ESSENTIAL NORMALITY OF PRINCIPAL SUBMODULES OF THE HARDY MODULE ON A STRONGLY PSEUDOCONVEX DOMAIN

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Dedicated to the memory of Ronald G. Douglas

ABSTRACT. Let Ω be a bounded, strongly pseudo-convex domain with smooth boundary in $\mathbb{C}^n$. Suppose that h is an analytic function defined on an open set containing $\overline{\Omega}$. We show that the principal submodule of the Hardy module $H^2(\Omega)$ generated by h is p-essentially normal for $p > n$.

1. INTRODUCTION

Throughout the paper, Ω will denote a bounded, strongly pseudo-convex domain with smooth boundary in $\mathbb{C}^n$. Associated with such a domain are the Bergman space $L_a^2(\Omega)$ and the Hardy space $H^2(\Omega)$. In contemporary multi-variate operator theory, these spaces are naturally considered as Hilbert modules [3] [8] over the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$. In this context, a linear subspace that is both closed and invariant under the multiplication by $z_1, \ldots, z_n$ is called a submodule. By taking orthogonal complement, each submodule also gives rise to a quotient module. In recent years, it has been discovered that these submodules and quotient modules lead to a lot of exciting mathematics and challenging problems.

One particular challenge is to determine the *essential normality* of these submodules and quotient modules. This problem stems from Arveson's famous conjecture [1] [2], which asserts that every graded submodule of $H_n^2 \otimes \mathbb{C}^m$ is p-essentially normal for $p > n$. Much progress has been made on this conjecture [9] [14] [17] [18] [19] [21] [27]. Later Douglas refined this conjecture for quotient modules, relating p to the complex dimension of the variety involved [6]. This more refined version is now called the Arveson-Douglas Conjecture, and a lot of work has been done along this line [7] [10] [12] [13] [30] [31].

Suppose that $\mathcal{M}$ is a submodule. Then we have the module operators

$$\mathcal{Z}_{\mathcal{M},i} = M_{z_i}|\mathcal{M},$$

$i = 1, \ldots, n$. The submodule $\mathcal{M}$ is said to be p-essentially normal if the commutators

$$[\mathcal{Z}_{\mathcal{M},i}^*, \mathcal{Z}_{\mathcal{M},j}], \quad i, j \in \{1, \ldots, n\},$$

all belong to the Schatten class $\mathcal{C}_p$. Essential normality is important because, for example, it leads to index theorems on the submodule and the corresponding quotient module [10] [30] [17] [19]. Indeed recent advances in the Arveson-Douglas Conjecture make it possible to even study the Helton-Howe trace invariants [20] on certain submodules and quotient modules [31].

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The focus of this paper will be on principal submodules. To motivate what we will do in this paper, let us first briefly review what has been shown for submodules. In [11], Douglas and K. Wang showed that in the case of the unit ball, for every polynomial $q \in \mathbb{C}[z_1, \dots, z_n]$, the principal submodule $[q]$ of the Bergman module $L^2_a(\mathbb{B}_n)$ is p-essentially normal for $p > n$. Once one knows what happens on the unit ball, it is natural to consider general Ω. But the case of general strongly pseudo-convex domain is considerably harder, because many of the techniques that work on $\mathbb{B}_n$, break down on general Ω.

So when the challenge of general strongly pseudo-convex domain was taken up in [7], a completely new approach had to be found. It was realized that most of the difficulties associated with a general Ω can be overcome with a new kind of inequality:

Theorem 1.1. [7] *Suppose Ω is a bounded strongly pseudo-convex domain with smooth boundary in $\mathbb{C}^n$ and h is a holomorphic function defined on a neighborhood of $\overline{\Omega}$. Then there exists an integer $N > 0$ such that $\forall w, z \in \Omega$ and $\forall f \in Hol(\Omega)$,*

$$|h(z)f(w)| \lesssim \frac{F(z,w)^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)||f(\lambda)|dv(\lambda).$$

Using this powerful tool, it was shown in [7] that the principal submodule $[h]$ of the Bergman module $L^2_a(\Omega)$ is p-essentially normal for all $p > n$.

Given the success on $L^2_a(\Omega)$, it is natural to ask, what about principal submodules of the Hardy module $H^2(\Omega)$? This obviously presents a new set of challenges, because $H^2(\Omega)$ is defined in terms of the surface measure on $\partial\Omega$. Using improved techniques and adapting ideas from [15], we will show that the analogous essential normality result indeed holds for the Hardy module. Here is the main result of the paper:

Theorem 1.2. *Suppose that h is an analytic function on an open set containing the closure of Ω. Let $[h]$ be the norm closure of $\{hf : f \in H^2(\Omega)\}$ in $H^2(\Omega)$. Then the principal submodule $[h]$ of the Hardy module is p-essentially normal for all $p > n$.*

Even with improved techniques, the proof of Theorem 1.2 still relies on Theorem 1.1. The main difference between this paper and [7] is that in the Hardy-space case, the gradient operator ∇ is heavily involved in the estimates.

Let us explain the two main steps in the proof of Theorem 1.2. First of all, our proof is based on the following fact: Suppose that L is a linear subspace of $H^2(\Omega)$ and T is a bounded operator on $L^2(\partial\Omega)$. If there is a $0 < C < \infty$ such that

$$\|Tf\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2_a(\Omega)}$$

for every $f \in L$, then TP_L is in the Schatten class $\mathcal{C}_p$ for $p > 2n$, where P_L is the orthogonal projection from $L^2(\partial\Omega)$ to the closure of L. This fact is known in the case of the unit ball [15]. But the unit-ball case is easy because one can take advantage of a convenient orthonormal basis. For a general Ω, these is no such convenient orthonormal basis, therefore the proof of this fact becomes a non-trivial undertaking. The proof of this fact involves equivalent norms in terms of ∇ for both $H^2(\Omega)$ and $L^2_a(\Omega)$. This first step takes up Section 3.

By a well-known argument, to prove Theorem 1.2, it suffices to show that

$$(1 - P)M_{\overline{z_i}}P \in \mathcal{C}_p \tag{1.1}$$

for all $p > 2n$ and $i \in \{1, \ldots, n\}$, where $P : L^2(\partial\Omega) \to [h]$ is the orthogonal projection. Let $\mathcal{O}(\overline{\Omega})$ denote the collection of analytic functions defined on some open set containing $\overline{\Omega}$. By the first step, (1.1) will follow if we can show that $\|(1-P)M_{\overline{z_i}}hf\|_{L^2(\partial\Omega)} \le C\|hf\|_{L^2_a(\Omega)}$ for every $f \in \mathcal{O}(\overline{\Omega})$. On the other hand, it is obvious that

$$(1-P)M_{\overline{z_i}}hf = (1-P)(\overline{z_i}hf - hg)$$

for every $g \in H^2(\Omega)$. Thus Theorem 1.2 will follow if we can show that

$$\inf\{\|\overline{z_i}hf - hg\|_{L^2(\partial\Omega)} : g \in H^2(\Omega)\} \le C\|hf\|_{L^2_a(\Omega)},$$

$f \in \mathcal{O}(\overline{\Omega})$. In the actual proof, we need to pick a $g \in H^2(\Omega)$ by formula for each given $f \in \mathcal{O}(\overline{\Omega})$. For this we use the weighted Bergman kernel K_l. We will show that for a sufficiently large $l \ge 1$, the formula

$$(T_i f)(z) = \int_\Omega \overline{w_i} f(w) K_l(z,w) |r(w)|^l dv(w)$$

gives us the right choice: we have

$$\|\overline{z_i}hf - hT_i f\|_{L^2(\partial\Omega)} \le C\|hf\|_{L^2_a(\Omega)}.$$

The proof of this inequality in Section 4 requires numerous applications of Theorem 1.1. Moreover, the gradient operator ∇ plays an essential role in the proof.

This paper is organized as follows. Section 2 contains various technical definitions and the necessary preliminaries for the proof of Theorem 1.2. Then the two main steps of the proof are carried out in Sections 3 and 4 as explained above.

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2. Preliminaries

We begin with a review of strongly pseudo-convex domains and their properties. We cite [16] [24] [25] [26] as our main references.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Suppose that there is a *defining function* $r \in C^\infty(\mathbb{C}^n)$ for Ω in the following sense:

(1) $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$.
(2) $|\nabla r(z)| \ne 0$ for all $z \in \partial\Omega$.

Then Ω is said to be a *bounded strongly pseudo-convex domain with smooth boundary* if there is a constant $c > 0$ such that

$$\sum_{i,j=1}^n \frac{\partial^2 r(p)}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j \ge c|\xi|^2$$

for all $p \in \partial\Omega$ and $\xi \in \mathbb{C}^n$.

For the rest of the paper, the symbol Ω will always denote a domain satisfying the conditions in the above definition. Furthermore, we fix a defining function $r(z)$ for Ω.

For a point $p \in \partial\Omega$, the *complex tangent space* [24] at p is defined by

$$T_p^{\mathbb{C}}(\partial\Omega) = \left\{(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r(p)}{\partial z_j}\xi_j = 0\right\}.$$

For each Ω, there is a $\delta > 0$ such that if $z \in \Omega_\delta := \{z \in \Omega : \ d(z, \partial\Omega) < \delta\}$, then there exists a unique point $\pi(z)$ in $\partial\Omega$ with $d(z, \pi(z)) = d(z, \partial\Omega)$. For such a z, we define the complex normal (respectively, tangential) direction at $\pi(z)$ to be the complex normal (respectively, tangential) direction at z. For $z \in \Omega_\delta$, we let $P_z(r_1, r_2)$ denote the polydisc centered at z with radius r_1 in the complex normal direction and radius r_2 in each complex tangential direction.

Notations: We use the symbols $\approx$, $\lesssim$ and $\gtrsim$ to denote relations "up to a constant (constants)" between positive scalars . For example, $A \approx B$ means there exist $0 < c < C < \infty$ such that $cB \leq A \leq CB$, and so on. For a point $z \in \Omega$, denote $\delta(z) = d(z, \partial\Omega)$, where d is the Euclidean distance. In the case when Ω is the unit ball $\mathbb{B}_n$, $\delta(z)$ is just $1 - |z|$. We write dv for the volume measure on Ω and $d\sigma$ for the surface measure on $\partial\Omega$.

Lemma 2.2. [24, Lemma 8] *Let Ω and r be as in Definition 2.1. Then there is a neighborhood U of $\overline{\Omega}$ such that*

$$|r(z)| \approx \delta(z) \quad \text{for} \quad z \in U.$$

For this reason, in most of our discussions we can use $|r(z)|$ and $\delta(z)$ interchangeably, and we will choose the function that is more convenient.

Definition 2.3. The *Bergman space* $L^2_a(\Omega)$ consists of all holomorphic functions on Ω which are square integrable with respect to the volume measure dv:

$$L^2_a(\Omega) = \left\{ f \in Hol(\Omega) : \int_\Omega |f(z)|^2 dv(z) < \infty \right\}.$$

With the defining function r already fixed, for each real number $\kappa > -1$, we define the *weighted Bergman space* $L^2_{a,\kappa}(\Omega)$ in a similar way:

$$L^2_{a,\kappa}(\Omega) = \left\{ f \in Hol(\Omega) : \int_\Omega |f(z)|^2 |r(z)|^\kappa dv(z) < \infty \right\}.$$

For $\epsilon > 0$, write

$$\Omega_\epsilon = \{z \in \Omega : \ r(z) < -\epsilon\}$$

and let $d\sigma_\epsilon$ be the surface measure on $\partial\Omega_\epsilon$. The *Hardy space* $H^2(\Omega)$ consists of holomorphic functions $f \in Hol(\Omega)$ such that

$$\sup_{\epsilon > 0} \int_{\partial\Omega_\epsilon} |f(z)|^2 d\sigma_\epsilon(z) < \infty.$$

Recall that each $f \in H^2(\Omega)$ uniquely determines an $f^* \in L^2(\partial\Omega) = L^2(\partial\Omega, d\sigma)$, and f is the Poisson integral of f^*. Also

$$\|f^*\|_{L^2(\partial\Omega)} \leq \sup_{\epsilon > 0} \left(\int_{\partial\Omega_\epsilon} |f(z)|^2 d\sigma_\epsilon(z) \right)^{1/2}.$$

We define $\|f^*\|_{L^2(\partial\Omega)}$ to be the Hardy space norm of f. With this norm, $H^2(\Omega)$ is a closed linear subspace of $L^2(\partial\Omega)$. We refer the reader to [28], [29] for these and other properties of the Hardy space.

Standard argument shows that the Hardy space, the Bergman space and weighted Bergman spaces are reproducing kernel Hilbert spaces. We use $S(z, w)$, $K(z, w)$ and $K_l(z, w)$ to denote their respective reproducing kernels.

As usual, we will need the familiar functions

$$X(z,w) = -r(w) - \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(z_j - w_j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 r(w)}{\partial w_j \partial w_k}(z_j - w_j)(z_k - w_k), \tag{2.1}$$

$$F(z,w) = |r(z)| + |r(w)| + |\mathrm{Im} X(z,w)| + |z-w|^2, \tag{2.2}$$

and

$$\rho(z,w) = |z-w|^2 + \left| \sum_{j=1}^{n} \frac{\partial r(z)}{\partial z_j}(w_j - z_j) \right|$$

associated with Ω and r.

Lemma 2.4. [16], [24] *There is a $\delta > 0$ such that*

$$|X(z,w)| \approx |r(z)| + |r(w)| + \rho(z,w) \approx F(z,w)$$

in the region

$$R_\delta := \{(z,w) \in \bar{\Omega} \times \bar{\Omega} : |r(z)| + |r(w)| + |z-w| < \delta\}.$$

On Ω, the infinitesimal Kobayashi metric (cf. [24] [22] [23]) is defined by the formula

$$F_K(p,\xi) = \inf\{\alpha > 0 : \exists f \in \mathbb{D}(\Omega) \text{ with } f(0) = p \text{ and } f'(0) = \xi/\alpha\}, \quad p \in \Omega,\ \xi \in \mathbb{C}^n,$$

where $\mathbb{D}(\Omega)$ is the set of holomorphic maps from the open unit disc $\mathbb{D}$ to Ω. For any C^1 curve $\gamma : [0,1] \to \Omega$, its Kobayashi length is given by the integral

$$L_K(\gamma) = \int_0^1 F_K(\gamma(x), \gamma'(x))dx.$$

If $p, q \in \Omega$, we write $\beta(p,q) = \inf\{L_K(\gamma)\}$, where the infimum is taken over all C^1 curves with $\gamma(0) = p$ and $\gamma(1) = q$. Then $\beta(p,q)$ is a complete metric and gives the usual topology on Ω. For $w \in \Omega$ and $t > 0$, define the Kobayashi ball

$$E(w,t) = \{z \in \Omega : \beta(z,w) < t\}.$$

Lemma 2.5. [24, Lemma 6] *Let $t > 0$ be given. Then there are constants a_1, a_2, b_1 and b_2 that depend only on Ω and t such that if $\delta > 0$ is small enough and if $w \in \{z \in \Omega : d(z, \partial\Omega) < \delta\}$, then*

$$P_w(a_1|r(w)|, b_1|r(w)|^{1/2}) \subset E(w,t) \subset P_w(a_2|r(w)|, b_2|r(w)|^{1/2}).$$

In particular, $v(E(w,t)) \approx |r(w)|^{n+1}$.

The next three lemmas can be found in [7].

Lemma 2.6. *There exists a $\delta > 0$ such that for $(z,w) \in R_\delta$,*

$$\rho(z,w) \approx \rho(w,z) \quad \textit{and} \quad |X(z,w)| \approx |X(w,z)|.$$

Lemma 2.7. *For a fixed $t > 0$, if $\beta(z,w) < t$, $z, w \in \Omega$, then $|r(z)| \approx |r(w)|$.*

Lemma 2.8. *Let $t > 0$ be given. Then there exists a $\delta > 0$ such that*

$$|X(z,\lambda)| \approx |X(w,\lambda)|$$

for $z, w, \lambda \in \Omega$ satisfying the conditions $(z,\lambda), (w,\lambda) \in R_\delta$ and $\beta(z,w) < t$. As a consequence, if $\beta(z,w) < t$, then $F(z,\lambda) \approx F(w,\lambda)$ for every $\lambda \in \Omega$.

The following integral estimates are standard:

Lemma 2.9. [25, Lemma 2.7] *Let $a \in \mathbb{R}$ and $\kappa > -1$. Then for $z \in \Omega$,*

$$\int_\Omega \frac{|r(w)|^\kappa}{F(z,w)^{n+1+\kappa+a}} dv(w) \approx \begin{cases} 1 & \text{if } a < 0 \\ \log\left\{|r(z)|^{-1}\right\} & \text{if } a = 0 \\ |r(z)|^{-a} & \text{if } a > 0 \end{cases}.$$

Our next lemma is a well-known fact. But since it will be used multiple times, we record it here for reference.

Lemma 2.10. *Let $\kappa \geq 0$. Then the operator*

$$(B_\kappa f)(z) = \int \frac{|r(w)|^\kappa}{F(z,w)^{n+1+\kappa}} f(w) dv(w), \quad f \in L^2(\Omega),$$

is bounded on $L^2(\Omega)$.

This follows from Lemma 2.9 by applying the Schur test with the test function $h(w) = |r(w)|^{-1/2}$.

Estimates involving the gradient will play a crucial role in this paper.

Lemma 2.11. *There exists a constant $C > 0$ such that for any $f \in Hol(\Omega)$ and $z \in \Omega$,*

$$|\nabla f(z)| \leq C \frac{1}{|r(z)|^{n+2}} \int_{E(z,1)} |f(w)| dv(w).$$

As a consequence, for each $\kappa > -1$,

$$\|\nabla f\|_{L^2_{a,\kappa+2}(\Omega)} \lesssim \inf_{c \in \mathbb{C}} \|f - c\|_{L^2_{a,\kappa}(\Omega)}.$$

Proof. It suffices to consider z that is close to the boundary. By Lemma 2.5, $E(z,1) \supset P_z(a|r(z)|, b|r(z)|^{1/2})$ for some $a, b > 0$. There is a unitary transformation $U_z : \mathbb{C}^n \to \mathbb{C}^n$ such that $U_z(1, 0, \ldots, 0)$ is the complex normal direction at z. Let

$$\tilde{f}(w) = f(z + U_z(a|r(z)|w_1, b|r(z)|^{1/2} w')),$$

where $w' = (w_2, \ldots, w_n)$. Then $\tilde{f}$ is defined on $\mathbb{B}_n$ and

$$\begin{aligned} |\nabla \tilde{f}(0)| &\lesssim \int_{\mathbb{B}_n} |\tilde{f}(w)| dv(w) \leq \frac{1}{a^2 b^{2(n-1)} |r(z)|^{n+1}} \int_{P_z(a|r(z)|, b|r(z)|^{1/2})} |f(\lambda)| dv(\lambda) \\ &\lesssim \frac{1}{|r(z)|^{n+1}} \int_{E(z,1)} |f(\lambda)| dv(\lambda). \end{aligned}$$

On the other hand, $|\nabla \tilde{f}(0)| \gtrsim |r(z)||\nabla f(z)|$. Therefore

$$|\nabla f(z)| \lesssim \frac{1}{|r(z)|^{n+2}} \int_{E(z,1)} |f(\lambda)| dv(\lambda),$$

proving the first assertion.

For the second assertion, when $\kappa > -1$, we have

$$\begin{aligned}
\int_\Omega |\nabla f(z)|^2 |r(z)|^{\kappa+2} dv(z) &\lesssim \int_\Omega |r(z)|^{-2(n+2)+\kappa+2} \Big(\int_{E(z,1)} |f(\lambda)| dv(\lambda) \Big)^2 dv(z) \\
&\le \int_\Omega |r(z)|^{\kappa-2(n+1)} \int_{E(z,1)} |f(\lambda)|^2 dv(\lambda) \cdot v(E(z,1)) dv(z) \\
&\lesssim \int_\Omega \int_{E(z,1)} |r(z)|^{\kappa-(n+1)} |f(\lambda)|^2 dv(\lambda) dv(z) \\
&= \int_\Omega \Big(\int_{E(\lambda,1)} |r(z)|^{\kappa-(n+1)} dv(z) \Big) |f(\lambda)|^2 dv(\lambda) \\
&\lesssim \int_\Omega |f(\lambda)|^2 |r(\lambda)|^\kappa dv(\lambda) = \|f\|^2_{L^2_{a,\kappa}(\Omega)}.
\end{aligned}$$

Since $\nabla(f - c) = \nabla f$ for every $c \in \mathbb{C}$, this completes the proof. □

We need the following crucial inequality from [7].

Theorem 2.12. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded strongly pseudo-convex domain with smooth boundary and h is a holomorphic function defined on a neighborhood of $\overline{\Omega}$. Then there exists a constant $N > 0$ such that $\forall w, z \in \Omega$ and $\forall f \in Hol(\Omega)$,*

$$|h(z)f(w)| \lesssim \frac{F(z,w)^N}{|r(w)|^{N+n+1}} \int_{E(w,1)} |h(\lambda)||f(\lambda)| dv(\lambda).$$

The proof of Theorem 2.12 was the bulk of the work in [7], and the essential normality result there depended on this theorem. Our proof of essential normality in this paper will also depend on Theorem 2.12. First of all, the combination of Lemma 2.11 and Theorem 2.12 allows us to control $f\nabla h$:

Lemma 2.13. *Under the same assumption as Theorem 2.12, there exists an $M > 0$ such that $\forall w, z \in \Omega$ and $\forall f \in Hol(\Omega)$,*

$$|\nabla h(z) f(w)| \lesssim \frac{F(z,w)^M}{|r(w)|^{n+2+M}} \int_{E(w,2)} |h(\lambda) f(\lambda)| dv(\lambda).$$

Proof. We apply Theorem 2.12 to $\partial_1 h, \ldots, \partial_n h$, which are holomorphic functions in a neighborhood of $\overline{\Omega}$. Thus there is an $M > 0$ such that

$$|\nabla h(z) f(w)| \lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} |\nabla h(\xi) f(\xi)| dv(\xi).$$

Then, applying Lemma 2.11 on the right-hand side, we have

$$
\begin{aligned}
|\nabla h(z)f(w)| &\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} \frac{1}{|r(\xi)|^{n+2}} \int_{E(\xi,1)} |h(\eta)|dv(\eta)|f(\xi)|dv(\xi) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,1)} \int_{E(\xi,1)} \frac{1}{|r(\xi)|^{n+2}} \frac{F(\eta,\xi)^N}{|r(\xi)|^{n+1+N}} \int_{E(\xi,1)} |h(\lambda)f(\lambda)|dv(\lambda)dv(\eta)dv(\xi) \\
&\leq \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,2)} \int_{E(\lambda,1)} \int_{E(\lambda,2)} \frac{F(\eta,\xi)^N}{|r(\xi)|^{2n+3+N}} dv(\eta)dv(\xi)|h(\lambda)f(\lambda)|dv(\lambda) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+1+M}} \int_{E(w,2)} \frac{1}{|r(\lambda)|} |h(\lambda)f(\lambda)|dv(\lambda) \\
&\lesssim \frac{F(z,w)^M}{|r(w)|^{n+2+M}} \int_{E(w,2)} |h(\lambda)f(\lambda)|dv(\lambda).
\end{aligned}
$$

In the above, the third $\lesssim$ calls for some explanation. In the preceding integral, since $\beta(\eta,\xi) < 3$, Lemmas 2.8 and 2.4 give us $F(\eta,\xi) \approx F(\xi,\xi) \approx |r(\xi)|$, while Lemma 2.7 gives us $|r(\xi)| \approx |r(\lambda)|$ because $\beta(\xi,\lambda) < 1$. □

Proposition 2.14. [5, Corollaire 1.7] *Let $K(z,w)$ be the Bergman kernel for Ω. Then*

$$K(z,w) = A(z,w)(-i\psi(z,w))^{-n-1} + B(z,w)\log(-i\psi(z,w)),$$

where $\psi \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ and $A, B \in C^\infty(\overline{\Omega} \times \overline{\Omega})$. Moreover, the function ψ has the following properties:

(1) $\psi(z,z) = -ir(z)$.
(2) $\psi(z,w) = -\overline{\psi(w,z)}$.
(3) *The Taylor expansion* $\psi(z,w) \sim -i\sum(\partial^\alpha r(w)/\partial w^\alpha)((z-w)^\alpha/\alpha!)$ *holds near the diagonal* $z = w$.
(4) $\mathrm{Im}\psi$ *is positive and* $\mathrm{Im}\{\psi(z,w)\} \gtrsim d(z,\partial\Omega) + d(w,\partial\Omega) + |z-w|^2$.

Lemma 2.15. *For each integer $l \geq 0$, let K_l be the corresponding weighted Bergman kernel for Ω. Then*

$$|K_l(z,w)| \lesssim \frac{1}{F(z,w)^{n+1+l}} \quad \textit{and} \quad |\nabla_z K_l(z,w)| \lesssim \frac{1}{F(z,w)^{n+2+l}}.$$

Proof. We use a standard trick from [25]. Define the domain

$$\tilde{\Omega} = \{(z,\xi) \in \mathbb{C}^n \times \mathbb{C}^l : r(z) + |\xi|^2 < 0\}.$$

(In the case $l = 0$, by $\tilde{\Omega}$ we mean the domain Ω itself.) Let $\tilde{K}$ be the Bergman kernel for $\tilde{\Omega}$. Then by the argument on page 230 of [25] we have

$$K_l(z,w) = \tilde{K}((z,0),(w,0)).$$

Now we apply Proposition 2.14 to $\tilde{\Omega}$ and $\tilde{K}$, which gives us

$$\tilde{K}(x,y) = \frac{A(x,y)}{(-i\psi(x,y))^{n+l+1}} + B(x,y)\log(-i\psi(x,y)).$$

Hence

$$K_l(z,w) = \frac{A((z,0),(w,0))}{\{-i\psi((z,0),(w,0))\}^{n+l+1}} + B((z,0),(w,0))\log\{-i\psi((z,0),(w,0))\} \tag{2.3}$$

for $z, w \in \Omega$. Note that
$$d((z,0), \partial\tilde{\Omega}) \gtrsim d(z, \partial\Omega)$$
for $z \in \Omega$. To see this, consider any (z_0, ξ_0) such that $r(z_0) + |\xi_0|^2 = 0$. We have
$$\begin{aligned} d((z,0),(z_0,\xi_0)) &= (|z-z_0|^2 + |\xi_0|^2)^{1/2} = (|z-z_0|^2 + |r(z_0)|)^{1/2} \\ &\gtrsim d(z,z_0) + d^{1/2}(z_0, \partial\Omega) \gtrsim d(z, \partial\Omega) \end{aligned}$$
as promised. From this and (4) in Proposition 2.14 we obtain
$$\mathrm{Im}\{\psi((z,0),(w,0))\} \gtrsim d(z,\partial\Omega) + d(w,\partial\Omega) + |z-w|^2.$$
It follows from the Taylor expansion of degrees $|\alpha| = 0$ and $|\alpha| = 1$ in Proposition 2.14(3) that
$$|\psi((z,0),(w,0))| + |z-w|^2 \gtrsim \left| r(w) + \sum_{j=1}^{n} \frac{\partial r(w)}{\partial w_j}(z_j - w_j) \right|.$$
These two inequalities together imply
$$|\psi((z,0),(w,0))| \gtrsim |X(z,w)| \approx F(z,w),$$
where the $\approx$ follows from Lemma 2.4. Combining this with (2.3), we obtain the desired upper bound for $|K_l(z,w)|$. Then, applying ∇_z to both sides of (2.3), the upper bound for $|\nabla_z K_l(z,w)|$ is similarly obtained. □

3. Equivalent Norms

We need various integral identities and inequalities.

Lemma 3.1 (Green's second identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. If φ and ψ are twice continuously differentiable in a neighborhood of $\overline{\Omega}$, then*
$$\int_\Omega (\psi\Delta\varphi - \varphi\Delta\psi)dv = \int_{\partial\Omega} (\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n})ds.$$

Boas and Straube [4] proved the following improved version of Poincaré inequality.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is locally the graph of a Hölder continuous function of exponent α, where $0 \le \alpha \le 1$, and suppose $1 \le p < \infty$. Let H be a cone in $W^{1,p}_{loc}(\Omega)$ such that the closure of $H \cap W^{1,p}(\Omega, \alpha)$ in $W^{1,p}(\Omega,\alpha)$ contains no nonzero constant function. Then there is a constant C such that*
$$\|u\|_p \le C\|\delta^\alpha \nabla u\|_p$$
for every function u in H, where δ denotes the distance to the boundary of Ω.

Corollary 3.3. *For $f \in L^2_a(\Omega)$ we have*
$$\|f - f_\Omega\|_{L^2_a(\Omega)} \approx \|\nabla f\|_{L^2_{a,2}(\Omega)}, \quad \text{where} \ \ f_\Omega = \frac{1}{v(\Omega)}\int_\Omega f dv.$$

Proof. The "$\gtrsim$" part follows from Lemma 2.11. Applying Theorem 3.2 to $H = \{f - f_\Omega : f \in L^2_a(\Omega)\}$, we obtain the "$\lesssim$" part. This completes the proof. □

Proposition 3.4. *For $f \in H^2(\Omega)$, we have*
$$\|f - f_\Omega\|_{H^2(\Omega)} \approx \|\nabla f\|_{L^2_{a,1}(\Omega)}.$$

Proof. It suffices to consider the case where f is holomorphic in a neighborhood of $\overline{\Omega}$. Also, since we can replace f with $f - f_\Omega$, we assume $f_\Omega = 0$. Now apply Lemma 3.1 with $\psi = |f|^2$ and $\varphi = r$. Since $r = 0$ on $\partial\Omega$, we have

$$\int_\Omega |f|^2 \Delta r dv + \int_\Omega (-r)\Delta|f|^2 dv = \int_{\partial\Omega} |f|^2 \frac{\partial r}{\partial n} d\sigma. \tag{3.1}$$

Note that $|r| = -r$ on Ω. We have $\Delta|f|^2 = 2|\nabla f|^2$ by the analyticity of f. Thus the second term on the left-hand side of (3.1) is exactly $2\|\nabla f\|^2_{L^2_{a,1}(\Omega)}$.

Since r is the defining function of Ω, $\partial r/\partial n > 0$ on $\partial\Omega$. By the compactness of $\partial\Omega$, there exists a $c > 0$ such that $\partial r/\partial n \geq c$ on $\partial\Omega$. Therefore

$$\|f\|^2_{H^2(\Omega)} \approx \int_{\partial\Omega} |f|^2 \frac{\partial r}{\partial n} d\sigma.$$

Since r is C^∞ on $\mathbb{C}^n$, we have

$$\left| \int_\Omega |f|^2 \Delta r dv \right| \lesssim \|f\|^2_{L^2_a(\Omega)}.$$

Since we assume $f_\Omega = 0$, Corollary 3.3 gives us

$$\|f\|^2_{L^2_a(\Omega)} \lesssim \|\nabla f\|^2_{L^2_{a,2}(\Omega)}.$$

Combining these facts with (3.1), we obtain

$$\|f\|^2_{H^2(\Omega)} \lesssim \|\nabla f\|^2_{L^2_{a,2}(\Omega)} + \|\nabla f\|^2_{L^2_{a,1}(\Omega)} \lesssim \|\nabla f\|^2_{L^2_{a,1}(\Omega)}.$$

An obvious rearrangement of the terms in (3.1) also yields

$$\|\nabla f\|^2_{L^2_{a,1}(\Omega)} \lesssim \|f\|^2_{L^2_a(\Omega)} + \|f\|^2_{H^2(\Omega)} \lesssim \|f\|^2_{H^2(\Omega)}.$$

This completes the proof. □

The next two propositions are known in the case of the unit ball [15], but need to be proved for the general Ω considered in this paper.

Proposition 3.5. *Let $I : H^2(\Omega) \to L^2_a(\Omega)$ be the natural embedding operator. Then, on the Hardy space $H^2(\Omega)$, the operator I^*I belongs to the Schatten class $\mathcal{C}_p$ for every $p > n$.*

Proof. The proof is largely a matter of keeping track of various operators and spaces, a "diagram chasing" of sort. Thus some convenient notation is necessary. First of all, for each $\kappa \geq 0$, let us write $L^2_\kappa(\Omega) = L^2(\Omega, |r|^\kappa dv)$. Under this notation, we have

$$L^2_{a,\kappa}(\Omega) = \{f \in L^2_\kappa(\Omega) : f \text{ is analytic on } \Omega\}.$$

Second, if $\mathcal{H}$ is a Hilbert space, let $\mathcal{H}^{[n]}$ denote the orthogonal sum of n copies of $\mathcal{H}$. Accordingly, if $A : \mathcal{H} \to \mathcal{H}'$ is an operator, then $A^{[n]} : \mathcal{H}^{[n]} \to \mathcal{H}'^{[n]}$ is the orthogonal sum of n copies of A.

Define $R : L^2_{a,1}(\Omega) \to L^2_1(\Omega)$ to be the operator of multiplication by the function $|r(z)|^{1/2}$. We first show that $R \in \mathcal{C}_p$ for every $p > 2n$. This is essentially the same as the proof of Lemma 5.1 in [12]. In fact, by the same interpolation argument, it suffices to verify that, for $p > 2n$, the quantity

$$\mathcal{I}_p = \iint |r(z)|^{p/2} |K_1(z,w)|^2 |r(w)| dv(w) |r(z)| dv(z)$$

is finite. By Lemmas 2.15 and 2.9, we have

$$\mathcal{I}_p \lesssim \iint \frac{|r(z)|^{p/2}}{F(z,w)^{2(n+2)}} |r(w)| dv(w) |r(z)| dv(z) \lesssim \int |r(z)|^{(p/2)-n-2} |r(z)| dv(z) < \infty.$$

Hence $R \in \mathcal{C}_p$ for $p > 2n$. This, of course, implies that $R^{[n]} \in \mathcal{C}_p$ for $p > 2n$.

Denote $\tilde{R} = R^{[n]} \oplus 1 : L^2_{a,1}(\Omega)^{[n]} \oplus \mathbb{C} \to L^2_1(\Omega)^{[n]} \oplus \mathbb{C}$. Since $\dim\mathbb{C} = 1$, we also have $\tilde{R} \in \mathcal{C}_p$ for $p > 2n$. Define the operator $X : H^2(\Omega) \to L^2_{a,1}(\Omega)^{[n]} \oplus \mathbb{C}$ by the formula

$$Xf = (\partial_1 f, \dots, \partial_n f, f_\Omega).$$

It follows from Proposition 3.4 that X is a bounded operator. Finally, define the operator $Y : L^2_a(\Omega) \to L^2_{a,2}(\Omega)^{[n]} \oplus \mathbb{C}$ by the formula

$$Yg = (\partial_1 g, \dots, \partial_n g, g_\Omega).$$

Then by Corollary 3.3, the operator Y^*Y is both bounded and invertible on $L^2_a(\Omega)$. For any $f \in H^2(\Omega)$, we have

$$\begin{aligned} \langle I^*Y^*YIf, f\rangle_{H^2(\Omega)} &= \|YIf\|^2_{L^2_{a,2}(\Omega)^{[n]}\oplus\mathbb{C}} = \|(\partial_1 f, \dots, \partial_n f)\|^2_{L^2_{a,2}(\Omega)^{[n]}} + |f_\Omega|^2 \\ &= \|(|r|^{1/2}\partial_1 f, \dots, |r|^{1/2}\partial_n f)\|^2_{L^2_1(\Omega)^{[n]}} + |f_\Omega|^2 \\ &= \|\tilde{R}Xf\|^2_{L^2_1(\Omega)^{[n]}\oplus\mathbb{C}} = \langle X^*\tilde{R}^*\tilde{R}Xf, f\rangle_{H^2(\Omega)}. \end{aligned}$$

This shows that $I^*Y^*YI = X^*\tilde{R}^*\tilde{R}X \in \mathcal{C}_p$ for $p > n$. Since Y^*Y is invertible, there is a $c > 0$ such that $Y^*Y \geq c$ on $L^2_a(\Omega)$. It follows that the operator inequality

$$I^*I \leq c^{-1} I^*Y^*YI$$

holds on the Hardy space $H^2(\Omega)$. Therefore $I^*I \in \mathcal{C}_p$ for $p > n$. □

Proposition 3.6. *Let L be a linear subspace of the Hardy space $H^2(\Omega)$. Suppose that T is a bounded operator on $L^2(\partial\Omega)$ for which there is a constant C such that*

$$\|Tf\|_{L^2(\partial\Omega)} \leq C\|f\|_{L^2_a(\Omega)} \quad \textit{for every } \ f \in L. \tag{3.2}$$

Then the operator TP_L belongs to the Schatten class $\mathcal{C}_p$ for every $p > 2n$, where P_L is the orthogonal projection from $L^2(\partial\Omega)$ onto the closure of L.

Proof. Using the embedding operator $I : H^2(\Omega) \to L^2_a(\Omega)$, by (3.2) we have

$$\langle T^*Tf, f\rangle_{L^2(\partial\Omega)} = \|Tf\|^2_{L^2(\partial\Omega)} \leq C^2\|f\|^2_{L^2_a(\Omega)} = C^2\|If\|^2_{L^2_a(\Omega)} = C^2\langle I^*If, f\rangle_{H^2(\Omega)}$$

for every $f \in L$. This implies that the operator inequality

$$(TP_L)^*TP_L = P_L T^*TP_L \leq C^2 P_L I^*IP_L$$

holds on $L^2(\partial\Omega)$. This and Proposition 3.5 together imply $(TP_L)^*TP_L \in \mathcal{C}_p$ for $p > n$. That is, $TP_L \in \mathcal{C}_{2p}$ for $p > n$. □

4. The Main Theorem

We are now ready to prove our main result:

Theorem 4.1. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded strongly pseudo-convex domain with smooth boundary and h is a holomorphic function defined in a neighborhood of $\overline{\Omega}$. Then the principal submodule $[h]$ of the Hardy module is p-essentially normal for every $p > n$.*

Proof. Given such an h, let $P : L^2(\partial\Omega) \to [h]$ be the orthogonal projection. By a standard argument (see, e.g., Proposition 4.1 in [1]), the desired essential normality for the submodule $[h]$ will follow if we can show that

$$(1 - P)M_{\overline{z_i}}P \in \mathcal{C}_p \tag{4.1}$$

for all $p > 2n$ and $i \in \{1, \dots, n\}$.

Let $\mathcal{O}(\overline{\Omega})$ be the collection of functions that are analytic some open set containing $\overline{\Omega}$. Fix a sufficiently large l and define

$$(T_i f)(z) = \int_\Omega \overline{w_i} f(w) K_l(z, w)|r(w)|^l dv(w)$$

for $f \in \mathcal{O}(\overline{\Omega})$. It is an easy consequence of Lemmas 2.15 and 2.9 that $(T_i f)(z) - \overline{z_i} f(z)$ is a bounded function on Ω. Thus T_i maps $\mathcal{O}(\overline{\Omega})$ into $H^2(\Omega)$. In particular, $hT_i f \in [h]$ for every $f \in \mathcal{O}(\overline{\Omega})$. Consequently,

$$\|(1 - P)M_{\overline{z_i}} hf\|_{L^2(\partial\Omega)} = \|(1 - P)(M_{\overline{z_i}} hf - hT_i f)\|_{L^2(\partial\Omega)} \le \|\overline{z_i} hf - hT_i f\|_{L^2(\partial\Omega)}.$$

Thus, applying Proposition 3.6 to the case where $L = \{hf : f \in \mathcal{O}(\overline{\Omega})\}$ and $T = (1 - P)M_{\overline{z_i}}$, (4.1) will follow if we can prove that

$$\|\overline{z_i} hf - hT_i f\|_{L^2(\partial\Omega)} \lesssim \|hf\|_{L^2_a(\Omega)} \tag{4.2}$$

for $f \in \mathcal{O}(\overline{\Omega})$.

To estimate the $d\sigma$-norm on the left-hand side of (4.2), let us denote

$$S(hf)(z) = \overline{z_i} h(z) f(z) - h(z)(T_i f)(z).$$

For $\epsilon > 0$, recall that $\Omega_\epsilon = \{z \in \Omega : \ r(z) < -\epsilon\}$. By Lemma 3.1,

$$\begin{aligned}\int_{\partial\Omega_\epsilon} |S(hf)(z)|^2 \frac{\partial r}{\partial n}(z) d\sigma_\epsilon(z) &= \int_{\partial\Omega_\epsilon} \frac{\partial |S(hf)|^2}{\partial n}(z) r(z) d\sigma_\epsilon(z) + \int_{\Omega_\epsilon} |S(hf)(z)|^2 \Delta r(z) dv(z) \\ &\quad - \int_{\Omega_\epsilon} \Delta(|S(hf)|^2)(z) r(z) dv(z) \\ &\lesssim \mathrm{I}_\epsilon + \mathrm{II} + \mathrm{III}.\end{aligned}$$

Here,

$$\begin{aligned}\mathrm{I}_\epsilon &= \epsilon \int_{\partial\Omega_\epsilon} \big|\nabla |S(hf)|^2(z)\big| d\sigma_\epsilon(z), \\ \mathrm{II} &= \int_\Omega |S(hf)(z)|^2 dv(z) \quad \text{and} \\ \mathrm{III} &= \int_\Omega \big|\Delta(|S(hf)|^2)(z)\big| |r(z)| dv(z).\end{aligned}$$

By direct calculation,

$$\Delta|S(hf)|^2 = \overline{S(hf)}\Delta S(hf) + S(hf)\Delta\overline{S(hf)} + 2|\nabla S(hf)|^2.$$

Further,

$$\Delta S(hf) = \Delta(\overline{z_i}hf) = 4\left(\partial_1\overline{\partial_1} + \cdots + \partial_n\overline{\partial_n}\right)(\overline{z_i}hf) = 4\partial_i(hf).$$

Similarly, $\Delta\overline{S(hf)} = 4\overline{\partial_i(hf)}$. Hence

$$\left|\Delta|S(hf)|^2\right| \lesssim |\nabla S(hf)|^2 + |S(hf)||\nabla(hf)|.$$

Consequently, we have

$$\mathrm{III} \lesssim \mathrm{IV} + \mathrm{V},$$

where

$$\mathrm{IV} = \int_\Omega |\nabla S(hf)(z)|^2|r(z)|dv(z) \quad \text{and}$$
$$\mathrm{V} = \int_\Omega |S(hf)(z)||\nabla(hf)(z)||r(z)|dv(z).$$

Recapping the above, we conclude that

$$\int_{\partial\Omega_\epsilon} |S(hf)(z)|^2\frac{\partial r}{\partial n}(z)d\sigma_\epsilon(z) \lesssim \mathrm{I}_\epsilon + \mathrm{II} + \mathrm{IV} + \mathrm{V}. \tag{4.3}$$

Let us estimate these quantities individually, beginning with II.

Using the reproducing property of K_l, we have

$$\begin{aligned} S(hf)(z) &= \overline{z_i}h(z)f(z) - h(z)\int_\Omega \overline{w_i}f(w)K_l(z,w)|r(w)|^l dv(w) \\ &= \int_\Omega (\overline{z_i} - \overline{w_i})h(z)f(w)K_l(z,w)|r(w)|^l dv(w). \end{aligned}$$

Therefore, by the first bound in Lemma 2.15,

$$\begin{aligned} |S(hf)(z)| &\le \int_\Omega |z-w||h(z)f(w)||K_l(z,w)||r(w)|^l dv(w) \\ &\lesssim \int_\Omega |h(z)f(w)|\frac{|r(w)|^l}{F(z,w)^{n+(1/2)+l}}dv(w). \end{aligned}$$

Applying Theorem 2.12 to the $|h(z)f(w)|$ above, we have

$$\begin{aligned} |S(hf)(z)| &\lesssim \int_\Omega \frac{F(z,w)^N}{|r(w)|^{n+1+N}}\int_{E(w,1)} |h(\lambda)f(\lambda)|dv(\lambda)\frac{|r(w)|^l}{F(z,w)^{n+(1/2)+l}}dv(w) \\ &= \int_\Omega\int_{E(\lambda,1)} \frac{|r(w)|^{l-N-n-1}}{F(z,w)^{n+(1/2)+l-N}}dv(w)|h(\lambda)f(\lambda)|dv(\lambda) \\ &\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+(1/2)+l-N}}|h(\lambda)f(\lambda)|dv(\lambda). \end{aligned}$$

Since the value of l is our choice, we can assume $l - N \ge 0$. Hence an application of Lemma 2.10 now gives us

$$\mathrm{II} \lesssim \|hf\|^2_{L^2_a(\Omega)}. \tag{4.4}$$

Next we consider IV.

Obviously,

$$\begin{aligned}\nabla S(hf)(z) &= \int_\Omega \nabla_z \overline{(z_i - w_i)} h(z) f(w) K_l(z,w) |r(w)|^l dv(w) \\ &\quad + \int_\Omega \overline{(z_i - w_i)} \nabla h(z) f(w) K_l(z,w) |r(w)|^l dv(w) \\ &\quad + \int_\Omega \overline{(z_i - w_i)} h(z) f(w) \nabla_z K_l(z,w) |r(w)|^l dv(w) \\ &= A(z) + B(z) + C(z).\end{aligned}$$

For $A(z)$, Lemma 2.15 and Theorem 2.12 give us

$$\begin{aligned}|A(z)| &\lesssim \int_\Omega |h(z)f(w)| \frac{|r(w)|^l}{F(z,w)^{n+1+l}} dv(w) \\ &\lesssim \int_\Omega \frac{F(z,w)^N}{|r(w)|^{n+1+N}} \int_{E(w,1)} |h(\lambda)f(\lambda)| dv(\lambda) \frac{|r(w)|^l}{F(z,w)^{n+1+l}} dv(w) \\ &= \int_\Omega \int_{E(\lambda,1)} \frac{|r(w)|^{l-N-n-1}}{F(z,w)^{n+1+l-N}} dv(w) |h(\lambda)f(\lambda)| dv(\lambda) \\ &\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1+l-N}} |h(\lambda)f(\lambda)| dv(\lambda).\end{aligned}$$

Thus another application of Lemma 2.10 leads to

$$\int |A(z)|^2 |r(z)| dv(z) \lesssim \|hf\|^2_{L^2_a(\Omega)}. \tag{4.5}$$

For $B(z)$, we apply Lemma 2.15 and Lemma 2.13, and the consequence of that is

$$\begin{aligned}|B(z)| &\lesssim \int_\Omega |\nabla h(z) f(w)| \frac{|r(w)|^l}{F(z,w)^{n+(1/2)+l}} dv(w) \\ &\lesssim \int_\Omega \frac{F(z,w)^{N'}}{|r(w)|^{n+2+N'}} \int_{E(w,2)} |h(\lambda)f(\lambda)| dv(\lambda) \frac{|r(w)|^l}{F(z,w)^{n+(1/2)+l}} dv(w) \\ &= \int_\Omega \int_{E(\lambda,2)} \frac{|r(w)|^{l-N'-n-2}}{F(z,w)^{n+(1/2)+l-N'}} dv(w) |h(\lambda)f(\lambda)| dv(\lambda) \\ &\lesssim \int_\Omega \frac{|r(\lambda)|^{l-N'-1}}{F(z,\lambda)^{n+(1/2)+l-N'}} |h(\lambda)f(\lambda)| dv(\lambda).\end{aligned}$$

Since $|r(z)|/F(z,\lambda) \lesssim 1$, from the above we obtain

$$|B(z)||r(z)|^{1/2} \lesssim \int_\Omega \frac{|r(\lambda)|^{l-N'-1}}{F(z,\lambda)^{n+l-N'}} |h(\lambda)f(\lambda)| dv(\lambda).$$

We can, of course, also assume that $l - N' - 1 \geq 0$. Since $n + l - N' = n + 1 + (l - N' - 1)$, an application of Lemma 2.10 with $\kappa = l - N' - 1$ gives us

$$\int |B(z)|^2 |r(z)| dv(z) \lesssim \|hf\|^2_{L^2_a(\Omega)}. \tag{4.6}$$

As for $C(z)$, it follows from the second bound in Lemma 2.15 that

$$|C(z)| \lesssim \int_\Omega |h(z)f(w)|\frac{|z-w||r(w)|^l}{F(z,w)^{n+2+l}}dv(w) \lesssim \int_\Omega |h(z)f(w)|\frac{|r(w)|^l}{F(z,w)^{n+(3/2)+l}}dv(w).$$

Applying Theorem 2.12 to $|h(z)f(w)|$ again, the above argument now yields

$$|C(z)| \lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+(3/2)+l-N}}|h(\lambda)f(\lambda)|dv(\lambda).$$

Again, $|r(z)|/F(z,\lambda) \lesssim 1$, which leads to

$$|C(z)||r(z)|^{1/2} \lesssim \int_\Omega \frac{|r(\lambda)|^{l-N}}{F(z,\lambda)^{n+1+l-N}}|h(\lambda)f(\lambda)|dv(\lambda).$$

Yet another application of Lemma 2.10 now results in

$$\int |C(z)|^2|r(z)|dv(z) \lesssim \|hf\|^2_{L^2_a(\Omega)}.$$

Combining this with (4.5) and (4.6), we obtain

$$\mathrm{IV} \lesssim \int (|A(z)|^2 + |B(z)|^2 + |C(z)|^2)|r(z)|dv(z) \lesssim \|hf\|^2_{L^2_a(\Omega)}. \tag{4.7}$$

For V, the Cauchy-Schwarz inequality gives us

$$\mathrm{V} \le \left(\int |S(hf)(z)|^2dv(z)\int |\nabla(hf)(z)|^2|r(z)|^2dv(z)\right)^{1/2} = \mathrm{II}^{1/2}\|\nabla(hf)\|_{L^2_{a,2}(\Omega)}.$$

By Lemma 2.11, we have $\|\nabla(hf)\|_{L^2_{a,2}(\Omega)} \lesssim \|hf\|_{L^2_a(\Omega)}$. Thus, recalling (4.4), we have

$$\mathrm{V} \lesssim \|hf\|^2_{L^2_a(\Omega)}. \tag{4.8}$$

Finally, let us consider I_ϵ, which is easier to handle because we can take advantage of various boundedness. As we have already mentioned, T_i maps $\mathcal{O}(\overline{\Omega})$ into $L^\infty(\Omega)$. Therefore for the given h and f, there is an $M < \infty$ that dominates

$$|S(hf)(z)|, \quad |h(z)f(w)| \quad \text{and} \quad |\nabla h(z)f(w)|$$

on Ω or on $\Omega \times \Omega$, as the case may be. We have

$$\left|\nabla|S(hf)|^2\right| = \left|\overline{S(hf)}\nabla S(hf) + S(hf)\nabla\overline{S(hf)}\right| \lesssim |S(hf)||\nabla S(hf)|.$$

Thus the above bound gives us

$$\left|\nabla|S(hf)|^2(z)\right| \lesssim M|\nabla S(hf)(z)| \le M(|A(z)| + |B(z)| + |C(z)|).$$

Using the same M, a review of the estimates of $|A(z)|$, $|B(z)|$, $|C(z)|$ now yields

$$|A(z)| \lesssim M\int_\Omega \frac{|r(w)|^l}{F(z,w)^{n+1+l}}dv(w),$$

$$|B(z)| \lesssim M\int_\Omega \frac{|r(w)|^l}{F(z,w)^{n+(1/2)+l}}dv(w) \quad \text{and}$$

$$|C(z)| \lesssim M\int_\Omega \frac{|r(w)|^l}{F(z,w)^{n+(3/2)+l}}dv(w).$$

Since $n+(3/2)+l$ is the dominant power for the three denominators, we have

$$\left|\nabla |S(hf)|^2(z)\right| \lesssim M\cdot M\int_\Omega \frac{|r(w)|^l}{F(z,w)^{n+(3/2)+l}}dv(w) \lesssim M^2|r(z)|^{-1/2},$$

where the second $\lesssim$ follows from Lemma 2.9. Hence

$$\mathrm{I}_\epsilon \lesssim \epsilon M^2\int_{\partial\Omega_\epsilon}|r(z)|^{-1/2}d\sigma_\epsilon(z) \lesssim \epsilon^{1/2}M^2.$$

Combining this estimate with (4.3), (4.4), (4.7) and (4.8), we find that

$$\limsup_{\epsilon\downarrow 0}\int_{\partial\Omega_\epsilon}|S(hf)(z)|^2\frac{\partial r}{\partial n}(z)d\sigma_\epsilon(z) \lesssim \lim_{\epsilon\downarrow 0}\mathrm{I}_\epsilon + \mathrm{II}+\mathrm{IV}+\mathrm{V} = \mathrm{II}+\mathrm{IV}+\mathrm{V} \lesssim \|hf\|^2_{L^2_a(\Omega)}.$$

Since $\partial r/\partial n > 0$ on $\partial\Omega$ and $\partial\Omega$ is compact, we have

$$\|S(hf)\|^2_{L^2(\partial\Omega)} \lesssim \limsup_{\epsilon\downarrow 0}\int_{\partial\Omega_\epsilon}|S(hf)(z)|^2\frac{\partial r}{\partial n}(z)d\sigma_\epsilon(z) \lesssim \|hf\|^2_{L^2_a(\Omega)}.$$

This proves (4.2) and completes the proof of the theorem. □

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