AN INTEGRAL FORMULA FOR SCHATTEN NORM ON THE HARDY SPACE: THE ONLY HIGH-DIMENSIONAL CASE

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Abstract. In [8], an exact integral formula was proved for the Schatten p-norm of Hankel operators on the one-variable Hardy space for $p = 2$, 4, and 6. Moreover, it was shown that these are the only values of p for which such a formula is possible. It was further pointed out in [8] that the only possibility for a meaningful multi-variable analogue of this formula is the case where $n = 2$ and $p = 6$. We show that the conjectured integral formula indeed holds in this one possible multi-variable case.

1. Introduction

This paper is inspired by the work [8] of Janson, Upmeier and Wallstén more than twenty-five years ago. In [8], the authors considered the Schatten norm of Hankel operators H_b on the Hardy space on the unit circle $\mathbf{T}$.

Recall that the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition $\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p} < \infty$. The Hilbert-Schmidt norm, i.e., the case $p = 2$, is the only one in this family of norms that is generally computable. For any $p \neq 2$, with the exception of trivial cases, it is always a challenge to determine the precise value of $\|A\|_p$. With that in mind, it is quite striking to see the following result in [8]:

Theorem 1.1. *The identity*

$$\|H_b\|_p^p = c_p \int_{\mathbf{T}} \int_{\mathbf{T}} \frac{|b(z) - b(w)|^p}{|z - w|^2} dm(z) dm(w) \tag{1.1}$$

holds for all conjugate analytic $b \in \mathrm{BMO}$ *and some constant* c_p *if and only if* p *equals* 2, 4 *or* 6; *here* $c_2 = 1$, $c_4 = 1/2$ *and* $c_6 = 1/6$.

Given Theorem 1.1, it was natural for the authors of [8] to raise the question about the analogue of (1.1) for Hankel operators on the unit ball in $\mathbf{C}^n$. As they pointed out on page 212 in [8], there is only one case that is of interest, namely the case where $n = 2$ and $p = 6$; for the other cases in the high-dimensional situation, one either finds $\infty = \infty$ or $0 = 0$, neither of which is very interesting.

In this paper we will settle this single unresolved case. That is, we will show that the analogue of (1.1) indeed holds in the case where $n = 2$ and $p = 6$.

Let us turn to the high-dimensional setting. In what follows we always assume $n \geq 2$; we allow $n > 2$ for ease of discussion. We write S for the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in

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$\mathbf{C}^n$. Let $d\sigma$ denote the standard spherical measure on S with the normalization $\sigma(S) = 1$. Then the easiest way to introduce the Hardy space $H^2(S)$ is to give it a description: it is the closure of the analytic polynomials $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Let

$$P : L^2(S, d\sigma) \to H^2(S)$$

be the orthogonal projection. Given a symbol function f, the Hankel operator H_f is defined by the formula

$$H_f g = (1 - P)(fg), \quad g \in H^2(S).$$

We will also view H_f as an operator on $L^2(S, d\sigma)$ and represent it in the form

$$H_f = (1 - P)M_f P.$$

We refer to [1-4,9,10] for the memberships of Hankel operators in Schatten classes, which are well-settled issues by now.

For any operator X that is not in the Schatten class $\mathcal{C}_p$, including the case where X is unbounded, we will interpret $\|X\|_p$ as infinity. Here is the main result of the paper:

Theorem 1.2. *Suppose that* $n = 2$. *Then the equality*

$$\|H_{\bar{h}}\|_6^6 = \frac{1}{6} \int_S \int_S \frac{|h(z) - h(w)|^6}{|1 - \langle z, w\rangle|^4} d\sigma(z) d\sigma(w) \tag{1.2}$$

holds for every $h \in H^2(S)$.

The rest of the paper consists of the proof of Theorem 1.2, for which we now give an outline. The proof relies on a Schatten-class membership for *double commutators*. Namely, we will show that if f, g are Lipschitz functions on S, then

$$[M_f, [M_g, P]] \in \mathcal{C}_p \tag{1.3}$$

for every $p > n$. Using (1.3) and a classic vanishing principle for trace due to Helton and Howe (see Lemma 2.1 below), we then show that when $n = 2$, if f, g and h are Lipschitz functions on S, then the commutators

$$[H_g^* H_g H_g^* H_g T_h, T_f], \ [T_h H_g^* H_g, T_f H_g^* H_g], \ [T_h H_g^* H_g, H_g^* H_g T_f],$$
$$[H_g T_h [H_g^* H_g, T_f], H_g^*], \ [H_g [T_h, H_g^* H_g], T_f H_g^*], \ \text{etc},$$

are in the trace class with *zero trace*. The vanishing of these traces allows us to refine the argument in [8] to prove Theorem 1.2.

2. Preliminaries

Our proof of Theorem 1.2 is based on a classic vanishing principle for trace:

Lemma 2.1. [7, Lemma 1.3] *Suppose that* X *is a self-adjoint operator and* C *is a compact operator. If* $[X, C]$ *is in the trace class, then* $\mathrm{tr}[X, C] = 0$.

In addition, we need some basic properties of Schatten classes. Given an operator A, write $s_1(A), \ldots, s_k(A), \ldots$ for its s-numbers [6]. Our next lemma is well known, but, for the convenience of the reader, we include it here anyway.

Lemma 2.2. *Let $A \in \mathcal{C}_{p_1}$ and $B \in \mathcal{C}_{p_2}$, where $p_1, p_2 \in [1, \infty)$. If $p_1p_2/(p_1+p_2) \geq 1$, then $AB \in \mathcal{C}_{p_1p_2/(p_1+p_2)}$ with*

$$\|AB\|_{p_1p_2/(p_1+p_2)} \leq \|A\|_{p_1}\|B\|_{p_2}.$$

If $p_1p_2/(p_1+p_2) < 1$, then $AB \in \mathcal{C}_1$.

Proof. By (7.9) on page 63 in [6], for every $k \in \mathbf{N}$, the inequality

$$s_1(AB) + \cdots + s_k(AB) \leq s_1(A)s_1(B) + \cdots + s_k(A)s_k(B)$$

holds. By [6, Lemma III.3.1], this implies that for each $1 \leq p < \infty$, we have

$$\sum_{j=1}^{\infty}\{s_j(AB)\}^p \leq \sum_{j=1}^{\infty}\{s_j(A)s_j(B)\}^p.$$

Then an application of appropriate Hölder's inequality completes the proof. □

Lemma 2.3. *Let $p_1, p_2 \in [1, \infty)$ be such that $p_1p_2/(p_1+p_2) \leq 1$. If $A \in \mathcal{C}_{p_1}$ and $B \in \mathcal{C}_{p_2}$, then we have $AB \in \mathcal{C}_1$, $BA \in \mathcal{C}_1$ and $\mathrm{tr}[A, B] = 0$.*

Proof. We have $A = A_1 + iA_2$, where A_1, A_2 are self-adjoint operators. Obviously, the membership $A \in \mathcal{C}_{p_1}$ implies $A_1 \in \mathcal{C}_{p_1}$ and $A_2 \in \mathcal{C}_{p_1}$. By Lemma 2.2, this means $A_jB \in \mathcal{C}_1$ and $BA_j \in \mathcal{C}_1$, $j = 1, 2$. Thus an application of Lemma 2.1 gives us $\mathrm{tr}[A_1, B] = 0$ and $\mathrm{tr}[A_2, B] = 0$. Since $A = A_1 + iA_2$, we obtain $\mathrm{tr}[A, B] = 0$ as promised. □

In addition to the Schatten classes, one often uses another convenient family of ideals. These are the ideals $\mathcal{C}_p^+$, which are defined as follows. For each $1 \leq p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k\geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}}$$

defines a symmetric norm for operators. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [6]. For our purpose, what matters is the inclusion relation $\mathcal{C}_p^+ \subset \mathcal{C}_{p'}$ for all $1 \leq p < p' < \infty$, which is well known.

Let $\mathrm{Lip}(S)$ denote the collection of functions on S that satisfy Lipschitz conditions with respect to the Euclidean metric. We recall the following fact:

Proposition 2.4. [3, Proposition 7.2] *If $f \in \mathrm{Lip}(S)$, then $[M_f, P] \in \mathcal{C}_{2n}^+$. Consequently, if $f \in \mathrm{Lip}(S)$, then $[M_f, P] \in \mathcal{C}_p$ for every $p > 2n$.*

Recall that for each $f \in L^\infty(S, d\sigma)$, the Toeplitz operator T_f is defined by the formula

$$T_f = PM_fP.$$

As it was the case in [8], we also need the "complementary Toeplitz operators". That is, for each $f \in L^\infty(S, d\sigma)$, we define

$$\tilde{T}_f = (1 - P)M_f(1 - P).$$

As an immediate consequence of Proposition 2.4 and Lemma 2.2, we have

Corollary 2.5. *If $f, g \in \mathrm{Lip}(S)$, then the commutators $[T_f, T_g]$ and $[\tilde{T}_f, \tilde{T}_g]$ are in the Schatten class $\mathcal{C}_p$ for every $p > n$.*

3. Double commutators

The focus of this section will be on double commutators of the form $[M_f, [M_g, P]]$.

Lemma 3.1. *Let an $\epsilon > 0$ be given. Then the formula*

$$(B_\epsilon f)(z) = \int \frac{f(w)}{|1 - \langle z, w\rangle|^{n-\epsilon}} d\sigma(w), \quad f \in L^2(S, d\sigma),$$

defines a bounded operator on $L^2(S, d\sigma)$.

Proof. By [11, Proposition 1.4.10], for each $\epsilon > 0$ there is a $0 < C_\epsilon < \infty$ such that

$$\int \frac{1}{|1 - \langle z, w\rangle|^{n-\epsilon}} d\sigma(w) \le C_\epsilon$$

for every $z \in S$. The easiest version of the Schur test then gives us $\|B_\epsilon\| \le C_\epsilon$. □

Given an operator A, for each $t > 0$ we define

$$N_A(t) = \mathrm{card}\{j \in \mathbf{N} : s_j(A) > t\},$$

where $s_1(A)$, $s_2(A)$, $\dots$, $s_j(A)$, $\dots$ are the s-numbers of A. It is well known that

$$N_{A+B}(t) \le N_A(t/2) + N_B(t/2) \tag{3.1}$$

for every $t > 0$. See, e.g., (7.1) in [3]. It is also well known [5, Lemma I.4.1] that

$$\sum_{j=1}^{\infty} (s_j(A))^p = p \int_0^\infty t^{p-1} N_A(t) dt, \quad 1 \le p < \infty. \tag{3.2}$$

Our next proposition is the key ingredient in the proof of Theorem 1.2.

Proposition 3.2. *If $f, g \in \mathrm{Lip}(S)$, then the double commutator $[M_f, [M_g, P]]$ is in the Schatten class $\mathcal{C}_p$ for every $p > n$.*

Proof. Let $f, g \in \mathrm{Lip}(S)$ be given. Then $[M_f, [M_g, P]]$ is the integral operator on $L^2(S, d\sigma)$ with the kernel function

$$Z(z, w) = \frac{(f(z) - f(w))(g(z) - g(w))}{(1 - \langle z, w\rangle)^n}.$$

Let $p > n$ also be given. Then we pick an $\epsilon > 0$ such that

$$2 + (p-2)(1-\epsilon) > n. \tag{3.3}$$

For each $t > 0$, define

$$E_t = \{(z,w) \in S \times S : |f(z) - f(w)||g(z) - g(w)| \le t^{1/(1-\epsilon)}\} \quad \text{and}$$
$$F_t = \{(z,w) \in S \times S : |f(z) - f(w)||g(z) - g(w)| > t^{1/(1-\epsilon)}\}.$$

Accordingly, we define the kernel functions

$$X_t(z,w) = \frac{(f(z) - f(w))(g(z) - g(w))}{(1 - \langle z,w\rangle)^n}\chi_{E_t}(z,w),$$
$$Y_t(z,w) = \frac{(f(z) - f(w))(g(z) - g(w))}{(1 - \langle z,w\rangle)^n}\chi_{F_t}(z,w),$$

and the integral operators

$$(G_t\varphi)(z) = \int X_t(z,w)\varphi(w)d\sigma(w),$$
$$(H_t\varphi)(z) = \int Y_t(z,w)\varphi(w)d\sigma(w),$$

$\varphi \in L^2(S, d\sigma)$. We have $|f(z) - f(w)| \le L(f)|z - w| \le \sqrt{2}L(f)|1 - \langle z,w\rangle|^{1/2}$, where $L(f)$ is the Lipschitz constant for f. A similar inequality holds for g. Therefore for each point $(z,w) \in E_t$, we have

$$\begin{aligned} &|f(z) - f(w)||g(z) - g(w)| \\ &\qquad = (|f(z) - f(w)||g(z) - g(w)|)^{1-\epsilon} \cdot (|f(z) - f(w)||g(z) - g(w)|)^{\epsilon} \\ &\qquad \le t \cdot \{2L(f)L(g)\}^{\epsilon}|1 - \langle z,w\rangle|^{\epsilon}. \end{aligned}$$

Consequently,

$$|X_t(z,w)| \le \frac{t\{2L(f)L(g)\}^{\epsilon}}{|1 - \langle z,w\rangle|^{n-\epsilon}}.$$

Applying Lemma 3.1, we obtain the inequality $\|G_t\| \le M_\epsilon t$, where

$$M_\epsilon = \{2L(f)L(g)\}^{\epsilon}\|B_\epsilon\|.$$

This obviously means that

$$\|G_{t/M_\epsilon}\| \le t \tag{3.4}$$

for every $t > 0$.

Since $Z(z,w) = X_{t/(3M_\epsilon)}(z,w) + Y_{t/(3M_\epsilon)}(z,w)$, we have

$$[M_f, [M_g, P]] = G_{t/(3M_\epsilon)} + H_{t/(3M_\epsilon)}.$$

By (3.4) we have $N_{G_{t/(3M_\epsilon)}}(t/2) = 0$. Applying (3.1), we obtain

$$N_{[M_f,[M_g,P]]}(t) \le N_{G_{t/(3M_\epsilon)}}(t/2) + N_{H_{t/(3M_\epsilon)}}(t/2) = N_{H_{t/(3M_\epsilon)}}(t/2). \tag{3.5}$$

On the other hand,

$$N_{H_{t/(3M_\epsilon)}}(t/2) \le \frac{4}{t^2}\|H_{t/(3M_\epsilon)}\|_2^2 = \frac{4}{t^2}\iint |Y_{t/(3M_\epsilon)}(z,w)|^2 d\sigma(z)d\sigma(w). \tag{3.6}$$

At this point, we remind the reader of our standing assumption that $n \ge 2$. Thus the condition $p > n$ in particular implies $p > 2$, which will be relevant in an integral below.

Combining (3.6) with (3.5) and (3.2), we find that

$$\begin{aligned}
&\|[M_f,[M_g,P]]\|_p^p = p\int_0^\infty t^{p-1}N_{[M_f,[M_g,P]]}(t)dt \\
&\le p\int_0^\infty t^{p-1}\frac{4}{t^2}\iint |Y_{t/(3M_\epsilon)}(z,w)|^2 d\sigma(z)d\sigma(w)dt \\
&= 4p\iint\int_0^\infty |Y_{t/(3M_\epsilon)}(z,w)|^2 t^{p-3}dtd\sigma(z)d\sigma(w) \\
&= 4p\iint\int_0^{3M_\epsilon\{|f(z)-f(w)||g(z)-g(w)|\}^{1-\epsilon}} t^{p-3}dt\frac{|f(z)-f(w)|^2|g(z)-g(w)|^2}{|1-\langle z,w\rangle|^{2n}}d\sigma(z)d\sigma(w) \\
&= \frac{4p(3M_\epsilon)^{p-2}}{p-2}\iint \frac{\{|f(z)-f(w)||g(z)-g(w)|\}^{2+(p-2)(1-\epsilon)}}{|1-\langle z,w\rangle|^{2n}}d\sigma(z)d\sigma(w),
\end{aligned}$$

where the third = follows from the definition of the set $F_{t/(3M_\epsilon)}$. Since $|f(z)-f(w)||g(z)-g(w)| \le 2L(f)L(g)|1-\langle z,w\rangle|$, we now have

$$\|[M_f,[M_g,P]]\|_p^p \le \frac{4p(3M_\epsilon)^{p-2}}{p-2}\iint \frac{\{2L(f)L(g)\}^{2+(p-2)(1-\epsilon)}}{|1-\langle z,w\rangle|^{2n-2-(p-2)(1-\epsilon)}}d\sigma(z)d\sigma(w).$$

Recalling (3.3), we have $2n-2-(p-2)(1-\epsilon) < n$. Hence the above is finite. That is, the double commutator $[M_f,[M_g,P]]$ is in the Schatten class $\mathcal{C}_p$ as promised. □

4. Commutators of Toeplitz operators and Hankel operators

We now consider the Schatten-class memberships of the kind of commutators mentioned in the title of the section.

Lemma 4.1. *Let $f, g \in \mathrm{Lip}(S)$. Then for every $p > 2n/3$, we have $[T_f, H_g^*H_g] \in \mathcal{C}_p$ and $[\tilde{T}_f, H_gH_g^*] \in \mathcal{C}_p$.*

Proof. From the relation $(1-P)P = 0$ we obtain

$$H_g^* H_g = PM_{\bar{g}}(1-P)M_g P = [P, M_{\bar{g}}](1-P)[M_g, P].$$

Therefore

$$[T_f, H_g^* H_g] = P[M_f, H_g^* H_g]P = P[M_f, [P, M_{\bar{g}}](1-P)[M_g, P]]P = P(A+B+C)P,$$

where

$$\begin{aligned} A &= [M_f, [P, M_{\bar{g}}]](1-P)[M_g, P], \\ B &= [P, M_{\bar{g}}][M_f, 1-P][M_g, P] \quad \text{and} \\ C &= [P, M_{\bar{g}}](1-P)[M_f, [M_g, P]]. \end{aligned}$$

Proposition 3.2 tells us that $[M_f, [P, M_{\bar{g}}]] \in \mathcal{C}_s$ for every $s > n$. By Proposition 2.4, we have $[P, M_g] \in \mathcal{C}_t$ for every $t > 2n$. Combining these two Schatten-class memberships with Lemma 2.2, we see that $A \in \mathcal{C}_p$ for every $p > 2n/3$. Similarly, we have $C \in \mathcal{C}_p$ for every $p > 2n/3$. Finally, B is the product of three commutators, each of which is in $\mathcal{C}_t$ for every $t > 2n$. From Lemma 2.2 we deduce $B \in \mathcal{C}_p$ for every $p > 2n/3$. Thus we conclude that $[T_f, H_g^* H_g] \in \mathcal{C}_p$ for every $p > 2n/3$.

For $[\tilde{T}_f, H_g H_g^*]$, note that

$$H_g H_g^* = (1-P)M_g P M_{\bar{g}}(1-P) = [1-P, M_g]P[M_{\bar{g}}, 1-P] = [P, M_g]P[M_{\bar{g}}, P]$$

and that

$$\begin{aligned} [\tilde{T}_f, H_g H_g^*] &= (1-P)[M_f, H_g H_g^*](1-P) = (1-P)[M_f, [P, M_g]P[M_{\bar{g}}, P]](1-P) \\ &= (1-P)(D+E+F)(1-P), \end{aligned}$$

where

$$\begin{aligned} D &= [M_f, [P, M_g]]P[M_{\bar{g}}, P], \\ E &= [P, M_g][M_f, P][M_{\bar{g}}, P] \quad \text{and} \\ F &= [P, M_g]P[M_f, [M_{\bar{g}}, P]]. \end{aligned}$$

By the argument at the end of the previous paragraph, we have $[\tilde{T}_f, H_g H_g^*] \in \mathcal{C}_p$ for every $p > 2n/3$. □

Lemma 4.2. *Suppose that $n = 2$. Then for all $f, g, h \in \mathrm{Lip}(S)$, the commutators*

$$[H_g H_g^* H_g H_g^* \tilde{T}_h, \tilde{T}_f] \quad \textit{and} \quad [H_g^* H_g H_g^* H_g T_h, T_f]$$

are in the trace class with zero trace.

Proof We have

$$[(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_f] = [H_gH_g^*, \tilde{T}_f]H_gH_g^*\tilde{T}_h + H_gH_g^*[H_gH_g^*, \tilde{T}_f]\tilde{T}_h + (H_gH_g^*)^2[\tilde{T}_h, \tilde{T}_f].$$

Since we now assume $n = 2$, we have $H_g \in \mathcal{C}_s$ for every $s > 4$. By Lemma 4.1, we have $[H_gH_g^*, \tilde{T}_f] \in \mathcal{C}_p$ for every $p > 4/3$. Also, Corollary 2.5 tells us that $[\tilde{T}_h, \tilde{T}_f] \in \mathcal{C}_t$ for every $t > 2$. Combining these facts with Lemma 2.2, we obtain the membership

$$[(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_f] \in \mathcal{C}_1.$$

For $f \in \mathrm{Lip}(S)$ we also have $\mathrm{Re}(f) \in \mathrm{Lip}(S)$ and $\mathrm{Im}(f) \in \mathrm{Lip}(S)$. Thus we have

$$[(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_{\mathrm{Re}(f)}] \in \mathcal{C}_1 \quad \text{and} \quad [(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_{\mathrm{Im}(f)}] \in \mathcal{C}_1.$$

Since $\tilde{T}_{\mathrm{Re}(f)}$ and $\tilde{T}_{\mathrm{Im}(f)}$ are self-adjoint operators and $(H_gH_g^*)^2\tilde{T}_h$ is compact, we can now apply Lemma 2.1 to obtain

$$\mathrm{tr}[(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_{\mathrm{Re}(f)}] = 0 \quad \text{and} \quad \mathrm{tr}[(H_gH_g^*)^2\tilde{T}_h, \tilde{T}_{\mathrm{Im}(f)}] = 0.$$

By the linearity of trace, we have $\mathrm{tr}[H_gH_g^*H_gH_g^*\tilde{T}_h, \tilde{T}_f] = 0$. This proves the lemma for the commutator $[H_gH_g^*H_gH_g^*\tilde{T}_h, \tilde{T}_f]$. The case for the commutator $[H_g^*H_gH_g^*H_gT_h, T_f]$ follows by a similar argument. □

Lemma 4.3. *Suppose that $n = 2$. Then for all $f, g, h \in \mathrm{Lip}(S)$, the commutators*

$$[T_hH_g^*H_g, T_fH_g^*H_g], \quad [T_hH_g^*H_g, H_g^*H_gT_f], \quad [\tilde{T}_hH_gH_g^*, \tilde{T}_fH_gH_g^*], \quad [\tilde{T}_hH_gH_g^*, H_gH_g^*\tilde{T}_f]$$

are in the trace class with zero trace.

Proof. Applying the "product rule" for commutators, we have

$$\begin{aligned}[T_hH_g^*H_g, T_fH_g^*H_g] &= [T_hH_g^*H_g, T_f]H_g^*H_g + T_f[T_hH_g^*H_g, H_g^*H_g] \\ &= [T_h, T_f]H_g^*H_gH_g^*H_g + T_h[H_g^*H_g, T_f]H_g^*H_g + T_f[T_h, H_g^*H_g]H_g^*H_g \\ &= A + B + C.\end{aligned}$$

Under the assumption $n = 2$, we have $H_g^*H_g \in \mathcal{C}_p$ for every $p > 2$. On the other hand, Lemma 4.1 tells us that $[H_g^*H_g, T_f] \in \mathcal{C}_s$ and $[T_h, H_g^*H_g] \in \mathcal{C}_s$ for every $s > 4/3$. Combining these Schatten-class memberships with Lemma 2.2, we see that $B, C \in \mathcal{C}_1$. By Corollary 2.5, we have $[T_h, T_f] \in \mathcal{C}_p$ for every $p > 2$. Hence another application of Lemma 2.2 leads to $A \in \mathcal{C}_1$. Thus $[T_hH_g^*H_g, T_fH_g^*H_g]$ is in the trace class.

For the commutator $[T_hH_g^*H_g, H_g^*H_gT_f]$, we have

$$\begin{aligned}[T_hH_g^*H_g, H_g^*H_gT_f] &= [T_hH_g^*H_g, H_g^*H_g]T_f + H_g^*H_g[T_hH_g^*H_g, T_f] \\ &= [T_h, H_g^*H_g]H_g^*H_gT_f + H_g^*H_g[T_h, T_f]H_g^*H_g + H_g^*H_gT_h[H_g^*H_g, T_f].\end{aligned}$$

Applying Lemma 4.1, Corollary 2.5 and Lemma 2.2 again, we reach the conclusion that the commutator $[T_h H_g^* H_g, H_g^* H_g T_f]$ is in the trace class.

If f is in $\mathrm{Lip}(S)$, then so is $\bar{f}$. Thus it follows from the last two paragraphs that

$$[T_h H_g^* H_g, T_f H_g^* H_g + H_g^* H_g T_{\bar{f}}] \in \mathcal{C}_1.$$

Note that the operator $T_f H_g^* H_g + H_g^* H_g T_{\bar{f}}$ is self-adjoint. Since $T_h H_g^* H_g$ is compact, we can now apply Lemma 2.1 to obtain

$$\mathrm{tr}[T_h H_g^* H_g, T_f H_g^* H_g + H_g^* H_g T_{\bar{f}}] = 0. \tag{4.1}$$

Similarly, we have

$$[T_h H_g^* H_g, i(T_f H_g^* H_g - H_g^* H_g T_{\bar{f}})] \in \mathcal{C}_1,$$

and the operator $i(T_f H_g^* H_g - H_g^* H_g T_{\bar{f}})$ is also self-adjoint. Thus another application of Lemma 2.1 leads to

$$\mathrm{tr}[T_h H_g^* H_g, T_f H_g^* H_g - H_g^* H_g T_{\bar{f}}] = 0. \tag{4.2}$$

Solving (4.1) and (4.2), we conclude that

$$\mathrm{tr}[T_h H_g^* H_g, T_f H_g^* H_g] = 0 \quad \text{and} \quad \mathrm{tr}[T_h H_g^* H_g, H_g^* H_g T_{\bar{f}}] = 0.$$

Since we can replace f by $\bar{f}$, this proves the lemma for the first pair of commutators. The conclusion for the second pair of commutators is proved by a similar argument. □

5. Proof of Theorem 1.2

(1) We first assume that $h \in H^2(S) \cap \mathrm{Lip}(S)$. Once such an h is given, we define

$$Q = [M_h, H_{\bar{h}}] \quad \text{and} \quad R = [M_{\bar{h}}, Q]$$

as in [8]. The analyticity of h leads to the identities

$$\tilde{T}_{\bar{h}}^* H_{\bar{h}} = H_{\bar{h}} T_{\bar{h}}^*, \quad H_{\bar{h}}^* \tilde{T}_{\bar{h}} = T_{\bar{h}} H_{\bar{h}}^*, \tag{5.1}$$

and $H_{\bar{h}} = [M_{\bar{h}}, P]$. Hence $Q = [M_h, [M_{\bar{h}}, P]]$. Thus Q and R are integral operators on $L^2(S, d\sigma)$ with the kernel functions

$$\frac{|h(z) - h(w)|^2}{(1 - \langle z, w\rangle)^2} \quad \text{and} \quad \frac{(\bar{h}(z) - \bar{h}(w))|h(z) - h(w)|^2}{(1 - \langle z, w\rangle)^2}$$

(recall that Theorem 1.2 assumes $n = 2$) respectively. Therefore

$$\mathrm{tr}(R^* R) = \iint \frac{|h(z) - h(w)|^6}{|1 - \langle z, w\rangle|^4} d\sigma(z) d\sigma(w). \tag{5.2}$$

As in [8], a simplification of notation is in order. For the rest of part (1), we will write

$$H = H_{\bar{h}}, \quad T = T_{\bar{h}} \quad \text{and} \quad \tilde{T} = \tilde{T}_{\bar{h}} \tag{5.3}$$

That is, for these three operators we simply drop the subscript $\bar{h}$ from the notation.

We decompose $L^2(S, d\sigma)$ in the form

$$L^2(S, d\sigma) = H^2(S) \oplus \{L^2(S, d\sigma) \ominus H^2(S)\}.$$

With respect to this decomposition, we have the matrix representation

$$M_{\bar{h}} = \begin{bmatrix} T & 0 \\ H & \tilde{T} \end{bmatrix}.$$

Using (5.1), straightforward matrix multiplication gives us

$$Q = \begin{bmatrix} T^* & H^* \\ 0 & \tilde{T}^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} T^* & H^* \\ 0 & \tilde{T}^* \end{bmatrix} = \begin{bmatrix} H^*H & 0 \\ 0 & -HH^* \end{bmatrix}.$$

Thus

$$R = \begin{bmatrix} T & 0 \\ H & \tilde{T} \end{bmatrix} \begin{bmatrix} H^*H & 0 \\ 0 & -HH^* \end{bmatrix} - \begin{bmatrix} H^*H & 0 \\ 0 & -HH^* \end{bmatrix} \begin{bmatrix} T & 0 \\ H & \tilde{T} \end{bmatrix} = \begin{bmatrix} [T, H^*H] & 0 \\ 2HH^*H & [HH^*, \tilde{T}] \end{bmatrix}$$

and

$$R^*R = \begin{bmatrix} [T, H^*H]^*[T, H^*H] + 4(H^*H)^3 & 2H^*HH^*[HH^*, \tilde{T}] \\ 2[HH^*, \tilde{T}]^*HH^*H & [HH^*, \tilde{T}]^*[HH^*, \tilde{T}] \end{bmatrix}. \tag{5.4}$$

Up to this point, the proof is a repeat of the argument in [8]. Next we bring the commutators from Section 4 into action.

By Lemma 4.1, we have $[T, H^*H], [HH^*, \tilde{T}] \in \mathcal{C}_p$ for every $p > 4/3$. Also, we know that $H \in \mathcal{C}_s$ for every $s > 4$. Combining these Schatten-class memberships with Lemma 2.2, we see that the four entries in matrix (5.4) are all in the trace class. Hence

$$\mathrm{tr}(R^*R) = \mathrm{tr}([T, H^*H]^*[T, H^*H] + [HH^*, \tilde{T}]^*[HH^*, \tilde{T}]) + 4\|H\|_6^6. \tag{5.5}$$

We need to figure out the first term on the right-hand side. We have

$$\begin{aligned} [T, H^*H]^*[T, H^*H] &= [H^*H, T^*][T, H^*H] \\ &= H^*HT^*TH^*H + T^*H^*HH^*HT - H^*HT^*H^*HT - T^*H^*HTH^*H \\ &= TH^*HH^*HT^* + T^*H^*HH^*HT - H^*HT^*H^*HT - T^*H^*HTH^*H + K_1 \\ &= H^*HH^*HT^*T + H^*HH^*HTT^* - H^*HT^*H^*HT - T^*H^*HTH^*H \qquad (5.6) \\ &\quad + K_1 + K_2 + K_3, \end{aligned}$$

where

$$K_1 = [H^*HT^*, TH^*H], \quad K_2 = [T, H^*HH^*HT^*] \quad \text{and} \quad K_3 = [T^*, H^*HH^*HT].$$

By Lemmas 4.3 and 4.2, K_1, K_2, K_3 are all in the trace class with zero trace. It follows from the analyticity of h that $TT^* - T^*T = H^*H$. Continuing with (5.6), we have

$$\begin{aligned}[T, H^*H]^*[T, H^*H] = (H^*H)^3 &+ 2H^*HH^*HT^*T - H^*HT^*H^*HT - T^*H^*HTH^*H \\ &+ K_1 + K_2 + K_3. \end{aligned} \tag{5.7}$$

Recall that $\tilde{T}^*H = HT^*$ and $H^*\tilde{T} = TH^*$ by (5.1). Therefore

$$\begin{aligned}[HH^*, \tilde{T}]^*[HH^*, \tilde{T}] &= [\tilde{T}^*, HH^*][HH^*, \tilde{T}] \\ &= \tilde{T}^*HH^*HH^*\tilde{T} + HH^*\tilde{T}^*\tilde{T}HH^* - \tilde{T}^*HH^*\tilde{T}HH^* - HH^*\tilde{T}^*HH^*\tilde{T} \\ &= \tilde{T}^*HH^*HH^*\tilde{T} + \tilde{T}HH^*HH^*\tilde{T}^* - HT^*TH^*HH^* - HH^*HT^*TH^* + L_1 \\ &= HH^*HH^*\tilde{T}\tilde{T}^* + HH^*HH^*\tilde{T}^*\tilde{T} - HT^*TH^*HH^* - HH^*HT^*TH^* \\ &\quad + L_1 + L_2 + L_3, \end{aligned}$$

where

$$L_1 = [HH^*\tilde{T}^*, \tilde{T}HH^*], \quad L_2 = [\tilde{T}^*, HH^*HH^*\tilde{T}] \quad \text{and} \quad L_3 = [\tilde{T}, HH^*HH^*\tilde{T}^*].$$

Again, by Lemmas 4.3 and 4.2, L_1, L_2, L_3 are all in the trace class with zero trace. We now use the relation $\tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^* = HH^*$. Continuing with the above, we have

$$\begin{aligned}[HH^*, \tilde{T}]^*[HH^*, \tilde{T}] &= (HH^*)^3 + 2HH^*HH^*\tilde{T}\tilde{T}^* - HT^*TH^*HH^* - HH^*HT^*TH^* \\ &\quad + L_1 + L_2 + L_3 \\ &= (HH^*)^3 + 2\tilde{T}^*HH^*HH^*\tilde{T} - HT^*TH^*HH^* - HH^*HT^*TH^* \\ &\quad + L_1 + L_2 + L_3 - 2L_2 \\ &= (HH^*)^3 + 2HT^*H^*HTH^* - HT^*TH^*HH^* - HH^*HT^*TH^* \\ &\quad + L_1 - L_2 + L_3. \end{aligned} \tag{5.8}$$

Obviously,

$$HT^*H^*HTH^* - HT^*TH^*HH^* = HT^*[H^*H, T]H^* = H^*HT^*[H^*H, T] + E, \tag{5.9}$$

where

$$E = [HT^*[H^*H, T], H^*].$$

By Lemma 4.1, we have $[H^*H, T] \in \mathcal{C}_p$ for every $p > 4/3$. Since $H \in \mathcal{C}_s$ for every $s > 4$, it follows from Lemmas 2.2 and 2.3 that E is in the trace class with zero trace. Similarly,

$$HT^*H^*HTH^* - HH^*HT^*TH^* = H[T^*, H^*H]TH^* = TH^*H[T^*, H^*H] + F, \tag{5.10}$$

where

$$F = [H[T^*, H^*H], TH^*].$$

Again, it follows from Lemmas 4.1, 2.2 and 2.3 that F is in the trace class with zero trace. Substituting (5.9) and (5.10) in (5.8), we obtain

$$\begin{aligned}[HH^*, \tilde{T}]^*[HH^*, \tilde{T}] = (HH^*)^3 + H^*HT^*[H^*H, T] + TH^*H[T^*, H^*H] \\ + L_1 - L_2 + L_3 + E + F.\end{aligned}$$

Combining this with (5.7), we see that

$$[T, H^*H]^*[T, H^*H] + [HH^*, \tilde{T}]^*[HH^*, \tilde{T}] = (H^*H)^3 + (HH^*)^3 + X + Y, \tag{5.11}$$

where

$$\begin{aligned}X = 2H^*HH^*HT^*T - H^*HT^*H^*HT - T^*H^*HTH^*H \\ + H^*HT^*[H^*H, T] + TH^*H[T^*, H^*H]\end{aligned}$$

and

$$Y = K_1 + K_2 + K_3 + L_1 - L_2 + L_3 + E + F.$$

We showed in the above that Y is in the trace class with zero trace. On the other hand,

$$\begin{aligned}X &= 2H^*HH^*HT^*T - H^*HT^*H^*HT - T^*H^*HTH^*H \\ &\quad + H^*HT^*H^*HT - H^*HT^*TH^*H + TH^*HT^*H^*H - TH^*HH^*HT^* \\ &= 2[H^*HH^*HT^*, T] + TH^*HH^*HT^* + [TH^*H, T^*H^*H] - H^*HT^*TH^*H \\ &= 2[H^*HH^*HT^*, T] + [TH^*H, H^*HT^*] + [TH^*H, T^*H^*H].\end{aligned}$$

It now follows from Lemmas 4.2 and 4.3 that X is in the trace class with zero trace. Substituting (5.11) in (5.5) and using the fact that $\text{tr}(X + Y) = 0$, we obtain

$$\text{tr}(R^*R) = \text{tr}((H^*H)^3 + (HH^*)^3) + 4\|H\|_6^6 = 6\|H\|_6^6.$$

Combining this with (5.2), we have proved (1.2) for $h \in H^2(S) \cap \text{Lip}(S)$.

(2) Now suppose that we have an $h \in H^2(S)$ which has the property that $H_{\bar{h}} \in \mathcal{C}_6$, i.e., $\|H_{\bar{h}}\|_6^6 < \infty$. We will show that (1.2) holds for such an h by using (1) and a standard smoothing technique.

First of all, for each $0 \le r < 1$, we defined the function

$$h_r(z) = h(rz).$$

For each τ in the unit circle $\mathbf{T}$, we define the unitary operator U_τ on $L^2(S, d\sigma)$ by the formula

$$(U_\tau f)(z) = f(\tau z), \quad z \in S,$$

$f \in L^2(S, d\sigma)$. By a straightforward calculation, for every $0 < r < 1$ we have

$$\int_{\mathbf{T}} \frac{1-r^2}{|1-r\tau|^2} U_\tau H_{\bar{h}} U_\tau^* dm(\tau) = H_{\bar{h}_r}.$$

From this identity it is elementary to deduce that $\|H_{\bar{h}_r}\|_6 \le \|H_{\bar{h}}\|_6$ and that

$$\lim_{r \uparrow 1} \|H_{\bar{h}_r} - H_{\bar{h}}\|_6 = 0. \tag{5.12}$$

By (1), we have

$$\|H_{\bar{h}_r}\|_6^6 = \frac{1}{6} \iint \frac{|h_r(z) - h_r(w)|^6}{|1 - \langle z, w \rangle|^4} d\sigma(z) d\sigma(w) \tag{5.13}$$

for every $0 < r < 1$. For $r, r' \in (0,1)$, since $H_{\bar{h}_r} - H_{\bar{h}_{r'}} = H_{\bar{h}_r - \bar{h}_{r'}}$, (1) also gives us

$$\|H_{\bar{h}_r} - H_{\bar{h}_{r'}}\|_6^6 = \frac{1}{6} \iint \frac{|\{h_r(z) - h_r(w)\} - \{h_{r'}(z) - h_{r'}(w)\}|^6}{|1 - \langle z, w \rangle|^4} d\sigma(z) d\sigma(w).$$

Taking the limit as $r' \to 1$ and applying Fatou's lemma, we obtain

$$\frac{1}{6} \iint \frac{|\{h_r(z) - h_r(w)\} - \{h(z) - h(w)\}|^6}{|1 - \langle z, w \rangle|^4} d\sigma(z) d\sigma(w) \le \|H_{\bar{h}_r} - H_{\bar{h}}\|_6^6. \tag{5.14}$$

From (5.13), (5.12) and (5.14) it is elementary to deduce

$$\|H_{\bar{h}}\|_6^6 = \frac{1}{6} \iint \frac{|h(z) - h(w)|^6}{|1 - \langle z, w \rangle|^4} d\sigma(z) d\sigma(w).$$

This proves (1.2) for any $h \in H^2(S)$ with the property $H_{\bar{h}} \in \mathcal{C}_6$.

(3) Now consider any $h \in H^2(S)$ for which the right-hand side of (1.2) is finite. But it is well known that the finiteness of the right-hand side of (1.2) implies $H_{\bar{h}} \in \mathcal{C}_6$. See, e.g., [3, Proposition 7.1]. Thus by (2), (1.2) holds in this case.

Summarizing (2) and (3) above, we have shown that for any $h \in H^2(S)$, if either side of (1.2) is finite, then (1.2) holds as an identity. Thus the only remaining possibility is that for a given $h \in H^2(S)$, neither side of (1.2) is finite, in which case (1.2) is an identity by convention. This completes the proof of Theorem 1.2. □

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