9.4. *Let $0 < s \le 1$ be given, and let $t > 0$ satisfy the condition $s(n+2t) > n$. Suppose that Φ is a symmetric gauge function satisfying condition* (DQK). *Then there exists a constant $0 < C_{9.4} < \infty$ such that for every regular Borel measure μ on $\mathbf{B}$ and every set of points $z(k,j) \in T_{k,j}$, $(k,j) \in I$, we have*

$$\Phi(\{\langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s\}_{(k,j)\in I}) \le C_{9.4} \Phi(\{\tilde{\mu}^s(Q_{k,j})\}_{(k,j)\in I}).$$

Proof. By Lemma 9.2, it suffices to show that

$$\Phi(\{\langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s\}_{(k,j)\in I}) \le C\Phi(\{2^{2snk} \mu^s(W_{k,j})\}_{(k,j)\in I}). \tag{9.9}$$

To prove this, in addition to the $H_{k,j}$ given by (9.7), we also need the set

$$H_{k,j}^{(\ell)} = \{(\ell, h) : (\ell, h) \in H_{k,j}\}$$

for each integer $\ell \in \{0, \dots, k\}$.

Let μ be a regular Borel measure on $\mathbf{B}$. For each triple of integers $0 \le \ell \le k$ and $1 \le j \le m(k)$, there is an element $(\ell, h(k,j;\ell)) \in H_{k,j}^{(\ell)}$ such that

$$\mu(W_{\ell,h(k,j;\ell)}) \ge \mu(W_{\ell,h}) \quad \text{for every} \quad (\ell, h) \in H_{k,j}^{(\ell)}.$$

Let $0 < s \le 1$ be given. Let $z(k,j) \in T_{k,j}$, $(k,j) \in I$. Lemma 9.3 implies that

$$\begin{aligned} \langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s &\le C_{9.3}^s \sum_{\ell=0}^{k} 2^{-2s(n+2t)(k-\ell)} 2^{2sn\ell} \mu^s(W_{\ell,h(k,j;\ell)}) \\ &= C_1 \sum_{\nu=0}^{k} 2^{-2s(n+2t)\nu} 2^{2sn(k-\nu)} \mu^s(W_{k-\nu,h(k,j;k-\nu)}) \end{aligned} \tag{9.10}$$

for each $(k,j) \in I$, where $C_1 = C^s_{9.3}$. Define

$$a_{k,j;\nu} = \begin{cases} 2^{2n(k-\nu)}\mu(W_{k-\nu,h(k,j;k-\nu)}) & \text{if} \quad \nu \le k \\ 0 & \text{if} \quad \nu > k \end{cases}$$

for all $(k,j) \in I$ and all $\nu \ge 0$. Thus we can rewrite (9.10) in the form

$$\langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s \le C_1 \sum_{\nu=0}^{\infty} 2^{-2s(n+2t)\nu} a^s_{k,j;\nu}.$$

Consequently, for each symmetric gauge function Φ we have

$$\Phi(\{\langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s\}_{(k,j)\in I}) \le C_1 \sum_{\nu=0}^{\infty} 2^{-2s(n+2t)\nu} \Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}). \tag{9.11}$$

Let us denote

$$\varphi_{k,j} = 2^{2nk}\mu(W_{k,j}) \tag{9.12}$$

for every $(k,j) \in I$. We obviously have $a_{k,j;\nu} = \varphi_{k-\nu,h(k,j;k-\nu)}$ when $\nu \le k$. Since $a_{k,j;\nu} = 0$ when $k < \nu$, for each $\nu \ge 0$ we have

$$\Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}) = \Phi(\{\varphi^s_{k-\nu,h(k,j;k-\nu)}\}_{(k,j)\in I^{(\nu)}}),$$

where $I^{(\nu)} = \{(k,j) : k \ge \nu, 1 \le j \le m(k)\}$.

For each $\nu \ge 0$, consider the map $G_\nu : I^{(\nu)} \to I$ defined by the formula

$$G_\nu(k,j) = (k-\nu, h(k,j;k-\nu)), \quad (k,j) \in I^{(\nu)}.$$

If $k \ne k'$, then, of course, $G_\nu(k,j) \ne G_\nu(k',j')$ for all possible j and j'. Now suppose that integers j and j' are in the set $\{1,\ldots,m(k)\}$ such that $G_\nu(k,j) = G_\nu(k,j')$. Then $h(k,j;k-\nu) = h(k,j';k-\nu)$. A chase of definitions gives us

$$B(u_{k-\nu,h(k,j;k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j}, 2^{-k}) \ne \emptyset \quad \text{and}$$
$$B(u_{k-\nu,h(k,j';k-\nu)}, 2^{-(k-\nu)}) \cap B(u_{k,j'}, 2^{-k}) \ne \emptyset.$$

Since $h(k,j;k-\nu) = h(k,j';k-\nu)$, we have $d(u_{k,j}, u_{k,j'}) \le 4 \cdot 2^{-(k-\nu)}$. Thus we conclude from (2.6) and (2.5) that there is an $N \in \mathbf{N}$ which depends only on n such that for all $\nu \le k$ and all $1 \le j \le m(k)$,

$$\operatorname{card}\{j' \in \{1,\ldots,m(k)\} : G_\nu(k,j') = G_\nu(k,j)\} \le N2^{2n\nu}.$$

That is, the map $G_\nu : I^{(\nu)} \to I$ is at most $N2^{2n\nu}$-to-1. Applying Lemma 4.8, we have

$$\Phi(\{a^s_{k,j;\nu}\}_{(k,j)\in I}) = \Phi(\{\varphi^s_{G_\nu(k,j)}\}_{(k,j)\in I^{(\nu)}}) \le N2^{2n\nu}\Phi(\{\varphi^s_{k,j}\}_{(k,j)\in I}).$$

Substituting this in (9.11) and recalling (9.12), we find that

$$\Phi(\{\langle T_\mu\psi_{z(k,j),t},\psi_{z(k,j),t}\rangle^s\}_{(k,j)\in I}) \le C_1N\sum_{\nu=0}^\infty 2^{-2\{s(n+2t)-n\}\nu}\Phi(\{\varphi^s_{k,j}\}_{(k,j)\in I})$$

$$= C_1N\sum_{\nu=0}^\infty 2^{-2\{s(n+2t)-n\}\nu}\Phi(\{2^{2snk}\mu^s(W_{k,j})\}_{(k,j)\in I}).$$

Since we assume $s(n+2t)>n$, (9.9) follows. This completes the proof. □

For each $(k,j)\in I$, we define the point

$$w_{k,j} = (1-2^{-2k-1})u_{k,j}.$$

Recalling (2.8) and (9.1), we have $w_{k,j}\in T_{k,j}\subset Q_{k,j}$ for every $(k,j)\in I$.

Lemma 9.5. [10,Lemma 6.1] *There is a $\tau_0>0$ such that $D(w_{k,j},\tau_0)\cap D(w_{t,h},\tau_0)=\emptyset$ for all $(k,j)\neq(t,h)$ in I. In other words, $\{w_{k,j}:(k,j)\in I\}$ is a τ_0-separated set in $\mathbf{B}$.*

Proof of Theorem 9.1. It is elementary that there is a bound $0<R<\infty$ for the Bergman-metric diameter of every $Q_{k,j}$, $(k,j)\in I$. Let $0<a\le b<\infty$ be given. By Lemmas 9.5 and 2.1, there is an $M\in\mathbf{N}$ such that

$$\text{card}\{(k',j')\in I:\beta(w_{k,j},w_{k',j'})\le 2b+2R\}\le M \tag{9.13}$$

for every $(k,j)\in I$. Suppose that Γ is an a,b-lattice in $\mathbf{B}$. If $z,z'\in\Gamma$ have the property that $D(z,b)\cap Q_{k,j}\neq\emptyset$ and $D(z',b)\cap Q_{k,j}\neq\emptyset$ for some $(k,j)\in I$, then

$$\beta(z,z')<R+2b.$$

Therefore, by Lemma 2.1, there is an $N\in\mathbf{N}$ such that

$$\text{card}\{z\in\Gamma:D(z,b)\cap Q_{k,j}\neq\emptyset\}\le N$$

for every $(k,j)\in I$. Let $0<s\le 1$ and μ be given. Then for every $(k,j)\in I$, there is a $z_{k,j}\in\Gamma$ satisfying the following two conditions:

(1) $D(z_{k,j},b)\cap Q_{k,j}\neq\emptyset$.

(2) If $z\in\Gamma$ and $D(z,b)\cap Q_{k,j}\neq\emptyset$, then $\tilde\mu(D(z,b))\le\tilde\mu(D(z_{k,j},b))$.

Since $\cup_{z\in\Gamma}D(z,b)=\mathbf{B}$, for every $(k,j)\in I$ we have

$$\tilde\mu(Q_{k,j})\le\sum_{\substack{z\in\Gamma\\ D(z,b)\cap Q_{k,j}\neq\emptyset}}\tilde\mu(D(z,b))\le N\tilde\mu(D(z_{k,j},b)).$$

Thus for any symmetric gauge function Φ, we have

$$\Phi(\{\tilde\mu^s(Q_{k,j})\}_{(k,j)\in I})\le N^s\Phi(\{\tilde\mu^s(D(z_{k,j},b))\}_{(k,j)\in I}). \tag{9.14}$$

By definition, for any pair of $(k, j), (k', j') \in I$ we have

$$D(z_{k,j}, b) \cap Q_{k,j} \neq \emptyset \quad \text{and} \quad D(z_{k',j'}, b) \cap Q_{k',j'} \neq \emptyset.$$

Thus if it happens that $z_{k,j} = z_{k',j'}$, then

$$\beta(w_{k,j}, w_{k',j'}) < R + b + b + R = 2b + 2R.$$

Combining this with (9.13), we see that the map $(k, j) \mapsto z_{k,j}$ from I into Γ is at most M-to-1. Thus an application of Lemma 4.8 gives us

$$\Phi(\{\tilde{\mu}^s(D(z_{k,j}, b))\}_{(k,j)\in I}) \leq M\Phi(\{\tilde{\mu}^s(D(z, b))\}_{z\in\Gamma}). \tag{9.15}$$

Since Γ is a-separated, Lemma 2.6 provides a partition $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_K$ such that for all $i \in \{1, \ldots, K\}$ and $(k, j) \in I$, we have $\text{card}(\Gamma_i \cap T_{k,j}) \leq 1$. Consider any $i \in \{1, \ldots, K\}$. Since $\cup_{(k,j)\in I} T_{k,j} = \mathbf{B}$, for each $z \in \Gamma_i$, there is a $(k, j) \in I$ such that $z \in T_{k,j}$. We now write $z(k, j)$ for this z. Then there is some $J_i \subset I$ such that $\Gamma_i = \{z(k, j) : (k, j) \in J_i\}$ and $z(k, j) \in T_{k,j}$ for every $(k, j) \in J_i$. Applying Lemma 9.4, (9.14) and (9.15) we have

$$\begin{aligned}\Phi(\{\langle T_\mu \psi_{z,t}, \psi_{z,t}\rangle^s\}_{z\in\Gamma_i}) &= \Phi(\{\langle T_\mu \psi_{z(k,j),t}, \psi_{z(k,j),t}\rangle^s\}_{(k,j)\in J_i}) \\ &\leq C_{9.4}\Phi(\{\tilde{\mu}^s(Q_{k,j})\}_{(k,j)\in I}) \leq C_{9.4} N^s M \Phi(\{\tilde{\mu}^s(D(z, b))\}_{z\in\Gamma})\end{aligned}$$

for every $i \in \{1, \ldots, K\}$. Hence

$$\Phi(\{\langle T_\mu \psi_{z,t}, \psi_{z,t}\rangle^s\}_{z\in\Gamma}) \leq C_{9.4} K N^s M \Phi(\{\tilde{\mu}^s(D(z, b))\}_{z\in\Gamma}).$$

Recalling (5.1) and applying Lemma 2.4, the lower bound in Theorem 9.1 follows.

For the upper bound, it suffices to note that for the given b, there is a $\delta > 0$ such that if $w \in D(z, b)$, then

$$\frac{1 - |z|^2}{|1 - \langle w, z\rangle|} \geq \delta \quad \text{and} \quad \frac{1 - |w|^2}{|1 - \langle w, z\rangle|} \geq \delta.$$

(This can be seen, for example, by writing $w = \varphi_z(u)$ with $u \in D(0, b)$). Hence

$$\tilde{\mu}(D(z, b)) \leq \delta^{-2n-2t}\langle T_\mu \psi_{z,t}, \psi_{z,t}\rangle$$

for every $z \in \mathbf{B}$. This proves the upper bound and completes the proof of Theorem 9.1. □

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