ON THE CONNECTEDNESS OF THE SPECTRA OF SELF-ADJOINT TOEPLITZ OPERATORS

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Abstract. We consider Toeplitz operators T_f on the Hardy space $H^2(S)$ of the unit sphere S in $\mathbf{C}^n$, $n \geq 2$. We show that if f is a bounded, real-valued measurable function that depends only on $|z_1|, \dots, |z_n|$, then the spectrum of T_f is connected.

1. Introduction

This paper concerns Toeplitz operators on the Hardy space of n variables. The problem we investigate goes back to the 1970s, when the investigations of these operators just began. To explain the problem, we first recall the basic setting.

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in $\mathbf{C}^n$. Let $d\sigma$ be the spherical measure on S. For convenience, we take the normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ can be simply described as the closure of the analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Let $P : L^2(S, d\sigma) \to H^2(S)$ be the orthogonal projection. Given an $f \in L^\infty(S)$, the Toeplitz operator T_f is defined by the formula

$$T_f h = P(fh), \quad h \in H^2(S).$$

The function f is usually called the *symbol* of the Toeplitz operator T_f.

In the case where $n = 1$, it is well known that both the spectrum and the essential spectrum of T_f are connected [2]. But when $n \geq 2$, we know from [1] that the spectrum and the essential spectrum of T_f can be disconnected. However, for the examples given in [1] with disconnected spectrum or essential spectrum, the Toeplitz operator T_f is not self-adjoint. Furthermore, it was conjectured in [1] (see page 359 of that paper) that if f is real valued, i.e., if T_f is self-adjoint, then the spectrum of T_f equals the interval

$$[\text{ess inf } f, \text{ess sup } f].$$

This conjecture is still open in its full generality. In fact, we are not aware of any progress on the conjecture in the decades since the publication of [1]. The purpose of this paper is to prove this conjecture for a special class of symbol functions.

Let us describe the set of symbol functions that we will deal with in this paper. Intuitively, these are the functions that depend only on $|z_1|, \dots, |z_n|$, but we can give a more rigorous definition as follows. We begin with the n-dimensional torus

$$\mathbf{T}^n = \{(\tau_1, \dots, \tau_n) \in \mathbf{C}^n : |\tau_1| = \cdots = |\tau_n| = 1\}.$$

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Given an $f \in L^1(S)$ and a $\tau = (\tau_1, \ldots, \tau_n) \in \mathbf{T}^n$, we define the function

$$f_\tau(z_1, \ldots, z_n) = f(\tau_1 z_1, \ldots, \tau_n z_n), \quad (z_1, \ldots, z_n) \in S. \tag{1.1}$$

Let $\mathcal{R}$ be the collection of functions $f \in L^\infty(S)$ satisfying the condition $f_\tau = f$ for every $\tau \in \mathbf{T}^n$. The main result of the paper is that for a real-valued function in $\mathcal{R}$, the spectrum of the corresponding Toeplitz operator is connected.

Theorem 1.1. *If f is a real-valued function in $\mathcal{R}$, then the spectrum of the Toeplitz operator T_f equals the interval* $[\text{ess inf } f, \text{ess sup } f]$.

For any $f \in \mathcal{R}$, the Toeplitz operator T_f is known to be diagonal with respect to the standard orthonormal basis $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ in $H^2(S)$. Thus T_f has eigenvalues $\{\langle fe_\alpha, e_\alpha\rangle : \alpha \in \mathbf{Z}_+^n\}$. To prove Theorem 1.1, we will show that these eigenvalues are dense in $[\text{ess inf } f, \text{ess sup } f]$. This density is proved in two steps.

As the first step, we show that for any open interval (a, b) satisfying the condition

$$\text{ess inf } f < a < b < \text{ess sup } f,$$

if $m \in \mathbf{N}$ is sufficiently large, then there are $\alpha_{(m)}, \alpha^{(m)} \in \mathbf{Z}_+^n$ with $|\alpha_{(m)}| = m = |\alpha^{(m)}|$ such that

$$\langle fe_{\alpha_{(m)}}, e_{\alpha_{(m)}}\rangle < a < b < \langle fe_{\alpha^{(m)}}, e_{\alpha^{(m)}}\rangle. \tag{1.2}$$

Furthermore, we show that $\alpha_{(m)}$, $\alpha^{(m)}$ can be chosen in such a way that their components are comparable to m. This first step takes up Section 3.

As the second step, we show in Section 4 that if m is sufficiently large, then $\alpha_{(m)}$ and $\alpha^{(m)}$ are "connected" by a chain of multi-indices $\gamma_1, \ldots, \gamma_k \in \mathbf{Z}_+^n$ satisfying the following two conditions:

(a) $\gamma_1 = \alpha_{(m)}$ and $\gamma_k = \alpha^{(m)}$.

(b) For every $1 \le \nu < k$, we have $|\langle fe_{\gamma_\nu}, e_{\gamma_\nu}\rangle - \langle fe_{\gamma_{\nu+1}}, e_{\gamma_{\nu+1}}\rangle| \le (b-a)/2$.

From (a), (b) and (1.2) it is clear that there is a $1 < j < k$ such that $\langle fe_{\gamma_j}, e_{\gamma_j}\rangle \in (a, b)$.

The proof of (1.2) involves multi-variable Bernstein polynomials. We prove the required convergence of these polynomials in the Appendix at the end of the paper.

2. Preliminaries

We adopt the standard multi-index notation [3, page 3]. Let $\{e_\alpha : \alpha \in \mathbf{Z}_+^n\}$ be the standard orthonormal basis for the Hardy space $H^2(S)$. Recall that

$$e_\alpha(z) = \left(\frac{(n-1+|\alpha|)!}{(n-1)!\alpha!}\right)^{1/2} z^\alpha,$$

$\alpha \in \mathbf{Z}_+^n$. See [3, Proposition 1.4.9]. It is easy to see that if $f \in \mathcal{R}$, then

$$\langle T_f e_\alpha, e_\beta\rangle = \langle fe_\alpha, e_\beta\rangle = 0 \quad \text{when} \quad \alpha \neq \beta.$$

Thus if $f \in \mathcal{R}$, then the Toeplitz operator T_f is a diagonal operator of the form

$$T_f = \sum_{\alpha \in \mathbf{Z}_+^n} \langle f e_\alpha, e_\alpha \rangle e_\alpha \otimes e_\alpha.$$

In other words, for any $f \in \mathcal{R}$, the Toeplitz operator T_f has eigenvalues $\{\langle f e_\alpha, e_\alpha \rangle : \alpha \in \mathbf{Z}_+^n\}$. It was shown in [1] that for a complex-valued $f \in \mathcal{R}$, the closure of the set $\{\langle f e_\alpha, e_\alpha \rangle : \alpha \in \mathbf{Z}_+^n\}$ can be disconnected. We will show that if f is a real-valued function in $\mathcal{R}$, then the set $\{\langle f e_\alpha, e_\alpha \rangle : \alpha \in \mathbf{Z}_+^n\}$ is dense in the interval $[\text{ess inf } f, \text{ess sup } f]$. Combining this statement with the obvious fact that the spectrum of such a T_f is contained in $[\text{ess inf } f, \text{ess sup } f]$, Theorem 1.1 follows.

Let Q denote the first quadrant of the closed unit ball in $\mathbf{R}^{n-1}$. That is,

$$Q = \{(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1} : x_1^2 + \cdots + x_{n-1}^2 \le 1 \ \text{ and } \ x_1 \ge 0, \dots, x_{n-1} \ge 0\}.$$

On Q we define the measure $d\mu$ by the formula

$$d\mu(x_1, \dots, x_{n-1}) = (n-1)! 2^{n-1} x_1 \cdots x_{n-1} dx_1 \cdots dx_{n-1}.$$

It is known that

$$\int_S \varphi(|z_1|, \dots, |z_{n-1}|) d\sigma(z_1, \dots, z_{n-1}, z_n) = \int_Q \varphi d\mu \tag{2.1}$$

for every $\varphi \in C(Q)$. See [4, page 1377].

We further introduce the set

$$\Delta = \{(t_1, \dots, t_{n-1}) \in \mathbf{R}^{n-1} : t_1 + \cdots + t_{n-1} \le 1 \ \text{ and } \ t_1 \ge 0, \dots, t_{n-1} \ge 0\}.$$

Let dV_{n-1} denote the standard Lebesgue measure on $\mathbf{R}^{n-1}$. Making the substitution $t_1 = x_1^2, \dots, t_{n-1} = x_{n-1}^2$, we have

$$\int_Q \xi(x_1^2, \dots, x_{n-1}^2) d\mu(x_1, \dots, x_{n-1}) = (n-1)! \int_\Delta \xi dV_{n-1}, \tag{2.2}$$

$\xi \in C(\Delta)$. For each $\xi \in C(\Delta)$, we define the function

$$(\mathcal{S}\xi)(z_1, \dots, z_{n-1}, z_n) = \xi(|z_1|^2, \dots, |z_{n-1}|^2), \quad (z_1, \dots, z_{n-1}, z_n) \in S. \tag{2.3}$$

Then it follows from (2.1) and (2.2) that

$$\int_S \mathcal{S}\xi d\sigma = (n-1)! \int_\Delta \xi dV_{n-1} \tag{2.4}$$

for every $\xi \in C(\Delta)$.

Proposition 2.1. *Given any $f \in L^\infty(S)$, there is a $g \in L^\infty(\Delta)$ such that*

$$\int_S f\mathcal{S}\xi d\sigma = (n-1)! \int_\Delta g\xi dV_{n-1} \tag{2.5}$$

for every $\xi \in C(\Delta)$. If $f \in \mathcal{R}$ and if f is real valued, then we have

$$\text{ess inf } g \le \text{ess inf } f \quad \text{and} \quad \text{ess sup } f \le \text{ess sup } g.$$

Proof. Define

$$\Phi(\xi) = \int_S f\mathcal{S}\xi d\sigma$$

for every $\xi \in C(\Delta)$. Note that (2.4) implies $\|\mathcal{S}\xi\|_{L^1(S,d\sigma)} = \|\xi\|_{L^1(\Delta,(n-1)!dV_{n-1})}$. Hence

$$|\Phi(\xi)| \le \|f\|_\infty \|\xi\|_{L^1(\Delta,(n-1)!dV_{n-1})}$$

for every $\xi \in C(\Delta)$. By the Hahn-Banach theorem, there is a bounded linear functional $\tilde{\Phi}$ on $L^1(\Delta,(n-1)!dV_{n-1})$ with $\|\tilde{\Phi}\| \le \|f\|_\infty$ such that $\tilde{\Phi}(\xi) = \Phi(\xi)$ for every $\xi \in C(\Delta)$. By the representation of $\tilde{\Phi}$, there is a $g \in L^\infty(\Delta)$ with $\|g\|_\infty \le \|f\|_\infty$ such that (2.5) holds.

To prove the second half of the proposition, we introduce the following device. Let dm_n denote the Lebesgue measure on $\mathbf{T}^n$ with the normalization $m_n(\mathbf{T}^n) = 1$. For every $\varphi \in L^1(S, d\sigma)$, we define

$$A\varphi = \int_{\mathbf{T}^n} \varphi_\tau dm_n(\tau),$$

where φ_τ is defined by (1.1).

Suppose that $f \in \mathcal{R}$ and that f is real valued. Let $\epsilon > 0$. Pick an $a \in \mathbf{R}$ such that

$$\text{ess inf } f + \epsilon - a < 0.$$

Since $\mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ is dense in $C(S)$ with respect to the norm $\|\cdot\|_\infty$ and since $C(S)$ is dense in $L^1(S, d\sigma)$ with respect to the L^1-norm, there is a $q \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ satisfying the conditions $q \ge 0$ on S, $\|q\|_{L^1(S,d\sigma)} = 1$, and $\int_S fqd\sigma \le \text{ess inf } f + \epsilon$. Thus

$$\int_S (f-a)qd\sigma \le \text{ess inf } f + \epsilon - a < 0.$$

By the invariance of $d\sigma$ and the membership $f \in \mathcal{R}$, for each $\tau \in \mathbf{T}^n$ we have

$$\int_S (f-a)qd\sigma = \int_S (f_\tau - a)q_\tau d\sigma = \int_S (f-a)q_\tau d\sigma.$$

Averaging over $\mathbf{T}^n$, we obtain

$$\int_S (f-a)Aqd\sigma = \int_S (f-a)qd\sigma \le \text{ess inf } f + \epsilon - a. \tag{2.6}$$

If u is a monomial of the form $z_1^{\alpha_1}\bar{z}_1^{\beta_1}\cdots z_n^{\alpha_n}\bar{z}_n^{\beta_n}$ for some $\alpha_1,\beta_1,\dots,\alpha_n,\beta_n \in \mathbf{Z}_+$, then $Au \neq 0$ only if $\alpha_j = \beta_j$ for every $1 \leq j \leq n$. Since $|z_n|^2 = 1 - |z_1|^2 - \cdots - |z_{n-1}|^2$ on S, we see that there is an $(n-1)$-variable polynomial p such that

$$Aq = \mathcal{S}p \quad \text{on} \quad S.$$

Substituting this in (2.6) and recalling (2.5) and (2.4), we now have

$$(n-1)!\int_\Delta (g-a)p dV_{n-1} = \int_S (f-a)\mathcal{S}p d\sigma \leq \text{ess inf } f + \epsilon - a \tag{2.7}$$

Since $q \geq 0$ on S, we have $Aq \geq 0$ on S. Hence $\mathcal{S}p \geq 0$ on S, and consequently $p \geq 0$ on Δ. Applying (2.4) again, we have

$$\|p\|_{L^1(\Delta,(n-1)!dV_{n-1})} = \|\mathcal{S}p\|_{L^1(S,d\sigma)} = \|Aq\|_{L^1(S,d\sigma)} \leq \|q\|_{L^1(S,d\sigma)} = 1. \tag{2.8}$$

Since $\text{ess inf } f + \epsilon - a < 0$, it follows from (2.7) and (2.8) that

$$\text{ess inf}(g-a) \leq \text{ess inf } f + \epsilon - a.$$

That is, $\text{ess inf } g \leq \text{ess inf } f + \epsilon$. Since $\epsilon > 0$ is arbitrary, if follows that $\text{ess inf } g \leq \text{ess inf } f$. The proof for the inequality $\text{ess sup } \; f \leq \text{ess sup } \; g$ is similar and will be omitted. □

3. Essential extrema

Given an $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbf{Z}_+^n$, we define the $(n-1)$-variable polynomial

$$u_\alpha(t_1,\dots,t_{n-1}) = \frac{(n-1+|\alpha|)!}{\alpha!} t_1^{\alpha_1}\cdots t_{n-1}^{\alpha_{n-1}}(1-t_1-\cdots-t_{n-1})^{\alpha_n}.$$

Obviously, for each $\alpha \in \mathbf{Z}_+^n$, u_α is a non-negative function on Δ, and by (2.4) we have

$$\int_\Delta u_\alpha dV_{n-1} = \int_S |e_\alpha|^2 d\sigma = 1. \tag{3.1}$$

If $f \in L^\infty(S)$ and $g \in L^\infty(\Delta)$ are a pair of functions satisfying (2.5), then

$$\int_S f|e_\alpha|^2 d\sigma = \int_\Delta g u_\alpha dV_{n-1} \tag{3.2}$$

for every $\alpha \in \mathbf{Z}_+^n$. The following is the first step in the proof of Theorem 1.1.

Lemma 3.1. *Let $g \in L^\infty(\Delta)$ be a real-valued function. Given any $\epsilon > 0$, there exist a positive number $0 < \delta \leq 1/2$ and an integer $0 < M < \infty$ such that the following holds true: For every $m \geq M$, there exist $\alpha_{(m)} = (\alpha_{m,1},\dots,\alpha_{m,n}) \in \mathbf{Z}_+^n$ and $\alpha^{(m)} = (\alpha_1^{(m)},\dots,\alpha_n^{(m)}) \in \mathbf{Z}_+^n$ satisfying the conditions*

(a) $|\alpha_{(m)}| = m = |\alpha^{(m)}|$.

(b) *For every* $1 \le j \le n$, *we have* $\delta m \le \alpha_{m,j} \le (1-\delta)m$ *and* $\delta m \le \alpha_j^{(m)} \le (1-\delta)m$.
(c) *We have*

$$\int_\Delta g u_{\alpha_{(m)}} dV_{n-1} \le \text{ess inf } g + \epsilon \quad \text{and} \quad \int_\Delta g u_{\alpha^{(m)}} dV_{n-1} \ge \text{ess sup } g - \epsilon.$$

Proof. For any $r > 0$, we define the subset

$$\Delta(r) = \{t \in \Delta : d(t, \mathbf{R}^{n-1} \backslash \Delta) > r\}$$

of Δ, where $d(t, \mathbf{R}^{n-1} \backslash \Delta) = \inf\{|t - x| : x \in \mathbf{R}^{n-1} \backslash \Delta\}$. Given an $\epsilon > 0$, we define

$$E = \{t \in \Delta : g(t) \le \text{ess inf } g + (\epsilon/10)\} \quad \text{and} \quad F = \{t \in \Delta : g(t) \ge \text{ess sup } g - (\epsilon/10)\}.$$

Then $V_{n-1}(E) > 0$ and $V_{n-1}(F) > 0$. This means that there is a $\delta > 0$ such that

$$V_{n-1}(E \cap \Delta(2\delta)) > 0 \quad \text{and} \quad V_{n-1}(F \cap \Delta(2\delta)) > 0.$$

Below we will explicitly produce $\alpha_{(m)}$; the details for producing $\alpha^{(m)}$ are similar and will not be repeated.

Since $V_{n-1}(E \cap \Delta(2\delta)) > 0$, there is a $t_* \in E \cap \Delta(2\delta)$ which is a Lebesgue point of g. Hence there is an $(n-1)$-dimensional cube B satisfying the conditions that $t_* \in B \subset \Delta(\delta)$ and that

$$\frac{1}{V_{n-1}(B)} \int_B |g - g(t_*)| dV_{n-1} \le \frac{\epsilon}{10}.$$

Since $t_* \in E$, we have $g(t_*) \le \text{ess inf } g + (\epsilon/10)$, and from the above inequality we obtain

$$\frac{1}{V_{n-1}(B)} \int_B g dV_{n-1} \le g(t_*) + \frac{\epsilon}{10} \le \text{ess inf } g + \frac{\epsilon}{5}. \tag{3.3}$$

There are continuous functions $\eta, \xi : \Delta \to [0, 1]$ satisfying the conditions that $0 \le \xi \le \chi_B \le \eta$ on Δ and that

$$\frac{1}{V_{n-1}(B)} \int_\Delta |\eta - \chi_B||g| dV_{n-1} \le \frac{\epsilon}{20} \quad \text{and} \quad \frac{1}{V_{n-1}(B)} \int_\Delta |\xi - \chi_B||g| dV_{n-1} \le \frac{\epsilon}{20}.$$

For $\varphi \in C(\Delta)$ and $m \in \mathbf{N}$, write φ_m for the m-th Bernstein polynomial of φ, as defined in the Appendix. We know from Proposition A.1 that $\lim_{m \to \infty} \|\varphi - \varphi_m\|_\infty = 0$. Combining this fact with the above inequalities, there is an $M_1 \in \mathbf{N}$ such that if $m \ge M_1$, then

$$\frac{1}{V_{n-1}(B)} \int_\Delta |\eta_m - \chi_B||g| dV_{n-1} \le \frac{\epsilon}{10} \quad \text{and} \quad \frac{1}{V_{n-1}(B)} \int_\Delta |\xi_m - \chi_B||g| dV_{n-1} \le \frac{\epsilon}{10}.$$

For $m \in \mathbf{N}$, we also write $(\chi_B)_m$ for the m-th Bernstein polynomial of χ_B. The condition $\xi \le \chi_B \le \eta$ implies $\xi_m \le (\chi_B)_m \le \eta_m$ for every $m \in \mathbf{N}$. Hence

$$\xi_m - \chi_B \le (\chi_B)_m - \chi_B \le \eta_m - \chi_B.$$

From this we see that the inequality

$$|(\chi_B)_m(t) - \chi_B(t)| \le \max\{|\eta_m(t) - \chi_B(t)|, |\xi_m(t) - \chi_B(t)|\}$$

holds for every $t \in \Delta$. Thus for each $m \ge M_1$, we have

$$\frac{1}{V_{n-1}(B)} \int_\Delta |(\chi_B)_m - \chi_B||g| dV_{n-1} \le \frac{\epsilon}{5}.$$

Combining this with (3.3), we find that

$$\frac{1}{V_{n-1}(B)} \int_\Delta (\chi_B)_m g dV_{n-1} \le \frac{1}{V_{n-1}(B)} \int_B g dV_{n-1} + \frac{\epsilon}{5} \le \text{ess inf } g + \frac{2\epsilon}{5}$$

for every $m \ge M_1$. By the definition of the Bernstein polynomial $(\chi_B)_m$ in the Appendix (see (A.2)), this means that if $m \ge M_1$, then

$$\sum_{|\beta| \le m} \chi_B(\beta/m) \frac{1}{V_{n-1}(B)} \int_\Delta \psi_{m,\beta} g dV_{n-1} \le \text{ess inf } g + \frac{2\epsilon}{5}, \tag{3.4}$$

where, for $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbf{Z}_+^{n-1}$,

$$\psi_{m,\beta}(t_1, \dots, t_{n-1}) = \frac{m!}{\beta!(m - |\beta|)!} t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}} (1 - t_1 - \cdots - t_{n-1})^{m-|\beta|}.$$

For each $m \in \mathbf{N}$, let k_m be the number of $\beta \in \mathbf{Z}_+^{n-1}$ satisfying the conditions $|\beta| \le m$ and $\beta/m \in B$. There is an $M_2 \ge M_1$ such that if $m \ge M_2$, then $k_m > 0$. For $m \ge M_2$ we can rewrite (3.4) in the form

$$\frac{1}{k_m} \sum_{|\beta| \le m} \chi_B(\beta/m) \frac{k_m}{V_{n-1}(B)} \int_\Delta \psi_{m,\beta} g dV_{n-1} \le \text{ess inf } g + \frac{2\epsilon}{5},$$

Hence for each $m \ge M_2$, there is a $\beta_{(m)} \in \mathbf{Z}_+^{n-1}$ satisfying the conditions $|\beta_{(m)}| \le m$, $\beta_{(m)}/m \in B$ and

$$\frac{k_m}{V_{n-1}(B)} \int_\Delta \psi_{m,\beta_{(m)}} g dV_{n-1} \le \text{ess inf } g + \frac{2\epsilon}{5}. \tag{3.5}$$

For each $m \ge M_2$, we now define the element $\alpha_{(m)} \in \mathbf{Z}_+^n$ by the formula

$$\alpha_{(m)} = (\beta_{(m)}, m - |\beta^{(m)}|).$$

Comparing the definitions of u_α and $\psi_{m,\beta}$, (3.5) can be rewritten as

$$\frac{k_m}{V_{n-1}(B) \prod_{j=1}^{n-1}(j+m)} \int_\Delta u_{\alpha_{(m)}} g dV_{n-1} \le \text{ess inf } g + \frac{2\epsilon}{5}.$$

From the definition of k_m it is clear that $k_m/m^{n-1} \to V_{n-1}(B)$ as $m \to \infty$. Hence

$$\lim_{m\to\infty} \frac{k_m}{V_{n-1}(B)\prod_{j=1}^{n-1}(j+m)} = 1.$$

Consequently, there is an $M_* \geq M_2$ such that if $m \geq M_*$, then

$$\int_\Delta u_{\alpha_{(m)}} g dV_{n-1} \leq \text{ess inf } g + \epsilon.$$

From the definition of $\alpha_{(m)}$ it is obvious that $|\alpha_{(m)}| = m$. Let us verify that $\alpha_{(m)}$ satisfies condition (b).

If $t = (t_1, \ldots, t_{n-1})$ belongs to $\Delta(\delta)$, then we have $t_j > \delta$ for every $1 \leq j \leq n-1$ and $1 - t_1 - \cdots - t_{n-1} > \delta$. Note that the latter condition implies that $1 - t_j > \delta$ for every $1 \leq j \leq n-1$. For each $m \geq M_1$, write $\beta_{(m)} = (\beta_{m,1}, \ldots, \beta_{m,n-1})$. Since $\beta_{(m)}/m \in B$ and $B \subset \Delta(\delta)$, we have $\beta_{m,j}/m > \delta$ and $1 - (\beta_{m,j}/m) > \delta$ for every $1 \leq j \leq n-1$, and $1 - (\beta_{m,1}/m) - \cdots - (\beta_{m,n-1}/m) > \delta$. From these inequalities it is easy to see that the multi-index $\alpha_{(m)} = (\beta_{(m)}, m - |\beta_{(m)}|)$ satisfies condition (b). This completes the proof. □

Lemma 3.1 motivates the following two definitions.

Definition 3.2. Let $0 < \delta \leq 1/2$. Then Z_δ denotes the collection of $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ satisfying the condition $\delta|\alpha| \leq \alpha_j \leq (1-\delta)|\alpha|$ for every $1 \leq j \leq n$.

Definition 3.3. Given $0 < \delta \leq 1/2$ and $m \in \mathbf{N}$, $Z_{\delta,m}$ denotes the collection of $\alpha \in Z_\delta$ satisfying the condition $|\alpha| = m$.

Proposition 3.4. *Let $f \in \mathcal{R}$ be a real-valued function. Given any $\epsilon > 0$, there exist a positive number $0 < \delta \leq 1/2$ and an integer $0 < M < \infty$ such that if $m \geq M$, then there exist $\alpha_{(m)}, \alpha^{(m)} \in Z_{\delta,m}$ which have the properties*

$$\langle f e_{\alpha_{(m)}}, e_{\alpha_{(m)}} \rangle \leq \text{ess inf } f + \epsilon \quad \textit{and} \quad \langle f e_{\alpha^{(m)}}, e_{\alpha^{(m)}} \rangle \geq \text{ess sup } f - \epsilon. \tag{3.6}$$

Proof. Given a real-valued $f \in \mathcal{R}$, by Proposition 2.1 and (3.2), there exist a real-valued $g \in L^\infty(\Delta)$ such that

$$\langle f e_\alpha, e_\alpha \rangle = \int_\Delta g u_\alpha dV_{n-1} \tag{3.7}$$

for every $\alpha \in \mathbf{Z}_+^n$ and such that

$$\text{ess inf } g \leq \text{ess inf } f \quad \text{and} \quad \text{ess sup } f \leq \text{ess sup } g. \tag{3.8}$$

Given an $\epsilon > 0$, Lemma 3.1 provides $0 < \delta \leq 1/2$ and $0 < M < \infty$ such that for $m \geq M$, there are $\alpha_{(m)}, \alpha^{(m)} \in Z_{\delta,m}$ which have the properties

$$\int_\Delta g u_{\alpha_{(m)}} dV_{n-1} \leq \text{ess inf } g + \epsilon \quad \text{and} \quad \int_\Delta g u_{\alpha^{(m)}} dV_{n-1} \geq \text{ess sup } g - \epsilon. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain (3.6). □

4. A chain of eigenvalues

For each $1 \le j \le n$, let ϵ_j be the element in $\mathbf{Z}_+^n$ whose j-th component is 1 and whose other components are 0. The purpose of this section is to estimate the L^1-norm on S of functions of the form $||e_\alpha|^2 - |e_{\alpha+\epsilon_i-\epsilon_j}|^2|$, where $i \ne j$. This will take two steps. The first step is the one-variable version of such estimates.

For each pair of $m \in \mathbf{N}$ and $j \in \{0, 1, \dots, m\}$, we define the one-variable polynomial

$$v_{m,j}(t) = \frac{(m+1)!}{j!(m-j)!} t^j (1-t)^{m-j}.$$

It is well known that for all integers $k \ge 0$ and $\ell \ge 0$, we have

$$\int_0^1 x^k (1-x)^\ell dx = \frac{k!\ell!}{(k+\ell+1)!}. \tag{4.1}$$

Lemma 4.1. *Given any $0 < \delta \le 1/2$, there is a constant $0 < C_{4.1}(\delta) < \infty$ such that if $m \in \mathbf{N}$ and $j \in \{0, 1, \dots, m\}$ satisfy the condition $\delta m \le j \le (1-\delta)m$, then*

$$\int_0^1 |v_{m,j}(t) - v_{m,j+1}(t)| dt \le \frac{C_{4.1}(\delta)}{m^{1/4}}.$$

Proof. By simple algebra,

$$v_{m,j}(t) - v_{m,j+1}(t) = \frac{(m+1)!}{(j+1)!(m-j)!} t^j (1-t)^{m-j-1}((j+1) - (m+1)t).$$

Therefore

$$(v_{m,j}(t) - v_{m,j+1}(t))^2 = h(t)((j+1)^2 - 2(j+1)(m+1)t + (m+1)^2 t^2),$$

where

$$h(t) = \left\{ \frac{(m+1)!}{(j+1)!(m-j)!} \right\}^2 t^{2j} (1-t)^{2(m-j-1)}.$$

Applying (4.1), we find that

$$\int_0^1 (v_{m,j}(t) - v_{m,j+1}(t))^2 dt = \left\{ \frac{(m+1)!}{(j+1)!(m-j)!} \right\}^2 (I_1 - I_2 + I_3),$$

where

$$I_1 = (j+1)^2 \frac{(2j)!(2(m-j-1))!}{(2m-1)!},$$

$$I_2 = 2(j+1)(m+1) \frac{(2j+1)!(2(m-j-1))!}{(2m)!} \quad \text{and}$$

$$I_3 = (m+1)^2 \frac{(2j+2)!(2(m-j-1))!}{(2m+1)!}.$$

By elementary manipulations,

$$
\begin{aligned}
I_1 - I_2 + I_3 &= \frac{(2j)!(2(m-j-1))!}{(2m-1)!}\biggl((j+1)^2 \\
&\qquad - 2(j+1)(m+1)\frac{2j+1}{2m} + (m+1)^2\frac{(2j+1)(2j+2)}{2m(2m+1)}\biggr) \\
&= \cdots\cdots \\
&= \frac{(2j)!(2(m-j-1))!}{(2m-1)!}\cdot\frac{(j+1)(m-j)}{2m+1}.
\end{aligned}
$$

Thus

$$
\int_0^1 (v_{m,j}(t) - v_{m,j+1}(t))^2 dt = \left\{\frac{(m+1)!}{(j+1)!(m-j)!}\right\}^2 \frac{(2j)!(2(m-j-1))!}{(2m-1)!}\cdot\frac{(j+1)(m-j)}{2m+1}.
$$

By the well-known asymptotic expansion of Stirling, $k! \approx k^{k+(1/2)}e^{-k}$. Applying this to the factorials in the above, we find that there is a constant $0 < C_1 < \infty$ which depends only on δ such that

$$
\int_0^1 (v_{m,j}(t) - v_{m,j+1}(t))^2 dt \le \frac{C_1}{m^{1/2}}
$$

for all $m \in \mathbf{N}$ and $j \in \{0, 1, \ldots, m\}$ satisfying the condition $\delta m \le j \le (1-\delta)m$. Combining this with the Cauchy-Schwarz inequality

$$
\int_0^1 |v_{m,j}(t) - v_{m,j+1}(t)| dt \le \left(\int_0^1 (v_{m,j}(t) - v_{m,j+1}(t))^2 dt\right)^{1/2},
$$

the lemma is proved. □

In the case $n = 2$, we can deduce Theorem 1.1 from Lemmas 3.1 and 4.1. But if $n \ge 3$, there is an additional step involved, which we will take next.

In the case $n \ge 3$, define

$$
\Sigma = \{(t_2, \ldots, t_{n-1}) \in \mathbf{R}^{n-2} : t_2 + \cdots + t_{n-1} \le 1 \ \text{ and } \ t_2 \ge 0, \ldots, t_{n-1} \ge 0\}.
$$

For all $\beta_1, \ldots, \beta_{n-1} \in \mathbf{Z}_+$, it follows from (3.1) that

$$
\int_\Delta t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}} dt_1 \cdots dt_{n-1} = \frac{\beta_1! \cdots \beta_n!}{(n-1+\beta_1+\cdots+\beta_{n-1})!}. \tag{4.2}
$$

An analogous identity holds on Σ. Combining this fact with (4.1), we also have

$$
\begin{aligned}
&\int_\Sigma \left(\int_0^1 s^{\beta_1}(1-s)^{\beta_2} ds\right) t_2^{\beta_1+\beta_2+1}\left(\prod_{2<j\le n-1} t_j^{\beta_j}\right) dt_2 \cdots dt_{n-1} \\
&\qquad = \frac{\beta_1!\beta_2!}{(\beta_1+\beta_2+1)!}\cdot\frac{(\beta_1+\beta_2+1)!\beta_3!\cdots\beta_{n-1}!}{(n-2+\beta_1+\beta_2+1+\sum_{2<j\le n-1}\beta_j)!} \\
&\qquad = \frac{\beta_1!\cdots\beta_{n-1}!}{(n-1+\beta_1+\cdots+\beta_{n-1})!} \qquad (4.3)
\end{aligned}
$$

for all $\beta_1, \ldots, \beta_{n-1} \in \mathbf{Z}_+$. (In the case $n = 3$, the $\prod_{2<j\le n-1} t_j^{\beta_j}$ above is interpreted to be 1, and the $\sum_{2<j\le n-1} \beta_j$ is interpreted to be 0.) By the Stone-Weierstrass approximation theorem, from (4.2) and (4.3) we deduce the identity

$$\int_\Delta \xi dV_{n-1} = \int_\Sigma \left(\int_0^1 \xi(st_2, (1-s)t_2, t_3, \ldots, t_{n-1}) ds \right) t_2 dt_2 \cdots dt_{n-1} \tag{4.4}$$

for every $\xi \in C(\Delta)$.

Lemma 4.2. *Given any $0 < \delta \le 1/2$, there is a constant $0 < C_{4.2}(\delta) < \infty$ such that if α and $\alpha + \epsilon_1 - \epsilon_2$ both belong to Z_δ (see Definition 3.2), then*

$$\int_\Delta |u_\alpha - u_{\alpha+\epsilon_1-\epsilon_2}| dV_{n-1} \le \frac{C_{4.2}(\delta)}{|\alpha|^{1/4}}.$$

Proof. If $n = 2$, then this lemma is just Lemma 4.1 stated in another way. Thus it suffices to consider the case $n \ge 3$. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_n)$ and denote

$$\varphi = |u_\alpha - u_{\alpha+\epsilon_1-\epsilon_2}|.$$

Then

$$\begin{aligned} \varphi(st_2, (1-s)t_2, \ldots, t_{n-1}) &= \frac{(n-1+|\alpha|)!}{(\alpha+\epsilon_1)!} t_2^{\alpha_1+\alpha_2} t_3^{\alpha_3} \cdots t_{n-1}^{\alpha_{n-1}} (1 - t_2 - \cdots - t_{n-1})^{\alpha_n} \\ &\quad \times |(\alpha_1+1)(1-s) - \alpha_2 s| s^{\alpha_1} (1-s)^{\alpha_2 - 1} \\ &= h(t_2, \ldots, t_{n-1}) w(s), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} h(t_2, \ldots, t_{n-1}) &= \frac{(n-1+|\alpha|)!}{(\alpha_1+\alpha_2+1)!\alpha_3! \cdots \alpha_n!} t_2^{\alpha_1+\alpha_2} t_3^{\alpha_3} \cdots t_{n-1}^{\alpha_{n-1}} (1 - t_2 - \cdots - t_{n-1})^{\alpha_n} \quad \text{and} \\ w(s) &= \frac{(\alpha_1+\alpha_2+1)!}{(\alpha_1+1)!\alpha_2!} |(\alpha_1+1)(1-s) - \alpha_2 s| s^{\alpha_1} (1-s)^{\alpha_2-1}. \end{aligned}$$

Note that

$$\int_0^1 w(s) ds = \int_0^1 |v_{\alpha_1+\alpha_2,\alpha_1}(s) - v_{\alpha_1+\alpha_2,\alpha_1+1}(s)| ds \le \frac{C_1}{(\alpha_1+\alpha_2)^{1/4}} \le \frac{C_2}{|\alpha|^{1/4}}$$

by Lemma 4.1. On the other hand, since $n - 1 + |\alpha| = n - 2 + |\alpha| + 1$, (3.1) implies

$$\int_\Sigma h(t_2, \ldots, t_{n-1}) t_2 dt_2 \cdots dt_{n-1} = 1.$$

Combining this with (4.4) and (4.5), we find that

$$\begin{aligned}\int_\Delta \varphi dV_{n-1} &= \int_\Sigma \left(\int_0^1 \varphi(st_2,(1-s)t_2,t_3,\dots,t_{n-1})ds\right) t_2 dt_2\cdots dt_{n-1} \\ &= \int_\Sigma h(t_2,\dots,t_{n-1})t_2dt_2\cdots dt_{n-1}\int_0^1 w(s)ds = \int_0^1 w(s)ds \le \frac{C_2}{|\alpha|^{1/4}}.\end{aligned}$$

This completes the proof. □

Proposition 4.3. *Let $0<\delta\le 1/2$ be given. For any $i\ne j$ in $\{1,\dots,n\}$, if α and $\alpha+\epsilon_i-\epsilon_j$ both belong to Z_δ, then*

$$\int_S ||e_\alpha|^2-|e_{\alpha+\epsilon_i-\epsilon_j}|^2|d\sigma \le \frac{C_{4.2}(\delta)}{|\alpha|^{1/4}},$$

where $C_{4.2}(\delta)$ is the constant provided by Lemma 4.2.

Proof. Performing a permutation of the variables $z_1,\dots,z_n$ if necessary, it suffices to consider the case where $i=1$ and $j=2$.

Define $\tilde{u}_\alpha = u_\alpha/(n-1)!$ for $\alpha\in\mathbf{Z}_+^n$. It is easy to verify that

$$||e_\alpha|^2-|e_{\alpha+\epsilon_1-\epsilon_2}|^2| = \mathcal{S}|\tilde{u}_\alpha-\tilde{u}_{\alpha+\epsilon_1-\epsilon_2}|$$

on S (see (2.3)). Thus by (2.4),

$$\begin{aligned}\int_S ||e_\alpha|^2-|e_{\alpha+\epsilon_1-\epsilon_2}|^2|d\sigma = \int_S \mathcal{S}|\tilde{u}_\alpha-\tilde{u}_{\alpha+\epsilon_1-\epsilon_2}|d\sigma &= (n-1)!\int_\Delta |\tilde{u}_\alpha-\tilde{u}_{\alpha+\epsilon_1-\epsilon_2}|dV_{n-1} \\ &= \int_\Delta |u_\alpha-u_{\alpha+\epsilon_1-\epsilon_2}|dV_{n-1}.\end{aligned}$$

Now an application of Lemma 4.2 completes the proof. □

Lemma 4.4. *Suppose that $\alpha,\beta\in Z_{\delta,m}$ for some $0<\delta\le 1/2$ and $m\in\mathbf{N}$. Also, suppose that $\alpha\ne\beta$. Then there exist $\gamma_1,\dots,\gamma_k\in Z_{\delta,m}$ which satisfy the following conditions*:

(a) *$\gamma_1=\alpha$ and $\gamma_k=\beta$.*

(b) *For every $1\le\nu<k$, there exist $i=i(\nu)$ and $j=j(\nu)$ such that $\gamma_{\nu+1}=\gamma_\nu+\epsilon_i-\epsilon_j$.*

Proof. First of all, for each $w=(w_1,\dots,w_n)\in\mathbf{Z}^n$, we define

$$|w| = |w_1|+\cdots+|w_n|.$$

We begin with $\gamma_1=\alpha$. By an induction on the $|\cdot|$ defined above, it suffices to find a $\gamma_2\in Z_{\delta,m}$ such that $\gamma_2=\gamma_1+\epsilon_i-\epsilon_j$ for some $i\ne j$ in $\{1,\dots,n\}$ and such that $|\gamma_2-\beta|<|\gamma_1-\beta|$.

Suppose that $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$. Since $|\alpha|=m=|\beta|$ and since $\alpha\ne\beta$, there exist $i\ne j$ in $\{1,\dots,n\}$ such that $\alpha_i<\beta_i$ and $\alpha_j>\beta_j$. We define

$$\gamma_2=(\gamma_{2,1},\dots,\gamma_{2,n})$$

such that $\gamma_{2,i} = \alpha_i + 1$, $\gamma_{2,j} = \alpha_j - 1$ and $\gamma_{2,\nu} = \alpha_\nu$ for every $\nu \in \{1, \dots, n\}\backslash\{i, j\}$. Obviously, $|\gamma_2| = |\alpha| = m$, $\gamma_2 = \gamma_1 + \epsilon_i - \epsilon_j$, and $|\gamma_2 - \beta| < |\gamma_1 - \beta|$. Thus the only thing that needs further verification is the membership $\gamma_2 \in Z_\delta$.

Since $\alpha, \beta \in Z_{\delta,m}$ and $\alpha_i < \beta_i$, we have

$$\delta m \le \alpha_i < \alpha_i + 1 = \gamma_{2,i} \le \beta_i \le (1 - \delta)m.$$

Since $\alpha, \beta \in Z_{\delta,m}$ and $\alpha_j > \beta_j$, we also have

$$\delta m \le \beta_j \le \alpha_j - 1 = \gamma_{2,j} < \alpha_j \le (1 - \delta)m.$$

Hence $\gamma_2 \in Z_\delta$. This completes the proof. □

With all this preparation, we can now prove our main result.

Proof of Theorem 1.1. Let f be a real-valued function in $\mathcal{R}$. It is obvious that the spectrum of T_f is contained in [ess inf f, ess sup f]. To prove the theorem, it suffices to show that for every pair of a, b satisfying the condition

$$\text{ess inf } f < a < b < \text{ess sup } f, \tag{4.6}$$

the interval (a, b) contains an eigenvalue of T_f. In other words, we need to find an $\alpha \in \mathbf{Z}^n_+$ such that $\langle f e_\alpha, e_\alpha \rangle \in (a, b)$.

Note that (4.6) means that there is an $\epsilon > 0$ such that

$$\text{ess inf } f + \epsilon < a < b < \text{ess sup } f - \epsilon.$$

For this ϵ, we apply Proposition 3.2. Thus there are $0 < \delta \le 1/2$ and $0 < M < \infty$ such that if $m \ge M$, then there exist $\alpha_{(m)}, \alpha^{(m)} \in Z_{\delta,m}$ which have the properties

$$\langle f e_{\alpha_{(m)}}, e_{\alpha_{(m)}} \rangle \le \text{ess inf } f + \epsilon \quad \text{and} \quad \langle f e_{\alpha^{(m)}}, e_{\alpha^{(m)}} \rangle \ge \text{ess sup } f - \epsilon.$$

Hence

$$\langle f e_{\alpha_{(m)}}, e_{\alpha_{(m)}} \rangle < a < b < \langle f e_{\alpha^{(m)}}, e_{\alpha^{(m)}} \rangle. \tag{4.7}$$

We now take an $m \ge M$ such that

$$\|f\|_\infty \frac{C_{4.2}(\delta)}{m^{1/4}} \le \frac{1}{2}(b - a), \tag{4.8}$$

where $C_{4.2}(\delta)$ is the constant provided by Lemma 4.2.

By (4.7), we obviously have $\alpha_{(m)} \ne \alpha^{(m)}$. Applying Lemma 4.4 to the pair $\alpha_{(m)}$, $\alpha^{(m)}$ in $Z_{\delta,m}$, we obtain $\gamma_1, \dots, \gamma_k \in Z_{\delta,m}$ satisfying the following conditions:

(i) $\gamma_1 = \alpha_{(m)}$ and $\gamma_k = \alpha^{(m)}$.

(ii) For every $1 \le \nu < k$, there exist $i(\nu)$ and $j(\nu)$ such that $\gamma_{\nu+1} = \gamma_\nu + \epsilon_{i(\nu)} - \epsilon_{j(\nu)}$.

By (4.7) and (i), we have

$$\langle fe_{\gamma_1}, e_{\gamma_1}\rangle < a < b < \langle fe_{\gamma_k}, e_{\gamma_k}\rangle. \tag{4.9}$$

Applying (ii), Proposition 4.3 and (4.8), for every $1 \le \nu < k$ we have

$$\begin{aligned}
|\langle fe_{\gamma_\nu}, e_{\gamma_\nu}\rangle - \langle fe_{\gamma_{\nu+1}}, e_{\gamma_{\nu+1}}\rangle| &\le \|f\|_\infty \int_S ||e_{\gamma_\nu}|^2 - |e_{\gamma_{\nu+1}}|^2| d\sigma \\
&= \|f\|_\infty \int_S ||e_{\gamma_\nu}|^2 - |e_{\gamma_\nu+\epsilon_{i(\nu)}-\epsilon_{j(\nu)}}|^2| d\sigma \\
&\le \|f\|_\infty \frac{C_{4.2}(\delta)}{m^{1/4}} \le \frac{1}{2}(b-a). \qquad (4.10)
\end{aligned}$$

Define $\nu_* = \max\{\nu : \langle fe_{\gamma_\nu}, e_{\gamma_\nu}\rangle \le a\}$. Then it follows from (4.9) that $\nu_* < k$. The definition of ν_* ensures that $\langle fe_{\gamma_{\nu_*+1}}, e_{\gamma_{\nu_*+1}}\rangle > a$. By the condition $\langle fe_{\gamma_{\nu_*}}, e_{\gamma_{\nu_*}}\rangle \le a$ and (4.10), we have $\langle fe_{\gamma_{\nu_*+1}}, e_{\gamma_{\nu_*+1}}\rangle < b$. That is,

$$\langle fe_{\gamma_{\nu_*+1}}, e_{\gamma_{\nu_*+1}}\rangle \in (a, b).$$

This completes the proof. □

Appendix

The purpose of this appendix is to justify the approximation of continuous functions on Δ by multi-variable Berntein polynomials in the proof of Lemma 3.1.

As defined in Section 2, we have the compact subset

$$\Delta = \{(t_1, \dots, t_{n-1}) \in \mathbf{R}^{n-1} : t_1 + \cdots + t_{n-1} \le 1 \ \text{ and } \ t_1 \ge 0, \dots, t_{n-1} \ge 0\}$$

of $\mathbf{R}^{n-1}$. Borrowing an idea from probability theory, each point $t = (t_1, \dots, t_{n-1}) \in \Delta$ gives rise to what might be called a "multinomial distribution", as follows.

Imagine that there is an n-sided coin which has the property that on a random toss, the probability of it landing on the j-th side is t_j for $1 \le j \le n-1$, and the probability of it landing on the n-th side is $1 - t_1 - \cdots - t_{n-1}$.

Now consider m tosses of this coin. For any $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbf{Z}_+^{n-1}$ with $|\beta| \le m$, the quantity

$$p_\beta = \frac{m!}{\beta!(m-|\beta|)!} t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}} (1 - t_1 - \cdots - t_{n-1})^{m-|\beta|}$$

is the probability of the coin landing on the j-th side β_j times for all $1 \le j \le n-1$. For each $1 \le j \le n-1$, let X_j be the random variable that counts the number of times of the coin landing on the j-th side in a random sequence of m tosses. For each $k \ge 1$, we have

$$\begin{aligned}
E(X_1^k) = \sum_{|\beta| \le m} \beta_1^k p_\beta &= \sum_{\nu=0}^m \nu^k t_1^\nu \frac{m!}{\nu!} \\
&\times \sum_{\beta_2+\cdots+\beta_{n-1} \le m-\nu} \frac{t_2^{\beta_2} \cdots t_{n-1}^{\beta_{n-1}} (1 - t_1 - \cdots - t_{n-1})^{m-\nu-\beta_2-\cdots-\beta_{n-1}}}{\beta_2! \cdots \beta_{n-1}!(m - \nu - \beta_2 - \cdots - \beta_{n-1})!} \\
&= \sum_{\nu=0}^m \nu^k \frac{m!}{\nu!(m-\nu)!} t_1^\nu (1-t_1)^{m-\nu}.
\end{aligned}$$

Obviously, the same calculation is valid for every $1 \le j \le n-1$. Therefore

$$E(X_j^k) = \sum_{\nu=0}^{m} \nu^k \frac{m!}{\nu!(m-\nu)!} t_j^{\nu}(1-t_j)^{m-\nu}$$

for all $1 \le j \le n-1$ and $k \ge 1$. That is, each X_j actually satisfies a *binomial* distribution. Thus we know from every textbook in probability theory that $E((X_j - mt_j)^2) = mt_j(1-t_j)$ for every $1 \le j \le n-1$. Equivalently,

$$E(((X_j/m) - t_j)^2) = \frac{t_j(1-t_j)}{m} \tag{A.1}$$

for every $1 \le j \le n-1$.

Note that (A.1) can also be proved by a much simpler argument, as follows. Obviously, the single-toss version of X_j has expectation value t_j and variance $t_j(1-t_j)$. Since the tosses are independent of each other, the variance in the case of m tosses equals $mt_j(1-t_j)$, i.e., $E((X_j - mt_j)^2) = mt_j(1-t_j)$.

For every pair of $m \in \mathbf{N}$ and $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbf{Z}_+^{n-1}$ satisfying the condition $|\beta| \le m$, we now define the $(n-1)$-variable polynomial

$$\psi_{m,\beta}(t_1, \dots, t_{n-1}) = \frac{m!}{\beta!(m-|\beta|)!} t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}}(1 - t_1 - \cdots - t_{n-1})^{m-|\beta|}.$$

For any function h on Δ, we define its m-th *Bernstein polynomial* by the formula

$$h_m(t) = \sum_{|\beta| \le m} h(\beta/m)\psi_{m,\beta}(t), \quad t \in \Delta. \tag{A.2}$$

Proposition A.1. *For every $f \in C(\Delta)$, we have*

$$\lim_{m\to\infty} \|f - f_m\|_\infty = 0.$$

Proof. We know that Lipschitz functions are dense in $C(\Delta)$ with respect to the norm $\|\cdot\|_\infty$. Combining this density with the obvious fact that $\|h_m\|_\infty \le \|h\|_\infty$ for $h \in C(\Delta)$, it suffices to prove the proposition for Lipschitz functions on Δ.

Let f be a Lipschitz function on Δ, and write L for its Lipschitz constant. Since

$$f_m(t) = \sum_{|\beta| \le m} f(\beta/m)\psi_{m,\beta}(t),$$

we have

$$\begin{aligned} |f(t) - f_m(t)| &= \left| \sum_{|\beta| \le m} (f(t) - f(\beta/m))\psi_{m,\beta}(t) \right| \le \sum_{|\beta| \le m} |f(t) - f(\beta/m)|\psi_{m,\beta}(t) \\ &\le L \sum_{|\beta| \le m} |t - (\beta/m)|\psi_{m,\beta}(t) \le L\Big(\sum_{|\beta| \le m} |t - (\beta/m)|^2 \psi_{m,\beta}(t)\Big)^{1/2}, \end{aligned} \tag{A.3}$$

where the last $\leq$ follows from the Cauchy-Schwarz inequality. For $t = (t_1, \ldots, t_{n-1})$ and $\beta = (\beta_1, \ldots, \beta_{n-1})$, we have

$$|t - (\beta/m)|^2 = (t_1 - (\beta_1/m))^2 + \cdots + (t_{n-1} - (\beta_{n-1}/m))^2.$$

Therefore for every $t = (t_1, \ldots, t_{n-1}) \in \Delta$, it follows from (A.3) and (A.1) that

$$\begin{aligned} |f(t) - f_m(t)| \leq L\biggl(\sum_{j=1}^{n-1} \sum_{|\beta| \leq m} (t_j - (\beta_j/m))^2 \psi_{m,\beta}(t)\biggr)^{1/2} &= L\biggl(\sum_{j=1}^{n-1} \frac{t_j(1-t_j)}{m}\biggr)^{1/2} \\ &\leq L\sqrt{\frac{n-1}{4m}}. \end{aligned}$$

This completes the proof. $\square$

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